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On Krull domains

By

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Introduction. One aim of this article is to provide for Krull domains a star-operation analogue of the following result: An integral domain D is a Dedekind domain if and only if each nonzero ideal A of D is strongly two generated. A nonzero ideal A of an integral domain D is called *strongly two generated* if for each $x \in A \backslash \{0\}$ there is $y \in A$ such that $A = xD + yD$. Lantz and Martin show in [17] that a strongly two generated ideal is invertible. Following this lead we define a strongly $*$-type 2 ideal, for a star-operation $*$, as a nonzero ideal A such that for each $x \in A \backslash \{0\}$, there is $y \in A^*$ such that $(x, y)^* = A^*$. Then in Section 1 we characterize Krull domains in terms of strongly $*$-type 2 ideals.

Recently there has been considerable activity [1, 7, 12, 26] (some of it inspired by an earlier preprint version of the present paper) in characterizing a Krull domain in terms of the $*$-invertibility of some or all fractional ideals of D. These results are interesting in that they indicate that most of the characterizations of Dedekind domains have $*$-operation analogues for Krull domains.

In Section 2 we continue this line of investigation by coordinating some of the recent results with some new characterizations of Krull domains in terms of $*$-invertibility.

1. Star operations and strongly $*$-type 2 ideals. Throughout this paper, we shall use D to denote an integral domain with quotient field K. Also $F(D)$ will denote the set of nonzero fractional ideals of D while $f(D)$ will denote the subset of finitely generated members of $F(D)$. We review the definition of star-operations (abbreviated $*$-operations).

Definition. A star-operation on D is a mapping $A \to A^*$ of $F(D)$ into $F(D)$ which satisfies, for each $a \in K \backslash \{0\}$ and each $A, B \in F(D)$, the following conditions

(1) $(a)^* = (a)$, and $(aA)^* = aA^*$.
(2) $A \subseteq A^*$, and $A^* \subseteq B^*$ whenever $A \subseteq B$.
(3) $(A^*)^* = A^*$.

A fractional ideal $A \in F(D)$ is called a $*$-ideal if $A = A^*$. Moreover, for $A \in F(D)$, A is $*$-finite if there is $B \in f(D)$ such that $A^* = B^*$ and A is strictly $*$-finite if $A^* = B^*$ and $B \subseteq A$. A star-operation $*$ on D is said to be of finite character if for each $A \in F(D)$,

$$A^* = \cup \{B^* \mid B \subseteq A \text{ and } B \in f(D)\}.$$

If $*$ is of finite character, then $*$-finite and strictly $*$-finite are equivalent properties.

If $A \in F(D)$, define $A^{-1} = \{x \in K \mid xA \subseteq D\}$. Then A is $*$-invertible if $(AA^{-1})^* = D$. If $*$ has finite character and A is $*$-invertible, then both A^* and $A^{-1} = (A^{-1})^*$ are strictly $*$-finite [14, p. 30].

R e m a r k s. Many elementary properties of star-operations can be found in [9, Section 32] where it is pointed out that the mapping $A \to A^*$ of $I(D)$, the set of nonzero integral ideals of D, into $I(D)$ that satisfies (1), (2), and (3) above has a unique extension to a $*$-operation on D. Moreover, for any star-operation $*$ on D, there is always an associated mapping $A \to A^{*_s}$ of $F(D)$ into $F(D)$ where $*_s$ is defined by $A^{*_s} = \{J^* \mid J \subseteqq A \text{ and } J \in f(D)\}$. The mapping $*_s$ is a star-operation of finite character and $*_s$ is equivalent to $*$, that is, $B^* = B^{*_s}$ for all $B \in f(D)$.

If $\{D_\alpha\}_{\alpha \in I}$ is a family of overrings of D such that $D = \cap D_\alpha$, then $\{D_\alpha\}$ induces a star-operation $A \to A^* = \cap AD_\alpha$. If, in addition, each D is a quotient ring of D, then we say that $*$ is induced by quotient rings of D.

