TOWARDS A THEORY OF GLOBAL SOLVABILITY ON [0, ∞) OF INITIAL-BOUNDARY VALUE PROBLEMS FOR THE EQUATIONS OF MOTION OF OLDROYD AND KELVIN–VOIGHT FLUIDS

A. P. Oskolkov and R. Shadiev UDC 517.9

Classical global solvability on [0, ∞) is proved for initial-boundary value problems (30), (32), (33), and (31), (32), (33) which describe two-dimensional motion of Oldroyd fluids and three-dimensional motion of Kelvin–Voight fluids of orders L = 2, 3, ...

1. As is well known, the governing equation for a linear viscoelastic fluid, which connects the stress deviator tensor σ and the strain rate tensor D has the form [1-4]:

$$\Big(1+\sum_{\ell=1}^{L}\lambda_\ell\frac{\partial^\ell}{\partial t^\ell}\Big)\sigma=2\nu\Big(1+\sum_{m=1}^{M}\varkappa_m\nu^{-1}\frac{\partial^m}{\partial t^m}\Big)D,\quad \nu,\lambda_L,\varkappa_M>0, \tag{1}$$

where L and M are related by one of three conditions: either M = L − 1, L = 1, 2, ... (Maxwell fluid of order L), M = L = 1, 2, ... (Oldroyd fluid of order L), or M = L + 1, L = 0, 1, 2, ... (Kelvin–Voight fluid of order L). A viscoelastic fluid is also called a viscous fluid.

By integrating the governing equation (1) over t with the natural condition $\sigma(t) \equiv 0$ for t < 0, we obtain:

for M = L − 1, L = 1, 2, ... (Maxwell fluid)

$$\sigma(t)=\int_0^t K_1(t-\tau)D(\tau)d\tau \tag{2}$$

for M = L = 1, 2, ... (Oldroyd fluid)

$$\sigma(t)=\mu_0^{(2)}D(t)+\int_0^t K_2(t-\tau)D(\tau)d\tau,\quad \mu_0^{(2)}=\varkappa_L\lambda_L^{-1},\quad \nu-\mu_0^{(2)}>0 \tag{3}$$

for m = L + 1, L = 1, 2, ... (Kelvin–Voight fluid or medium)

$$\sigma(t)=\mu\frac{\partial D}{\partial t}+\mu_0^{(3)}D(t)+\int_0^t K_3(t-\tau)D(\tau)d\tau,\quad \mu=\varkappa_{L+1}\cdot\lambda_L^{-1},\ \mu_0^{(3)}>0. \tag{4}$$

In (2)-(4), the kernel $K_i(t)$, i = 1, 2, 3 satisfies the differential equation

$$\sum_{\ell=0}^{L}\lambda_\ell\frac{\partial^\ell K_i}{\partial t^\ell}=0,\quad t>0 \tag{5}$$

and the Cauchy initial conditions

Translated from Zapiski Nauchnykh Seminarov Leningradskogo Otdeleniya Matematicheskogo Instituta im. V. A. Steklova, Akademii Nauk SSSR, Vol. 180, pp. 121-141, 1990.

 1072-1964/94/6802-0240$12.50

$$\sum_{l=0}^{n}\lambda_{L-n+l}\frac{\partial^l K_i}{\partial t^l}(0)=\varkappa_{L-n-1}-\mu\lambda_{L-n-2}\delta_{3i}-\mu_0^{(i)}\lambda_{L-n-1}(1-\delta_{1i}),\quad n=0,1,\ldots,L-1, \tag{6}$$

in which δ_{ij} is the Kronecker delta symbol and for simplicity, $\lambda_{-1}\equiv 0$, $\lambda_0 \equiv 1$, $\varkappa_0 \equiv \nu$.

Substituting (2)-(4) into the equations of motion of a continuous, incompressible medium in the Cauchy form

$$\frac{dv}{dt}+\operatorname{grad}p=f+\operatorname{div}\sigma,\quad \operatorname{div}v=0,\ \frac{dv}{dt}=\frac{\partial v}{\partial t}+v_k\frac{\partial v}{\partial x_k}, \tag{7}$$

we obtain the integrodifferential equations of motion for linear viscoelastic fluids:

for M = L − 1, L = 1, 2, ... (Maxwell fluid of order L)

$$\frac{dv}{dt}-\int_0^t K_1(t-\tau)\Delta v\,d\tau+\operatorname{grad}p=f,\ \operatorname{div}v=0; \tag{8}$$

for M = L = 1, 2, ... (Oldroyd fluid of order L)

$$\frac{dv}{dt}-\mu_0^{(2)}\Delta v-\int_0^t K_2(t-\tau)\Delta v\,d\tau+\operatorname{grad}p=f,\ \operatorname{div}v=0; \tag{9}$$

for M = L + 1, L = 1, 2, ... (Kelvin–Voight fluid of order L)

$$\frac{dv}{dt}-\mu\frac{\partial\Delta v}{\partial t}-\mu_0^{(3)}\Delta v-\int_0^t K_3(t-\tau)\Delta v\,d\tau+\operatorname{grad}p=f,\ \operatorname{div}v=0. \tag{10}$$

System (8)-(10) is solved in $Q_T \equiv \Omega\times(0, T)$, where Ω is a bounded domain from E^2 or E^3, $0 < T \le \infty$, with initial-boundary conditions:

$$v\big|_{t=0}=v_0(x),\ x\in\Omega;\quad v\big|_{\partial\Omega}=0,\ t\geq 0. \tag{10*}$$

2. In the works of A. P. Oskolkov and his students N. A. Karazeevii and A. A. Kotsiolis (see [2-4]) it was shown that the system of integrodifferential equations (8), (9), and (10) for the motion of Maxwell, Oldroyd, and Kelvin–Voight fluids of order L = 1, 2, ... can be reduced by several methods to a system of differential equations. Specifically:

2.1. We assume that the characteristic equation $Q(p) \equiv 1 + \sum_{l=1}^{L}\lambda_l p^l = 0$ has the simple roots $\{\alpha_m\}$, m = 1, ..., L: $Q(\alpha_m) = 0$, $Q'(\alpha_m) \neq 0$, and that these roots are real and negative: $\alpha_m < 0$, m = 1, ..., L. Then kernels $K_i(t)$ have the form:

$$K_i(t)=\sum_{l=1}^{L}\beta_l^{(i)}e^{\alpha_l t},\quad i=1,2,3, \tag{11}$$

where the coefficients $\{\beta_l^{(i)}\}$ are determined from the Cauchy initial conditions (6) and are written out explicitly in [4]. We will assume that these coefficients are positive: $\beta_l^{(i)} > 0$, i = 1, 2, 3; m = 1, ..., L.

We substitute (11) into (8)-(10) and assume that

$$u_m(t)\equiv\int_0^t e^{\alpha_m(t-\tau)}v(\tau)\,d\tau,\quad m=1,\ldots,L. \tag{12}$$

It is easy to see that $\{u_m(t)\}$ satisfy the differential equations

$$\frac{\partial u_m}{\partial t}-v-\alpha_m u_m=0,\quad m=1,\ldots,L. \tag{13}$$

Then we obtain the equations of motion for Maxwell, Oldroyd, and Kelvin–Voight fluids of order L = 1, 2, ... in the form of the following system of differential equations:

$$\left.\begin{array}{l}\frac{dv}{dt}-\sum_{m=1}^{L}\beta_m^{(1)}\Delta u_m+\operatorname{grad}p=f,\ \operatorname{div}v=0\end{array}\right\}\frac{d}{dt}=\frac{\partial}{\partial t}+v_K\frac{\partial}{\partial x_K}\tag{14}$$

plus (13);

$$\left.\frac{dv}{dt}-\mu_0^{(2)}\Delta v-\sum_{m=1}^{L}\beta_m^{(2)}\Delta u_m+\operatorname{grad}p=f,\ \operatorname{div}v=0\right\}\tag{15}$$

plus (13);

$$\left.\frac{dv}{dt}-\mu\frac{\partial\Delta v}{\partial t}-\mu_0^{(3)}\Delta v-\sum_{m=1}^{L}\beta_m^{(3)}\Delta u_m+\operatorname{grad}p=f,\operatorname{div}v=0\right\}\tag{16}$$

plus (13).

Systems (14)-(160 are solved in $Q_\infty \equiv \Omega(0, \infty)$, $\Omega \in E^n$, $n = 2, 3$, for initial-boundary conditions

$$v\big|_{t=0}=v_0(x),\ \ u_m\big|_{t=0}=0,\ \ x\in\Omega;\ \ v\big|_{\partial\Omega}=u_m\big|_{\partial\Omega}=0,\ t\geq 0.\tag{17}$$

2.2. We now assume that the characteristic equation $Q(p) = 0$ has nonsimple roots $\{\alpha_m\}$ of multiplicity n_m: $m = 1, \ldots, N < L$ $\sum_m n_m = L_m$. As before, we will assume that the roots $\{\alpha_m\}$ are real and negative: $\alpha_m < 0$, $m = 1, \ldots, N$.

Then the kernels $K_i(t)$ have the form:

$$K_i(t)\equiv\sum_{m=1}^{N}\sum_{s=0}^{n_m-1}\beta_{ms}^{(i)}t^s e^{\alpha_m t},\tag{18}$$

the coefficients $\{\beta_{ms}^{(i)}\}$ are determined as before from the initial Cauchy conditions (6) and are assumed to be positive: $\beta_{ms}^{(i)} > 0$.

