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Convergence of solutions to the mean curvature flow with a Neumann boundary condition

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Abstract. This work continues our considerations in [15], where we discussed existence and regularity results for the mean curvature flow with homogenious Neumann boundary data. We study the long time evolution of compact, smooth, immersed manifolds with boundary which move under the mean curvature flow in Euclidian space. On the boundary, a Neumann condition is prescribed in a purely geometric manner by requiring a vertical contact angle between the unit normal fields of the immersions and a given, smooth hypersurface Σ. We deduce estimates for the curvature of the immersions and, in a special case, we obtain a precise description of the possible singularities.

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1 Introduction and main results

During the last years, hypersurfaces evolving under various geometric flows have been studied intensively; besides questions concerning existence and uniqueness, there has always been great interest in determining the "long time behaviour" of solutions. In the case of compact, convex hypersurfaces *without boundary*, the hypersurfaces will typically shrink to a point within a finite time T_C. Hereby the growth rate of the second fundamental form A as t approaches T_C can be used to classify the possible singularities: If there is a constant K such that $|A|^2 \leq K(T_C - t)\,\forall t$, the singularity is said to be *of type I*, otherwise it is called a *type-II*-singularity. Huisken proved [10] that convex hypersurfaces moving by mean curvature end up in a type-I-singularity, whereas it was shown by Angenent [3] that the curve shortening flow for plane curves developing a cusp produces a type-II-singularity. Rescaling the hypersurfaces appropriately often yields a better description of what actually happens as $t \to T_C$: For the mean curvature flow Huisken [10] established the convergence of the rescaled hypersurfaces to a "round sphere" in the C^∞-topology. Later [11] he obtained a complete classification of the possible type-I limit surfaces with nonnegative mean curvature. Further results concerning this topic are due to Andrews [2], Hamilton [7], Stone [13], and others.

Evolving hypersurfaces *with boundary* often have been described as graphs of a scalar function $w : \Omega \times [0,T] \longrightarrow \mathbb{R}$. For given Neumann boundary data on $\partial\Omega \times [0,T]$, a solution will typically converge to a hypersurface which moves by translation; the speed of the translation is determined essentially by the angle at which the hypersurface meets the boundary. This result was established in the two-dimensional case by Altschuler, Wu [4]; in the special case of an orthogonal contact angle (but in arbitrary dimensions), Huisken [12] proved the existence of a unique solution which converges to a minimal surface as $t \longrightarrow \infty$. Further results concerning a wider class of evolution equations are due to Guan [6].

This work continues our previous considerations in [15], where we discussed existence and regularity results for the mean curvature flow with a homogenious Neumann boundary condition. We prescribe boundary data on an arbitrary smooth hypersurface $\boldsymbol{\Sigma}$ and study the long time behaviour of solutions to given smooth initial and support hypersurfaces; in a special case, we obtain a precise description of the possible singularities.

The results presented here are part of the author's Dissertation [14], the research upon which has been carried out during the years 1992-1994 and under the excellent tutorship of G. Huisken.

1.1 Preliminaries

In our notation, we follow our previous work [15]. Thus, M^n denotes a compact, smooth, orientable n-manifold ($n \geq 1$) with compact, smooth boundary ∂M^n, $\boldsymbol{\Sigma}$ a smooth hypersurface in $\mathbb{R}^{n+1}$ and $\boldsymbol{F}_0 : M^n \longrightarrow \mathbb{R}^{n+1}$ a smooth immersion with $M_0 := \boldsymbol{F}_0(M^n)$ and

$$\partial M_0 = M_0 \cap \boldsymbol{\Sigma}\ , \quad \langle \boldsymbol{\nu}_0, \boldsymbol{\mu} \circ \boldsymbol{F}_0 \rangle(x) = 0 \quad \forall x \in \partial M^n, \tag{1.1}$$

where $\boldsymbol{\nu}_0, \boldsymbol{\mu}$ are unit normal fields to M_0, $\boldsymbol{\Sigma}$, respectively. We consider a family of smooth immersions $\boldsymbol{F} : M^n \times [0, T_C) \longrightarrow \mathbb{R}^{n+1}$; as usual, we denote the induced metric by (g_{ij}) and the second fundamental form by $A = (h_{ij})$; $H := g^{ij}\, h_{ij}$ is the mean curvature, $\boldsymbol{H} := -H\, \boldsymbol{\nu}$ the mean curvature vector of the immersions and $\Delta := g^{ij}\, \nabla_i\, \nabla_j$ the Laplace-Beltrami operator on M^n. We characterize quantities on $\boldsymbol{\Sigma}$ by an upper or lower index $_{\Sigma}$. The Einstein summation convention is used; hereby, the range of greek indices is from 1 to $n+1$, whereas capital arabic indices range from 1 to $n-1$ and small arabics from 1 to n. Summation indices on $\boldsymbol{\Sigma}$ are denoted by $\hat{\imath}, \hat{\jmath}, \hat{k}, \ldots$ We then define

Definition 1.1 *(Mean curvature flow with Neumann boundary condition) Let $\boldsymbol{\Sigma}$, $\boldsymbol{F}_0$ and $\boldsymbol{F}$ be as before. $\boldsymbol{F}$ is said to move under the mean curvature flow with homogenious Neumann boundary condition, if*

$$(1.2)\qquad \begin{cases} \frac{d}{dt}\boldsymbol{F}(x,t) &=& \boldsymbol{H}(x,t) & \forall (x,t)\in M^n\times[0,T_C),\\ \boldsymbol{F}(.,0) &=& \boldsymbol{F}_0, & \\ \boldsymbol{F}(\partial M^n,t) &\subset& \boldsymbol{\Sigma} & \forall (x,t)\in \partial M^n\times[0,T_C),\\ \langle \boldsymbol{\nu},\boldsymbol{\mu}\circ\boldsymbol{F}\rangle(x,t) &=& 0 & \forall (x,t)\in \partial M^n\times[0,T_C). \end{cases}$$

1.2 Main results

In [15], we proved the following existence result:

Theorem 1.2 *Let Σ, M_0 be as before. Then there exists a unique solution to equation 1.2 on a maximal time interval $[0, T_C)$. This solution is smooth for $t > 0$ and in the class $C^{2+\alpha,1+\alpha/2}$ (with arbitrary $0 < \alpha < 1$) for $t \geq 0$. Moreover, if $T_C < \infty$, then*

$$sup\{|A|^2(x,t) : x \in M^n\} \to \infty \quad as \quad t \to T_C.$$

This work is devoted to the study of the singularity which develops as $t \to T_C$, if Σ and M_0 are (strictly) convex hypersurfaces. Under the further restriction that Σ is *umbilic*, i.e. the boundary of a ball B_R^{n+1} or a hyperplane, we can prove that convexity of the M_t as well as a pinching condition for the eigenvalues of the second fundamental form A are preserved under the mean curvature flow equation 1.2. Modifying an idea of Tso [16], this yields an upper bound on the growth rate of the second fundamental form of the M_t. Using an appropriate rescaling technique and applying results of Hamilton [8] (concerning hypersurfaces with pinched second fundamental form) and Huisken [10] (on the behaviour of compact hypersurfaces without boundary), we finally arrive at the following

Theorem 1.3 *Let $n \geq 2$, $\mathcal{G} = B_R^{n+1}$ ($R = \infty$ allowed), $\Sigma = \partial\mathcal{G}$, and let $M_0 \subset \mathcal{G}$ be a strictly convex, imbedded hypersurface satisfying equations 1.1. Then the hypersurfaces M_t of the solution to equation 1.2 shrink to a single point on Σ as $t \to T_C < \infty$, ending up in a type-I-singularity. Moreover, there is a sequence $t_k \to T_C$ such that, rescaling the M_{t_k} appropriately, the rescaled hypersurfaces converge in the C^∞-topology to a hemisphere with boundary on a hyperplane.*

