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C^2-Regularity for Partially Free Minimal Surfaces*

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1. Introduction

Consider a configuration in Euclidean N-space consisting of a smooth Jordan arc C having its end points P_1 and P_2 on a smooth hypersurface S, but no other points in common with S. Let $B := \{(u,v) | u^2+v^2<1\}$ and denote by $\partial^+ B$ ($\partial^- B$) its boundary portions in $v>0$ ($v<0$). Let $Z(C,S)$ be the set of all surfaces $x = x(u,v) = (x_1(u,v), \ldots, x_N(u,v)) \in C^0(B) \cap H^{1,2}(B)$ which are bounded by C and S in the following sense: x maps $\partial^+ B$ continuously and in weakly monotonic manner onto C such that $x(-1,0)=P_1$, $x(1,0)=P_2$ and $x(0,1)=P_3$ for some fixed third point on C, different from P_1 and P_2. The free boundary condition is expressed as

$$\lim_{(u,v)\to(u_0,v_0)} \inf_{y\in S} |x(u,v)-y| = 0$$

for $(u_0,v_0) \in \partial^- B$. This implies the continuity of the distance function $\operatorname{dist}(x,S)$ but not the continuity of x itself. It is well known (see VI in [2]) that the variational problem

$$\int_B |y_u|^2 + |y_v|^2 \, du\, dv \to \text{Min} \quad (y \in Z(C,S)) \tag{1}$$

has at least one solution $x \in Z(C,S)$ which is a minimal surface, i.e. it solves the system

$$\Delta x = 0, \qquad x_u \cdot x_v = 0, \qquad |x_u| = |x_v| \tag{2}$$

on B. Note that in general a minimal surface does not represent a minimum of the corresponding variational problem.

There is satisfactory information concerning the behaviour of a solution x of (1) at the "fixed" boundary portion; see [13], pp. 281–325. Roughly spoken the result is that, up to $\partial^+ B$, x is as smooth as the curve C itself is. The crucial point in proving regularity at the free boundary portion $\partial^- B$ is to show continuity of x on $B \cup \partial^- B$. For solutions of (1), i.e. for area minimizing

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minimal surfaces, regularity at the free boundary has been proved by J.C.C. Nitsche in [12], K.H. Goldhorn and S. Hildebrandt in [6] and by W. Jäger in [10]. These results are to the effect that $x \in C^{k,\mu}(B \cup \partial^- B)$ if the supporting surface S is of that class and is admissable in some sense. For $k \geqq 3$ this has been proved by W. Jäger in [10]. The case $S \in C^{2,\mu}$ has been investigated by J.C.C. Nitsche in [14]. M. Grüter, S. Hildebrandt and J.C.C. Nitsche in [8] and the author in [4] proved regularity of the free boundary of stationary points x of the variational problem (1) under slightly different assumptions. In both papers the supporting surface S was assumed to be of class C^3.

The aim of this paper is to manage the cases $S \in C^{2,\mu}$ and $S \in C^2(C^{1,1})$ which means that we prove: $x \in C^{2,\mu}(B \cup \partial^- B)$ in the first case and $x \in C^{1,\mu}(B \cup \partial^- B)$ for every $\mu \in (0,1)$ in the second case. The proof consists in reflecting a given stationary solution x at the supporting surface S. W. Jäger proved in [10] weak transversality for such stationary solutions and this we use to prove that the reflected solution solves a quasilinear elliptic system of second order. In general there is no continuity for solutions of such systems. But using the conformality relations in (2) we may apply a method developed by M. Grüter in [7] to prove regularity.

2. Results

First of all we have to specify what an admissable supporting surface S is. We adopt W. Jäger's notions since they are easy to verify. For example every compact hypersurface S in $\mathbb{R}^N$ given by $f(x)=0$, $f \in C^{m,\mu}(\mathbb{R}^N, \mathbb{R})$, $\nabla f \neq 0$ on S, represents such an admissable supporting surface.