We substitute kernel (18) into the integrodifferential equations (8)-(10) and assume

$$u_{ms}(t)\equiv\int_0^t(t-\tau)^s e^{\alpha_m(t-\tau)}v(\tau)d\tau,\ \ s=0,1,\ldots,n_m-1;\ m=1,\ldots,N.\tag{19}$$

It is easy to see that the functions $\{u_{ms}\}$ satisfy the following system of differential equations

$$\frac{\partial u_{m0}}{\partial t}-v-\alpha_m u_{m0}=0,\ \ \frac{\partial u_{ms}}{\partial t}-su_{m,s-1}-\alpha_m u_{ms}=0,\ s=1,\ldots,n_m-1.\tag{20}$$

Then we obtain the equations of motion for Maxwell, Oldroyd, and Kelvin–Voight fluids of order $L = 1, 2, \ldots$, respectively, in the form of the following systems for differential equations

$$\left.\frac{dv}{dt}-\sum_{m=1}^{N}\sum_{s=0}^{n_m-1}\beta_{ms}^{(1)}\Delta u_{ms}+\operatorname{grad}p=f,\ \operatorname{div}v=0,\right\}\tag{21}$$

plus (20);

$$\left.\frac{dv}{dt}-\mu_0^{(2)}\Delta v-\sum_{m=1}^{N}\sum_{s=0}^{n_m-1}\beta_{ms}^{(2)}\Delta u_{ms}+\operatorname{grad}p=f,\ \operatorname{div}v=0\right\}\tag{22}$$

plus (20);

$$\left.\frac{dv}{dt}-\mu\frac{\partial\Delta v}{\partial t}-\mu_0^{(3)}\Delta v-\sum_{m=1}^{N}\sum_{s=0}^{n_m-1}\beta_{ms}^{(3)}\Delta u_{ms}+\operatorname{grad}p=f,\operatorname{div}v=0\right\}\tag{23}$$

plus (20).

Systems (21)-(23) are solved with the initial-boundary conditions:

$$v\big|_{t=0}=v_0(x),\quad u_{ms}\big|_{t=0}=0,\quad x\in\Omega;\quad v\big|_{\partial\Omega}=u_{ms}\big|_{\partial\Omega}=0,\quad t\geqslant 0. \tag{24}$$

The network of equations (20), which relate v and $\{u_{ms}\}$, s = 0, 1, .., $n_m - 1$, is easily transformed to the set of equations

$$v=((s-1)!)^{-1}\sum_{K=0}^{S+1}(-1)^K C_{S+1}^K \alpha_m^K \frac{\partial^{S+1-K}u_{ms}}{\partial t^{S+1-K}},\quad s=0,1,\dots,n_m-1, \tag{25}$$

only v and one of u_{ms}, s = 0, 1, ..., $n_m - 1$, enters into each of these equations. In (25), $C_{s+1}{}^k$ are the binomial coefficients, and for consistency, $(-1)! \equiv 0! \equiv 1$. It follows from (24) and (20) that the S-th equation in (25) is solved for initial conditions

$$\frac{\partial^p u_{ms}}{\partial t^p}\bigg|_{t=0}=0,\quad x\in\Omega,\quad p=0,1,\dots,s;\quad s=0,1,\dots,n_m-1. \tag{26}$$

2.3. Another variant for reducing the system of integrodifferential equations (8)-(10) to a system of coupled differential equations was given in the works of A. P. Oskolkov [2, 3]. Specifically, we introduce into (8), (9), and (10) new unknown functions $w_i(x, t)$, i = 1, 2, 3, with the help of

$$\sum_{S=0}^{L-1}\alpha_s^{(i)}\frac{\partial^s w_i}{\partial t^s}=\int_0^t K_i(t-\tau)v(x,\tau)d\tau,\quad \alpha_0^{(i)}=1;\quad \sum_{S=0}^{L}\gamma_s^{(i)}\frac{\partial^s w_i}{\partial t^s}=v. \tag{27}$$

By differentiating by t the first of equations (27) l times, l = 0, 1, ..., L, multiplying the resulting equation by λ_l, summing over l = 0, 1, ..., L, and using (5) for $K_i(t)$ and the second equation from (27), we obtain the identity

$$\sum_{l=1}^{L}\lambda_l\sum_{p=0}^{l-1}\frac{\partial^p K_i}{\partial t^p}(0)\sum_{S=0}^{L}\gamma_s^{(i)}\frac{\partial^{s+l-p-1}w_i}{\partial t^{s+l-p-1}}=\sum_{l=0}^{L}\lambda_l\sum_{S=0}^{L-1}\alpha_s^{(i)}\frac{\partial^{s+l}w_i}{\partial t^{s+l}}.$$

From this identity, by equating the null coefficients for derivatives

$$\frac{\partial^\tau w_i}{\partial t^\tau},\quad \tau=0,1,\dots,2L-1,$$

we obtain the following algebraic system for the coefficient $\{\alpha_s^{(i)}\}$, s = 1, ..., L − 1, and $\{\gamma_s^{(i)}\}$, s = 0, 1, ..., L;

$$\alpha_0^{(i)}=1,\quad \sum_{i+j+s-1=l}\lambda_i\frac{\partial^j K_i}{\partial t^j}(0)\gamma_s^{(i)}=\sum_{i+j=l}\lambda_l\alpha_j^{(i)},\quad l=0,1,\dots,2L-1. \tag{28}$$

Likewise, the motion of Maxwell, Oldroyd, and Kelvin–Voight fluids of order L = 1, 2, ... can be described respectively by the following systems of coupled differential equations:

$$\frac{\partial v}{\partial t}+v_K\frac{\partial v}{\partial x_K}-\Delta\sum_{S=0}^{L-1}\alpha_s^{(1)}\frac{\partial^s w_1}{\partial t^s}+\operatorname{grad}p=f_1,$$
$$v=\sum_{S=0}^{L}\gamma_s^{(1)}\frac{\partial^s w_1}{\partial t^s},\quad \operatorname{div}v=\operatorname{div}w_1=0; \tag{29}$$

$$\frac{\partial v}{\partial t}+v_K\frac{\partial v}{\partial x_K}-\mu_0^{(2)}\Delta v-\Delta\sum_{S=0}^{L-1}\alpha_s^{(2)}\frac{\partial^s w_2}{\partial t^s}+\operatorname{grad}p=f_2$$
$$v=\sum_{S=0}^{L}\gamma_s^{(2)}\frac{\partial^s w_2}{\partial t^s},\quad \operatorname{div}v=\operatorname{div}w_2=0 \tag{30}$$

$$\frac{\partial v}{\partial t}+v_K\frac{\partial v}{\partial x_K}-\mu_1\frac{\partial\Delta v}{\partial t}-\mu_0^{(3)}\Delta v-\Delta\sum_{S=0}^{L-1}\alpha_s^{(3)}\frac{\partial^s w_3}{\partial t^s}+\operatorname{grad}p=f_3$$
$$v=\sum_{S=0}^{L}\gamma_s^{(3)}\frac{\partial^s w_3}{\partial t^s},\quad \operatorname{div}v=\operatorname{div}w_3=0 \tag{31}$$

in which the coefficients $\{\alpha_s^{(i)}\}$, $s = 1, \ldots, L-1$ and $\{\gamma_s^{(i)}\}$, $s = 0, 1, \ldots, L$, $i = 1, 2, 3$, are determined from algebraic system (28).

Systems (29)-(31) are solved with the initial-boundary conditions

$$v\big|_{t=0}=v_0(x);\ \frac{\partial^s w_i}{\partial t^s}\Big|_{t=0}=0,\ s=0,1,\ldots,L-1,\ x\in\Omega;\ v\big|_{\partial\Omega}=w_i\big|_{\partial\Omega}=0,\ t\geqslant 0. \tag{32}$$

3. The theory of classical solvability of the initial-boundary value problems (9), (10*), and (10), (10*) is constructed in the works of A. P. Oskolkov for the most general integrodifferential equations of motion of Oldroyd fluids of order $L = 1, 2, \ldots$, and for Kelvin–Voight fluids of order $L = 0, 1, 2, \ldots$ on the finite time interval $0 < t \leq Y < \infty$. (For a complete bibliography of Oskolkov's works on this problem, see [2-4].)

In the works of N. A. Karazeevii, A. A. Kotsiolis, and A. P. Oskolkov, the theory of global classical solvability on $[0, \infty)$ is constructed for the initial-boundary value problems (15), (17), and (16), (17) for two-dimensional flow of Oldroyd fluids of order $L = 1, 2, \ldots$ and three-dimensional flow of Kelvin–Voight fluids of order $L = 0, 1, 2, \ldots$ (see [4] for a detailed bibliography of these works). On this basis, they construct the theory of attractors and the dynamic systems which give rise to two-dimensional initial-boundary value problem (15), (17), and three-dimensional flow of Kelvin–Voight fluids of order $L = 2, 3, \ldots$. This is done for the case when the coefficients $\{\alpha_s^{(i)}\}$, $s = 0, 1, \ldots, L-1$ and $\{\gamma_s^{(i)}\}$, $s = 0, 1, \ldots, L$ of systems (30) and (31) satisfy the following conditions:

$$\alpha_0^{(i)}>\alpha_1^{(i)}>\ldots>\alpha_{L-1}^{(i)}>0,\quad 0<\gamma_L^{(i)}<\gamma_{L-1}^{(i)}<\ldots<\gamma_1^{(i)}<\gamma_0^{(i)},\ i=2,3. \tag{33}$$

Conditions (33) are by no means necessary and can be considerably weakened.