We emphasize that the requirement of Σ being umbilic is needed exclusively to establish the conservation of the convexity and pinching condition. Consequently, the Theorem holds for *arbitrary convex support hypersurfaces* Σ whenever conservation of convexity and pinching can be guaranteed for the solutions of equation 1.2. Furthermore, all techniques (except for the rescaling procedure, where Hamilton's compactness result requires $n \geq 2$) are valid for plane curves. Since - as we shall see - the conservation of a positive lower bound on H already follows from the convexity of Σ, we obtain

Proposition 1.4 *Let Σ be the smooth boundary of a convex domain $\mathscr{G} \subset \mathbb{R}^2$, and let Σ have uniformly bounded curvature. Furthermore, let $M_0 \subset \bar{\mathscr{G}}$ be a convex, imbedded curve satisfying equations 1.1. Then the curves of the solution to equation 1.2 shrink to a single point on Σ as $t \to T_C < \infty$.*

2 The Neumann boundary condition

We begin with analyzing the given Neumann boundary condition:

$$(2.1) \qquad \langle \nu, \mu \circ F \rangle (x,t) = 0 \quad \forall (x,t) \in \partial M^n \times [0,T] .$$

Differentiating this equation with respect to time yields

Proposition 2.1 *Let $p = F(x,t) \in \partial M^n \times (0,T]$. Then*

$$\langle \nabla H, \mu \rangle (p) = (H \cdot {}^{\Sigma}\!A(\nu,\nu))(p) .$$

Proof. $\frac{d}{dt}\nu = \nabla H$ for $t > 0$, according to [10], Lemma 3.3. Moreover,

$$\tfrac{d}{dt}\mu(F(x,t)) = d\mu(\tfrac{d}{dt}F) = d\mu(H) = -H\, d\mu(\nu) .$$

Hence, from equation 2.1 we obtain

$$0 = \langle \nabla H, \mu \rangle - H \,\langle \nu, d\mu(\nu)\rangle = \langle \nabla H, \mu \rangle - H\, {}^{\Sigma}\!A(\nu,\nu) .$$

□

Next, we want to differentiate equation 2.1 with respect to a tangential space direction. This is best done using an orthonormal moving frame on M_t for a fixed $t \geq 0$: We define orthonormal vector fields $e_1, \dots, e_{n+1}$ in $\mathbb{R}^{n+1}$ such that for $p \in \Sigma \cap M_t$

$$e_1(p), \dots, e_{n-1}(p) \in T_p(\Sigma \cap M_t) , \quad e_n(p) = \mu(p) , \quad e_{n+1}(p) = \nu(p) .$$

This means that the e_α, e_i, e_I are an orthonormal basis for the tangent bundles $T\mathbb{R}^{n+1}$, TM_t, $T(\Sigma \cap M_t)$, respectively[1]. We describe the covariant derivative in $\mathbb{R}^{n+1}$ by

$${}^0\nabla_{e_\alpha} e_\beta =: \Gamma^{\sigma}_{\alpha\beta} e_\sigma .$$

Note that the $\Gamma^{\sigma}_{\alpha\beta}$ are *not* the usual Christoffel symbols which are deduced by using local coordinates; especially they are not symmetric with respect to the lower indices. Instead, using the relations between covariant derivatives and the second fundamental form:

$$(2.2) \qquad \begin{aligned} {}^0\nabla_{e_i} e_j &= \nabla_{e_i} e_j - A(e_i,e_j)\nu , & {}^0\nabla_{e_i} \nu &= A(e_i,e_p)e_p , \\ {}^0\nabla_{e_{\hat\imath}} e_{\hat\jmath} &= {}^{\Sigma}\nabla_{e_{\hat\imath}} e_{\hat\jmath} - {}^{\Sigma}\!A(e_{\hat\imath},e_{\hat\jmath})\mu , & {}^0\nabla_{e_{\hat\imath}} \mu &= {}^{\Sigma}\!A(e_{\hat\imath},e_{\hat p})e_{\hat p} , \end{aligned}$$

[1] Recall the range of the various summation indices!

we obtain the following equations[2]:

$$\text{(2.3)} \qquad \Gamma^{\sigma}_{\alpha\beta} = -\Gamma^{\beta}_{\alpha\sigma}\,, \quad \Gamma^{\nu}_{ij} = -A(e_i, e_j)\,, \quad \Gamma^{\mu}_{\hat{\imath}\hat{\jmath}} = -{}^{\Sigma}\!A(e_{\hat{\imath}}, e_{\hat{\jmath}})\,.$$

Finally, a straightforward calculation yields for the covariant derivative of symmetric tensors T, T_Σ on M_t, Σ, respectively:

$$\text{(2.4)} \qquad \begin{aligned} (\nabla_{e_k} T)(e_i, e_j) &= d(T(e_i, e_j))(e_k) - \Gamma^{p}_{ki}\, T(e_p, e_j) - \Gamma^{q}_{kj}\, T(e_i, e_q)\,, \\ ({}^{\Sigma}\nabla_{e_{\hat{k}}} T_\Sigma)(e_{\hat{\imath}}, e_{\hat{\jmath}}) &= d(T_\Sigma(e_{\hat{\imath}}, e_{\hat{\jmath}}))(e_{\hat{k}}) - \Gamma^{\hat{p}}_{\hat{k}\hat{\imath}}\, T_\Sigma(e_{\hat{p}}, e_{\hat{\jmath}}) - \Gamma^{\hat{q}}_{\hat{k}\hat{\jmath}}\, T_\Sigma(e_{\hat{\imath}}, e_{\hat{q}})\,. \end{aligned}$$

Using these relations, we can differentiate the Neumann boundary condition in a tangential space direction: From equations 2.1 and 2.2, we derive

$$0 = \langle {}^{0}\nabla_{e_I} \nu, \mu \rangle + \langle \nu, {}^{0}\nabla_{e_I} \mu \rangle = A(e_I, \mu) + {}^{\Sigma}\!A(e_I, \nu)\,.$$

Since $e_1(p), \ldots, e_{n-1}(p)$ span the tangent space $T_p(\Sigma \cap M_t)$, we obtain

Proposition 2.2 *Let $p \in \Sigma \cap M_t$, $v \in T_p M_t$ and $w := v - \langle v, \mu \rangle \mu \in T_p(M_t \cap \Sigma)$ be the projection of v onto $T_p \Sigma$. Then*

$$\begin{aligned} A(w, \mu) &= -{}^{\Sigma}\!A(w, \nu) \quad \textit{and} \\ A(v, \mu) &= -{}^{\Sigma}\!A(w, \nu) + \langle v, \mu \rangle A(\mu, \mu)\,. \end{aligned}$$

□

Moreover, we have a relation between the covariant derivatives of the second fundamental forms of M_t and Σ in tangential directions:

Proposition 2.3 *Let $p \in \Sigma \cap M_t$ and v, $w \in T_p(\Sigma \cap M_t)$. Then*

$$\begin{aligned} (\nabla_v A)(\mu, w) &= -({}^{\Sigma}\nabla_v {}^{\Sigma}\!A)(\nu, w) \\ &\quad - A(v, e_P)\, {}^{\Sigma}\!A(e_P, w) - A(w, e_P)\, {}^{\Sigma}\!A(e_P, v) \\ &\quad + A(\mu, \mu)\, {}^{\Sigma}\!A(v, w) \quad + A(v, w)\, {}^{\Sigma}\!A(\nu, \nu)\,. \end{aligned}$$

