International Journal of Game Theory (1996) 25: 189–198

The Nakamura Theorem for Coalition Structures of Quota Games

RAJAT DEB and SHLOMO WEBER

Department of Economics, Richard B. Johnson Center for Economic Studies, Dedman College, Southern Methodist University, Dallas, Texas 75275-0496, USA

EYAL WINTER

The Hebrew University of Jerusalem, Israel

Abstract: This paper considers a model of society $\mathscr{S}$ with a finite number of individuals, n, a finite set off alternatives, Ω, effective coalitions that must contain an *a priori* given number q of individuals. Its purpose is to extend the Nakamura Theorem (1979) to the quota games where individuals are allowed to form groups of size q which are smaller than the grand coalition. Our main result determines the upper bound on the number of alternatives which would guarantee, for a given n and q, the existence of a stable coalition structure for any profile of complete transitive preference relations. Our notion of stability, $\mathscr{S}$-equilibrium, introduced by Greenberg–Weber (1993), combines both *free entry* and *free mobility* and represents the natural extension of the core to improper or non-superadditive games where coalition structures, and not only the grand coalition, are allowed to form.

1 Introduction

In his 1979 paper Nakamura determined the upper bound on the cardinality of the set of alternatives which would guarantee the existence of the core of the simple game associated with the society $\mathscr{S}$ for any profile of individuals' preferences. The purpose of this paper is to extend the Nakamura Theorem (1979) to the improper quota[1] games, where individuals are allowed to form groups of size q which are smaller than the grand coalition. We consider a model of society $\mathscr{S}$ with a finite number of individuals, n, finite set of alternatives, Ω, where the *effective* coalitions[2] must contain, at least, an *a priori* given number q of individuals. Every effective coalition can impose any alternative for its members while every "ineffective" coalition, has no power whatsoever.

The important distinction from the Nakamura model is that we allow for an effective coalition to contain less than a half of the individual ($q < \frac{1}{2}$), which covers the case of improper or non-superadditive games. Thus, to define "stable

[1] There are several different ways in which the term "quota" is used in game theory. We use this term in the same way as in Moulin (1988).

[2] We use the term "effective" coalition rather than "winning" for the simple reason that the winning coalitions are usually associated with proper games in which the complement of a winning coalition cannot be a winning coalition itself. In contrast, in our model of improper games it might be the case that both a coalition and its complement are winning or effective in the sense that they could choose any alternative for its members.

coalition structures" we use the concept of $\mathcal{S}$-equilibrium presented in Greenberg–Weber (1993), which represents the generalization of the core concept used by Nakamura and combines two properties, *free mobility* and *free entry*. Free mobility[3] implies that each individual is free to join any coalition which adopts the alternative she prefers, whereas free entry means that every effective coalition is allowed to form and to adopt any alternative its members wish to choose. In equilibrium, the society is partitioned into effective coalitions each choosing an alternative in such a way that each individual belongs to a coalition whose offered alternative is the one that she likes best among all offered alternatives and there is no effective coalition which can choose to offer an alternative to make all its members better off than they currently are.

Our main result determines, for any number of individuals n and the quota q, the maximal number of alternatives which guarantee the existence of an $\mathcal{S}$-equilibrium for any profile of individuals' preferences. It turns out that in almost all the cases it is possible to find an example of a society with the space of alternatives consisting of three or four alternatives, for which $\mathcal{S}$-equilibrium fails to exist. Only in one case, with four individuals where all two-person coalitions are effective, there always exists an $\mathcal{S}$-equilibrium for any arbitrary (finite) number of alternatives.

The paper is organized as follows: In the next section we formally describe the model and state our main result, the proof of which is presented in Section 3. In Section 3 we prove some Lemmas which are used in the proof of our theorem.

2 The Model

Let Ω be a finite of potential alternatives and $N=\{1,2,\ldots,n\}$ be a set of individuals. We assume that each individual $i \in N$ has a complete, transitive, weak preference relation, $\succeq_i$, over elements of Ω (we shall use $\succ_i$ to denote the asymmetric component of $\succeq_i$). Let W be a set of *effective* coalitions where an effective coalition can choose any alternative in Ω while the choice set of a non-effective coalition is empty. In this paper we consider *quota games*, i.e. there exists an integer q, $1 \leq q \leq n$ such that

$$W=\{S \subset N \,\|\, |S| \geq q\}.$$

The society $\mathcal{S}$ is represented by a quadruple $(N, \Omega, q, \succeq)$, where $\succeq$ is the profile of individuals' preferences.