Three particular $*$-operations have special significance: the d-, v-, and t-operations. The d-operation is just the identity operation, that is, $A^d = A$ for each $A \in F(D)$. On the other hand, the mapping $A \to (A^{-1})^{-1} = A_v =$ the intersection of all principal fractional ideals that contain A, is called the v-operation. The t-operation is defined by $t = v_s$, or equivalently, for each $A \in F(D)$,

$$A_t = \{B_v \mid B \subseteqq A \text{ and } B \in f(D)\}.$$

Commonly v-ideals are called divisorial ideals or reflexive ideals.

D e f i n i t i o n. If $A \in F(D)$, and $*$ is a star-operation on D, then A is called a strongly $*$-type 2 ideal of D if for each nonzero $\alpha \in A$, there exists $b \in A^*$ such that $A^* = (a, b)^*$.

First we extend a result of Lantz and Martin [17].

Theorem 1.1. *If $*$ is a star-operation induced by quotient rings of D, then every strongly $*$-type 2 ideal of D is $*$-invertible.*

P r o o f. Let $*$ be induced by the family of quotient rings $\{D_{s_\alpha}\}_{\alpha \in I}$ and let $A \in F(D)$ be such that A is a strongly $*$-type 2 ideal of D. Then as $A^*D = AD_{s_\alpha}$ for each $\alpha \in I$ [9, p. 396], we conclude that AD_{s_α} is a strongly two generated ideal of D_{s_α}. But then according to [17], AD_{s_α} is an invertible ideal of D_{s_α}.

Now consider $(A^{-1}A)^* D_{s_\alpha} = (A^{-1}A) D_{s_\alpha} = (A^{-1}D_{s_\alpha})(AD_{s_\alpha})$. Since A is a strongly $*$-type 2 ideal of D, it follows that A is a strongly v-type 2 ideal of D. Hence $A^* = (a, b)^*$ implies that $A_v = (a, b)_v$ for $a, b \in A_v$. Therefore, $A^{-1} = A_v^{-1} = (a, b)_v^{-1} = (a, b)^{-1} = D : (a, b)$ and $A^{-1} D_{s_\alpha} = D_{s_\alpha} : (a, b) D_{s_\alpha} = ((a, b) D_{s_\alpha})^{-1} \subseteqq (AD_{s_\alpha})^{-1}$. But since the opposite containment is obvious, $A^{-1} D_{s_\alpha} = (AD_{s_\alpha})^{-1}$ for each α. Hence, $\cap (A^{-1} D_{s_\alpha})(AD_{s_\alpha}) = \cap (AD_{s_\alpha})^{-1}(AD_{s_\alpha}) = \cap D_{s_\alpha} = D$. Thus, $(A^{-1}A)^* = D$ and A is $*$-invertible.

Now the characterization.

Proposition 1.2. *An integral domain D is a Krull domain if and only if*

(i) *the family $\{D_M \mid M \in t - \max(D)\}$ induces the v-operation on D and*
(ii) *each $A \in F(D)$ is strongly v-type 2.*

P r o o f. Suppose that (i) and (ii) hold and let $A \in F(D)$. Then $A_v = \cap AD_M$. So $A_v D_M = AD_M$ for each maximal t-ideal M. By (ii) for each non-zero $a \in AD_M$ there is $b \in AD_M (= A_v D_M)$ such that $AD_M = (a, b) D_M$. That is AD_M is strongly two generated and thus principal. On the other hand as $A_v = (a, b)_v$ for some $a, b \in A_v$ (by (ii)) and as $A^{-1} = (A_v)^{-1}$ we have $A^{-1} = (1/a) \cap (1/b)$. But then

$$\begin{aligned} A^{-1} D_M &= (1/a) D_M \cap (1/b) D_M \\ &= ((a, b) D_M)^{-1} = (((a, b)_v D_M)_v)^{-1} = (AD_M)^{-1}. \end{aligned}$$

Clearly as AD_M is principal so is $(AD_M)^{-1}$. But then $\bigcap_{M \in t-\max D} (AA^{-1})D_M = D$, which means that AA^{-1} belongs to no maximal t-ideal of D, thus implying that A is t-invertible. Now according to Jaffard [14] D is Krull if and only if each non-zero ideal of D is t-invertible.

