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Additive reduction of algebraic tori

By

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Let K be a number field and $T_{/K}$ a group scheme admitting a Néron model $\mathcal{T}$ over $\mathcal{O}$, the ring of integers of K. The connected components of the finite fibers of $\mathcal{T}$ are interesting arithmetic invariants of T. In the case of bad reduction, the description of these finite fibers is sometimes difficulted by the presence of unipotent components. If T is an algebraic torus and $\mathfrak{p}$ is a finite prime of K, the reduction of $\mathcal{T}^0$, the connected component of $\mathcal{T}$, modulo $\mathfrak{p}$ is an affine, connected, smooth group scheme over a finite field; hence, it has a canonical decomposition:

$$\mathcal{T}_{\mathfrak{p}}^0 := \mathcal{T}^0 \otimes_{\mathcal{O}} \mathcal{O}/\mathfrak{p} = T_{\mathfrak{p}} \times U,$$

where $T_{\mathfrak{p}}$ is a torus and U is unipotent. Since T is completely determined by an integral Galois representation:

$$\varrho : \mathrm{Gal}(\bar{K}/K) \to GL_d(\mathbb{Z}),$$

it should be possible to describe $T_{\mathfrak{p}}$ and U in terms of ϱ. The description of $T_{\mathfrak{p}}$ is easy (see Section 1), whereas the description of U in full generality is much more difficult to deal with.

We consider in this note an easier question: when is U isomorphic to a power of $\mathbb{G}_a$? Sometimes the fact that all these unipotent components are additive, enables one to carry on local-to-global processes. For instance, assuming additivity of the unipotent components and that the torus splits by an abelian extension of K, in [3] it is shown how to construct from the L-series of T an explicit formal group law for the formal completion of $\mathcal{T}$ along the zero section. Our aim is to prove the following:

(0.1) Theorem. *Let e be the ramification index of $\mathfrak{p}$ in the splitting field of T and let p be the prime number lying under $\mathfrak{p}$. Then*:

$$p > e \Rightarrow U \cong \mathbb{G}_a \times \cdots \times \mathbb{G}_a .$$

The proof is based on a theorem of Ono [6] stablishing an isogeny between a power of T and certain products of Weil restrictions of $\mathbb{G}_m$.

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1. Generalities. The toric component. It is clear that the study of $\mathcal{T}_{\mathfrak{p}}^0$ can be reduced to the local case. Therefore, we fix the prime number p once and for all and we assume throughout that K is a finite extension of $\mathbb{Q}_p$, $\mathcal{O}$ its ring of integers, $\mathfrak{p}$ the maximal ideal of $\mathcal{O}$ and k the residue field.

Let S be a scheme. A group scheme $\mathcal{T}$ over S is called a d-dimensional torus if there exists a surjective étale morphism, $S' \to S$, such that $\mathcal{T} \otimes_S S' \cong \mathbb{G}^d_{m,S'}$. The d-dimensional tori are thus classified by:

$$H^1(\pi_1(S, \bar{s}), \operatorname{Aut}(\mathbb{G}_m^d)) = \operatorname{Hom}(\pi_1(S, \bar{s}), GL(d, \mathbb{Z}));$$

that is, by continuous integral representations:

$$\varrho : \pi_1(S, \bar{s}) \to GL(d, \mathbb{Z}).$$

Now, let S denote either Spec (K), Spec $(\mathcal{O})$ or Spec (k). By the well-known canonical isomorphisms between $\pi_1(S, \bar{s})$ and respective Galois groups, we have a commutative diagram of functors:

$$\begin{array}{ccccc} \underline{k - tori} & \leftarrow & \underline{\mathcal{O} - tori} & \rightarrow & \underline{K - tori} \\ \downarrow & & \downarrow & & \downarrow \\ \underline{G_k - mods} & \leftarrow & \underline{G_{K^{nr}} - mods} & \rightarrow & \underline{G_K - mods} \end{array},$$

where $G_k = \operatorname{Gal}(\bar{k}, k)$, $G_K = \operatorname{Gal}(\bar{K}, K)$, $G_{K^{nr}} = \operatorname{Gal}(K^{nr}/K)$ and K^{nr} is the maximal unramified extension of K. In the upper horizontal row we have the base-change functors, in the lower horizontal row the natural functors deduced from the canonical identifications:

$$G_k \cong G_{K^{nr}} \cong G_K/I_K,$$

where I_K is the inertia subgroup. The vertical functors are the equivalence of categories:

$$X : \underline{S - tori} \to \underline{\pi_1(S, \bar{s}) - mods},$$

where $X(\mathcal{T})$ is the character group of $\mathcal{T}$; that is, the $\pi_1(S, \bar{s})$-module associated to the étale sheaf $\underline{Hom}(\mathcal{T}, \mathbb{G}_m)$. In particular, the functor $\underline{\mathcal{O} - tori} \to \underline{k - tori}$ is an equivalence of categories. Also, base change by $j : \operatorname{Spec}(K) \to \operatorname{Spec}(\mathcal{O})$ stablishes an equivalence between $\underline{\mathcal{O} - tori}$ and the full subcategory of $\underline{K - tori}$ of the tori with good reduction (see (1.1) below).