Nakamura (1979) determined the upper bound on the cardinality of the set of alternatives which would guarantee the existence of the core of the simple game associated with the society $\mathcal{S}$ for any profile of complete and transitive individuals' preferences. In our paper we wish to generalize the Nakamura Theorem by

[3] See Caplin–Nalebuff (1992) for a discussion on free mobility in voting models.

considering a natural extension of the core, which consists of alternatives adopted by the *grand* coalition N, which are not blocked by any effective coalition. We allow for more than one (pairwise disjoint) effective coalitions to be formed where every effective coalition can impose a *different* alternative for its members. For this purpose we use the concept of $\mathscr{S}$-equilibrium offered in Greenberg–Weber (1993), which combines both Nash-like and core-like conditions. However, for $q > \frac{1}{2}$ no two disjoint coalitions may be formed, hence only one alternative may be offered and the set of $\mathscr{S}$-equilibria coincides with the core. In the case where $q \leq \frac{1}{2}$, the Nakamura Theorem implies that if the set of alternatives consists of, at least, two elements, there is always a profile of individuals' preferences for which the core is empty, and we are able to characterize the class of societies which have an empty core but nevertheless admit an $\mathscr{S}$-equilibrium.

Let us introduce some notation and definitions. A collection P of pairwise disjoint effective coalitions $C_1, C_2, \ldots, C_j$, is called an *effective partition* of N if $\bigcup_{j=1}^{J} C_j = N$. The set of all effective partitions of N is denoted by Π.

Definition 1: (P, A) is called an *$\mathscr{S}$-equilibrium*, where $P = (C_1, C_2, \ldots, C_J) \in \Pi$ and $A = \{a_1, \ldots, a_J\} \subseteq \Omega$ (we will say that C_j offers a_j) if

(D.1.1) For all $j, h = 1, \ldots, J$, $a_j \succsim_i a_h$ whenever $i \in C_j$. That is, each individual in C_j (weakly) prefers a_j over all other alternatives in A.

(D.1.2) There exists no $C \in W$ and $\omega \in \Omega$ such that $\omega \succ_i a$ for all $i \in C$ and all $a \in A$. That is, C does not block (P, A) via ω.

(D.1.1) (*free mobility*) is a Nash-type condition: No individual can be made better off by migrating to another (existing) coalition. This condition is trivially satisfied for $q > 1/2$ as there is only one alternative offered by the grand coalition. For the case where $q \leq \frac{1}{2}$ it is however important to allow individuals to choose the best among the offered alternatives. (D.1.2) (*free entry*) is a core-type condition: There is no (effective) group of individuals who can form a coalition and choose an alternative which makes each of its members better off than they currently are.

In order to determine the cardinality of the alternative space which would yield the existence of an $\mathscr{S}$-equilibrium for any profile of preferences, we shall use the following result:

Lemma 1: If there exists a profile of individuals' preferences over the set of alternatives Ω with $|\Omega| = k_0$ such that an $\mathscr{S}$-equilibrium does not exist, then for any set of alternatives $\bar{\Omega}$ with $|\bar{\Omega}| > k_0$ there exists a profile of individuals' preferences over $\bar{\Omega}$ for which an $\mathscr{S}$-equilibrium will not exist.

Proof of Lemma 1: Let (n, q) be given and assume that $k = |\Omega|$ is such that there exists an ordering $\succsim$ for which an $\mathscr{S}$-equilibrium does not exist. Let $\tilde{\Omega}$ be a finite set of alternatives which do not belong to Ω and let us extend the ordering $\succsim$ to $\Omega \cup \tilde{\Omega}$ by requiring that for all $i \in N$ $a \succ_i b$ for all $a \in \Omega$, $b \in \tilde{\Omega}$, and $b \succsim_i c$ for all $b, c \in \tilde{\Omega}$. Then if there exists an equilibrium for the "extended" society no alternative from $\tilde{\Omega}$ can be offered in equilibrium. Since there exists no $\mathscr{S}$-equilibrium it follows that the addition of the "bottom" alternatives to Ω does not rescue the non-existence of an equilibrium. □