For the converse recall from [5, p. 485, Corollary 1] the following statement: "Let A be a Krull domain, K its field of fractions and $\underset{\sim}{a}$, $\underset{\sim}{b}$ and $\underset{\sim}{c}$ three divisorial ideals of A such that $\underset{\sim}{a} \subseteq \underset{\sim}{b}$. There exists $x \in K$ such that $\underset{\sim}{a} = \underset{\sim}{b} \cap x\underset{\sim}{c}$." Now let $\underset{\sim}{a}$ be a divisorial ideal and $x \in \underset{\sim}{a}\backslash(0)$ then $\underset{\sim}{a}^{-1} \subseteq (1/x)$ and by the above result there is $y \in K$ such that $\underset{\sim}{a}^{-1} = (1/x) \cap yD$. So that $\underset{\sim}{a} = (x, y^{-1})_v$. That is, each divisiorial ideal $\underset{\sim}{a}$ is strongly v-type 2. Indeed to say that every divisorial ideal is strongly v-type 2 is equivalent to saying that every non-zero ideal is strongly v-type 2. Combining this with the fact that in a Krull domain $t - \max(D)$ induces the v-operation, we get the result.

Remarks. We have observed that Corollary 1 of [5, p. 485] can be used to conclude that every divisorial ideal of a Krull domain is strongly v-type 2. This raises two questions:

(1) Does the statement "Every divisorial ideal is strongly v-type 2" characterize Krull domains?

(2) Are Krull domains characterized by the property announced in Corollary 1 of [5, p. 485]; that is, are Krull domains characterized by the statement: "For divisorial ideals A, B, and C of D such that $A \subseteq B$, there is an element $x \in K$, the quotient field of D, such that $A = B \cap xC$"?

The answer to both questions is no, unless the v-operation is induced by localizations at maximal t-ideals of D. To explain this asssertion let us recall some definitions. In [5, p. 551] an integral domain D is called pseudo principal if the group of divisibility of D is a complete lattice ordered group. In such an integral domain every divisorial ideal is principal and hence strongly of v-type 2. Indeed the statement of Corollary 1 of [5, p. 485] is also satisfied by pseudo principal ideal domains. But there are non-Krull examples: a valuation domain with value group R [4, p. 551], the ring of entire functions [25] and the polynomial rings thereof [25] and [2].

On the other hand, if we decide to give Krull domains a closer look we can prove the following slightly stronger result.

Proposition 1.3. *Let A be a non-zero ideal of a Krull domain D and let $x \in A\backslash\{0\}$. Then there exists $y \in A$ such that $(x, y)_v = A_v$.*

To prove this proposition we need the following lemma which is a slight deviation from a well-known statement for divisorial ideals.

Lemma 1.4. *Let A be a non-zero integral ideal of a Krull domain D such that*

$$A_v = P_1^{(n_1)} \cap P_2^{(n_2)} \cap \ldots \cap P_r^{(n_r)}$$

where each P_i is a rank one prime ideal of D and each n_i is a positive integer. Then there is at least one $z \in A$ such that

$$(z) = P_1^{(n_1)} \cap P_2^{(n_2)} \cap \ldots \cap P_r^{(n_r)} \cap Q_1^{(m_1)} \cap \ldots \cap Q_t^{(m_t)}$$

for some (possibly empty) set of rank one prime ideals $\{Q_j\}$ where each m_j is a positive integer.