Proof. Using equations 2.2 through 2.4 with $T = A$, we obtain

$$\begin{aligned} (\nabla_{e_I} A)(\mu, e_J) &= d(A(\mu, e_J))(e_I) - {}^{\Sigma}\!A(e_I, e_P) A(e_P, e_J) \\ &\quad - \Gamma^{Q}_{IJ}\, A(\mu, e_Q) + {}^{\Sigma}\!A(e_I, e_J) A(\mu, \mu)\,. \end{aligned}$$

Analogously, for $T_\Sigma = {}^{\Sigma}\!A$:

$$\begin{aligned} ({}^{\Sigma}\nabla_{e_I} {}^{\Sigma}\!A)(\nu, e_J) &= d({}^{\Sigma}\!A(\nu, e_J))(e_I) - A(e_I, e_P)\, {}^{\Sigma}\!A(e_P, e_J) \\ &\quad - \Gamma^{Q}_{IJ}\, {}^{\Sigma}\!A(\nu, e_Q) + A(e_I, e_J)\, {}^{\Sigma}\!A(\nu, \nu)\,. \end{aligned}$$

Now, according to Proposition 2.2, we have $A(\mu, e_J) + {}^{\Sigma}\!A(\nu, e_J) \equiv 0$ on the boundary and hence $d\left(A(\mu, e_J) + {}^{\Sigma}\!A(\nu, e_J)\right)(e_I) = 0$. Thus we obtain the result by summing the above equations. □

[2] Here and in the sequel we replace the indices $_{n}, _{n+1}$ by the somewhat more suggestive notation $_{\mu}, _{\nu}$

We finish this section by summarizing the results. If we define

$$h_{ij}(x,t) := A(e_i, e_j)(x,t), \quad \nabla_k h_{ij}(x,t) := (\nabla_{e_k} A)(e_i, e_j)(x,t)$$

etc., we have:

Theorem 2.4 *On the boundary $\Sigma \cap M_t$, the components of the second fundamental forms of M_t and Σ are related by the equations*

$$\begin{array}{lrcl} (i) & h_{\mu I} & = & -h^{\Sigma}_{\nu I}, \\ (ii) & \nabla_I h_{\mu J} & = & -{}^{\Sigma}\nabla_I h^{\Sigma}_{\nu J} - h_{IP} h^{\Sigma}_{PJ} - h_{JP} h^{\Sigma}_{PI} + h_{\mu\mu} h^{\Sigma}_{IJ} + h_{IJ} h^{\Sigma}_{\nu\nu}, \\ (iii) & \nabla_\mu H & = & H\, h^{\Sigma}_{\nu\nu}. \end{array}$$

□

Note that the equations $(i), (ii)$ are valid for $t \geq 0$, whereas equation (iii) requires $t > 0$.

Remark 2.5 Given a point $p = F(x,t) \in \Sigma \cap M_t$, we can introduce local coordinates around p such that the vector fields $e_I(p) := \frac{\partial}{\partial x^I} F(x,t)$, $e_n(p) := \frac{\partial}{\partial x^n} F(x,t) = \mu(p)$ and $e_{n+1} := \nu(x,t)$ are orthonormal at p; this relates the formalisms of orthonormal moving frames and local coordinate systems.

3 Curvature bounds

This section is devoted to the study of the mean curvature of the hypersurfaces M_t of a given solution to equation 1.2. The evolution equation for H is (cf. [10], Corollary 3.5)

$$\frac{\partial}{\partial t} H = \Delta H + |A|^2 H. \tag{3.1}$$

3.1 Estimates from below

Theorem 3.1 *Let $H(.,0) \geq 0$. Then $H \geq 0$ on $M^n \times [0,T]$; moreover, if $H(.,0) \not\equiv 0$, we have $H > 0$ $\forall t > 0$.*

Proof. According to Proposition 2.1, we have $\langle \nabla H, \mu \rangle(p) = (H \cdot {}^{\Sigma}\!A(\nu,\nu))(p)$. Thus, regarding the evolution equation 3.1, the conditions of the weak maximum principle proved in [15], Theorem 3.1, are fulfilled for H; hence $H \geq 0$ is preserved. Now the strong maximum principle (e.g. [15], Corollary 3.2) yields the result. □

Theorem 3.2 *Let Σ be convex with respect to the exterior*[3] *normal field μ, and let $H(.,0) \geq C_0 > 0$. Then $H \geq C_0$ for all $t \in [0, T_C)$, and the maximal time interval $[0, T_C)$ for the solution is bounded by $T_C \leq t_0 := \frac{n}{2C_0^2}$.*

[3] For a definition of *exterior*, see [15], section 1.

Proof. The solution $u(t)$ of the ordinary initial value problem

$$\tfrac{d}{dt}u = \tfrac{1}{n}\,u^3\,, \quad u(0) = C_0$$

is given by $u(t) = C_0 \cdot (1 - 2\,C_0^2\,t/n)^{-1/2}$. Now, using the simple algebraic identity $|A|^2 \geq \frac{1}{n}H^2$, we see that the function $\varphi(x,t) := H(x,t) - u(t)$ fulfils the following differential equation:

$$(\tfrac{\partial}{\partial t} - \Delta)\varphi \;\geq\; \tfrac{1}{n}(H^3 - u^3) \;=\; \tfrac{1}{n}(H^2 + H\,u + u^2)(H - u) \;=:\; c(x,t)\,\varphi\,.$$

Moreover, by the convexity of Σ and since $u > 0$, we have on the boundary

$$\langle \nabla\varphi, \mu\rangle \;=\; {}^{\Sigma}\!A(\nu,\nu)\,H \;\geq\; {}^{\Sigma}\!A(\nu,\nu)\,H - {}^{\Sigma}\!A(\nu,\nu)\,u \;=\; {}^{\Sigma}\!A(\nu,\nu)\,\varphi\,.$$

Thus, the result again is a consequence of the above cited maximum principles. □

It can easily be seen that the above estimate on the maximal existence interval is sharp for convex hypersurfaces *without* boundary, since the sphere S^n_R homothetically shrinks to a point in $T_C = \frac{R^2}{2n} = \frac{n}{2C_0^2} = t_0$. Thus, t_0 can also be reached in the boundary-case: Let $\mathcal{G}$ be a cone, $\Sigma = \partial\mathcal{G}$ and M_0 part of a sphere S^n_R such that the center of the sphere coincides with the vertex of the cone. Note that the boundary condition is fulfilled as M_t shrinks homothetically to the vertex of the cone. Now, the natural question is whether this is the *only* setup with maximal existene time of the solution. We prove:

Theorem 3.3 *Let $\mathcal{G} \subset \mathbb{R}^{n+1}$ be a convex domain with smooth boundary Σ. Let F be the solution of equation 1.2, where M_0 has strictly positive mean curvature. Then $T_C = t_0$ if and only if $\mathcal{G}$ is a cone with vertex p and $M_0 \subset S_R(p)$.*

Proof. We define $U(t) := \min_{M^n} H(x,t)$ and obtain from the proof of Theorem 3.2

$$U(t) \geq \frac{C_0}{\sqrt{1 - 2\,C_0^2\,t/n}}\,.$$

Now, using Theorem 3.2, we conclude that in order to obtain $T_C = t_0$, actually equality must hold; thus $U(t)$ fulfils the ordinary initial value problem

$$\tfrac{d}{dt}U = \tfrac{1}{n}\,U^3\,, \quad U(0) = C_0\,.$$

Now we consider the function $\varphi(x,t) := H(x,t) - U(t)$. Since for every t due to the compactness of M^n there exists a zero of $\varphi(.,t)$, by the strong maximum principle we conclude $\varphi \equiv 0\;\forall t$; thus $H(.,t) \equiv U(t)$ on M_t. This in turn implies $\nabla H \equiv 0$ and Proposition 2.1 yields ${}^{\Sigma}\!A(\nu,\nu) \equiv 0$, since $H \geq C_0 > 0$. Furthermore, from $\Delta H \equiv 0$ and the evolution equations for H and U we conclude $|A|^2 = \frac{1}{n}H^2$ and thereby that the M_t are spheres. Finally the equation $\frac{\partial}{\partial t}\nu = \nabla H$ ([10], Lemma 3.3) implies $\nu(.,t) = const(t)$, and so these spheres must be concentric; the result follows. □

We remark that this proof equally holds for the boundaryless case: Here the only way to obtain the maximal existence interval of a solution which starts with strictly positive mean curvature is M_0 being a sphere.