Remark: Lemma 1 implies that for any pair of n, q there exists a positive integer $K(n, q)$ (not necessarily finite) such that an $\mathscr{S}$-equilibrium exists for all preference profiles if and only if $k < K(n, q)$. In particular, if $q = 1$, i.e. every coalition is effective, each individual is allowed to choose her top alternative (which is not necessarily unique) and the partition of the set N into singletons will constitute an $\mathscr{S}$-equilibrium, yielding $K(n, 1) = \infty$ for any number of players n. On the other hand, if $q > (n/2)$ then the set of $\mathscr{S}$-equilibria coincides with the core. It is well-known (see Moulin (1988) for references) that the core is nonempty if and only if the number of alternatives is less than $v(n, q) = \left\lceil \frac{n}{n-q} \right\rceil$, the smallest integer greater than or equal to $\frac{n}{n-q}$, which is the minimal number of effective coalitions whose intersection is empty.[4]

The goal of this paper is, therefore, to determine the value of $K(n, q)$ for any pair of n players and quota q which satisfy $2 \leq q \leq (n/2)$. Our result can be stated as follows:

Theorem: Let $2 \leq q \leq (n/2)$. For $q = 2$

$$K(n, 2) = \begin{cases} \infty & \textit{if} \quad n = 4 \\ 4 & \textit{if} \quad n > 4 \end{cases}$$

For $q \geq 3$

$$K(n, q) = \begin{cases} 3 & \textit{if} \quad 2q \leq n \leq 3q - 3 \\ 4 & \textit{if} \quad 3q - 2 \leq n \end{cases}$$

3 Proof of the Theorem

Let us first introduce some additional notation. For any pair x, y of alternatives in Ω denote by $W_{x,y}(S_{x,y})$ the set of individuals who weakly (strongly, respectively) prefer x over y, i.e.

$$W_{x,y} = \{i \in N \mid x \succsim_i y\}$$
$$S_{x,y} = \{i \in N \mid x \succ_i y\}$$

For each alternative $x \in \Omega$ denote by $W_x(S_x)$ the set of individuals who weakly

[4] It is easy to verify (see Moulin (1988)) that $v(n, q) = [n/n - q]$, where $[x]$ denotes the smallest integer which is greater than or equal to x.

(strongly, respectively) prefer x over all other alternatives in Ω, i.e.

$$W_x = \bigcap_{y \in \Omega - \{x\}} W_{x,y}$$

$$S_x = \bigcap_{y \in \Omega - \{x\}} S_{x,y}$$

To prove the theorem stated in the previous section we shall use the following lemmas, the proofs of which are presented in the next section:

Lemma 2: Suppose that set of alternatives Ω consists of three elements a, b and c. If there exists an alternative $x \in \Omega$ which is weakly preferred over the other two by at least q individuals, i.e. $|W_x| \geq q$, then there exists an $\mathscr{S}$-equilibrium.

Lemma 3: If $|\Omega| = 4$ and there is an integer $r \geq q$ such that $v(n-r, q) = 3$ then there is a profile $\succcurlyeq$ for which there exists no $\mathscr{S}$-equilibrium.

Lemma 4: Suppose that n, q are such that there exist three positive integers n_1, n_2, n_3 satisfying $n_1 + n_2 + n_3 = n$, $\max[n_1, n_2, n_3] < q$ and $\min[n_1 + n_2, n_1 + n_3, n_2 + n_3] \geq q$. Then $K(n, q) = 3$.

Lemma 5: $K(7, 3) = 4$.

Lemma 6: $K(4, 2) = \infty$.

Proof of the Theorem: Let $q = 2$. By Lemma 6, $K(4, 2) = \infty$. For $n > 4$ and $|\Omega| = 3$ there exists an alternative which is a top choice for at least two individuals. Thus, by Lemmas 1 and 2, $K(n, 2) \geq 4$ for $n > 4$. Moreover, since $v(3, 2) = [\frac{3}{1}] = 3$, Lemma 3 implies that for all $n > 4$ and $|\Omega| = 4$ there is a profile of preferences $\succcurlyeq$ for which there is no $\mathscr{S}$-equilibrium, yielding $K(n, 2) = 4$ for $n > 4$.