THEOREM 1. Let the conditions be satisfied: Ω is a bounded domain from E^2; $\partial\Omega \in C^{2+\alpha}$; $v_0(x) \in C^{2+\alpha}(\bar\Omega) \cap H(\Omega)$; $f(x, t) \in L_\infty(0, T; C^\alpha(\bar\Omega)) \cap L_2(Q_T)$, $0 < \alpha < 1$; $f_t \in L_2(Q_T)$, $0 < T \leq \infty$, and let (33) be satisfied. Then the initial-boundary value problem (30), (32) has a unique solution $(v; 2_2)$:

$$\Big(v, w_2, \frac{\partial w_2}{\partial t}, \ldots, \frac{\partial^L w_2}{\partial t^4}\Big)\in L_\infty(0,T;C^{2+\alpha}(\bar\Omega)\cap H(\Omega))\cap W^1_\infty(0,T;C^\alpha(\bar\Omega)\cap\mathring{J}(\Omega)) \tag{34}$$

and the following estimate holds for this solution:

$$\begin{gathered}\Big\|v; w_2, \frac{\partial w_2}{\partial t}, \ldots, \frac{\partial^L w_2}{\partial t^4}\Big\|_{L_\infty(0,T;C^{2+\alpha}(\bar\Omega))}+\Big\|v_t; \frac{\partial^{L+1} w_2}{\partial t^{L+1}}\Big\|_{L_\infty(0,T;C^\alpha(\bar\Omega))}\leqslant\\ \leqslant C_1\big(\|v_0\|^{(2+\alpha)}_{\bar\Omega}; \|f\|_{L_\infty(0,T;C^\alpha(\bar\Omega))}; \|f,f_t\|_{2,Q_T}; \mu^{-1}; \min_s(\alpha_s^{(2)}-\alpha_{s+1}^{(2)}); \min_s(\gamma_{s+1}^{(2)}-\gamma_s^{(2)})\big),\end{gathered} \tag{35}$$

in which the constant C_1 does not depend on $T \leq \infty$.

THEOREM 2. Let the conditions be satisfied: Ω is a bounded domain from E^3; $\partial\Omega \in C^{2+\alpha}$; $v_0(x) \in C^{2+\alpha}(\bar\Omega) \cap H(\Omega)$; $f(x, t) \in L_\infty(0, T; C^\alpha(\bar\Omega)) \cap L_2(Q_T)$, $0 < \alpha < 1$; $f_t \in L_2(Q_T)$, $0 < T \leq \infty$, and let (33) be satisfied. Then the initial-boundary value problem (31), (32) has a unique solution (v, w_3):

$$\Big(v; w_2, \frac{\partial w_3}{\partial t}, \ldots, \frac{\partial^L w_3}{\partial t^4}\Big)\in L_\infty(0,T;C^{2+\alpha}(\bar\Omega)\cap H(\Omega)) \tag{36}$$

and the following estimate holds for this solution:

$$\Big\|v; w_2, \frac{\partial w_3}{\partial t}, \ldots, \frac{\partial^L w_3}{\partial t^4}\Big\|_{L_\infty(0,T;C^{2+\alpha}(\bar\Omega))}\leqslant C_2\big(\|v_0\|^{(2+\alpha)}_{\bar\Omega}; \|f\|_{L_\infty(0,T;C^\alpha(\bar\Omega))}; \|f,f_t\|_{2,Q_T}; \mu_1^{-1}; \min_s(\alpha_s^{(3)}-\alpha_{s+1}^{(3)}); \min_s(\gamma_{s+1}^{(3)}-\gamma_s^{(3)})\big), \tag{37}$$

where, as before, the constant C_2 does not depend on $T < \infty$.

Analysis of the proof of the theorem of global classical solvability on $[0, \infty)$ of the initial-boundary value problems for the Navier–Stokes equations and the equations of motion for linear viscoelastic fluids (see [2-5]), shows that it is sufficient to prove the following to establish theorems 1 and 2. One must prove the existence on the whole on $[0, \infty)$ of a unique generalized solution in the sense of O. A. Ladyzhenskaya [5] of problem (30), (32) or problem (31), (32), respectively. In other words, the existence of the generalized solution with finite norms $\max_{0\le t\le T}\|v_t;\ \partial w_{2x}/\partial t,\ \ldots,\ \partial^L w_{2x}/\partial t^4\|_{2,\Omega}$ + $\|v_{tx}\partial w_{2x}/\partial t,\ \ldots,\ \partial^L w_{2x}/\partial t^4\|_{2,Q_T}$ for problem (30), (32) and the generalized solution with finite norms $\max_{0\le t\le T}\|v_{tx};\ \partial w_{3x}/\partial t,\ \ldots,\ \partial^L w_{3x}/\partial t^4\|_{2,\Omega}$ + $\|v_{tx};\ \partial w_{3x}/\partial t,\ \ldots,\ \partial^L w_{3x}/\partial t^4\|_{2,Q_T}$ for problem (31), (32) must be proved. Having done this, the embedding theorem of S. L. Sobolev and the theorem of classical solvability of the corresponding linearized problems with constant coefficients whose solution norms do not depend on $T \le \infty$ can be used to prove theorems 1 and 2 throughout the total volume.

In turn, to prove the existence theorems on the whole on $[0, \infty)$ of the unique, generalized solution in the sense of O. A. Ladyzhenskaya for problems (30), (32) or (31), (32), it is known to be sufficient to obtain an a priori estimate on the whole for the norms which enter into the definition of the generalized solution in the sense of Ladyzhenskaya. We now turn to the problem of obtaining these a priori estimates.

We multiply the first of the equations of (30) by v, integrate over Ω and over t from 0 to $t \le T$ and make use of the second equation in (30). By integrating by parts over x, we obtain:

$$\begin{gathered}\frac{1}{2}\|v\|^2_{2,\Omega_t}+\mu\|v_x\|^2_{2,Q_t}+\sum_{s=0}^{L-1}\alpha_s^{(2)}\gamma_s^{(2)}\left\|\frac{\partial^s w_{2x}}{\partial t^s}\right\|^2_{2,Q_T}+\frac{1}{2}\sum_{s=0}^{L-2}(\alpha_s^{(2)}\gamma_s^{(2)}+\alpha_{s+1}^{(2)}\gamma_s^{(2)})\left\|\frac{\partial^s w_{2x}}{\partial t^s}\right\|^2_{2,\Omega_t}+\\ +\frac{1}{2}\alpha_{L-1}^{(2)}\gamma_L^{(2)}\left\|\frac{\partial^{L-1}w_2}{\partial t^{L-1}}\right\|^2_{2,\Omega_t}+\sum_{s=0}^{L-1}\sum_{m=0}^{L}\alpha_s^{(2)}\gamma_m^{(2)}\iint_{Q_T}\frac{\partial^s w_{2x}}{\partial t^s}\frac{\partial^m w_{2x}}{\partial t^m}\,dx\,dt=\frac{1}{2}\|v_0\|^2_{2,\Omega}+(f,v)_{2,Q_T},\quad 0<t\le T \\ s\ne m,\quad s\ne m+1,\quad m\ne s+1.\end{gathered}\tag{38}$$

We integrate the last group of integrals on the left by parts over time t, and estimate the resultant integrals by using the Hölder and Cauchy inequalities. Then by using (33), and applying the Hölder, Friedrich and Cauchy inequalities for the estimate of $(f, v)_{2,Q_T}$, and maximizing over $t \in [0, T]$, we obtain the estimate:

$$\begin{gathered}\max_{0\le t\le T}\Big(\|v\|^2_{2,\Omega_t}+C_3(\min_s(\alpha_s^{(2)}-\alpha_{s+1}^{(2)}),\min_s(\gamma_{s+1}^{(2)}-\gamma_s^{(2)}))\sum_{s=0}^{L-1}\left\|\frac{\partial^s w_{2x}}{\partial t^s}\right\|_{2,\Omega_t}\Big)+\\ +C_4(\min_s(\alpha_s^{(2)}-\alpha_{s+1}^{(2)}),\min_s(\gamma_{s+1}^{(2)}-\gamma_s^{(2)}))\sum_{s=0}^{L-1}\left\|\frac{\partial^s w_{2x}}{\partial t^s}\right\|^2_{2,Q_T}\le\|v_0\|^2_{2,\Omega}+(\mu C_\Omega)^{-1}\|f\|^2_{2,Q_T}=A_1.\end{gathered}\tag{39}$$