Definition 1. Let S be an $(N-1)$-dimensional $C^{m,\mu}$-manifold in $\mathbb{R}^N$ with the following properties: For every point $x_0 \in S$ there is some neighbourhood U_0 of x_0 in $\mathbb{R}^N$ and some real-valued function $f_0 \in C^{m,\mu}(U_0)$ $(m \geqq 2,\ 0 \leqq \mu \leqq 1)$ with $\nabla f_0 \neq 0$ in U_0 and $f(x)=0$ iff $x \in S \cap U_0$. There is some positive number d and there are functions ξ, a, $n \in C^{m-1,\mu}(U_d)$ in the strip $U_d = \{x \in \mathbb{R}^N \mid \operatorname{dist}(x,S) < d\}$ such that every $x \in U_d$ can be written as

$$x = a(x) + \xi(x)\, n(x)$$

where $a(x) \in S$, $n(x)$ is normal to S in $a(x)$ $(|n(x)|=1)$ and $|\xi(x)| = \operatorname{dist}(x,S)$. In addition to that we assume:

$$\sup_{U_d} |\nabla n| \leqq C_1.$$

Under these conditions we call S an admissable supporting surface of class $C^{m,\mu}$.

Let us look now at the basic variational problem (1) presented in the introduction. Since we are concerned with the free boundary portion only we may localize the situation by cutting out a neighbourhood of a point in $\partial^- B$ and we may achieve the following situation by conformal mapping. At this point it is obvious that our restriction to the special boundary configuration (C, S) is only for the sake of simplicity.

Definition 2. Let $x \in C^0(\overline{Q^+} \setminus \bar{I})$ be a solution of (2) on $Q^+ = \{(u,v) \mid |u| < 1, 0 < v < 1\}$ where $I = (-1,1) \times \{0\}$. We call $y \in C^0(\overline{Q^+} \setminus \bar{I}) \cap H^{1,2}(Q^+)$ admissable relative to x, if

$$\lim_{(u,v) \to (u_0, 0)} \operatorname{dist}(y(u,v), S) = 0 \tag{3}$$

for every $u_0 \in [-1, 1]$ and if $y = x$ on $\partial Q^+ \setminus \bar{I}$. x is called stationary if

$$\frac{d}{d\varepsilon} \int_{Q^+} |y_u^{(\varepsilon)}|^2 + |y_v^{(\varepsilon)}|^2 \, du \, dv \bigg|_{\varepsilon = 0} = 0 \tag{4}$$

exists for admissable $y^{(\varepsilon)} = x + \varepsilon z^{(\varepsilon)}$ $(|\varepsilon| < \varepsilon_0)$.

We now are able to formulate our results:

Theorem. *Let S be an admissable supporting surface of class $C^{m,\mu}(m \in \mathbb{N},\ m \geqq 2,\ 0 \leqq \mu \leqq 1)$ and let x be a stationary minimal surface on Q^+. Then*

$$x \in C^{m,\mu}(Q^+ \cup I), \qquad \text{if } 0 < \mu < 1,$$

$$x \in C^{m-1,\nu}(Q^+ \cup I) \quad \text{for every } \nu \in (0,1), \text{ if } \mu = 0$$

and

$$x \in C^{m,\nu}(Q^+ \cup I) \quad \text{for every } \nu \in (0,1), \text{ if } \mu = 1.$$

As the reader will see, the classes $C^{m-1,1}$ for S generate the same regularity properties for x as $S \in C^{m,0}$ does. From now on we will assume the assumptions of the theorem to be satisfied. Because of (3) we find some $\delta > 0$ such that $x(\bar{Q}_\delta^+ \setminus I) \subset U_d$, where $Q_\delta^+ = \{(u,v) \mid |u| < 1, 0 < v < \delta\}$.