Let $q = 3$. Since $n = 6$ and $q = 3$ sastisfy the conditions of Lemma 4, if follows that $K(6, 3) = 3$. By Lemma 5, $K(7, 3) = 4$. Let now $n \geq 8$. Since $v(5, 3) = [\frac{5}{2}] = 3$, we can choose $r = n - 5 \geq 3$ so that Lemma 3 would yield $K(n, 3) \leq 4$. If the Ω consists of three alternatives, then, by Lemma 2, there exists an $\mathscr{S}$-equilibrium for a society with at least eight individuals. Thus, $K(n, 3) = 4$ for all $n \geq 8$.

Let $q \geq 4$. If $2q \leq n \leq 3q - 3$ we may choose $n_1 = n - (2q - 2)$, $n_2 - n_3 = q - 1$ which satisfy $n_1 \leq n_2 = n_3, n_1 + n_2 + n_3 = n$ with $n_1 + n_2 > q$. Thus, Lemma 4 yields $K(n, q) = 3$. If $n \geq 3q - 2$ and $|\Omega| = 3$, then there always exists an alternative which is a top alternative for at least q individuals. Then, by Lemma 2, $K(n, q) \geq 4$ for all $n \geq 3q - 2$. On the other hand, since $v(2q - 2, q) = [(2q - 2)/(q - 2)] = 3$ for $q \geq 4$, we may choose $r = n - (2q - 2) \geq q$ so that, by Lemma 3, $K(n, q) \leq 4$. Thus, together with the previous inequality this yields $K(n, q) = 4$ for all $n \geq 3q - 2$. □

4 Proof of Lemmas 2–6

Proof of Lemma 2: Consider the set of alternaives $\Omega = \{a, b, c\}$ and assume, without loss of generality, that $|\{i \in N \,|\, a \succsim_i \omega \text{ for all } \omega \in \Omega\}| \geq q$. Consider a partition of the set N into three pairwise disjoint subsets, A, B and C, where

$$A = W_a = W_{a,b} \cap W_{a,c} = \{i \in N \,|\, a \succsim_i b \text{ and } a \succsim_i c\}.$$

$$B = S_{b,a} \cap W_{b,c} = \{i \in N \,|\, b \succsim_i c \text{ and } b \succ_i a\}.$$

$$C = S_c = S_{c,a} \cap S_{c,b} = \{i \in N \,|\, c \succ_i a \text{ and } c \succ_i b\}.$$

There are several cases to consider:

(i) $|C| \geq q$ and $|B| \geq q$: Then $((A, B, C), (a; b, c))$ is an $\mathcal{S}$-equilibrium. Indeed, each of the alternatives a, b, c is supported by coalitions A, B, C, respectively, and since $|A| \geq q$, every alternative is supported by the set of individuals who choose it as their top alternative.

(ii) $|B| \geq q > |C|$: Then $((W_{a,b}, S_{b,a}), (a, b))$ is an $\mathcal{S}$-equilibrium. Indeed, since $A \subset W_{a,b}$ and $B \subset S_{b,a}$, both coalitions $W_{a,b}$ and $S_{b,a}$ are effective. Moreover, no individual would switch from $W_{a,b}$ to $S_{b,a}$ and vice versa. It remains, therefore, to observe that the cardinality of the set C of individuals who strictly prefer c over other two alternatives is less than q.

(iii) $|C| \geq q > |B|$: This case is similar to (ii).

(iv) $|B| < q$, $|C| < q$, $|B \cup C| \geq q$: If $|S_{b,a}| \geq q$ then, as in case (ii), $((W_{a,b}, S_{b,a}), (a, b))$ is an $\mathcal{S}$-equilibrium. Similarly, if $|S_{c,a}| \geq q$ then $((W_{a,c}, S_{c,a}), (a, c))$ is an $\mathcal{S}$-equilibrium. If both $|S_{b,a}|$ and $|S_{c,a}|$ are less than q then there is no effective coalition which can block a via either b or c, yielding $((N), (a))$ as an $\mathcal{S}$-equilibrium.