Proof. If the total number of rank one primes of D is finite, then every ideal of D is principal and the proposition holds vacuously. So we assume that the number of rank one primes of D is infinite. Also if A is divisorial the existence of $z \in A$ with the claimed property is well-known [8]. So we assume that A is not divisorial. Now as

$$A_v = \bigcap_{P \in X^1(D)} AD_P \quad \text{and as} \quad A_v = \cap P_i^{(n_i)};$$

we conclude that for each $i\,(=1,\ldots,r)$ there is, $x_i \in A$ such that $x_i\, Dp_i = A\,D_{P_i} = P_i^{n_i}\,D_{P_i}$. In short $Vp_i(x_i) = n_i$ for $i = 1,\ldots,r$ where Vp_i is the valuation corresponding to the ring D_{P_i}. By rearranging the prime ideals P_i we may assume that $Vp_1(x_1) = n_1$, $Vp_2(x_1) = n_2, \ldots, Vp_s(x_1) = n_s$. If $s = r$, we are done. If $s < r$ we find $y \in A$ such that $Vp_i(y) = n_i$, $i = 1, 2, \ldots, s+1$. For this pick $k_{s+1} \in J = P_1^{(n_1+1)} \cap \ldots \cap P_s^{(n_s+1)} \cap A$ with $Vp_{s+1}(k_{s+1}) = n_{s+1}$. But then $x_1 + k_{s+1} \in A$ and $Vp_i(x_1 + k_{s+1}) = n_i$ for $i = 1, \ldots, s+1$. Proceeding in this manner we can find $z \in A$ with the required property.

Proof of Proposition 1.3. We consider two cases according to whether or not $A_v = D$. If $A_v = D$ and $x \in A\backslash\{0\}$, then $(x) = P_1^{(a_1)} \cap P_2^{(a_2)} \cap \ldots \cap P_n^{(a_n)}$ where each P_i is a rank one prime ideal of D and each a_i is a positive integer. Since $A \nsubseteq P_1 \cup \ldots \cup P_n$, there is a $y \in A\backslash(P_1 \cup \ldots \cup P_n)$ [16, Theorem 124]. But then (x, y) is contained in no rank one prime ideal of D and so $(x, y)_v = D$.

If, on the other hand, $A_v \neq D$, then let $A_v = P_1^{(n_1)} \cap \ldots \cap P_r^{(n_r)}$ where each P_i is a rank one prime ideal of D and each n_i is a positive integer. Then by Proposition 1.2 there exists $z \in A$ such that

$$(z) = P_1^{(n_1)} \cap \ldots \cap P_r^{(n_r)} \cap Q_1^{(m_1)} \cap \ldots \cap Q_s^{(m_s)}$$

for some (possibly empty) set of rank one prime ideals $\{Q_j\}$ where each m_j is a positive integer. If

$$(x) = P_1^{(a_1)} \cap \ldots \cap P_r^{(a_r)} \cap T_1^{(b_1)} \cap \ldots \cap T_t^{(b_t)} \subseteqq A$$

where each T_k is a rank one prime ideal and each a_i and each b_j is a positive integer. Then $a_i \geqq n_i$ for $i = 1, \ldots, r$.

Suppose that by rearranging the P_i's, $a_i = n_i$ for $i = 1, \ldots q$ and $a_i > n$ for $i > q$. Invoking [16, Theorem 124] again we can select $u \in P_1^{(2n_1)} \cap \ldots \cap P_q^{(2n_q)}$ so that $u \notin P_{q+1}, \ldots, P_r$. Then (x, uz) is such that $(x, uz)\, D_{pi} = A\, D_{pi}$ for $i = 1, \ldots, r$. Note that $uzD + A = A \nsubseteq T_1 \cup \ldots \cup T_t$ so, by [16, Theorem 124], there exists $a \in A$ such that $uz + a \notin T_1 \cup \ldots \cup T_t$. But then $(x, uz + a)_v = A_v$.

These results lead to the following conclusion.

Corollary 1.5. *Let D be an integral domain such that* $\{D_M \mid M \in t - \max(D)\}$ *induces the v-operation on D. Then the following equivalent:*

(1) *D is a Krull domain.*
(2) *For each* $A \in F(D)$ *and for each* $a \in A\backslash\{0\}$ *there exists* $b \in A$ *such that* $A_v = (a, b)_v$.
(3) *Each* $A \in F(D)$ *is strongly v-type* 2.
(4) *Each divisorial* $A \in F(D)$ *is strongly v-type* 2.
(5) *For every triple of divisorial ideals A, B, C of D with* $A \subseteqq B$ *there exists* $x \in K$ *such that* $A = B \cap xC$.