By definition, the Néron model of a smooth group scheme T over K is the sheaf $j_*(T)$ with respect to the smooth topology. Since j is smooth, $T \cong j^* j_*(T)$. By a theorem of Raynaud [4] (cf. also [1] 10.1), if T is a torus over K, then its Néron model is representable by a smooth group scheme $\mathcal{T}$ locally of finite type over $\mathcal{O}$. Hence, there is a group-scheme isomorphism:

$$\psi : \mathcal{T} \otimes_{\mathcal{O}} K \xrightarrow{\sim} T,$$

and functorial group isomorphisms, compatible with ψ:

$$\mathcal{T}(\mathcal{X}) \xrightarrow{\sim} T(\mathcal{X} \otimes_{\mathcal{O}} K),$$

for any smooth scheme $\mathscr{X}$ over $\mathcal{O}$. For instance, the Néron model $\mathscr{G}$ of $\mathbb{G}_m$ fits into the exact sequence:

$$1 \to \mathbb{G}_{m,\mathcal{O}} \to \mathscr{G} \to i_* \mathbb{Z} \to 1,$$

where $i: \mathrm{Spec}(k) \to \mathrm{Spec}(\mathcal{O})$ is the natural morphism. The connected component $\mathscr{T}^0$ of $\mathscr{T}$ is then an affine [5, Lemme IX 2.2] smooth group scheme over $\mathcal{O}$ of finite type and we have a canonical decomposition over k:

$$\mathscr{T}_{\mathfrak{p}}^0 := \mathscr{T}^0 \otimes_{\mathcal{O}} k = T_{\mathfrak{p}} \times U,$$

where $T_{\mathfrak{p}}$ is a torus and U is unipotent. The toric component is easy to describe. Let us see first the case of good reduction:

(1.1) Proposition-Definition. *Let $T_{/K}$ be a torus and $\mathscr{T}_{/\mathcal{O}}$ its Néron model. T has good reduction when it satisfies any of the following equivalent conditions:*

(1) *$\mathscr{T}_{\mathfrak{p}}^0$ is a torus over k;*
(2) *$\mathscr{T}^0$ is a torus over $\mathcal{O}$;*
(3) *there exists a torus over $\mathcal{O}$ with generic fiber isomorphic to T;*
(4) *I_K acts trivially on $X(T)$;*
(5) *T splits over an unramified extension of K.*

In this case, $X(\mathscr{T}_{\mathfrak{p}}^0)$ is isomorphic to $X(T)$ as G_k-module.

Proof. By [2, X, 8.2], $\mathscr{T}^0$ is a torus if and only if all its fibers are tori; hence, (1) is equivalent to (2). (2) $\Rightarrow$ (3) is clear and (3) $\Rightarrow$ (4) is a consequence of the commutative diagram of functors above. (4) $\Leftrightarrow$ (5) is also clear. Finally, (5) $\Rightarrow$ (2) is a consequence of the fact that the Néron model is stable by étale basis change [4]. □

In general, the toric component of $\mathscr{T}_{\mathfrak{p}}^0$ can be described as the reduction of the maximal subtorus of T with good reduction. This is well defined:

(1.2) Proposition. *Let T be a torus over K with splitting field L. Given a normal subgroup H of* Gal(L/K)*, there exists a unique subtorus T_H of T, maximal with the property that H acts trivially on $X(T_H)$. Moreover, $X(T_H) \cong X(T)/\ker(tr)$, where:*

$$tr: X(T) \to X(T)^H,$$

is the homomorphism defined by $tr(x) = \sum_{\sigma \in H} x^\sigma$.

Proof. Imitate [8, 7.4]. □

(1.3) Theorem. *Let T_0 be the maximal subtorus of T with good reduction. Then, $T_{\mathfrak{p}}$ is isomorphic to the reduction of the connected component of the Néron model of T_0. In particular,*

$$X(T_{\mathfrak{p}}) \cong X(T_0) \cong X(T)/\ker(X(T) \xrightarrow{tr} X(T)^{I_K}),$$

as G_k-modules.