Carrying out the same procedure for (31), (32), we obtain the estimate:

$$\begin{gathered}\max_{0\le t\le T}\Big(\|v\|^2_{2,\Omega_t}+\mu\|v_x\|^2_{2,\Omega_t}+C_3\sum_{s=0}^{L-1}\left\|\frac{\partial^s w_{3x}}{\partial t^s}\right\|^2_{2,Q_T}\Big)+C_4\sum_{s=0}^{L-1}\left\|\frac{\partial^s w_{3x}}{\partial t^s}\right\|^2_{2,Q_T}+\mu_1\|v_x\|^2_{2,Q_T}\le\\ \le\|v_0\|^2_{2,\Omega}+\mu\|v_{0x}\|^2_{2,\Omega}+(\mu,C_\Omega)^{-1}\|f\|^2_{2,Q_T}=A_2.\end{gathered}\tag{40}$$

We further differentiate the equations in (31) by t, multiply the first resultant equation by $\partial v/\partial t \equiv \sum_{s=0}^{L}\gamma_s^{(3)}\partial^{s+1}w_3/\partial t^{s+1}$ and integrate over Ω and t from 0 to $t \le T$. By integrating by parts over x, we obtain:

$$\begin{gathered}\frac{1}{2}\|v_t\|^2_{2,\Omega_t}+\frac{\mu}{2}\|v_{xt}\|^2_{2,\Omega_t}+\mu_1\|v_{xt}\|^2_{2,Q_T}+\sum_{s=0}^{L-1}\alpha_s^{(3)}\gamma_s^{(3)}\left\|\frac{\partial^{s+1}w_{3x}}{\partial t^{s+1}}\right\|^2_{2,Q_T}+\\ +\frac{1}{2}\alpha_{L-1}^{(3)}\gamma_L^{(3)}\left\|\frac{\partial^L w_{3x}}{\partial t^L}\right\|^2_{2,\Omega_t}+\frac{1}{2}\sum_{s=0}^{L-2}(\alpha_s^{(3)}\gamma_{s+1}^{(3)}+\alpha_{s+1}^{(3)})\left\|\frac{\partial^{s+1}w_{3x}}{\partial t^{s+1}}\right\|^2_{2,\Omega_t}+\\ +\sum_{s=0}^{L-1}\sum_{m=0}^{L}\alpha_s^{(3)}\gamma_m^{(3)}\iint_{Q_t}\frac{\partial^{s+1}w_{3x}}{\partial t^{s+1}}\frac{\partial^{m+1}w_{3x}}{\partial t^{m+1}}\,dx\,dt=-\iint_{Q_t}v_{xt}v_t v_{x_x}\,dx\,dt+\\ +(f_t,v_t)_{2,Q_T}+\frac{1}{2}(\|v_t(x,0)\|^2_{2,\Omega}+\mu\|v_{tx}(x,0)\|^2_{2,\Omega}+\alpha_{L-1}^{(3)}\|v_{0x}\|^2_{2,\Omega}).\end{gathered}\tag{41}$$

We estimate the integral $\mathcal{J}_t = -\int_{\Omega_t} v_{kt} v_t v_{x_k} dx$ with the help of the Hölder and Young inequalities, and the well-known inequality of Ladyzhenskaya [5, Chap. 1]

$$\|v\|^4_{4,\Omega_t} \leq \left(\tfrac{4}{3}\right)^{\frac{3}{2}} \|v\|_{2,\Omega_t} \|v_x\|^3_{2,\Omega_t}, \quad \forall v \in \overset{\circ}{W}{}^1_2(\Omega),\ \Omega_t \in E^3, \tag{42}$$

and the already proved estimate (40), in view of which $\max\limits_{0\leq t\leq T} \|v_x\|_{2,\Omega_t} \leq (A_2\mu^{-1})^{1/2} = B_2$:

$$|\mathcal{J}_t| \leq \sqrt{3}\,\|v_x\|_{2,\Omega_t} \|v_t\|^2_{4,\Omega_t} \leq \sqrt{3}\left(\tfrac{4}{3}\right)^{3/4} \|v_x\|_{2,\Omega_t} \|v_t\|^{1/2}_{2,\Omega_t} \|v_{xt}\|^{3/2}_{2,\Omega_t} \leq$$
$$\leq \frac{\mu_1}{2}\|v_{tx}\|^2_{2,\Omega_t} + c_{\text{num}} B_2^2 \|v_x\|^2_{2,\Omega_t} \mu_1^{-3} \|v_t\|^2_{2,\Omega_t}, \quad 0 \leq t \leq T. \tag{43}$$

Having done this, we now integrate the last group of integrals on the left in (41) by parts over t, and estimate the resultant integrals with the help of the Hölder and Cauchy inequalities. Then by using (33), and applying the Holder, Friedrich and Cauchy inequality for an estimate of $(f_t, v_t)_{2,\Omega_t}$, using (43) and the easily proved estimate for the solution of (31), (32) (see [5, Chap. VI], [6])

$$\|v_t(x,0)\|^2_{2,\Omega} + \mu\|v_{tx}(x,0)\|^2_{2,\Omega} \leq C_5\left(\|f(x,0)\|_{2,\Omega}, \|v_0\|^{(2)}_{2,\Omega}\right) \tag{44}$$

we obtain the inequality:

$$\|v_t\|^2_{2,\Omega_t} + \mu\|v_{xt}\|^2_{2,\Omega_t} + \mu_1\|v_{tx}\|^2_{2,Q_T} + c_3\sum_{s=0}^{L-1}\left\|\frac{\partial^{s+1} w_{3x}}{\partial t^{s+1}}\right\|^2_{2,\Omega_t} +$$
$$+ c_4\sum_{s=0}^{L-1}\left\|\frac{\partial^{s+1} w_{3x}}{\partial t^{s+1}}\right\|^2_{2,Q_t} \leq c_{\text{num}} B_2^2 \mu_1^{-3}\|v_x\|^2_{2,\Omega_t}\|v_t\|^2_{2,\Omega_t} + (\mu_1 c_\Omega)^{-1}\|f_t\|^2_{2,Q_t} +$$
$$+ c_6\left(\|f(x,0)\|_{2,\Omega}, \|v_0\|^{(2)}_{2,\Omega}\right) \leq c_7\|v_x\|^2_{2,\Omega_t}\|v_t\|^2_{2,\Omega_t} + (\mu_1 c_\Omega)^{-1}\|f_t\|^2_{2,Q_T} + c_6 \equiv$$
$$\equiv c_7\|v_x\|^2_{2,\Omega_t}\|v_t\|^2_{2,\Omega_t} + c_8, \quad 0 \leq t \leq T. \tag{45}$$

From (45) and (40) with the help of the "integral" version of Gronwall's lemma [see below (53) and (54)], we obtain the estimate

$$\|v_t\|^2_{2,\Omega_t} \leq c_8 \exp\left\{c_7\int_0^T \|v_x\|^2_{2,\Omega_\tau} d\tau\right\} \leq c_8 \exp(\mu_1^{-1} A_2 c_7), \tag{46}$$

and then from (45) and (46), by maximizing over t, we find the estimate:

$$\max_{0\leq t\leq T}\left(\|v_t\|^2_{2,\Omega_t} + \mu\|v_{tx}\|^2_{2,\Omega_t} + c_3\sum_{s=0}^{L-1}\left\|\frac{\partial^{s+1} w_{3x}}{\partial t^{s+1}}\right\|^2_{2,\Omega_t}\right) + c_4\sum_{s=0}^{L-1}\left\|\frac{\partial^{s+1} w_{3x}}{\partial t^{s+1}}\right\|^2_{2,Q_T} + \mu_1\|v_{tx}\|^2_{2,Q_T} \leq$$
$$\leq c_8\left\{1 + c_7\|v_t\|^2_{2,\Omega_t}\int_0^t \|v_x\|^2_{2,\Omega_\tau} d\tau\right\} \leq c_8\left(1 + \mu_1^{-1} A_2 c_7 \exp(\mu_1^{-1} A_2 c_7)\right) \equiv c_9. \tag{47}$$

Estimate (47) is in fact the required global a priori estimate for the norms with enter into the definition of the generalized solution in the sense of Ladyzhenskaya for initial-boundary value problem (31), (32).