3. Preliminary Lemmata

In [10], Lemma 1 W. Jäger proves weak transversality of stationary free minimal surfaces on the free boundary:

1. Lemma. *For every $\Phi \in C_0^1(Q^+ \cup I)$*

$$\lim_{\substack{v \to 0 \\ v > 0}} \int_{-1}^{1} (x_v - n(x) \cdot x_v n(x)) \cdot \Phi \, du = 0. \tag{5}$$

Let us reflect the solution surface x at the supporting surface S

$$y(u,v) = \begin{cases} x(u,v) & \text{in } (u,v) \in \bar{Q}_\delta^+ \\ x(u,-v) - 2\xi(x(u,-v))\, n(x(u,-v)) & \text{if } (u,-v) \in Q_\delta^+ \end{cases} \tag{6}$$

and claim that y solves some quasilinear elliptic system on $Q_\delta = \{(u,v) \mid |u| < 1, |v| < \delta\}$. We introduce some abbreviations related to the reflection mapping $y = F(x) = x - 2\xi(x)\, n(x)$. Define the matrices H, B and T by $H_{ik} = \xi_{x_i x_k}$, $T_{ik} = \delta_{ik} - 2n_i n_k$ $(i, k = 1, \ldots, N)$ and $B = (I - 2\xi H)^{-1}$. Then the following relations are easily verified: $Hn = 0$, $HT = TH = H$, $\dfrac{\partial F}{\partial x} = TB^{-1}$, $\det \dfrac{\partial F}{\partial x} = -1 + O(d)$ for small d, $F = F^{-1} \in C^{m-1,\mu}(U_d)$, $\xi \circ F = -\xi$, $n \circ F = n$, $H \circ F = BH$, $B \circ F = B^{-1}$.

2. Lemma. *There is some $\delta>0$ such that y is in $H^{1,2}(Q_\delta)\cap C^1(Q_\delta\setminus I)$ and for every test function $\Phi\in \mathring{H}^{1,2}(Q_\delta)\cap L^\infty(Q_\delta)$ we have:*

$$\int_{Q_\delta} A y_u\cdot\Phi_u+A y_v\cdot\Phi_v\,du\,dv=\int_{Q_\delta} f\cdot\Phi\,du\,dv \tag{7}$$

and

$$A y_u\cdot A y_v=0,\qquad |A y_u|=|A y_v| \tag{8}$$

on $Q_\delta\setminus I$, where we have set $A=I$, $f=0$ on $\bar{Q}_\delta^+$ and $A=B(y)^{-1}$,

$$f=2(H(y)y_u\cdot B(y)^{-1}y_u+H(y)y_v\cdot B(y)^{-1}y_v)\,n(y)$$
$$-2(n(y)\cdot y_u H(y)y_u+n(y)\cdot y_v H(y)y_v)$$

on $Q_\delta^-=\{(u,v)\,|\,|u|<1,\ -\delta<v<0\}$.

Proof: Since $\xi\circ x=0$ on I continuously we may chose $\delta>0$ small enough for our calculations. For $\Phi\in C_0^1(Q_\delta)$ and small $\varepsilon>0$ we derive from the harmonicity of $x=y$ on Q^+:

$$\int_{Q_\delta^+\setminus Q_\varepsilon^+} y_u\cdot\Phi_u+y_v\cdot\Phi_v\,du\,dv=-\int_{-1}^{1} x_v\cdot\Phi\,du|_{v=\varepsilon}. \tag{9}$$

On the lower half plane the situation is more complicated. Let us agree for the following lines to take x, x_u and x_v in $(u,-v)$. On Q_δ^-:

$$y_u=T(x)\,B(x)^{-1}x_u,\qquad y_v=-T(x)\,B(x)^{-1}x_v.$$

From this and $x_{uu}+x_{vv}=0$ we infer:

$$\int_{Q_\delta^-\setminus Q_\varepsilon^-} y_u\cdot\Phi_u+y_v\cdot\Phi_v\,du\,dv=2\int_{Q_\delta^-\setminus Q_\varepsilon^-}((H(x)x_u\cdot x_u+H(x)x_v\cdot x_v)\,n(x)$$
$$+n(x)\cdot x_u H(x)x_u+n(x)\cdot x_v H(x)x_v)\cdot\Phi-\xi(x)\,H(x)x_u\cdot\Phi_u$$
$$+\xi(x)\,H(x)x_v\cdot\Phi_v\,du\,dv-\int_{-1}^{1}(x_v-2n(x)\cdot x_v n(x))(u,\varepsilon)\cdot\Phi(u,-\varepsilon)\,du.$$