Proof. (1) → (5) is Corollary 1 of [5, p. 485] (5) → (4) as observed in the proof of Proposition 1.2, (4) → (3) is trivial and (3) → (1) is Proposition 1.2. Now (1) → (2) is Proposition 1.3 and (2) → (3) is trivial.

Remarks. If the family $\{M_\alpha\}$ is the set of maximal ideals of D, then $\{D_{M_\alpha}\}$ induces the identity star-operation, that is, the so-called d-operation. Moreover, a strongly d-type 2 ideal is strongly two generated in the sense of Lantz and Martin [17] and, therefore, must be invertible. An obvious corollary is that D is a Dedekind domain if and only if every nonzero ideal of D is a strongly d-type 2 ideal.

2. Krull domains and *t*-invertibility. In this section we collect several other equivalent formulations for a Krull domain stated in terms of t-invertibility. First we need some terminology and one simple proposition. If $*_1$ and $*_2$ are two $*$-operations defined on D, then $*_1$ is finer than $*_2$ if $A_{*_1} \subseteqq A_{*_2}$ for each $A \in F(D)$. In particular, if $*_1$ is finer than $*_2$, then any $*_2$ ideal is a $*_1$-ideal for if $A = A_{*_2}$, then $A \subseteqq A_{*_1} \subseteqq A_{*_2} = A$. Furthermore, if $*_1$ is finer than $*_2$ and $*_2$ is of finite character then $*_1$ is of finite character.

Proposition 2.1. *Suppose $*_1$ and $*_2$ are $*$-operations defined on D where $*_1$ is finer than $*_2$.*

(1) *If P is an integral $*_2$-ideal of D maximal among the set of non-$*_1$-invertible $*_2$-ideals of D, then P is a prime ideal of D.*
(2) *If $*_2$ is a $*$-operation of finite character, then the set of $*_2$-ideals that are non-$*_1$-invertible is inductive.*

Proof. (1) If P is not a prime ideal of D, then there are elements $a, b \in D \backslash P$ such that $ab \in P$. Let $I = (P, a)_{*_2}$ and $J = P : I$. Then I and J are $*_2$-ideals of D properly containing P since $a \in I$ and $b \in J$. Therefore, by the maximality of P, I and J are $*_1$-invertible.

Next we assert that $P = (IJ)_{*_1}$. To see this we need only prove that $J = (PI^{-1})_{*_1}$ because $(PI^{-1})_{*_1} = J$ implies that $((PI^{-1})_{*_1} I)_{*_1} = (JI)_{*_1} = (PI^{-1}I)_{*_1} = (P(I^{-1}I)_{*_1})_{*_1} = (P)_{*_1} \subseteqq P_{*_2} = P$. But $P \subseteqq (P)_{*_1}$ so $P_{*_1} = P_{*_2} = P$ and $P = ((PI^{-1})_{*_1} I)_{*_1}$. Thus, we show $(PI^{-1})_{*_1} = J$. Since $P \subseteqq I$ implies $PI^{-1} \subseteqq II^{-1} \subseteqq D$ which implies $PII^{-1} \subseteqq P$, $PI^{-1} \subseteqq J$. Therefore, $(PI^{-1})_{*_1} \subseteqq J_{*_1} \subseteqq J_{*_2} = J$. On the other hand, the definition of J implies $JI = (P:I)I \subseteqq P$. Hence, $(JI)_{*_1} \subseteqq P_{*_1} \subseteqq P_{*_2} = P$ and $((JI)_{*_1} I^{-1})_{*_1} \subseteqq (PI^{-1})_{*_1}$. But $((JI)_{*_1} I^{-1})_{*_1} = (J(II^{-1})_{*_1})_{*_1} = J_{*_1}$ since I is $*_1$-invertible. Also since J is a $*_2$-ideal, $J_{*_1} = J$. Therefore, $J = (JI_{*_1} I^{-1})_{*_1} \subseteqq (PI^{-1})_{*_1} \subseteqq J$ and we conclude $(PI^{-1})_{*_1} = J$ and, therefore, $P = (IJ)_{*_1}$.