Similarly, by differentiating the equations in (30) by t, multiplying the first of the resultant equations by $\partial v/\partial t$, integrating over Ω and over t from 0 to $t \leq T$, and then integrating by parts over x, we obtain the equality:

$$\frac{1}{2}\|v_t\|^2_{2,\Omega_t} + \mu\|v_{xt}\|^2_{2,Q_t} + \sum_{s=0}^{L-1}\alpha_s^{(2)}\gamma_s^{(2)}\left\|\frac{\partial^{s+1} w_{2x}}{\partial t^{s+1}}\right\|^2_{2,Q_t} + \frac{1}{2}\alpha_{L-1}^{(2)}\gamma_L^{(2)}\left\|\frac{\partial^{L} w_{2x}}{\partial t^{L}}\right\|^2_{2,\Omega_t} +$$
$$+ \frac{1}{2}\sum_{s=0}^{L-2}\left(\alpha_s^{(2)}\gamma_{s+1}^{(2)} + \alpha_{s+1}^{(2)}\gamma_s^{(2)}\right)\left\|\frac{\partial^{s+1} w_{2x}}{\partial t^{s+1}}\right\|^2_{2,\Omega_t} + \sum_{s=0}^{L-1}\sum_{m=0}^{L}\alpha_s^{(2)}\gamma_s^{(2)}\iint_{Q_t}\frac{\partial^{s+1} w_{2x}}{\partial t^{s+1}}\frac{\partial^{m+1} w_{2x}}{\partial t^{m+1}}dxdt =$$

$$-\iint_{Q_t} v_{kt} v_t v_{x_k}\,dx\,dt + (f_t, v_t)_{2,Q_t} + \frac{1}{2}\left(\|v_t(x,0)\|^2_{2,\Omega} + \alpha^{(2)}_{L-1}\|v_{0x}\|^2_{2,\Omega}\right),$$
$$0 < t \leq T. \tag{48}$$

We now estimate the integral $\mathcal{J}_t = -\int_{\Omega_t} v_{kt} v_t v_{x_k} dx$ with the help of the Hölder and Cauchy inequalities and Ladyzhenskaya's inequality [5, Chap. 1]

$$\|v\|^4_{4,\Omega_t} \leq 2\|v\|^2_{2,\Omega_t} \cdot \|v_x\|^2_{2,\Omega_t}, \quad \forall v \in \overset{0}{W}{}^1_2(\Omega), \ \Omega \in E^2; \tag{49}$$

$$|\mathcal{J}_t| \leq \sqrt{2}\,\|v_x\|_{2,\Omega_t}\|v_t\|_{2,\Omega_t}\|v_{xt}\|^2_{2,\Omega_t} \leq \frac{\mu}{2}\|v_{xt}\|^2_{2,\Omega_t} + \mu^{-1}\|v_x\|^2_{2,\Omega_t}\|v_t\|^2_{2,\Omega_t}$$
$$0 < t \leq T. \tag{50}$$

Then, we integrate over time t the last group of integrals on the left in (48) an estimate, as before, the resultant integrals with the help of Hölder and Cauchy inequalities. Then using (33), we once again apply the Hölder, Friedrich and Cauchy inequalities for an estimate of $(f_t, v_t)_{2,\Omega_t}$, use (50) and the easily proved estimate for the solution of (30), (32) (see [5, Chap. VI])

$$\|v_t(x,0)\|^2_{2,\Omega} \leq c_9\left(\|f(x,0)\|_{2,\Omega}, \|v_0\|^{(2)}_{2,\Omega}\right), \tag{51}$$

to obtain the inequality (compare to (45)):

$$\|v_t\|^2_{2,\Omega_t} + \frac{\mu}{2}\|v_{xt}\|^2_{2,\Omega_t} + c_3\sum_{s=0}^{L-1}\left\|\frac{\partial^{s+1} w_{2x}}{\partial t^{s+1}}\right\|^2_{2,Q_t} \leq \mu^{-1}\int_0^t \|v_x\|^2_{2,\Omega_\tau}\|v_t\|^2_{2,\Omega_\tau} d\tau +$$

$$+(\mu c_\Omega)^{-1}\|f_t\|^2_{2,Q_t} + c_{10}\left(\|f(x,0)\|_{2,\Omega}, \|v_0\|^{(2)}_{2,\Omega}\right) \leq \tag{52}$$

$$\leq \mu^{-1}\int_0^t \|v_x\|^2_{2,\Omega_\tau}\|v_\tau\|^2_{2,\Omega_\tau} d\tau + c_{11}\left(\|f_t\|_{2,Q_T}; c_{10}\right), \ 0 < t \leq T.$$

It is easy to see that Gronwall's lemma admits the following equivalent ("integral") formulation: if the function y(t) satisfies on (0, T) the inequality

$$y(t) \leq \int_0^t F(\tau)\,y(\tau)\,d\tau + c_{11}, \quad F_1(t) \in L_1(0,T), \tag{53}$$

then

$$y(t) \leq c_{11}\exp\left(\int_0^t F(\tau)\,d\tau\right), \quad 0 \leq t \leq T. \tag{54}$$

By applying this lemma to $y(t) \equiv \|v_t\|_{2,\Omega_t}{}^2$, $F(t) \equiv \mu^{-1}\|v_x\|_{2,\Omega_t}{}^2$ and using the already proved inequality (39), in view of which $\|v_x\|_{2,Q_T}{}^2 \leq \mu^{-1}A_1$, we obtain from (52) the estimate

$$\|v_t\|^2_{2,\Omega_t} \leq c_{11}\exp(\mu^{-2}A_1), \quad 0 \leq t < T, \tag{55}$$

and then from (52) and (55), by maximizing over t, we have:

$$\max_{0 \leq t \leq T}\left(\|v_t\|^2_{2,\Omega_t} + c_3\sum_{s=0}^{L-1}\left\|\frac{\partial^{s+1} w_{2x}}{\partial t^{s+1}}\right\|^2_{2,\Omega_t}\right) + \mu\|v_{xt}\|^2_{2,Q_T} + c_4\sum_{s=0}^{L-1}\left\|\frac{\partial^{s+1} w_{2x}}{\partial t^{s+1}}\right\|^2_{2,Q_T} \leq c_{11}\left(1 + \mu^{-2}A_1\exp\{\mu^{-2}A_1\}\right). \tag{56}$$

Estimate (56) is the required global a priori estimate for the norms which enter into the definition of the generalized solution in the sense of Ladyzhenskaya of the initial-boundary value problem (30), (32).

4. The solvability of the initial boundary value problems for the hydrodynamic equations on the infinite time interval $[0, \infty)$ is necessary not only for construction of a theory of attractors, that is, the global theory of hydrodynamic stability [7], but also for the construction of a local theory of hydrodynamic stability, that is, the stability theory for steady-state and periodic (and also quasiperiodic) solutions of the hydrodynamic equations. The principle of linearization, or the first Lyapunov method, have been used since the time of A. M. Lyapunov and A. Poincaré for the construction of local stability theory for nonlinear equations. Lyapunov was concerned with operator differential equations of the first order

$$\frac{du}{dt} + A(t)u + K(u) = f(t), \quad u(0) = u_0 \tag{57}$$

with bounded operators A(t) and K(u) in Banach spaces [8], and with the Navier–Stokes equations and those of magneto-hydrodynamics [9, 10], which lead to (57) with unbounded operators A(t) and K(u) acting in a Hilbert space H. In his classical works, based on the principal of linearization of such systems of differential equations, Lyapunov made it clear that the justifying principle for linearization in local stability theory for (57) is based on three facts:

1. local single-valued solvability of the nonlinear problem (57) in the neighborhood of the fundamental solution (being studied for stability) on the entire semiaxis $t \geq 0$;

2. the correctness of nonlinear problem (57), which means that its solution u(t) belongs to the same space $H_1 \subset H$, that it belongs to at $t = 0$, and continuously changes in the norm of H_1 for continuous variation in this norm of the initial conditions u_0 (for example, for the Navier–Stokes equations, $H \equiv J(\Omega)$, $H_1 \equiv H(\Omega)$) [53];

3. the single-valued solvability of the linearized (on the fundamental solution) problem corresponding to (57), on the entire semiaxis $t \geq 0$ in this same space H_1, and the knowledge that the solution operator Z(t, s) of this linearized problem has the properties of an analytical semigroup.

Having in mind the construction in the future of a local stability theory for the equation of motion of a Kelvin–Voight fluid, we will show in this work the global, single-valued solvability on the entire semiaxis $t \geq 0$ of the initial boundary value problems (16), (17), and (31)-(33). This is done for the equations of motion of a Kelvin–Voight fluid in the most natural class of functions from the point of view of the theory of hydrodynamic stability [9, 10]:

$$v, v_t, v_x, v_{xt}, v_{xx}, v_{xxt} \in L_2(Q_\infty) \tag{58}$$

and v(x, t) satisfies either (16) or (31) almost everywhere in Q_∞. Strictly speaking, we will prove the necessity of a priori estimates of the solutions of the initial boundary value problems (16), (17), and (31)-(33), on the basis of which the existence of solution (58) to these problems is easily proved by the Galerkin method [5]. It will also be clear from the estimates we obtain, that the initial boundary value problems (16), (17) and (31)-(33) are correctly posed in $H(\Omega)$ and $W_2^2(\Omega) \cap H(\Omega)$.