Proof. It suffices to show that:

$$\mathscr{T}_m \otimes_{\mathcal{O}} k \cong \mathscr{T}_{\mathfrak{p}}, \quad \mathscr{T}_m \otimes_{\mathcal{O}} K \cong T_0,$$

where $\mathscr{T}_m$ is the maximal subtorus of $\mathscr{T}^0$. More generally, there are bijections:

$\{\text{subtori of } \mathscr{T}_{\mathfrak{p}}^0\} \leftrightarrow \{\text{subtori of } \mathscr{T}^0\} \leftrightarrow \{\text{subtori of } T \text{ with good reduction}\}$.

For the first one see [2, XII]. The second mapping from left to right is injective by (1.1). It remains to show that given a subtorus of T with good reduction, $T' \hookrightarrow T$, the corresponding map between the connected components of the Néron models, $\mathscr{T}'^0 \to \mathscr{T}^0$, is also injective. As a map between two sheafs for the smooth topology it is clearly injective because of the left-exactness of j_*; but this is not sufficient in general. In our case where $\mathscr{T}'^0$ is a torus over $\mathcal{O}$, the assertion is clear because the kernel is a group-scheme of multiplicative type with trivial generic fiber. □

Remark. The most natural torus over k which can be obtained from T is the one determined by the G_k-module $X(T)^{I_K}$. It is easy to check that this torus is isomorphic to $((T^\vee)_{\mathfrak{p}})^\vee$, where $^\vee$ indicates dual. The dual torus satisfies $X(T^\vee) = X(T)^\vee$ by definition.

2. Weil restriction. In this paragraph we collect some results we need about the Weil restriction functor.

Recall that for any scheme S, a S-functor is a covariant functor from $\underline{S - Sch}$ to $\underline{Sets}$. Given a morphism $u: S' \to S$ of schemes, the Weil restriction $R_{S'/S}$ is the right-adjoint functor of the scalar-extension functor. That is, for any S'-functor X, $R_{S'/S}(X)$ is the S-functor defined by:

$$R_{S'/S}(X)(Y) = X(Y \times_S S'),$$

for any S-scheme Y. The following properties of $R_{S'/S}$ are easy (see [1, 7.6 Thm 4] for (2.1)).

(2.1) Proposition. *If $S = \operatorname{Spec}(R)$, $S' = \operatorname{Spec}(R')$ are affine, R' is projective and of finite type as R-module and X is representable by an affine group scheme, then $R_{S'/S}(X)$ is also representable by an affine group scheme.*

(2.2) Proposition. *Let $S' \to S$ be a finite étale Galois covering of S and $\Gamma = \operatorname{Gal}(S'/S)$. Let X be a S'-functor and for any $\sigma \in \Gamma$, let X^σ be the S'-functor defined by:*

$$X^\sigma(Y) := X(Y \times_{S'} \underset{\sigma}{\nearrow} S').$$

Then, there is a canonical isomorphism:

$$R_{S'/S}(X) \times_S S' \xrightarrow{\sim} \prod_{\sigma \in \Gamma} X^\sigma.$$

If, moreover, X is defined over S, then we obtain an isomorphism:

$$R_{S'/S}(X) \times_S S' \xrightarrow{\sim} X^{\# \Gamma}.$$

In particular, the Weil restriction of a torus by a finite étale morphism is again a torus.

(2.3) Proposition. *Suppose that we have morphisms of schemes: $S' \to S \to S''$. Let T be a scheme over S, $T' = T \times_S S'$ and let X, X' be arbitrary S'-functors. Then, there are canonical isomorphisms:*

(1) $R_{S'/S}(X) \times_S T = R_{T'/T}(X \times_{S'} T')$

(2) $R_{S'/S''}(X) = R_{S/S''}(R_{S'/S}(X))$

(3) $R_{S'/S}(X \times_{S'} X') = R_{S'/S}(X) \times_S R_{S'/S}(X')$.