Since I and J are both $*_1$-invertible it follows that P is $*_1$-invertible. This contradiction completes the proof of (1).

(2) Let (C_α) be a chain of $*_2$-ideals that are each non-$*_1$-invertible. Let $C = \cup C_\alpha$. We know that C is a $*_2$-ideal because $*_2$ is of finite character. For if B is any finite set contained in C, $B \subseteqq C_{\alpha 0}$ for some $\alpha 0$ and then $B_{*_2} \subseteqq (C_{\alpha 0})_{*_2} = C_{\alpha 0} \subseteqq C$. Therefore $C = \cup \{B_{*_2} \mid B$ is a finitely generated ideal contained in $C\}$. If C is $*_1$-invertible, then obviously C is $*_2$-invertible. But then, as $*_2$ is of finite character $C = (c_1, \ldots, c_k)_{*_2}$ [14, p. 30]. This leads to the conclusion that $C = C_\alpha$ for some α. But this contradicts the fact that each C_α is non-$*_1$-invertible.

Corollary 2.2. *Let $*$ be a $*$-operation defined on an integral domain D. If P is an integral $*$-ideal maximal among the set of non-$*$-invertible ideals of D, then P is a prime ideal of D.*

Proof. Let $*_1 = * = *_2$ in the statement of Proposition 2.1.

Corollary 2.3. *If P is an integral v-ideal of D maximal among the set of all non-invertible v-ideals of D, then P is a prime ideal of D.*

Proof. Let $*_1$ be the d-operation where $A_d = A$ and let $*_2$ be the v-operation.

Corollary 2.4. *If there is one non-t-invertible t-ideal of D, then there is a non-t-invertible prime t-ideal of D.*

Proof. Apply the above remark to the t-operation and use the fact that t is a $*$-operation of finite character to obtain a maximal non-t-invertible ideal of D by Proposition 2.1 (2).

An integral domain D is a Mori domain if D satisfies the ascending chain condition (ACC) on integral v-ideals. Nishimura [21, Lemma 1] originally published a result later rediscovered by Querre [22, Theorem 1], namely: an integral domain D is a Mori domain if and only if every $A \in F(D)$ is strictly v-finite.

In [18] we obtained an analogue of Cohen's Theorem for a v-domain D. We showed that if each prime t-ideal of a v-domain D is a t-ideal of finite type, then each t-ideal of D is of finite type and, therefore, D is a Mori domain. One might attempt to prove that result by weakening the assumption on D say, for example, by assuming only that D is integrally closed. The following example shows that our result in [18] is the best possible.

Example. It is well-known that $Q + xR[x]$ is a Mori domain [3] where Q and R are respectively the fields of rational and real numbers. In a similar fashion we can establish that $T = \bar{Q} + xR[x]$, where $\bar{Q}$ is the algebraic closure of Q in R, is an integrally closed Mori domain. Moreover we claim that $D = \bar{Q}[\pi] + xR[x]$ is an integral domain such that:

(1) Every prime t-ideal is either principal or equal to $xR[x]$. (See [20].)
(2) $xR[x] = (x/\alpha, x/\beta)_v$ where α and β are algebraically independent over $\bar{Q}(\pi)$.

Proof. Observe that $\left(\frac{x}{\alpha}, \frac{x}{\beta}\right)^{-1} = \dfrac{\left(\frac{x}{\alpha}\right) \cap \left(\frac{x}{\beta}\right)}{\frac{x^2}{\alpha}\beta}$. Now every element of $(x/\alpha) \cap (x/\beta)$ has to be of the type $x^2 f(x)$ where $f(x) \in R[x]$. So $\dfrac{\left(\frac{x}{\alpha}\right) \cap \left(\frac{x}{\beta}\right)}{\frac{x^2\beta}{\alpha}} = R[x]$. Consequently, $\left(\frac{x}{\alpha}, \frac{x}{\beta}\right)_v = xR[x]$. Nevertheless the ring D is not a Mori domain since $\{(x/\pi^i)D\}_{i=0}^{\infty}$ forms an infinite ascending chain of v-ideals of D.