3.2 Estimate from above

As we have seen, it is not hard to establish lower bounds on H under reasonably weak conditions. In contrast, it is much harder to control the increasing of H from above; this can be done if the M_t are guaranteed to be *convex*. Now convexity of the M_t is preserved in the special case of Σ being umbilic, cf. Theorem 4.4. Thus in the sequel we will require Σ to be the boundary of a sphere or a hyperplane, but we emphasize that the following method also applies to *arbitrary* convex hypersurfaces Σ, if by any means we can guarantee that convexity of the M_t is preserved.

In this section, we apply a technique due to Tso [16] who studied closed hypersurfaces moving under the Gauss-Kronecker-flow.

Assumption 3.4 *For the remainder of this section we let* $\mathcal{G} = B_R^{n+1}$ *or* $\mathcal{G}$ *be a half space and* $\Sigma = \partial\mathcal{G}$. *Furthermore,* F *is the solution of equation 1.2 with* $M_0 \subset \bar{\mathcal{G}}$ *being an imbedded, convex hypersurface.*

Thus all M_t will be imbedded hypersurfaces due to [15], Corollary 4.5, and since $H(.,0) \geq 0$, this will equally hold for $t > 0$ (Theorem 3.1). Finally, convexity of M_t for $t > 0$ follows from Theorem 4.4 below.

We start our considerations by establishing evolution equations for some auxiliary functions: First, there exists $\delta_0 > 0$ and a neighbourhood $\mathcal{U} := \{p \in \mathbb{R}^{n+1} : \mathrm{dist}_{n+1}(p, \Sigma) < \delta_0\}$ of Σ such that the distance function

$$g : \mathcal{U} \longrightarrow \mathbb{R}, \quad g(p) := \mathrm{dist}_{n+1}(p, \Sigma)$$

is well defined on $\mathcal{U}$ and smooth on $\mathcal{U}\backslash\Sigma$ and fulfils the estimates

$$\|Dg\| \leq 1, \quad |D^2 g| < C_1, \tag{3.2}$$

where $C_1 = C_1({}^{\Sigma}\!A) > 0$ depends only on the curvature of Σ. Moreover, if $p \in \Sigma$, we have

$$\langle Dg, \mu\rangle(p) = -1. \tag{3.3}$$

Lemma 3.5 *(The distance function) The function* $d : M^n \times [0, T] \longrightarrow \mathbb{R}$, $d(x,t) := (g\circ F)(x,t)$ *fulfils the evolution inequality*

$$|(\tfrac{\partial}{\partial t} - \Delta)d(x,t)| \leq n\, C_2,$$

where the constant $C_2 > 0$ *depends only on* ${}^{\Sigma}\!A$. *Furthermore,*

$$\langle \nabla d, \mu\circ F\rangle = -1 \quad on \quad \partial M^n \times [0,T], \quad and \quad \|\nabla d\| \leq 1 \quad on \quad \mathcal{U}.$$

Proof. A straightforward calculation yields

$$(\tfrac{\partial}{\partial t} - \Delta)d(x,t) = -g^{ij} \cdot (D^2 g\circ F)(\tfrac{\partial}{\partial x^i}F, \tfrac{\partial}{\partial x^j}F)(x,t);$$

thus the evolution inequality for d follows from equation 3.2. Analogously, the boundary behaviour is a consequence of

$$\langle \nabla d, \boldsymbol{\mu}\circ \boldsymbol{F}\rangle(x,t) = \langle Dg\circ \boldsymbol{F}, \boldsymbol{\mu}\circ \boldsymbol{F}\rangle(x,t) .$$

Finally, $\|\nabla d\|^2 = \|Dg\|^2$ on $\mathscr{U}$; again, equation 3.2 yields the result. □

We remark that for convex hypersurfaces $\boldsymbol{\Sigma}$ we have $(\frac{\partial}{\partial t} - \Delta)d \geq 0$; unfortunately, this turns out to be the wrong sign for the desired estimates.

Lemma 3.6 *(The support function) The function* $s : M^n \times [0,T] \longrightarrow \mathbb{R}$, $s(x,t) := \langle \boldsymbol{\nu}, \boldsymbol{F}\rangle(x,t)$ *fulfils*

$$\begin{aligned} (\tfrac{\partial}{\partial t} - \Delta)s(x,t) &= -|A|^2 + 2\,s\,H \quad \forall t > 0\,, \\ \langle \nabla s, \boldsymbol{\mu}\circ \boldsymbol{F}\rangle &= A(\boldsymbol{F}^T, \boldsymbol{\mu}\circ\boldsymbol{F}) \quad \textit{on the boundary}\,; \end{aligned}$$

hereby $\boldsymbol{F}^T$ *is the projection of* $\boldsymbol{F}$ *onto the tangent hyperplane of* M_t.

Proof. The evolution equation was proved by Ecker and Huisken [5]. The boundary condition is obtained by a straightforward calculation. □

Now we let $\boldsymbol{p} \in \mathscr{S}$ be the origin of a coordinate system in $\mathbb{R}^{n+1}$, and we choose $\sigma > 0$ and $T_\sigma > 0$ such that $s(.,t) \geq 2\,\sigma$ on $[0, T_\sigma]$. (This is possible due to the convexity of the imbeddings M_t.) We then consider the function

$$f : M^n \times [0, T_\sigma] \longrightarrow \mathbb{R}\,,\ f(x,t) := H\,\frac{\psi \circ d}{s - \sigma}(x,t)\,,$$

$\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ being a C^2-function which will be chosen later.

Lemma 3.7 *For* $t > 0$, *we have*

$$\begin{aligned} (\tfrac{\partial}{\partial t} - \Delta)f &\leq -\tfrac{\sigma\,(s-\sigma)}{n\,(\psi\circ d)^2} f^3 + \left(\tfrac{2}{\psi\circ d} + 2\,\|\boldsymbol{F}\|\,|\tfrac{\psi'}{\psi^2}\circ d|\right) f^2 \\ &\quad + \left(n\,C_2\,|\tfrac{\psi'}{\psi}\circ d| + |\tfrac{2\,\psi'^2 - \psi\,\psi''}{\psi^2}\circ d|\right) f \\ &\quad + \langle \nabla Q, \nabla f\rangle\,, \\ \langle \nabla f, \boldsymbol{\mu}\rangle &\leq f\,\left(C_\Sigma - \tfrac{\psi'}{\psi}(0) + \tfrac{1}{s-\sigma}\,C_\Sigma\,\|\boldsymbol{F}\|\right) \quad \textit{on the boundary}\,; \end{aligned}$$

hereby $Q := 2\,\log(\frac{s-\sigma}{\psi\circ d})$ *and* $C_\Sigma := \|{}^{\Sigma}\!A\|_\infty = \sup\{\, |{}^{\Sigma}\!A|(\boldsymbol{p}) : \boldsymbol{p} \in \boldsymbol{\Sigma}\,\}$.