4.1. We first examine the initial boundary value problem (16), (17). The following estimate was obtained for its solution in [4]:

$$\begin{gathered}\frac{1}{2}\max_{t\geq 0}\Big(\|v\|^2_{2,\Omega_t}+\mu\|v_x\|^2_{2,\Omega_t}+\sum_{l=1}^{L}\beta_l\|u_{lx}\|^2_{2,\Omega_t}\Big)+\mu_1\|v_x\|^2_{2,Q_\infty}+\sum_{l=1}^{L}|\alpha_l|\beta_l\|u_{lx}\|^2_{2,Q_\infty}\leq\\ \leq\frac{1}{2}\big(\|v_0\|^2_{2,\Omega}+\mu\|v_{0x}\|^2_{2,\Omega}\big)+\Big[\big(\|v_0\|^2_{2,\Omega}+\mu\|v_{0x}\|^2_{2,\Omega}\big)^{1/2}+\|f\|_{2,1;Q_\infty}\Big]\|f\|_{2,1;Q_\infty}\equiv A_1.\end{gathered} \tag{59}$$

Let $\tilde{\Delta}$ be the Stokes operator [5, Chap. 1]. For the solution of (16), (17), for $\forall t \geq 0$, the equation

$$J_1(v;\{u_l\})\equiv\Big(v_t+v_xv_{x_k}-\mu\tilde{\Delta}v_t-\mu_1\tilde{\Delta}v-\sum_{l=1}^{L}\beta_l\tilde{\Delta}u_l,\tilde{\Delta}v\Big)_{2,\Omega_t}=(f,\tilde{\Delta}v)_{2,\Omega_t} \tag{60}$$

is valid. We integrate this by parts over x, using the Hölder and Cauchy inequalities, and apply the embedding theorem for $W_2^2(\Omega)$ in $C(\bar{\Omega})$ (recalling that $\Omega \in E^3$) and the second fundamental inequality for the Stokes operator $\tilde{\Delta}$ [5]. In view of these, the following inequality holds:

$$\Big|\int_{\Omega_t}v_kv_{x_k}\tilde{\Delta}v\,dx\Big|\leq\max_{\Omega_t}|v|\cdot\|v_x\|_{2,\Omega_t}\|\tilde{\Delta}v\|_{2,\Omega_t}\leq C(\Omega)\|v_x\|_{2,\Omega_t}\|\tilde{\Delta}v\|^2_{2,\Omega_t}\leq \tag{61}$$

$$\leq \frac{\mu_1}{2}\|\tilde{\Delta} v\|^2_{2,\Omega_t} + \frac{c^2(\Omega)}{2\mu_1}\|v_x\|^2_{2,\Omega_t}\|\tilde{\Delta} v\|^2_{2,\Omega_t}, \quad t \geq 0.$$

Then, by assuming $y(t) = \|v_x\|^2_{2,\Omega_t} + \mu\|\tilde{\Delta} v\|^2_{2,\Omega_t} + \sum_{\ell=1}^{L}\beta_\ell\|\tilde{\Delta} u_\ell\|^2_{2,\Omega_t}$ we obtain the inequality:

$$y'(t) + \mu_1\|\tilde{\Delta} v\|^2_{2,\Omega_t} + \sum_{\ell=1}^{L}|\alpha_\ell|\beta_\ell\|\tilde{\Delta} u_\ell\|^2_{2,\Omega_t} \leq c_1(\Omega, \mu^{-1}, \mu_1^{-1})\|v_x\|^2_{2,\Omega_t} y(t) + c_2(\mu_1^{-1})\|f\|^2_{2,\Omega_t}, \quad t \geq 0. \tag{62}$$

Integrating (62) over t from 0 to t and using the Cauchy initial conditions (17), we obtain the inequality:

$$\begin{aligned} y(t) &\leq c_1\int_0^t \|v_x\|^2_{2,\Omega_\tau} y(\tau)d\tau + c_2\|f\|^2_{2,\Omega_t} + (\|v_{0x}\|^2_{2,\Omega} + \mu\|\tilde{\Delta} v_0\|^2_{2,\Omega}) \leq \\ &\leq c_1\int_0^t \|v_x\|^2_{2,\Omega_\tau} y(\tau)d\tau + c_3(\|f\|_{2,Q_\infty}; \|v_{0x}\|_{2,\Omega}, \|\tilde{\Delta} v_0\|^2_{2,\Omega}), \quad t > 0, \end{aligned} \tag{63}$$

and to this, using the fact that because of (59), $\|v_x\|^2_{2,\Omega_t} \in L_1(0, \infty)$ and $\|v_x\|^2_{2,Q_\infty} \leq \mu_1^{-1}A_1$, we apply the "integral" variant of Gronwall's lemma (mentioned in Sec. 3) to obtain:

$$y(t) \leq c_3 \exp\left(\int_0^t c_1\|v_x\|^2_{2,\Omega_\tau} d\tau\right) \leq c_3 \exp(c_1\mu_1^{-1}A_1) = A_2, \quad t > 0. \tag{64}$$

Then, by integrating (62) once again over t from 0 to ∞ over the whole volume, using (59) and (64), and maximizing over $t \geq 0$, we obtain the estimate:

$$\begin{aligned} &\max_{t \geq 0}\left(\|v_x\|^2_{2,\Omega_t} + \mu\|\Delta v\|^2_{2,\Omega_t} + \sum_{\ell=1}^{L}\beta_\ell\|\Delta u_\ell\|^2_{2,\Omega_t}\right) + \mu_1\|\Delta v\|^2_{2,Q_\infty} + \\ &+ \sum_{\ell=1}^{L}|\alpha_\ell|\beta_\ell\|\Delta u_\ell\|^2_{2,Q_\infty} \leq c_1\mu_1^{-1}A_1A_2 + c_2\|f\|^2_{2,Q_\infty} + \|v_{0x}\|^2_{2,\Omega} + \mu\|\Delta v_0\|^2_{2,\Omega} = A_3. \end{aligned} \tag{65}$$

Furthermore, for $\forall t \geq 0$, the equality

$$J_2(v; \{u_\ell\}) = \left(v_t + v_K v_{x_K} - \mu\tilde{\Delta} v_t - \mu_1\tilde{\Delta} v - \sum_{\ell=1}^{L}\beta_\ell\tilde{\Delta} u_\ell, \tilde{\Delta} v_t\right)_{2,\Omega_t} = (f, \tilde{\Delta} v_t)_{2,\Omega_t}, \tag{66}$$

is also valid for the solution of (16), (17). We now integrate (66) by parts over x and use the Hölder and Cauchy inequality and the embedding theorem for $W_2^2(\Omega)$ in $C(\Omega)$ and the estimate which results from (65) $\|\Delta v\|^2_{2,\Omega_t} \leq \mu^{-1}A_3$, $t \geq 0$, thanks to which the inequality

$$\begin{aligned} &\left|\int_{\Omega_t} v_K v_{x_K}\Delta v_t\, dx\right| \leq \max_{\Omega_t}|v| \cdot \|v_x\|_{2,\Omega_t}\|\Delta v_t\|_{2,\Omega_t} \leq c(\Omega)\|\Delta v\|_{2,\Omega_t} \cdot \\ &\cdot \|v_x\|_{2,\Omega_t}\|\Delta v_t\|_{2,\Omega_t} \leq \frac{\mu}{2}\|\Delta v_t\|^2_{2,\Omega_t} + c(\Omega)\mu^{-1})A_3\|v_x\|^2_{2,\Omega_t}, \quad t \geq 0, \end{aligned} \tag{67}$$

is valid. Then, integrating over t from 0 to ∞ and using (59), in view of which $\|v_x\|^2_{2,Q_\infty} \leq \mu_1^{-1}A_1$, we obtain the estimate:

$$\begin{aligned} &\|v_{xt}\|^2_{2,Q_\infty} + \mu\|\Delta v_t\|^2_{2,Q_\infty} + \sum_{\ell=1}^{L}\beta_\ell\|\Delta u_{\ell t}\|^2_{2,Q_\infty} + \max_{t \geq 0}\left(\mu_1\|\Delta v\|^2_{2,\Omega_t} + \right. \\ &\left. + \sum_{\ell=1}^{L}|\alpha_\ell|\beta_\ell\|\Delta u_\ell\|^2_{2,\Omega_t}\right) \leq c_4(\mu^{-1})\|f\|^2_{2,Q_\infty} + c(\mu^{-1}, \mu_1^{-1})A_1A_3 + \mu_1\|\Delta v_0\|^2_{2,\Omega} = A_4. \end{aligned} \tag{68}$$

It follows from (65), (68) and Friedrich's inequality that the following global a priori estimate for the solution of (16), (17) is valid:

$$\max_{t \geq 0}\|v, v_x, v_{xx}\|^2_{2,\Omega_t} + \|v, v_x, v_t, v_{xt}, v_{xx}, v_{xxt}\|^2_{2,Q_\infty} \leq c(A_1 - A_4) = \tag{69}$$

$$=A_5(\|f\|_{2,1;Q_\infty}, \|f\|_{2,Q_\infty}; \|v_{0x}\|_{2,\Omega}, \|\Delta v_0\|_{2,\Omega}; \mu^{-1}, \mu_1^{-1}).$$

From (69) follows the existence of a unique global solution on the entire semiaxis $t \geq 0$ of initial boundary value problem (16), (17). This solution has properties (58) and satisfies (16) almost everywhere in Q_∞. It exists for the following conditions on the problem data: Ω is a bounded domain from E^3 with boundary $\partial\Omega \subset C^2$; $f(x, t) \in L_2(Q_\infty) \cap L_{2,1}(Q_\infty)$; $v_0(x) \in W_2^2(\Omega) \cap H(\Omega)$.