(v) $|B \cup C| < q$: Then $((N), (a))$ is an $\mathcal{S}$-equilibrium. Indeed, the number of individuals for which a is not a top alternatives is less than q. Thus, a belongs to the core, and, therefore $((N), (a))$ is an $\mathcal{S}$-equilibrium. □

Proof of Lemma 3: Suppose that there exists an integer $r \geq q$ such that $v(n - r, q) = 3$. We shall construct the preference profile over four alternatives for which an $\mathcal{S}$-equilibrium does not exist. Let $R = \{1, 2, \ldots, r\}$. Since $v(n - r, q) = 3$, by the Nakamura theorem, there exists a preference profile of individuals in $N \setminus R$ over alternatives (a, b, c) such that the core of $N \setminus R$ is empty. Let alternative d be the (strictly) worst for all members of $N \setminus R$ and the (strictly) best for all individuals in R. (Note that $|R| = r \geq q$.) Suppose, in negation, that there is an $\mathcal{S}$-equilibrium. Then it should include two alternatives: d and one of the alternatives among a, b or c, say a. But, by the construction, there exists an effective subset of $N \setminus R$ which would block a via either b or c, a contradiction, showing that there is no $\mathcal{S}$-equilibrium. □

Proof of Lemma 4: Let Ω consist of three alternatives a, b, c and assume that there exists a partition of the set N into three pairwise disjoint sets N_1, N_2 and N_3, such that $|N_l| < q$ for $l = 1,2,3$ and $|N_l \cup N_m| \geq q$ for all $l, m = 1,2,3$, $l \neq m$. Let $a \succ_i b \succ_i c$ for all $i \in N_1$, $b \succ_i c \succ_i a$ for all $i \in N_2$ and $c \succ_i a \succ_i b$ for all $i \in N_3$. Then it is easy to verify that an $\mathscr{S}$-equilibrium fails to exist, yielding $K(n,q) \leq 3$. On the other hand, if Ω consist of two alternatives, the core is, obviously, nonempty, yielding $K(n,q) \geq 2$, which together with the previous argument implies $K(n,q) = 3$. □

Proof of Lemma 5: Lemma 2 implies that for $n = 7$, $q = 3$, $|\Omega| = 3$ there exists an $\mathscr{S}$-equilibrium. We shall now show that there is a society with four alternatives for which there is no $\mathscr{S}$-equilibrium. Define the preferences profile as follows (all the preferences are strict):

1	2	3	4	5	6	7
a	a	b	b	c	c	d
b	c	d	d	d	d	a
c	b	a	a	a	a	b
d	d	c	c	b	b	c

Since individuals 5, 6 and 7 prefer d over both a and b, neither $((N), (a))$ nor $((N), (b))$ is an $\mathscr{S}$-equilibrium, and, moreover the pair (a, b) will be blocked via d and hence cannot be supported as an $\mathscr{S}$-equilibrium. Since individuals 1, 3 and 4 prefer b over both c and d, it follows that neither $((N), (c))$ nor $((N), (d))$ is an $\mathscr{S}$-equilibrium, and the pair (c, d) will be blocked via b and, hence, cannot be supported as an $\mathscr{S}$-equilibrium.

We consider the four remaining pairs of alternatives and show that none of them can be supported in $\mathscr{S}$-equilibrium. Individuals 1, 2 and 7 prefer a over both b and c, whereas 2, 5 and 6 prefer c over both b and d. Moreover, given the alternatives a and c, individuals 1, 2, 3, 4 and 7 prefer a over c and, given the alternatives a and d, individuals 3, 4, 5, 6 and 7 prefer d over a, implying that there is no $\mathscr{S}$-equilibrium. □

Before proceeding with the proof of Lemma 6 we need additional notation and definitions. Recall that $\mathfrak{R}^N$ denotes the Euclidean space of dimension n, and for $x \in \mathfrak{R}^N$ and $C \subset N$, $\mathbf{x}^C$ denotes the projection of $\mathbf{x}$ on C. For $\mathbf{x}, \mathbf{y} \in \mathfrak{R}^C$ we denote $\mathbf{x} \gg \mathbf{y}$ if $x^i > y^i$ for all $i \in C$. Represent preferences of each individual $i \in N$ by the positive-valued utility function u_i and consider the following coalitional form game without side payments (N, V), associated with the $\mathscr{S}$, where the characteristic function of V is given by:

$$V(C) = \begin{cases} \{\mathbf{x} \in \mathfrak{R}^N | x^i \leq 0 \ \ \forall i \in C\} & \text{if } |C| = 1 \\ \{\mathbf{x} \in \mathfrak{R}^N | \exists \omega \in \Omega \text{ s.t. } u_i(\omega) \geq x^i \ \ \forall i \in C\} & \text{if } |C| \geq 2. \end{cases}$$