Proof. Using equation 3.1 and Lemmata 3.5, 3.6, we obtain by a straightforward calculation

$$\begin{aligned} (\tfrac{\partial}{\partial t} - \Delta)f &\leq f\,\tfrac{1}{s-\sigma}\,(-\sigma\,|A|^2 + 2\,H) + f\,|\tfrac{\psi'}{\psi}\circ d|\,n\,C_2 \\ &\quad - f\,\tfrac{2}{s-\sigma}\,\left(\tfrac{\psi'}{\psi}\circ d\right)\langle \nabla s, \nabla d\rangle + f\,\left(\tfrac{2\,\psi'^2 - \psi\,\psi''}{\psi^2}\circ d\right)\,\|\nabla d\|^2 \\ &\quad + 2\,\tfrac{\psi\circ d}{s-\sigma}\,\langle \nabla \tfrac{s-\sigma}{\psi\circ d}, \nabla f\rangle\,. \end{aligned}$$

Moreover, we compute $\langle \nabla s, \nabla d\rangle = A(\boldsymbol{F}^T, \nabla d)$, and thereby

$$|\langle \nabla s, \nabla d\rangle| \leq |A|\,\|\boldsymbol{F}\|\,\|\nabla d\|\,.$$

Putting this into the above evolution equation for f and using the inequality $\frac{1}{n}H^2 \le |A|^2 \le H^2$ (which is valid since the M_t are *convex* hypersurfaces), we obtain

$$\begin{aligned}(\tfrac{\partial}{\partial t} - \Delta)f \;&\le\; f\,\tfrac{1}{s-\sigma}(-\tfrac{\sigma}{n}H^2 + 2H) + f\,|\tfrac{\psi'}{\psi}\circ d|\,n\,C_2 \\ &+\; f\,\tfrac{2}{s-\sigma}|\tfrac{\psi'}{\psi}\circ d|\,H\,\|F\| + f\,|\tfrac{2\psi'^2-\psi\psi''}{\psi^2}\circ d| \\ &+\; \langle \nabla Q, \nabla f\rangle .\end{aligned}$$

Finally, using $H = f\,\frac{s-\sigma}{\psi\circ d}$, the evolution equation follows.

On the boundary, we have

$$\nabla f = \frac{\psi(0)}{s-\sigma}\nabla H + H\left(\frac{\psi'(0)}{s-\sigma}\nabla d - \frac{\psi(0)}{(s-\sigma)^2}\nabla s\right) .$$

By Proposition 2.1 and Lemmata 3.5, 3.6, we arrive at

$$\langle \nabla f, \mu\rangle = \frac{\psi(0)}{s-\sigma}\,{}^{\Sigma}\!A(\nu,\nu)H - H\left(\frac{\psi'(0)}{s-\sigma} + \frac{\psi(0)}{(s-\sigma)^2}A(F^T,\mu)\right) .$$

Corollary 2.2 gives

$$A(F^T,\mu) = \langle F^T,\mu\rangle A(\mu,\mu) - {}^{\Sigma}\!A(F^T - \langle F^T,\mu\rangle\mu\,,\nu)\,;$$

here, $A(\mu,\mu) \ge 0$, as the M_t are convex. Furthermore, Σ bounds a convex domain, and thus the boundary condition $\langle \mu,\nu\rangle = 0$ yields[4] $\langle F^T,\mu\rangle = \langle F,\mu\rangle \ge 0$. Hence

$$A(F^T,\mu) \ge -C_\Sigma\,\|F^T - \langle F^T,\mu\rangle\mu\| \ge -C_\Sigma\,\|F\| ,$$

and putting this into the above expression for $\langle \nabla f, \mu\rangle$ leads to the stated boundary condition. □

Before stating the Theorem, we observe that, with respect to the choice of our coordinates, the hypersurfaces M_t "shrink" as t increases; more precisely, we have

$$\text{(3.4)}\qquad \sup\{\|F(x,t)\| : (x,t) \in M^n \times [0,T_\sigma]\} = \sup\{\|F(x,0)\| : x \in M^n\} .$$

This follows from equation 1.2 by calculating $\frac{d}{dt}\|F\|^2(x,t) = -2H\,s \le 0$.

We now show that the mean curvature H is bounded from above as long as the support function s is strictly positive; geometrically this latter condition means that the convex body bounded by M_t and Σ contains a small ball.

Theorem 3.8 *(An upper bound on the growth of H) Let* $\mathcal{G} \subset \mathbb{R}^{n+1}$ *be a ball or a half space,* $\Sigma := \partial\mathcal{G}$, $C_\Sigma := \|{}^{\Sigma}\!A\|_\infty$ *and* $M_0 \subset \bar{\mathcal{G}}$ *a convex imbedded hypersurface. Let* $F : M^n \times [0,T] \longrightarrow \mathbb{R}^{n+1}$ *be a solution of equation 1.2 with* $F(M^n,0) = M_0$*. Finally, choose*[5] $\sigma \in (0\,,\sup_{M^n}\|F(.,0)\|\cdot\min\{C_\Sigma\,\delta_0, 1\})$ *and* $T_\sigma \in (0,T]$ *such that* $s(.,t) \ge 2\sigma\ \ \forall t \in [0,T_\sigma]$*. Then*

[4] Recall that the origin of our coordinate system is *in* $\bar{\mathcal{G}}$!

[5] δ_0 is as defined at the beginning of this subsection.

$$H(x,t) \le \max\{ \sup_{M^n} H(.,0) \, , \; C_0 \tfrac{n}{\sigma^3} \} \quad \forall (x,t) \in M^n \times [0, T_\sigma] \, ,$$

where C_0 is a positive constant depending only on the curvature bound C_Σ and the initial imbedding M_0.

Proof. Define $k_0 := \sup_{M^n} \|F(.,0)\|$, $\delta := \frac{\sigma}{C_\Sigma k_0} < \delta_0$, $C := \frac{1}{4\delta^3}$ and

$$\psi(z) := \begin{cases} C\,(z-\delta)^3 + 1/2 \, , & \text{if } z \le \delta \, , \\ 1/2 \, , & \text{otherwise.} \end{cases}$$

Then $\psi \in C^2(\mathbb{R}_+)$ and $\psi \in [1/4, 1/2]$. We now consider the function f of Lemma 3.7:

(*i*) Using equation 3.4 and $\sigma < k_0$, the boundary behaviour is

$$\langle \nabla f, \mu \rangle \;\le\; f\left(C_\Sigma - \tfrac{3\,C\,\delta^2}{1/2 - C\,\delta^3} + C_\Sigma \tfrac{k_0}{\sigma} \right) \;<\; f\left(C_\Sigma \tfrac{2\,k_0}{\sigma} - \tfrac{3}{\delta} \right) \;<\; 0 \, ;$$

hence a maximum of f cannot occur on the boundary.

(*ii*) For any point $(x,t) \in [0, T_\sigma]$ with $\mathrm{dist}_{n+1}(F(x,t), \Sigma) > \delta$, we have $\psi \equiv 1/2$ and thus, using $s \ge 2\sigma$ on $[0, T_\sigma]$, the evolution equation for f yields

$$(\tfrac{\partial}{\partial t} - \Delta) f \;\le\; 4 f^2 (-\tfrac{\sigma^2}{n} f + 1) \;+\; \langle \nabla Q, \nabla f \rangle \, .$$

Hence f cannot develop a new maximum at (x,t) with $f \ge \frac{n}{\sigma^2}$.