4.2. We now examine problem (31)-(33). For $\forall t \geq 0$, the inequality

$$J_3(v; w) \equiv \iint\limits_{Q_T}(v_t + v_K v_{x_K} - \mu\frac{\partial\tilde{\Delta}v}{\partial t} - \mu_1\tilde{\Delta}v - \tilde{\Delta}\sum_{\ell=0}^{L-1}\gamma_\ell^{(3)}\frac{\partial^\ell w_3}{\partial t^\ell})\tilde{\Delta}v\,dQ_t = \iint\limits_{Q_t} f\tilde{\Delta}v\,dQ_t \tag{70}$$

is valid for the solution of this problem. From this, using the equation $\tilde{\Delta}v = \sum_{l=0}^{L}\alpha_l^{(3)}(\partial^l\tilde{\Delta}w_3/\partial t^l)$, integrating by parts over x, and by parts over time t in those terms with w_3, using the Hölder and Cauchy inequality, and using (33) and (61) as well, we obtain:

$$\|v_x\|^2_{2,\Omega_t} + \mu\|\Delta v\|^2_{2,\Omega_t} + \mu_1\iint\limits_{Q_t}|\tilde{\Delta}v|^2 dQ + C_{3,3}\sum_{s=0}^{L-1}\left\|\frac{\partial^s\Delta w_3}{\partial t^s}\right\|^2_{2,\Omega_t} + C_{4,3}\sum_{s=0}^{L-1}\left\|\frac{\partial^s\Delta w_3}{\partial t^s}\right\|^2_{2,Q_t} \leq C_1(\Omega, \mu_1^{-1})\int_0^t\|v_x\|^2_{2,\Omega_t}\|\tilde{\Delta}v\|^2_{2,\Omega_t}dt +$$

$$C_2(\mu_1^{-1})\|f\|^2_{2,Q_t} + (\|v_{0x}\|^2_{2,\Omega} + \mu\|\tilde{\Delta}v_0\|^2_{2,\Omega} + \alpha_{L-1}^{(3)}\|\Delta v_0\|^2_{2,\Omega}) \leq c_1\int_0^t\|v_x\|^2_{2,\Omega_t}\|\tilde{\Delta}v\|^2_{2,\Omega_t}dt + c_3(\|f\|_{2,Q_\infty};$$

$$\|v_{0x}\|_{2,\Omega}, \|\tilde{\Delta}v_0\|_{2,\Omega}; \mu_1^{-1}), \quad t > 0. \tag{71}$$

From (71), assuming $y(t) \equiv \|\tilde{\Delta}v\|_{2,\Omega_t}^2$, we obtain the inequality

$$y(t) \leq \mu^{-1}c_1\int_0^t\|v_x\|^2_{2,\Omega_\tau}y(\tau)d\tau + c_3\mu^{-1}, \tag{72}$$

and from this, by using $\|v_x\|_{2,\Omega_t}^2 \in L_1(0, \infty)$ and $\|v_x\|_{2,Q_\infty}^2 \leq \mu_1^{-1}A_{2,3}$ which follows from (40), and by applying the "integral" version of Gronwal's lemma, we have the inequality:

$$\|\tilde{\Delta}v\|^2_{2,\Omega_t} \leq \frac{c_3}{\mu}\exp\left(\int_0^t\mu^{-1}c_1\|v_x\|^2_{2,\Omega_t}d\tau\right) \leq \frac{c_3}{\mu}\exp(\mu^{-1}c_1\mu_1^{-1}A_{2,3}) \equiv A_2, \quad t > 0. \tag{73}$$

We now use (39) and (73) in inequality (71), and maximize the latter over $t \geq 0$ to obtain the estimate:

$$\max_{t\geq 0}(\|v_x\|^2_{2,\Omega_t} + \mu\|\tilde{\Delta}v\|^2_{2,\Omega_t}) + \mu_1\iint\limits_{Q_\infty}\|\tilde{\Delta}v\|^2 dx\,dt +$$

$$+C_{3,3}\sum_{s=0}^{L-1}\max_{t\geq 0}\left\|\frac{\partial^s\Delta w_3}{\partial t^s}\right\|^2_{2,\Omega_t} + C_{4,3}\sum_{s=0}^{L-1}\left\|\frac{\partial^s\Delta w_3}{\partial t^s}\right\|^2_{2,Q_\infty} \leq c_1A_2\mu_1^{-1}A_{2,3} + C_3 \equiv A_3. \tag{74}$$

Furthermore, for $\forall t \geq 0$ the equality

$$J_4(v; w) \equiv \int_0^t(v_{tt} + v_K v_{x_K t} + v_{K_t}v_{x_K} - \mu\frac{\partial^2\tilde{\Delta}v}{\partial t^2} - \mu_1\tilde{\Delta}v_t - \sum_{\ell=0}^{L-1}\alpha_\ell^{(3)}\frac{\partial^{\ell+1}\tilde{\Delta}w_3}{\partial t^{\ell+1}}, \tilde{\Delta}v_t)_{2,\Omega_t}dt = \int_0^t(f_t, \tilde{\Delta}v_t)_{2,\Omega_t}dt \tag{75}$$

is also valid for the solution of (31)-(33). We integrate (75) by parts over x, and using $\tilde{\Delta}v_t = \sum_{s=0}^{L}\gamma_s^{(3)}(\partial^{s+1}\tilde{\Delta}w_3/\partial t^{s+1})$, we integrate by parts over t those terms with w_3. We then use the Hölder and Cauchy inequality condition (33) and also apply inequality (44) and an estimate which comes from (44)

$$\|\Delta v_t(x,0)\|^2_{2,\Omega} \le c_4(\mu^{-1}; \|f(x,0)\|_{2,\Omega}; \|v_0\|^2_{2,\Omega}) . \tag{75'}$$

Finally, applying the already proved estimates (40), (47), and (74), in view of which

$$\|v_x\|^2_{2,Q_\infty} \le \mu_1^{-1}A_{2,3}; \quad \|v_{xt}\|^2_{2,Q_\infty} \le \mu_1^{-1}c_{2,3}; \quad \max_{t\ge 0}\|\tilde{\Delta} v\|^2_{2,\Omega_t} \le \mu^{-1}A_3 \tag{76}$$

and using inequalities coming from (76):

$$\iint_{Q_t} v_k v_{x_k t}\Delta v_t\,dx\,dt \le \int_0^t \max_{\Omega_t}|v|\cdot\|v_{xt}\|_{2,\Omega_t}\|\tilde{\Delta} v_t\|_{2,\Omega_t}dt \le \tag{77}$$

$$\le \int_0^t C(\Omega)\|\Delta v\|_{2,\Omega_t}\|\tilde{\Delta} v\|_{2,\Omega_t}\|v_{xt}\|_{2,\Omega_t}dt \le \frac{\mu_1}{2}\|\tilde{\Delta} v_t\|^2_{2,Q_T} + \frac{(C(\Omega)\mu^{-1}A_3}{2\mu_1}\|v_{xt}\|^2_{2,Q_t} = \frac{\mu_1}{2}\|\tilde{\Delta} v_t\|^2_{2,Q_t} + c_6 ,$$

$$\iint_{Q_T} v_{kt}v_{x_k}\tilde{\Delta} v_t\,dx\,dt \le \int_0^t \max_{\Omega_t}|v_t|\cdot\|v_x\|_{2,\Omega_t}\|\Delta v_t\|_{2,\Omega_t}dt \le c(\Omega)\int_0^t \|v_x\|_{2,\Omega_t}\times \tag{78}$$

$$\times\|\Delta v_t\|^2_{2,\Omega_t}dt \le \frac{\mu_1}{2}\|\Delta v_t\|^2_{2,Q_t} + \frac{c^2(\Omega)}{2\mu_1}\int_0^t \|v_x\|^2_{2,\Omega_\tau}\|\Delta v_t\|^2_{2,\Omega_\tau}d\tau ,$$

we obtain the inequality:

$$\|v_{xt}\|^2_{2,\Omega_t} + \mu\|\tilde{\Delta} v_t\|^2_{2,\Omega_t} + \mu_1\|\tilde{\Delta} v_t\|^2_{2,Q_t} + c_{3,3}\sum_{\ell=0}^{L-1}\left\|\frac{\partial^{\ell+1}\Delta w_3}{\partial t^{\ell+1}}\right\|^2_{2,\Omega_t} + c_{4,3}\sum_{s=0}^{L-1}\left\|\frac{\partial^{s+1}\Delta w_3}{\partial t^{s+1}}\right\|^2_{2,Q_t} \le c_{5,3} + \mu c_4 + c_7(\mu^{-1})\|f_t\|^2_{2,Q_\infty} + c_6 , \tag{79}$$

$$c_8\int_0^t \|v_x\|^2_{2,\Omega_\tau}\|\Delta v_t\|^2_{2,\Omega_\tau}d\tau \equiv c_8\int_0^t \|v_x\|^2_{2,\Omega_\tau}\|\Delta v_t\|^2_{2,\Omega_\tau}d\tau + c_9 .$$

With the help of Gronwall's lemma and (40), we obtain from (79) the estimate:

$$\|\tilde{\Delta} v_t\|^2_{2,\Omega_t} \le \mu^{-1}c_9\exp(\mu^{-1}\mu_1^{-1}c_8A_{2,3}) , \qquad 0<t\le\infty , \tag{80}$$

and then from (79) and (8), by maximizing over $t \in [0, \infty)$, we have the estimate:

$$\max_{0\le t\le\infty}\left\{\|v_{xt}\|^2_{2,\Omega_t} + \mu\|\tilde{\Delta} v_t\|^2_{2,\Omega_t} + c_{3,3}\sum_{s=0}^{L-1}\left\|\frac{\partial^{s+1}\Delta w_3}{\partial t^{s+1}}\right\|^2_{2,\Omega_t}\right\} + \tag{81}$$

$$+ c_{4,3}\sum_{s=0}^{L-1}\left\|\frac{\partial^{s+1}\Delta w_3}{\partial t^{s+1}}\right\|^2_{2,Q_\infty} + \mu_1\|\tilde{\Delta} v_t\|^2_{2,Q_\infty} \le c_9(1+\mu^{-1}\exp\{\mu^{-1}\mu_1^{-1}c_8A_{2,3}\}).$$