We combine this with (9) and get:

$$\int_{Q_\delta\setminus Q_\varepsilon} A y_u\cdot\Phi_u+A y_v\cdot\Phi_v\,du\,dv=\int_{Q_\delta\setminus Q_\varepsilon} f\cdot\Phi\,du\,dv$$
$$-\int_{-1}^{1} x_v(u,\varepsilon)\cdot(\Phi(u,\varepsilon)-\Phi(u,-\varepsilon))\,du$$
$$-2\int_{-1}^{1}(x_v-n(x)\cdot x_v n(x))(u,\varepsilon)\cdot\Phi(u,-\varepsilon)\,du.$$

Transversality (5) completes the proof for $\varepsilon\to 0$.

M. Grüter proved in [7], (2.5) Proposition a fundamental inequality for surfaces given in conformal parameters. In our situation y is a smooth function on $Q_\delta\setminus I$ closely related to x. Here this inequality is derived easily by using that dx/dw is holomorphic and $\sum_{k=1}^{3}(dx_k/dw)^2\equiv 0$ on Q^+, if x is not a constant.

3. Lemma. *Assume that $x(\overline{Q_\delta^+\setminus I})\subset U_{d/2}$. Then for every ball $B(w_1,r)=\{w=(u,v)\,|\,|(u,v)-(u_1,0)|<r\}\Subset Q_\delta$ $(w_1=(u_1,0)\in I)$ and for every w_0*

$=(u_0, v_0)\in B(w_1, r)\setminus I$ *we have:*

$$\limsup_{\rho\to 0,\rho>0} \rho^{-2} \int_{B(w_1,r)\cap\{w \mid |y(w)-y(w_0)|<\rho\}} |y_u|^2+|y_v|^2\,du\,dv \geqq \left(\frac{1-2dC_1}{1+2dC_1}\right)^2 2\pi. \quad (10)$$

Now we are going to prove the main Lemma of this paper. It will enable us to prove continuity of y in $w_1\in I$. The most important step is to control the integral in (10) from above uniformly in w_0.

4. Lemma. *Let w_1, $B(w_1,r)$ and w_0 be as in 3. Lemma and assume that $x(\overline{Q_\delta^+}\setminus I)\subset U_{d/6}$. Then there is some $R_0>0$ such that for every $R\in[0,R_0]$ from*

$$\inf_{\partial B(w_1,r)} |y-y(w_0)|>R \quad (11)$$

it follows that

$$R\leqq C_2\Big(\int_{B(w_1,r)} |y_u|^2+|y_v|^2\,du\,dv\Big)^{1/2}.$$

The constants R_0 and C_2 depend on d and C_1 only.

Proof. Chose $\lambda\in C^1(\mathbb{R})$ such that $0\leqq\lambda\leqq 1$, $\lambda'\geqq 0$, $\lambda=0$ on $(-\infty,0]$, $\lambda=1$ on $[\varepsilon,\infty)$, $\lambda>0$ on $(0,\infty)$ where ε is some positive number to be specified later. Let us define

$$z=\begin{cases} y-y(w_0) & \text{on } \overline{Q_\delta^+} \\ y-y(w_0)-2\xi(y)(n(y)-n(y(w_0))) & \text{on } Q_\delta^- \end{cases}$$

and introduce the test function

$$\Phi=\begin{cases} z\lambda(\rho-|z|) & \text{on } B(w_1,r) \\ 0 & \text{elsewhere,} \end{cases}$$