(*iii*) Now let $\mathrm{dist}_{n+1}(F(x,t), \Sigma) \le \delta$. Here we have $2\sigma \le s \le \|F\| \le k_0$ and $1/4 \le \psi \le 1/2$, and thereby

$$\begin{aligned} (\tfrac{\partial}{\partial t} - \Delta) f \;\le\; & -\tfrac{4\sigma^2}{n} f^3 \;+\; (8 + 96\,C\,\delta^2 k_0) f^2 \\ & + 12\,C\,\delta\,(n\,\delta\,C_2 + 16\,C\,\delta^3 + 4) f \\ & + \langle \nabla Q, \nabla f \rangle \, . \end{aligned}$$

By Young's inequality,

$$(8 + 96\,C\,\delta^2 k_0) f \;\le\; \tfrac{2\sigma^2}{n} f^2 \;+\; 8\,\tfrac{n}{\sigma^2} (1 + 12\,C\,\delta^2 k_0)^2 \, ,$$

and using $\delta = \frac{\sigma}{C_\Sigma k_0}$, $\sigma < k_0$ and $C\,\delta^3 = 1/4$, we arrive at

$$(\tfrac{\partial}{\partial t} - \Delta) f \;\le\; f\left(-\tfrac{2\sigma^2}{n} f^2 + 2n\,C_3^2 \tfrac{1}{\sigma^4} \right) \;+\; \langle \nabla Q, \nabla f \rangle$$

with a constant $C_3 > 0$ depending only on M_0 and C_Σ. Thus, the occurence of new maxima with $f \ge C_3 \frac{n}{\sigma^3}$ can be ruled out.

(*iv*) From (*i*)-(*iii*) we conclude that H cannot develop a new maximum with

$$H \ge \tfrac{n\,C_3}{\sigma^3} 4\,k_0 \;=:\; C_0 \tfrac{n}{\sigma^3}$$

within the time interval $[0, T_\sigma]$. □

Remark 3.9 The above proof and thus the Theorem hold for arbitrary convex domains $\mathscr{G}$, if we can guarantee that convexity is preserved for the hypersurfaces M_t.

Remark 3.10 Since for convex hypersurfaces $|A|^2 \le H^2$, by controlling the mean curvature we automatically control the entire second fundamental form of the M_t.

4 Convexity and pinching

Here we study the properties of the second fundamental form of a given solution to the mean curvature flow equation 1.2: We show that (under suitable conditions on Σ) convexity and a pinching estimate are preserved. More precisely, we require Σ to be an umbilic hypersurface in $\mathbb{R}^{n+1}$, as in this case by the results of section 2 we can completely control the behaviour of the various tensors on the boundary. We begin by summarizing some evolution equations from [10]:

$$\frac{\partial}{\partial t} g_{ij} = -2H\, h_{ij}\,, \tag{4.1}$$

$$\frac{\partial}{\partial t} h_{ij} = \Delta h_{ij} - 2H\, h_{ik}\, g^{kl}\, h_{lj} + |A|^2\, h_{ij}\,, \tag{4.2}$$

$$\frac{\partial}{\partial t} |A|^2 = \Delta |A|^2 - 2\,|\nabla A|^2 + 2\,|A|^4\,. \tag{4.3}$$

We will make extensive use of the following maximum principle for symmetric tensors M_{ij} on M^n proved in [15], section 3:

Theorem 4.1 *(Weak maximum principle for symmetric tensors) Let M_{ij} be a continuous symmetric tensor on $M^n \times [0,T]$ which is of the class $C^{2,1}$ for $t > 0$. Let $N_{ij} := p(M_{ij}, g_{ij})$ be a polynomial in M_{ij} such that $N_{ij}\, v^i\, v^j \geq 0$ for all zero eigenvectors of M_{ij} ("zero eigenvector condition"). Let*

$$L(M_{ij}) := \frac{\partial}{\partial t} M_{ij} - \Delta M_{ij} - u^k\, \nabla_k M_{ij} - N_{ij}$$

be a parabolic operator with bounded vector fields u^k, w^k on $M^n \times [0,T]$ and $\partial M^n \times [0,T]$ respectively and $g(u,\mu) \geq 0$, $g(w,\mu) > 0$ on the boundary. Suppose the following conditions are fulfilled ($\delta > 0$ being an arbitrary, small constant):

$$\begin{array}{llll} (i) & M_{ij}(.,0) & \geq 0, & \\ (ii) & L(M_{ij}) & \geq 0 & \text{on } M^n \times (0,T], \\ (iii) & (\nabla_w M_{ij}) v^i v^j (x,t) & \geq 0 & \forall (x,t) \in \partial M^n \times (0,T],\ \forall \text{ minimal eigen-} \\ & & & \text{vectors } v \text{ of } M_{ij} \text{ with eigenvalue} \in [-\delta, 0]. \end{array}$$

Then $M_{ij} \geq 0$ on $M^n \times [0,T]$.

Furthermore, if at a point $(x,t) \in \partial M^n \times [0,T]$ the tensor M_{ij} decomposes, i.e. $M_{I\mu}(x,t) = M_{\mu I}(x,t) = 0 \quad \forall\, 1 \leq I \leq (n-1)$, the same conclusion holds if condition (iii) is replaced by

$$\begin{array}{llll} (iii') & (\nabla_w M_{ij}) v^i v^j (x,t) & \geq 0 & \forall \text{ minimal eigenvectors } v \in T_x \partial M^n \\ & & & \text{of } M_{ij} \text{ with eigenvalue} \in [-\delta, 0]\,, \\ (iii)'' & (\nabla_w M_{\mu\mu})(x,t) & \geq 0 & \text{if } M_{\mu\mu} \text{ min. and } -\delta \leq M_{\mu\mu}(x,t) \leq 0\,. \end{array}$$

Assumption 4.2 *Throughout this section, let Σ be the boundary of a ball $B^{n+1}_{1/\alpha}$ or a hyperplane in $\mathbb{R}^{n+1}$. In the former case, let $M_0 \subset \bar{B}^{n+1}_{1/\alpha}$, thereby Σ being convex.*

Thus we have

$$^{\Sigma}\!A(v,w)(p) = \alpha\, {}^{\Sigma}g(v,w)(p) \quad \forall p \in \Sigma\,, \quad \forall v,w \in T_p\Sigma\,,$$

or, equivalently, for the components with respect to a parametrisation,

$$h^{\Sigma}_{\hat{\imath}\hat{\jmath}}(p) = \alpha\, {}^{\Sigma}g_{\hat{\imath}\hat{\jmath}}(p) \quad \forall p \in \Sigma\,, \quad \forall \hat{\imath},\hat{\jmath}\,.$$

Thereby $\alpha \geq 0$ due to the convexity of Σ.

Theorem 4.3 *Let Σ be umbilic, $p \in \Sigma \cap M_t$, and choose local coordinates around p as discussed in Remark 2.5. Then the following relations hold:*

$$\begin{array}{llcl}
(i) & h_{\mu I} & = & 0\,, \\
(ii') & \nabla_\mu h_{IJ} & = & \alpha(-h_{IJ} + h_{\mu\mu}\,\delta_{IJ})\,, \\
(ii'') & \nabla_\mu h_{\mu\mu} & = & \alpha(2H - n\,h_{\mu\mu})\,, \\
(iii) & \nabla_\mu H & = & \alpha H\,.
\end{array}$$

Proof. (i) follows immediately from $h^{\Sigma}_{\nu I} = \alpha\,\delta_{\nu I} = 0$ and Theorem 2.4,(i). Moreover, by Ricci's Lemma, ${}^{\Sigma}\nabla_{\hat{\imath}}\, {}^{\Sigma}g_{\hat{p}\hat{q}} \equiv 0$; hence ${}^{\Sigma}\nabla_I\, h^{\Sigma}_{\nu J} = \alpha\, {}^{\Sigma}\nabla_I\, {}^{\Sigma}g_{\nu J} = 0$. Theorem 2.4,$(ii)$ thus yields $\nabla_I\, h_{\mu J} = \alpha(-h_{IJ} + h_{\mu\mu}\,\delta_{IJ})$, and $(ii)'$ is a consequence of the *Codazzi* equations $\nabla_I\, h_{\mu J} = \nabla_\mu\, h_{IJ}$. (iii) follows from $h^{\Sigma}_{\nu\nu} = \alpha\,\delta_{\nu\nu} = \alpha$, and combining $(ii)'$ and Theorem 2.4, (iii) yields (ii). □

Thus if Σ is umbilic, the *normal* derivative of the second fundamental form of M_t can be expressed essentially in terms of the curvature operator itself. Furthermore, in this case the second fundamental form A decomposes on Σ.