Intuitively, the projection $V(C)$ on C is the set of all C-attainable utility levels. The coalition structure core (Aumann–Dreze (1974)) of the game (N, V) is defined as follows:

Definition 2: Let P be a partition of N. *The P-core of the game* (N, V) is given by:

$$Core_P V = \{\mathbf{x} \in \bigcap_{C \in P} V(C) \mid \nexists\, T \subseteq N \text{ and } \mathbf{y} \in V(T) \text{ s.t. } \mathbf{y}^T \gg \mathbf{x}^T\}.$$

The coalition structure core of the game (N, V), $Core_\Pi V$ is given by:

$$Core_\Pi V = \bigcup_{P \in \Pi} Core_P V.$$

That is, the game (N, V) has a nonempty coalition structure core if there exists a partition P for which the P-core is nonempty.

It is obvious that if the set of $\mathscr{S}$-equilibria is nonempty then so is the coalition structure core, but the opposite is not necessarily true. Indeed, the coalition structure core requires only free entry, whereas $\mathscr{S}$-equilibrium imposes both free entry and free mobility. It is convenient to introduce the *super-additive cover game* $(N, \tilde{V})$

$$\tilde{V}(C) = \begin{cases} \tilde{V}(C) = V(C) & \text{if } C \neq N \\ \bigcup_{P \in \Pi} \bigcap_{D \in P} V(D) & \text{if } C = N. \end{cases}$$

As shown in Greenberg–Weber (1986), the core of $(N, \tilde{V})$ coincides with the coalition structure core of the game (N, V).

Following Bondareva (1962) we define

Definition 3: A collection of subsets of N, $\delta = \{B_1, B_2, \ldots, B_K\}$ is called *balanced* if there exist positive numbers $\gamma_1, \gamma_2, \ldots, \gamma_K$, *called balancing weights,* such that

$$\sum_{B_k \in \delta(i)} \gamma_k = 1 \quad \text{for all } i \in M,$$

where $\delta(i) = \{C \in \delta \mid i \in C\}$. A balanced collection of coalitions is *minimal* if none of its proper subcollections is balanced.

The non-sidepayment game (N, v) is *balanced* if for any balanced collection $\delta = \{B_1, \ldots, B_K\}$ and any $\mathbf{x} \in \Re^N$

$$\mathbf{x} \in v(B_k) \quad \text{for } k = 1, 2, \ldots, K, \text{ imply } x \in v(N).$$

Scarf's (1967) theorem implies that every non-sidepayment balanced game has a nonempty core. Although Le Breton (1989) has shown that, in general, the games of the type considered here are not balanced, we are able to show that:

Result: If $n = 4$ and $q = 2$, the game $(N, \tilde{V})$ is balanced.

Proof: In order to prove the balancedness of the non-sidepayment game $(N, \tilde{V})$, it suffices to consider only minimal balanced collections (see Shapley (1972)). Let $\delta = \{B_1, \ldots, B_K\}$ be a minimal balanced collection and $\mathbf{x} \in \tilde{V}(B_k)$ for all k. If δ is a partition then we are done. Assume, therefore, that δ is not a partition. First, consider the case where δ contains a three-person coalition. Without loss of generality, assume that $B_1 = \{1, 2, 3\} \in \delta$, offers an alternative a. Balancedness implies that there is a coalition $B_k \in \delta$ which contains individual 4. Let b_k offer an alternative b. Since δ is not a partition, $|B_k| > 1$, and suppose B_h contains also 1. Then we have a partition $\{\{1, 4\}, \{2, 3\}\}$ where coalition $\{1, 4\}$ offers b and coalition $\{2, 3\}$ offers a to guarantee each individual i at least x^i, which shows that $\mathbf{x} \in \tilde{V}(N)$.