where $\rho\in(0,(1-dC_1/3)R)$. *Φ is a test function for* (7) *since $\Phi=0$* for $|z|\geqq\rho$ and $|z|\geqq|y-y(w_0)|-d/3\,|n(y)-n(y(w_0))|\geqq(1-dC_1/3)|y-y(w_0)|$ for $|z|<\rho$, so that (11) gives us $\Phi=0$ on $\partial B(w_1,r)$. Thus we may insert Φ into (7) and estimate the integrands separately on Q_δ^+ and Q_δ^-. On $Q_\delta^+\cap B(w_1,r)$ we have:

$$\begin{aligned} Ay_u\cdot\Phi_u+Ay_v\cdot\Phi_v &=(|y_u|^2+|y_v|^2)\,\lambda(\rho-|z|)-\left(\left(\frac{z}{|z|}\cdot y_u\right)^2+\left(\frac{z}{|z|}\cdot y_v\right)^2\right)|z|\,\lambda'(\rho-|z|) \\ &\geqq(|y_u|^2+|y_v|^2)(\lambda(\rho-|z|)-\tfrac{1}{2}\rho\lambda'(\rho-|z|)). \end{aligned} \quad (12)$$

Here we used the conformality relations (8) with $A=I$. On $Q_\delta^-\cap B(w_1,r)$ we get:

$$\begin{aligned} &Ay_u\cdot\Phi_u+Ay_v\cdot\Phi_v \\ &=\{|B(y)^{-1}y_u|^2+|B(y)^{-1}y_v|^2-2(n(y)\cdot y_u B(y)^{-1}y_u+n(y)\cdot y_v B(y)^{-1}y_v) \\ &\quad\cdot(n(y)-n(y(w_0)))\}\,\lambda(\rho-|z|)-\left\{\left(\frac{z}{|z|}\cdot B(y)^{-1}y_u\right)^2+\left(\frac{z}{|z|}\cdot B(y)^{-1}y_v\right)^2\right. \\ &\quad-2\left(n(y)\cdot y_u\frac{z}{|z|}\cdot B(y)^{-1}y_u+n(y)\cdot y_v\frac{z}{|z|}\cdot B(y)^{-1}y_v\right)\frac{z}{|z|} \\ &\quad\left.\cdot(n(y)-n(y(w_0)))\right\}|z|\,\lambda'(\rho-|z|) \\ &\geqq(|B(y)^{-1}y_u|^2+|B(y)^{-1}y_v|^2)((1-C_3\rho)\,\lambda(\rho-|z|)-\tfrac{1}{2}\rho(1+C_3\rho)\,\lambda'(\rho-|z|)), \end{aligned} \quad (13)$$

$C_3 = 2C_1/(1-dC_1/3)$. In this estimate we have used (8) with $A = B(y)^{-1}$ and $|n(y) - n(y(w_0))| \leqq C_1 |y - y(w_0)| \leqq (C_3/2)\rho$ since $|z| \leqq \rho$.

The right hand side of system (7) can be estimated from above as follows:

$$f \cdot \Phi \leqq C_4(|B(y)^{-1} y_u|^2 + |B(y)^{-1} y_v|^2)\, \rho\, \lambda(\rho - |z|). \tag{14}$$

If we write

$$\Psi(\rho) := \int_{B(w_1, r)} (|A y_u|^2 + |A y_v|^2)\, \lambda(\rho - |z|)\, du\, dv,$$

estimates (12), (13) and (14) give us:

$$(1 - C_3 \rho)\, \Psi(\rho) - \tfrac{1}{2}\rho(1 + C_3 \rho)\, \Psi'(\rho) \leqq C_4 \rho\, \Psi(\rho),$$

whence

$$-\frac{d}{d\rho}(\rho^{-2}(1 + C_3 \rho)\, \Psi(\rho)) \leqq C_5 \rho^{-2}(1 + C_3 \rho)\, \Psi(\rho)$$

and integration from ρ_1 to ρ_2, $0 < \rho_1 < \rho_2 < (1 - dC_1/3)\, R$ yields

$$\rho_1^{-2}(1 + C_3 \rho_1)\, \Psi(\rho_1) \leqq e^{C_5(\rho_2 - \rho_1)} \rho_2^{-2}(1 + C_3 \rho_2)\, \Psi(\rho_2).$$

The right hand side of this inequality can be estimated from above by

$$C_6 \rho_2^{-2} \int_{B(w_1, r)} |y_u|^2 + |y_v|^2\, du\, dv$$

and the left hand side from below by

$$C_7 \rho^{-2} \int_{B(w_1, r)} \lambda(\rho - |y - y(w_0)|)(|y_u|^2 + |y_v|^2)\, du\, dv,$$

where we have written $\rho = \rho_1/(1 + dC_1/3)$.