A prime ideal P of an integral domain D where P is minimal over an integral t-ideal is itself a t-ideal; in particular, if P is minimal over an ideal A of the type $A = aD : bD$ where $(0) \neq A \neq D$, then P is a prime t-ideal, called an associated prime of (a principal ideal aD of) D [6].

Theorem 2.5. *Let D be an integral domain. Then the following are equivalent:*

(1) *D is a Krull domain.*
(2) *Every $A \in F(D)$ is t-invertible. (In the terminology of Jaffard* [14], *D is $t - \beta$ total.)*
(3) *Each prime t-ideal of D is t-invertible.*
(4) *Each t-ideal $A \in F(D)$ is t-invertible.*
(5) *Each associated prime ideal of D is t-invertible.*
(6) *D is completely integrally closed and each maximal t-ideal of D is t-invertible.*
(7) *D is completely integrally closed and each maximal t-ideal of D is divisorial.*
(8) *D is completely integrally closed and each t-ideal $A \in F(D)$ is divisorial.*
(9) *D is a Mori domain and each prime v-ideal of D is v-invertible.*
(10) *D is a Mori domain and each v-ideal of D is v-invertible.*
(11) *D is a Mori domain and completely integrally closed.*
(12) *D is a Mori domain and PD_P is principal for each prime t-ideal of D.*

Proof. (1) $\Rightarrow$ (2) by [14, p. 82]. Obviously (2) $\Rightarrow$ (3) and Corollary 2.4 shows (3) $\Rightarrow$ (4). Clearly (4) $\Rightarrow$ (5).

Now let us prove (5) $\Rightarrow$ (6). Using (b) of the proof of Theorem 1 of [18] we conclude that D is completely integrally closed and using (c) of the same theorem we conclude that every associated prime ideal of D is rank one. Because each associated prime is t-invertible (and hence strictly v-finite) we conclude, from Lemma 1.4 [18], that every associated prime is a maximal t-ideal. This in turn implies that every maximal t-ideal must be an associated prime (since it contains one). Hence each maximal t-ideal of D is t-invertible.

(6) $\Rightarrow$ (7) is easy since if a t-ideal is t-invertible, it is strictly v-finite and hence is a v-ideal (divisorial ideal). To show (7) $\Rightarrow$ (2) all we need prove is that $(AA^{-1})_v = D$ for all $A \in F(D)$. Suppose on the contrary that for some $A \in F(D)$, $(AA^{-1})_t \neq D$. Then, there must exist some maximal t-ideal P such that $(AA^{-1})_t \subseteqq P$. But then $AA^{-1} \subseteqq P$ implies that $(AA^{-1})_v \subseteqq P_v = P$, and this contradicts the fact that D is completely integrally closed.

Clearly (1) $\Rightarrow$ (8) $\Rightarrow$ (7). Moreover, (2) $\Rightarrow$ (9) since (2) implies each $A \in F(D)$ must be strictly v-finite.

Next to show that (9) $\Rightarrow$ (10) we note that in a Mori domain every t-ideal is a v-ideal.

Clearly (10) $\Rightarrow$ (11) by [9, p. 421]. Next we show (11) $\Rightarrow$ (12). For each $A \in F(D)$, $(AA^{-1})_v = D$ since D is completely integrally closed. But since D is a Mori domain A and A^{-1} are strictly v-finite so $(AA^{-1})_t = (AA^{-1})_v = D$.

(1) $\Rightarrow$ (12) is well known since a prime t-ideal P of a Krull domain D is such that D_P is a DVR. Conversely (12) $\Rightarrow$ (1) since for each prime t-ideal P of D, D_P is a DVR [4].