4.1 Convexity

Theorem 4.4 *Let Σ be convex and umbilic. Then convexity is preserved for any solution of the mean curvature flow equation 1.2.*

Proof. We want to apply Theorem 4.1. To this end, we define $M_{ij} := h_{ij}$, $N_{ij} := -2H\, h_{ik}\, g^{kl}\, h_{lj} + |A|^2\, h_{ij}$, $u^k = 0$ and observe that the required evolution equation and the zero eigenvector condition are fulfilled. Furthermore, $h_{ij}(.,0) \geq 0$ by assumption, and thus in order to apply Theorem 4.1 we only have to check the boundary condition. Let $p := F(x_0,t_0) \in \Sigma$ and choose local coordinates around p as in Remark 2.5. As A decomposes on Σ (cf. Theorem 4.3,(i)), we can use the second part of Theorem 4.1:

(iii') If $v := v^I\,\partial_I$ is a minimal eigenvector of h_{ij} with $h_{ij}v^iv^j = h_{IJ}v^Iv^J = -\delta\|v\|^2 \leq 0$, we compute from Theorem 4.3,$(ii)'$ $\nabla_\mu\, h_{ij}v^iv^j = \alpha(\delta + h_{\mu\mu})\,\|v\|^2$. As v is minimal, $h_{\mu\mu} \geq -\delta$ and thus $\nabla_\mu\, h_{ij}\, v^i\, v^j \geq 0$.

(iii'') Now let $h_{\mu\mu} \leq 0$ be minimal. Then Theorem 4.3,(ii'') yields $\nabla_\mu\, h_{\mu\mu} \geq 2\,\alpha H$. By assumption, $H(.,0) \geq 0$, and by Theorem 3.1, this is preserved; thus $\nabla_\mu\, h_{\mu\mu} \geq 0$.

Hence the result follows from Theorem 4.1. □

Proposition 4.5 *Let Σ be convex and umbilic and $C \leq \frac{2}{n} \min_{M^n} H(.,0)$. Then the estimate*

$$h_{ij} \geq C\, g_{ij}$$

is preserved for any solution of equation 1.2.

Proof. Let $M_{ij} := h_{ij} - C\, g_{ij}$. A straightforward calculation using equations 4.1, 4.2 yields

$$(\tfrac{\partial}{\partial t} - \Delta) M_{ij} \geq -2 H\, M_{ik}\, g^{kl}\, M_{lj} + (|A|^2 - 2\, CH)\, M_{ij} \ ,$$

thus a "suitable" evolution equation (with respect to Theoren 4.1) and the zero eigenvector condition are fulfilled. By assumption, $M_{ij}(.,0) \geq 0$; thus again the only thing to check is the boundary condition of Theorem 4.1. Using Theorem 4.3 and Ricci's Lemma, we compute

$$\begin{aligned} \nabla_\mu M_{IJ} &= \alpha(-M_{IJ} + M_{\mu\mu}) \ , \\ \nabla_\mu M_{\mu\mu} &= \alpha(2H - n\, M_{\mu\mu} - n\, C) \ . \end{aligned}$$

Now, if $v = v^I\, \partial_I$ is a minimal eigenvector with $M_{IJ}\, v^I\, v^J = -\delta\, \|v\|^2 \leq 0$, we have $M_{\mu\mu} \geq -\delta$ and thus $\nabla_\mu M_{IJ}\, v^I\, v^J = \alpha(\delta + M_{\mu\mu}) \|v\|^2 \geq 0$. If, on the other hand, $M_{\mu\mu} \leq 0$ is minimal, we compute $\nabla_\mu M_{\mu\mu} \geq 2\,\alpha(H - \min_{M^n} H(.,0))$. As Σ is convex, we have $H \geq \min_{M^n} H(.,0)$ by Theorem 3.2; thus $\nabla_\mu M_{\mu\mu} \geq 0$.

Hence we can apply Theorem 4.1 and obtain the result. □

4.2 Pinching

Now we want to establish various pinching estimates. Huisken [10] in the boundaryless case proved that for any $0 < \varepsilon \leq 1/n < \kappa < 1$ the estimate

$$\varepsilon H\, g_{ij} \leq h_{ij} \leq \kappa H\, g_{ij}$$

is preserved. Note that this estimate is optimal as ε and κ cannot exceed the above stated range for any convex hypersurface. Now, in our case, such an optimal estimate cannot hold: Let M_0 be a subset of S^n_R and $\Sigma = S^n_{1/\alpha}$ such that the Neumann boundary condition is fulfilled. Then $h_{ij} = \frac{1}{n} H\, g_{ij}$, i.e. $\varepsilon = 1/n$; but an elementary geometric consideration shows that under the mean curvature flow M_t will *not* be a sphere for $t > 0$; thus the optimal pinching is not preserved.

Theorem 4.6 *Let Σ be convex and umbilic, and let $\kappa \geq \frac{2}{n+1}$. Then equation 1.2 preserves the pinching condition*

$$h_{ij} \leq \kappa H\, g_{ij} \ .$$

Proof. Let $M_{ij} := \kappa H\, g_{ij} - h_{ij}$. By a straightforward calculation,

$$(\tfrac{\partial}{\partial t} - \Delta)M_{ij} = 2\,H\, M_{ik}\, g^{kl}\, M_{lj} + (|A|^2 - 2\,\kappa H^2)\, M_{ij}\ ,$$

i.e. the evolution equation and the zero eigenvector condition are fulfilled. By assumption $M_{ij}(.,0) \geq 0$, and thus it remains to check the boundary condition. Here, Theorem 4.3 yields

$$\nabla_\mu M_{IJ} = \alpha\,(\kappa H\, \delta_{IJ} - M_{IJ} + M_{\mu\mu}\, \delta_{IJ})\ ,$$
$$\nabla_\mu M_{\mu\mu} = \alpha\, \left(-n\, M_{\mu\mu} + ((n+1)\kappa - 2)\, H\right)\ .$$

Thus, if $v = v^I\, \partial_I$ is a minimal eigenvector with $M_{IJ} v^I v^J = -\delta\, \|v\|^2 \leq 0$, we have $\nabla_\mu M_{IJ} v^I v^J = \alpha\,(\kappa H + \delta + M_{\mu\mu})\, \|v\|^2 \geq 0$, as $H \geq 0$ is preserved. On the other hand, if $M_{\mu\mu} \leq 0$ is minimal, we obtain $\nabla_\mu M_{\mu\mu} \geq \alpha\ ((n+1)\kappa - 2)\ H \geq 0$, as $\kappa \geq \frac{2}{n+1}$. Thus Theorem 4.1 yields the result. □

Now we would like to proceed as in [10] and establish a pinching estimate from below using the function $M_{ij} = h_{ij} - \varepsilon H\, g_{ij}$; unfortunately, the normal derivative of this function at the boundary is $\nabla_\mu M_{IJ} v^I v^J \geq -\alpha\, \varepsilon\, H\, \|v\|^2$, if v is a minimal tangential eigenvector; this is not sufficient to guarantee for the requirements of Theorem 4.1. It turns out that the negative term occurs from the boundary behaviour of H - thus we must replace H by a function with a "better" boundary behaviour.