From estimates (40) and (74), valid for $\forall t \ge 0$, and from (81), valid for $\forall T \le \infty$, follows the existence on the entire semiaxis $t \ge 0$ of a unique, global solution of the initial boundary value problem (31)-(33), which has the following properties:

$$\max_{t\ge 0}\|v, v_x, v_{xx}\|^2_{2,\Omega_t} + \|v_x, v_{xx}\|^2_{2,Q_\infty} + \max_{0\le t\le\infty}\|v_{xt}, \Delta v_t\|^2_{2,\Omega_t} + \|\Delta v_t\|^2_{2,Q_\infty} < \infty. \tag{69$_*$}$$

This solution exists for the following conditions on the problem data: Ω is a bounded domain from E^3 with boundary $\partial\Omega \subset C^2$,

$$f(x,t) \in L_2(Q_\infty)\cap L_{2,1}(Q_\infty),\ f_t \in L_2(Q_\infty),\ v_0(x) \in W_2^2(\Omega)\cap H(\Omega).$$

4.3. The results obtained in Secs. 4.1 and 4.2 have yet another aspect. Based on the results of Sec. 4.2 and methods, well-known in the theory of partial differential equations, for increasing the smoothness of generalized solutions of boundary value and initial-boundary value problems of mathematical physics [5, 11, 2-4], theorem 2 of our work can once again be proved on the global classical solvability of the initial-boundary value problem (31)-(33) on the semiaxis $t \geq 0$. And, on the basis of the results of Sec. 4.1, we can prove theorem 2.1 of [4] on the global classical solvability of (16), (17) on the semiaxis $t \geq 0$.

4.4. As already noted above, to justify the principle of linearization in the local theory of hydrodynamic stability, it is necessary to have at least local single-valued solvability of the corresponding nonlinear initial-boundary value problem in the neighborhood of the fundamental solution $v^*(x, t)$ (that being studied for stability) on the entire semiaxis $t \geq 0$. In this section, we will show that:

1. initial-boundary value problem (16), (17) has a single-valued, global solution on $t \geq 0$ in the class of solutions (69) in the neighborhood of any fundamental solution $v^*(x, t)$ from class (69);

2. initial-boundary value problem (31)-(33) has a single-valued, global solution on the semiaxis $t \geq 0$ in the class of solutions (69_*) in the neighborhood of any fundamental solution $v^*(x, t)$ from class (69_*).

If $v^*(x, t)$ is the fundamental solution of (16), (17) from class (69), then by seeking the solution of (16), (17) in the form $v \equiv v^* + u(x, t)$, we obtain for $u(x, t)$ the following initial-boundary value problem:

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} - \mu\frac{\partial \Delta u}{\partial t} - \mu_1 \Delta u + u_k\frac{\partial u}{\partial x_k} + v_k^*\frac{\partial u}{\partial x_k} + u_k\frac{\partial v^*}{\partial x_k} - \sum_{l=1}^{L}\beta_l \Delta u_l + \operatorname{grad} p = 0, \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \operatorname{div} u = 0 \\ &u = \frac{\partial u_s}{\partial t} - \alpha_s u_s, \quad s = 1, \ldots, L, \quad (x, t) \in Q_\infty \end{aligned}\right\} \tag{82}$$

$$u|_{t=0} = u_0(x), \quad u_s|_{t=0} = 0, \; x \in \Omega; \; u|_{\partial\Omega} = u_s|_{\partial\Omega} = 0, \quad t \geq 0. \tag{83}$$

Similarly, if $v^*(x, t)$ is the fundamental solution of the initial-boundary value problem (31)-(33) from class (69_*), then by seeking the solution of (31)-(33) in the form $v \equiv v^* + u$, we obtain for $u(x, t)$ the following initial-boundary value problem:

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} - \mu\frac{\partial \Delta u}{\partial t} - \mu_1 \Delta u + u_k\frac{\partial u}{\partial x_k} + v_k^*\frac{\partial u}{\partial x_k} + u_k\frac{\partial v^*}{\partial x_k} - \sum_{l=0}^{L-1}\alpha_l^{(3)}\frac{\partial^l \Delta w}{\partial t^l} + \\ &\qquad\qquad\qquad\qquad + \operatorname{grad} p = 0, \quad \operatorname{div} u = 0 \\ &u = \sum_{s=0}^{L}\gamma_s^{(3)}\frac{\partial^s w}{\partial t^s}, \quad (x, t) \in Q_\infty \end{aligned}\right\} \tag{84}$$

$$\begin{aligned} &u|_{t=0} = u_0(x); \quad \left.\frac{\partial^s w}{\partial t^s}\right|_{t=0} = 0, \quad s = 0, 1, \ldots, L-1, \quad x \in \Omega; \\ &\qquad\qquad\qquad\qquad u|_{\partial\Omega} = w|_{\partial\Omega} = 0, \quad t \geq 0. \end{aligned} \tag{85}$$

The following is true:

THEOREM 3. Let Ω be a bounded domain from E^3, $\partial\Omega \in C^2$, $u_0(x) \in W_2^2(\Omega) \cap H(\Omega)$. Then:

1) for $\forall v^*(x, t)$ from class (69), the initial-boundary value problem (82), (83) has a unique global solution $u(x, t)$ from class (69) on the semiaxis $t \geq 0$;

2) for $\forall v^*(x, t)$ from class (69_*), the initial-boundary value problem (84), (85), (33) has a unique global solution $u(x, t)$ from class (69_*) on the semiaxis $t \geq 0$.

To prove theorem 3, it is sufficient to obtain global a priori estimates for the solution of (82), (83) in class (69) and global a priori estimates for the solution of (84), (85), (33) in class (69_*), after which the existence of the solutions is easily proved by Galerkin's method [5, 11]. The proof of these estimates is basically analogous to the proof of (69) for the solution of (16), (17), and to that of (81) for the solution of (31)-(33) respectively. This is because equations (82) and (84) differ from (16) and (31) respectively, only in the presence of linear terms $v_x^*(\partial u/\partial x_k) + (\partial v^*/\partial x_k)u_k$ with "good" $v^*(x, t)$ and therefore are omitted (see [12]).

In conclusion, we thank O. A. Ladyzhenskaya, corresponding member of the Academy of Sciences of the USSR, for her stimulating attention to our work.

REFERENCES

1. R. B. Bird, R. C. Armstrong, and O. Hassager, Dynamics of Polymeric Liquids, Volume 1: Fluid Mechanics, Wiley, New York (1977).
2. A. P. Oskolkov, "Initial boundary value problems for the equations of motion of Kelvin–Voight and Oldroyd fluids," Works of the Mathematical Institute of the Academy of Sciences of the USSR, **179**, 126-164 (1987).
3. A. P. Oskolkov, "Well-posedness of initial-boundary value problems for the equations of motion of linear, visco-elastic fluids," in: Problems in the Dynamic Theory of the Propagation of Seismic Waves [in Russian], Vol. 26, Leningrad (1987), pp. 100-126.
4. N. A. Karazeeva, A. A. Kotsiolis, and A. P. Oskolkov, "On attractors and dynamics systems which arise from initial-boundary value problems for the equations of motion of linear, viscoelastic fluids," LOMI (Leningrad Branch of the V. A. Steklov Institute of Mathematics) Preprint P-10-88, Leningrad (1988).
5. O. A. Ladyzhenskaya, Mathematical Problems in the Dynamics of Viscous, Incompressible Fluids, Second edition [in Russian], Moscow (1970).
6. A. P. Oskolkov, "On certain unsteady linear and quasilinear systems encountered in the study of viscous fluid motion," in Boundary-Value Problems in Mathematical Physics and Related Problems in the Theory of Functions, 9 [in Russian], Notes from the LOMI Scientific Seminars, Vol. 59 (1976), pp. 133-177.
7. O. A. Ladyzhenskaya, "On determining the minimal global attractors for the Navier–Stokes equations and other partial differential equations," Usp. Mat. Nauk, **42**, No. 6, 25-60 (1987).
8. Yu. L. Daletskii, and M. G. Krein, Stability of Solutions of Differential Equations in a Banach Space [in Russian], Moscow (1970).
9. O. A. Ladyzhenskaya and V. A. Solonnikov, "On the principle of linearization and invariant manifolds for magneto-hydrodynamic problems," in: Boundary-Value Problems in Mathematical Physics and Related Problems in the Theory of Functions, 7 [in Russian], Notes from the LOMI Scientific Seminars, Vol. 38 (1973), pp. 46-89.
10. V. I. Yudovich, The Method of Linearization in Hydrodynamic Stability Theory [in Russian], Doctoral Dissertation (1984).
11. O. A. Ladyzhenskaya, Boundary-Value Problems of Mathematical Physics [in Russian], Moscow (1973).
12. A. P. Oskolkov, A. A. Kotsiolis, and R. Shadiev, "Global a priori estimates on the semiaxis $t \geq 0$, asymptotic stability and periodicity in time of "small" solutions of the equations of motion of Oldroyd and Kelvin–Voight fluids," LOMI Preprint P-10-89, Leningrad (1989).