Remarks. The equivalence of (1) and (11) of Theorem 2.5 was proved originally by Nishimura [21]. In Theorem 2.5, we have improved the proof of one of our own earlier results. We showed in [18] that an integral domain D is a Krull domain if and only if each associated prime ideal of D is t-invertible. There we proved the sufficientcy in two steps. First, we showed that if every associated prime ideal of D is t-invertible, then D is a Prüfer v-multiplication domain where every maximal t-ideal of D has rank one. Then, using the notion of Kronecker function ring, we established that indeed D is a Krull domain. In the present proof we avoid the Kronecker function ring detour. Another proof of this result also appears in [12]. Finally, (1) $\Rightarrow$ (6), was improved by Kang [15] to: D is Krull if and only if every minimal prime of a principal ideal of D is t-invertible.

Corollary 2.6. *An integral domain D is Krull domain if and only if every maximal t-ideal is t-invertible and rank one.*

Proof. The implication ($\Rightarrow$) is well-known. For the converse note that under the condition every associated prime is a maximal t-ideal. Next apply (5) of Theorem 2.5.

Remarks. Now it is easy to see how the proofs of the following statements should go.

(1) D is a UFD if and only if every maximal t-ideal of D is principal and of rank one.
(2) D is a locally factorial Krull domain if and only if every maximal t-ideal of D is invertible and of rank one.
(3) D is almost factorial if and only if every maximal t-ideal P is of rank one and for each such prime ideal P there is n such that $(T^n)_t$ is principal. (See [24] or [8] for the definition of almost factorial domains.)
(4) D is locally factorial almost factorial domain if and only if every maximal t-ideal P is of rank one and for each such prime ideal P there is n such that P^n is principal.
(5) D is a Dedekind domain if and only if every maximal t-ideal of D is of rank one, maximal, and t-invertible.

Statement (5) can be restated as: D is Dedekind if and only if D is a field or every maximal ideal of D is invertible and of rank one.

Note that in each case the condition requiring each maximal t-ideal P to be of rank one is important.

Example. In $R = Z + xQ[x]$ every maximal ideal is a t-ideal and is principal. But because maximal t-ideals of the type $pZ + xQ[x]$ are not rank one, R cannot be a UFD or a PID. Another example: let R be a valuation ring where the maximal ideal M is principal and $\dim R > 1$. Then the maximal t-ideal M is t-invertible yet R is not a UFD or a PID.

We have shown that if every associated prime ideal P of an integral domain D is t-invertible, then D is a Krull domain. We now ask the question: Suppose that every maximal t-ideal of D is t-invertible, what additional conditions on D ensure that D is a Krull domain? We listed one condition in Theorem 2.5 (6). The Mori condition is sufficient, but this condition can be weakened to one concerning the ascending chain condition on principal ideals. We say that D satisfies the strong ACCP if D and D_M satisfy the ascending chain condition on principal ideals for each maximal t-ideal M of D.

Proposition 2.7. *If D satisfies the strong* ACCP *and if every maximal t-ideal M of D is t-invertible, then D is a Krull domain.*

Proof. By Corollary 2.6 all we need show is that each maximal t-ideal of D is of rank one. For this we note that if M is a maximal t-ideal of D, then MD_M is principal because M is t-invertible. Let $MD_M = pD_M$ where $p \in M$. Now the ACCP in D_M ensures that MD_M is of rank one for if $x \in \cap (MD_M)^n$ and $x \neq 0$, then we have the infinite ascending chain of principal ideals in $D_M : (x) \subsetneqq (x/p) \subsetneqq (x/p^2) \ldots$. Thus, $x = 0$ and MD_M is of rank one.

Corollary 2.8. *A Mori domain D is a Krull domain if and only if each maximal t-ideal of D is t-invertible.*

Remark. It is easy to see that an integral domain D with the ascending chain condition on principal ideals is a UFD if every maximal t-ideal of D is principal. This observation raises a question about the use of the strong ACCP property in Proposition 2.7. It is plausible that Proposition 2.7 may need only the ascending chain condition on principal ideals of D.

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