It remains to consider the case where δ contains only one- or two-person coalitions. Using the result of Balinski (1970), the only minimal balanced collection that needs to be considered is that consisting of a singleton and the three two-person coalitions which do not contain this singleton. Consider, without loss of generality, the minimal balanced collection $\{\{1, 2\}, \{2, 3\}, \{1, 3\}, \{4\}\}$, where coalitions $\{1, 2\}$, $\{2, 3\}$ and $\{1, 3\}$ offer the alternatives a, b and c, respectively. Then the partition $\{\{1, 2\}, \{3, 4\}\}$, where coalition $\{1, 2\}$ offers a and coalition $\{3, 4\}$ offers c, guarantees to each individual i at least x^i. Thus, $\mathbf{x} \in \tilde{V}(N)$. □.

Proof of Lemma 6: The Result above guarantees that the core of $(N, \tilde{V})$ is nonempty. If there is an element in the core of $(N, \tilde{V})$ where all individuals choose the same alternative ω then the pair $((N), \omega)$ is, obviously, an $\mathscr{S}$-equilibrium. Suppose, therefore, that in all elements of the core of $(N, \tilde{V})$ the individuals choose two different alternatives. Take an element in the core of $(N, \tilde{V})$, say $u = (u_1(a), u_2(a), u_3(b), u_4(b))$, and assume, without loss of generality, that it is Pareto efficient: there is no other element in the core of $(N, \tilde{V})$, which would guarantee all individuals at least the same level of utility and a strictly higher level of utility for at least one individual. If this is an $\mathscr{S}$-equilibrium, we are done. Otherwise, assume, without loss of generality, that $u_2(b) > u_2(a)$.

Since (being a singleton) alternative b is not in the core of the game $(N, \tilde{V})$, it is blocked by a two-person coalition C via alternative c. Since u is in the core of $(N, \tilde{V})$, by Pareto efficiency, C must include individual 1. Moreover, the Pareto optimality assumption also implies $c \neq a$. Assume that there is no alternative d and a two-person coalition D such that D blocks u via d, where $d \succ_1 c$. Examine the following two cases:

(i) $C = \{1, 2\}$: Since $c \neq b$ consider the vector $(u_1(c), u_2(c), u_3(b), u_4(b))$. It satisfies free mobility and if it is not an $\mathscr{S}$-equilibrium, it should be blocked only by a two-person coalition including individual 1, a contradiction to the choice of c.

(ii) $C=\{1,3\}$: Since $c \neq b$ consider the vector $(u_1(c), u_2(b), u_3(c), u_4(b))$. It satisfies free mobility and if it is not an $\mathscr{S}$-equilibrium, it should be blocked only by a two-person coalition including individual 1, a contradiction to the choice of c. □

5 References

Aumann RJ, Dreze JH (1974) Cooperative games with coalition structures. International Journal of Game Theory 3: 217–237

Balinski M (1970) On the maximum matching, minimal covering. In: Kuhn HW (ed) Proceedings of the Princeton Symposium on Mathematical Programming. Princeton NJ, Princeton University Press

Bondareva ON (1962) Theory of the core of the n-person game (in Russian). Vestnik Leningradskogo Universiteta 13: 141–142

Caplin A, Nalebuff B (1992) Individuals and institutions, American Economic Review (Papers and Proceedings) 82: 317–322

Greenberg J, Weber S (1986) Strong tiebout equilibrium under restricted preferences domain. Journal of Economic Theory. 38: 101–117

Greenberg J, Weber S (1993) Stable coalition structures with unidimensional set of alternatives. Journal of Economic Theory. 60: 62–82

Le Breton M (1989) A note on balancedness and nonemptiness of the core in voting games. International Journal of Game Theory. 18: 111–117

Moulin H (1988) Axioms of cooperative decision making. Econometric Society Monographs 15. Cambridge University Press

Nakamura K (1979) The vetoers in simple game with ordinal preferences. International Journal of Game Theory. 8: 55–61

Scarf H (1967) The core of an n-person game. Econometrica 35: 50–69

Shapley LS (1972) On balanced sets and cores. Naval Logistics Quarterly 14: 453–460

Received April 1993
Revised version August 1994
Final version November 1994