The Weil restriction functor does not commute with the connected component. For instance, if L/K is a finite extension of local fields and $A_{/L}$ is an abelian variety with good reduction, then its Néron model, $\mathscr{A}$ is connected, but $R_{\mathcal{O}_L/\mathcal{O}_K}(\mathscr{A})$, which is the Néron model of $R_{L/K}(A)$, may be disconnected, since $R_{L/K}(A)$ may have bad reduction. Nevertheless we have the following:

(2.4) Proposition. *Let $S' \to S$ be a finite morphism and let T be a torus over S'. Then, $R_{S'/S}(T)$ is connected.*

Proof. By (2.3) we can assume that S is the spectrum of an algebraically closed field κ. Then, S' is the spectrum of a finite dimensional κ-algebra A. Since A is a product of strictly henselian rings, we have $T = \mathbb{G}_m^d$, and $R_{A/\kappa}(\mathbb{G}_m)$ is clearly connected. In fact,

$$R_{A/\kappa}(\mathbb{G}_m) = \operatorname{Spec}(\kappa[X_1, \dots, X_n, Y]/Y \cdot N(X_1, \dots, X_n) - 1),$$

where $n = \dim_\kappa A$ and $N(X_1, \dots, X_n)$ is the polynomial obtained by computing the determinant of the endomorphism of A given by multiplication by $X_1 e_1 + \dots + X_n e_n$, for a fixed κ-basis $e_1, \dots, e_n$ of A. $\square$

3. The unipotent component. Let K, $\mathcal{O}$, $\mathfrak{p}$, k be as in Section 1. Let L be a finite extension of K with ring of integers $\mathcal{O}_L$ and residue field k_L. Let e, f be the ramification index and residual degree of L/K.

We prove first Theorem (0.1) for the torus $R_{L/K}(\mathbb{G}_m)$. We begin with the following observation:

(3.1) Lemma. *$R_{\mathcal{O}_L/\mathcal{O}_K}(\mathbb{G}_m)$ is the connected component of the Néron model of $R_{L/K}(\mathbb{G}_m)$.*

Proof. Let $\mathscr{G}$ be the Néron model of $\mathbb{G}_m$ over $\mathcal{O}_L$. Clearly, the Weil restriction functor commutes with j_*; hence, $R_{\mathcal{O}_L/\mathcal{O}_K}(\mathscr{G})$ is the Néron model of $R_{L/K}(\mathbb{G}_m)$. By (2.4) we have:

$$R_{\mathcal{O}_L/\mathcal{O}_K}(\mathbb{G}_m) = R_{\mathcal{O}_L/\mathcal{O}_K}(\mathscr{G}^0) \hookrightarrow R_{\mathcal{O}_L/\mathcal{O}_K}(\mathscr{G})^0.$$

Since, on the other hand, the Weil restriction functor preserves open and closed immersions [1, 7.6 Prop 2], the last morphism must be an isomorphism. $\square$

(3.2) Proposition. *Let $T_\mathfrak{p}$, U be the toric and unipotent component of the finite fiber of $R_{\mathcal{O}_L/\mathcal{O}_K}(\mathbb{G}_m)$. Then, $T_\mathfrak{p}$ is the f-dimensional torus $R_{k_L/k}(\mathbb{G}_m)$. Moreover U is additive ($U \cong \mathbb{G}_a^{(e-1)f}$) if and only if $p \geqq e$.*

Proof. Assume first that L/K is totally ramified. Then L is defined by an Eisenstein polynomial:

$$\mathcal{O}_L \cong \mathcal{O}[X]/(X^e + p \cdot q(X)), \deg(q(X)) < e.$$

Denoting by $s: \mathrm{Spec}(k) \to \mathrm{Spec}(\mathcal{O})$ the finite fiber of $\mathcal{O}$, we have:

$$R_{\mathcal{O}_L/\mathcal{O}_K}(\mathbb{G}_m)_s(A) = R_{\mathcal{O}_L \times s/s}(\mathbb{G}_m)(A) = (A[X]/X^e)^*,$$

for any k-algebra A. Let $B = A[X]/X^e$; we have a split exact sequence:

$$1 \to 1 + XB \to B^* \to A^* \to 1.$$

If $p < e$, $U(A) = 1 + XB$ is not additive because it is not annihilated by p. Whereas if $p \geqq e$, there is a functorial-in-A isomorphism:

$$1 + XB \xrightarrow{\log} XB \cong A^{e-1},$$

given by the logarithm:

$$\log(1 + q(X)) = \sum_{i=1}^{\infty} (-1)^{i+1} (q(X)^i)/i.$$

In the general case, if K^{nr} is the maximal unramified subextension of L/K, with ring of integers $\mathcal{O}^{nr}$ and finite fiber $s_0: \mathrm{Spec}(k_L) \to \mathrm{Spec}(\mathcal{O}^{nr})$, we have by (2.3):

$$\begin{aligned} R_{\mathcal{O}_L/\mathcal{O}_K}(\mathbb{G}_m)_s &= R_{\mathcal{O}^{nr}/\mathcal{O}_K}(R_{\mathcal{O}_L/\mathcal{O}^{nr}}(\mathbb{G}_m))_s = R_{k_L/k}(R_{\mathcal{O}_L/\mathcal{O}^{nr}}(\mathbb{G}_m)_{s_0}) \\ &= R_{k_L/k}(\mathbb{G}_m \times U_0) = R_{k_L/k}(\mathbb{G}_m) \times R_{k_L/k}(U_0). \end{aligned}$$