We take $\varepsilon \in (0, \rho)$ and employ the properties of λ to arrive at

$$\rho^{-2} \int_{B(w_1, r) \cap \{w \mid |y(w) - y(w_0)| < \rho - \varepsilon\}} |y_u|^2 + |y_v|^2\, du\, dv \leqq C_8 \rho_2^{-2} \int_{B(w_1, r)} |y_u|^2 + |y_v|^2\, du\, dv.$$

For $\varepsilon \to 0$, $\rho \to 0$ and $\rho_2 \to (1 - dC_1/3) R$ we infer from 3. Lemma for some constant C_2 the inequality

$$C_2^{-2} \leqq R^{-2} \int_{B(w_1, r)} |y_u|^2 + |y_v|^2\, du\, dv$$

which proves the Lemma.

4. Proof of the Theorem

First of all let us recall that 4. Lemma states the smallness of $\inf |y - y(w_0)|$ taken over $\partial B(w_1, r)$ for small Dirichlet integral uniformly in $w_0 \in B(w_1, r) \setminus I$. As we shall show now this and the Courant-Lebesgue Lemma imply the continuity of y in w_1.

5. Lemma. *$y \in C^{0,\nu}(Q_\delta)$ for every $\nu \in (0,1)$.*

Proof. Let $w_1 \in I$ and let $\delta > 0$ be small enough. Assume that $B(w_1, r_1) \Subset Q_\delta$. For given $R \in (0, R_0]$ we chose r_2 from $(0, r_1]$ such that

$$C_2\Big(\int_{B(w_1,r_2)} |y_u|^2 + |y_v|^2\, du\, dv\Big)^{\frac{1}{2}} < R.$$

According to the Lemma of Courant-Lebesgue (see [9]) there is some r_3 from $[r_2/2, r_2]$ with

$$\operatorname*{osc}_{\partial B(w_1,r_3)} y \leqq (\pi/\log 2)^{\frac{1}{2}}\Big(\int_{B(w_1,r_2)} |y_u|^2 + |y_v|^2\, du\, dv\Big)^{\frac{1}{2}}.$$

5. Lemma gives us for $r = r_3$ and $w_0 \in B(w_1, r_3) \setminus I$:

$$\inf_{\partial B(w_1,r_3)} |y - y(w_0)| \leqq R.$$

On account of

$$|y(w_{01}) - y(w_{02})| \leqq |y(w_{01}) - y(w')| + |y(w') - y(w'')| + |y(w'') - y(w_{02})|$$

$(w_{01}, w_{02} \in B(w_1, r_3),\ w',\ w'' \in \partial B(w_1, r_3))$, we obtain that

$$|y(w_{01}) - y(w_{02})| \leqq C_9 R$$

or

$$\operatorname*{osc}_{B(w_1,r_3)} y \leqq C_9 R$$

and y is continuous in w_1.

The system (7) can be written as

$$\int_{Q_\delta} y_u \cdot \Phi_u + y_v \cdot \Phi_v\, du\, dv = \int_{Q_\delta} f \cdot \Phi + g_1 \cdot \Phi_u + g_2 \cdot \Phi_v\, du\, dv$$

with suitable $|g_1| + |g_2| \leqq C_{10}(|y_u| + |y_v|) \sup_{Q_\delta} |\xi(y)|$. Since the oscillation of y on small balls is small we may apply Theorem 3 from [3] to get Hölder continuity for every Hölder exponent.