If $p < e$, then U_0 is not annihilated by p, hence, $U = R_{k_L/k}(U_0)$ has the same property. If $p \geqq e$ we have seen that $U_0 = \mathbb{G}_a^{(e-1)}$, and it is clear that $R_{k_L/k}(\mathbb{G}_a) = \mathbb{G}_a^f$. □

We can now deduce Theorem (0.1) from the theorem of Ono [6, 1.5]:

(3.3) Proof of Theorem (0.1). Let L be the splitting field of T and K^{nr}, $\mathcal{O}^{nr}$, s, s_0, k_L as above. Since the Néron model is stable by étale basis change, $\mathcal{T} \otimes_{\mathcal{O}} \mathcal{O}^{nr}$ is the Néron model of $T^{nr} := T \otimes_K K^{nr}$ and:

$$(\mathcal{T} \otimes_{\mathcal{O}} \mathcal{O}^{nr})^0_{s_0} = (\mathcal{T}^0 \otimes_{\mathcal{O}} \mathcal{O}^{nr})_{s_0} = \mathcal{T}^0_s \otimes_k k_L.$$

If the theorem were true for T^{nr}, we would have:

$$U \otimes_k k_L \cong \mathbb{G}_a \times \cdots \times \mathbb{G}_a,$$

but since $\mathbb{G}_a$ admits no torsors [2, XVII, 4.1.5], U must be already additive. Hence, we can reduce the proof to the case L/K totally (and tamely) ramified. By the theorem of Ono, we have an isogeny between the two following tori:

$$\alpha: T^m \times \prod_v R_{K_v/K}(\mathbb{G}_m)^{m_v} \to \prod_v R_{K_v/K}(\mathbb{G}_m)^{n_v},$$

where K_v runs over all subextensions of L/K and m, m_v, n_v are uniquely determined non-negative integers. Let $\hat{\alpha}$ be the dual isogeny and let n be the degree of α, so that:

$$(*)\ \hat{\alpha} \circ \alpha = n\cdot, \quad \alpha \circ \hat{\alpha} = n\cdot.$$

Since $p > e$ (in fact, for any prime number not dividing $e = [L:K]$), we can choose α so that p doesn't divide n (cf. the proof of [6, 1.3.3]). By the universal property, we have morphisms α, $\hat{\alpha}$ between the respective Néron models:

$$\alpha : \mathscr{T}^m \times \prod_v R_{\mathcal{O}_{K_v}/\mathcal{O}}(\mathscr{G})^{m_v} \leftrightarrows \prod_v R_{\mathcal{O}_{K_v}/\mathcal{O}}(\mathscr{G})^{n_v} : \hat{\alpha},$$

still satisfying (*). By (3.1), taking connected components we get morphisms:

$$\alpha : (\mathscr{T}^0)^m \times \prod_v R_{\mathcal{O}_{K_v}/\mathcal{O}}(\mathbb{G}_m)^{m_v} \leftrightarrows \prod_v R_{\mathcal{O}_{K_v}/\mathcal{O}}(\mathbb{G}_m)^{n_v} : \hat{\alpha}.$$

Now, by (3.2) we have:

$$R_{\mathcal{O}_{K_v}/\mathcal{O}}(\mathbb{G}_m)_s = T_v \times \mathbb{G}_a^{r_v},$$

where T_v is a torus and r_v is an integer depending on K_v. Therefore, by taking finite fiber and unipotent component we have morphisms:

$$\alpha : U^m \times \mathbb{G}_a^r \leftrightarrows \mathbb{G}_a^t : \hat{\alpha},$$

still satisfying (*). Since p does not divide n, multiplication by n on $U^m \times \mathbb{G}_a^r$ is a monomorphism and:

$$0 = \hat{\alpha} \circ (p\cdot) \circ \alpha = np\cdot \Rightarrow (p\cdot) = 0,$$

hence p annihilates U and this property characterizes additivity among the unipotent, connected, smooth group schemes over a perfect field (see [7, 2.6.7] for algebraically closed fields and apply again that $\mathbb{G}_a$ has no torsors). □

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