Thus we have proved the crucial starting regularity of the free boundary. For $S \in C^{2,\mu}$ $(\mu > 0)$ higher regularity follows from well known regularity results. But since we assume S only to be of class $C^2(C^{1,1})$ we have to prove Hölder continuity of the first derivatives of x. This will be done by reflection of the complex derivative of x.

6. Lemma. *$x \in C^{1,\nu}(Q^+ \cup I)$ for every $\nu \in (0,1)$.*

Proof. First of all we observe that $x \in H^{1,p}(Q_{(\delta)} \cap Q^+)$ for every $p \in [1, \infty)$, where $Q_{(\delta)} = \{(u,v) \mid |u| < 1 - \delta,\ |v| < 1 - \delta\}$. This follows easily from the fact that x is harmonic on Q^+ and Hölder continuous on $Q^+ \cup I$ for every Hölder exponent:

$$|x_u(w_0)| + |x_v(w_0)| \leqq \frac{2}{v_0} \sup_{\{w \mid |w - w_0| < v_0\}} |x(w) - x(w_0)|.$$

But then $\xi(x)\in H^{1,p}(Q_{(\delta)}\cap Q^+)$ and $\Delta\xi(x)\in L^p(Q_{(\delta)}\cap Q^+)$ because of (15). $\xi(x)$ vanishes on I and from Theorem 15.1 in [11] we obtain:

$$\xi(x)\in C^{1,\nu}(\bar{Q}_{(\delta')}\cap\overline{Q^+}) \quad \text{for every } \nu\in(0,1) \text{ and } \delta'>\delta.$$

In order to prove the same regularity for x itself we employ a reflection trick for the complex derivative $dx/dw=(x_u-ix_v)/2$, $(w=u+iv)$.

We define

$$F(w)=\begin{cases}\dfrac{dx}{dw}(w) & \text{on } Q^+\\[2ex] T(x(\bar{w}))\dfrac{dx}{d\bar{w}}(\bar{w}) & \text{on } Q^-.\end{cases}$$

and show that for every complex valued function Φ from $C_0^1(Q_{(\delta')})$

$$\int_{Q_{(\delta')}} F\Phi_{\bar{w}}\,du\,dv=\int_{Q_{(\delta')}} G\Phi\,du\,dv$$

where $G=0$ on Q^+ and

$$G(w)=-\sum_{j=1}^{N}\frac{\partial T}{\partial x_j}(x(\bar{w}))\frac{dx_j}{dw}(\bar{w})\frac{dx}{d\bar{w}}(\bar{w})$$

on Q^-. This is done in the same way as in the proof of 2. Lemma, if we make use of transversality (5) and of $\xi(x)_u=n(x)\cdot x_u=0$ on the real axis. Theorem 1.17 in [15] says that

$$F(w)=F_0(w)-\frac{1}{\pi}\int_{Q_{(\delta'')}}\frac{G(z)}{z-w}\,d\alpha\,d\beta$$

$(z=\alpha+i\beta)$ almost everywhere on $Q_{(\delta'')}$ and F_0 is some holomorphic function. Since $G\in L^p$ for every $p<\infty$, Theorem 1.20 in [15] gives us: $F\in C^{0,\nu}(Q_{(\delta''')})$ $(\delta'''>\delta'')$ for every $\nu\in(0,1)$. Thus 6. Lemma is proved. The *proof of* our *theorem* follows directly from 6. Lemma if we use well known results on boundary regularity of Dirichlet and Neumann problems (see Theorem 12.1 in [11], Theorem 6.26 in [5] and Theorem 6.2 in [1]. We only have to formulate the boundary value problems. x is a solution of the problem

$$\Delta x=0 \text{ on } Q^+, \quad x_v=\xi(x)_v\,n(x) \text{ on } I$$

and $\xi(x)$ solves

$$\Delta\xi(x)=H(x)x_u\cdot x_u+H(x)x_v\cdot x_v \text{ on } Q^+, \quad \xi(x)=0 \text{ on } I. \tag{15}$$

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