Lemma 4.7 *Let Σ be convex and umbilic. Furthermore, let M_0 be strictly convex. Then*

$$(\tfrac{\partial}{\partial t} - \Delta)|A| \leq |A|^3\ ,$$

and on the boundary (with respect to the "usual" coordinates)

$$\nabla_\mu |A| = \alpha\, |A|^{-1}\, (3\,H\, h_{\mu\mu} - |A|^2 - n\, h_{\mu\mu}^2)\ .$$

Proof. First note that $|A|^2 > 0$ is preserved (Theorem 4.5). Thus $|A|$ is a smooth function, and we compute

$$(\tfrac{\partial}{\partial t} - \Delta)|A| = \tfrac{1}{|A|}\, (\|\nabla |A|\|^2 - |\nabla A|^2) + |A|^3\ .$$

The evolution equation then follows from the Cauchy-Schwarz inequality $\|\nabla |A|\|^2 \leq |\nabla A|^2$.

Furthermore, using Theorem 4.3, a simple calculation yields

$$\nabla_\mu |A|^2 = 2\,\alpha\,(3\,H\, h_{\mu\mu} - |A|^2 - n\, h_{\mu\mu}^2)$$

on Σ, and the result follows from $\nabla_\mu |A| = \frac{1}{2|A|}\, \nabla_\mu |A|^2$. □

Theorem 4.8 *Let Σ be convex and umbilic, and let M_0 be strictly convex. Then for $\varepsilon \leq \min\{\frac{1}{3\sqrt{n}}, \frac{2}{n}\}$, equation 1.2 preserves the pinching estimate*

$$h_{ij} \geq \varepsilon\, |A|\, g_{ij}\ .$$

Proof. We define $M_{ij} := h_{ij} - \varepsilon\,|A|\,g_{ij}$ and compute, using the above Lemma

$$(\tfrac{\partial}{\partial t} - \Delta)M_{ij} \geq -2H\,M_{ik}\,g^{kl}\,M_{lj} + (|A|^2 - 2\,\varepsilon H\,|A|)M_{ij}\ ,$$

and at a boundary point

$$\begin{aligned}\nabla_\mu M_{IJ} &= \alpha\,\big(-M_{IJ} + M_{\mu\mu}(1 - 3\,\varepsilon H\,|A|^{-1})\,\delta_{IJ} \\ &\quad + \varepsilon(-3\,\varepsilon H + |A| + n\,|A|^{-1}\,h^2_{\mu\mu})\,\delta_{IJ}\big)\ , \\ \nabla_\mu M_{\mu\mu} &= \alpha\,\big(-(n + 3\,\varepsilon H\,|A|^{-1})\,M_{\mu\mu} + (2 - 3\,\varepsilon^2)\,H \\ &\quad - \varepsilon\,(n-1)\,|A| + n\,\varepsilon\,|A|^{-1}\,h^2_{\mu\mu}\big)\ .\end{aligned}$$

As convexity is preserved, we have $\frac{1}{n}H^2 \leq |A|^2 \leq H^2$; furthermore, by assumption, $\varepsilon \leq \frac{1}{3\sqrt{n}}$ and thus $\lambda := 1 - 3\,\varepsilon H\,|A|^{-1} \in [0,1]$.

Now, if $v = v^I\,\partial_I$ is a minimal eigenvector with $M_{IJ}\,v^I\,v^J = -\delta\,\|v\|^2 \leq 0$,

$$\nabla_\mu M_{ij} v^i v^j \geq \alpha\,\big(\delta + M_{\mu\mu}\,\lambda\big)\ \|v\|^2\ ,$$

and $M_{\mu\mu} \geq -\delta$ (by minimality of $-\delta$) and $0 \leq \lambda \leq 1$ yield $\nabla_\mu M_{ij} v^i v^j \geq 0$.

If $M_{\mu\mu} = -\delta \leq 0$ is minimal, we have

$$\nabla_\mu M_{\mu\mu} \geq \alpha\,|A|\ \big(2 - 3\,\varepsilon^2 - \varepsilon\,(n-1)\big) \geq 0\ ,$$

by the assumption on ε.

Again the result follows from Theorem 4.1. □

Corollary 4.9 *Let Σ be convex and umbilic, M_0 strictly convex and $\varepsilon \leq \min\{\frac{1}{3\sqrt{n}}, \frac{2}{n}\}$. Then the estimate $h_{ij}(.,0) \geq \varepsilon\,(H\,g_{ij})(.,0)$ implies*

$$h_{ij} \geq \tfrac{\varepsilon}{\sqrt{n}} H\,g_{ij} \quad on \quad M^n \times [0,T]\ .$$

Proof. This follows directly from Theorem 4.8 and the estimate $\frac{1}{n}H^2 \leq |A|^2 \leq H^2$ for convex hypersurfaces. □

5 Convergence

We finally turn to the study of the singularity which develops as $t \longrightarrow T_C$. The main results have been stated in Theorem 1.3 and Proposition 1.4; thus it essentially remains to give the proofs. As before, we let Σ be the boundary of a ball or a half space in $\mathbb{R}^{n+1}$; however, in this section we will *not* use the fact that Σ is umbilic. Thus this latter property is needed only to preserve convexity and pinching of the solutions M_t, cf. section 4. We begin with summarizing the assumptions we will need throughout this section:

Assumption 5.1 *Let Σ be convex and umbilic and M_0 a strictly convex, imbedded hypersurface. Furthermore, let $\boldsymbol{F} : [0, T_C) \longrightarrow \mathbb{R}^{n+1}$ be a solution to equation 1.2 on a maximal time interval and $\boldsymbol{F}(M^n, 0) = M_0$.*

5.1 Convergence to a point

Here we generalize an idea of Tso [16] to the boundary case. Note that, by Proposition 4.5 and [15], Theorem 4.5, the M_t will be imbedded and strictly convex with a uniform lower bound on the second fundamental form, i.e. there exists $\varepsilon > 0$ such that

$$(5.1) \qquad h_{ij} \geq \varepsilon g_{ij} \quad \forall (x,t) \in M^n \times [0, T_C) .$$

Now for all t the hypersurfaces M_t and Σ bound a convex domain $\mathscr{K}_t \subset \mathbb{R}^{n+1}$, and from equation 3.4 we immediately conclude

$$\mathscr{K}_{t_2} \subset \mathscr{K}_{t_1} \quad \text{if} \quad t_2 \geq t_1 .$$

We set

$$\mathscr{K} := \bigcap \{\mathscr{K}_t : 0 \leq t < T_C\}$$

and observe that $\mathscr{K}$ is convex as this holds for each $\mathscr{K}_t$. Now suppose that $\sigma > 0$ and $\boldsymbol{p} \in \mathscr{K}$ such that the ball $B^{n+1}_{2\sigma}(\boldsymbol{p}) \subset \mathbb{R}^{n+1}$ is contained in $\mathscr{K}$. Then, choosing coordinates in $\mathbb{R}^{n+1}$ with origin in $\boldsymbol{p}$, we obtain $s \geq 2\sigma$ on $M^n \times [0, T_C)$ for the support function of our solution $\boldsymbol{F}$. According to Theorem 3.8 this implies an upper bound on H and thus, by convextiy, on $|A|$. Hence, by theorem 1.2, we can extend $\boldsymbol{F}$ to a solution on $[0, T_C + \delta)$ which contradicts the maximaltiy of T_C.

We therefore conclude that $\mathscr{K}$ cannot contain any ball B^{n+1}_r. But this implies that $\mathscr{K}$ has zero volume, $V(\mathscr{K}) = 0$, as $\mathscr{K}$ is a bounded, convex domain. Now we can prove:

Theorem 5.2 *(Convergence to a point) Under the assumptions 5.1 the hypersurfaces M_t converge to a point on Σ as $t \to T_C$.*

Proof. (cf. [16], Theorem 4.2) Suppose $\mathscr{K}$ is not a point, i.e. $\mathrm{diam}(\mathscr{K}) > 0$. Let $\boldsymbol{p}, \boldsymbol{q}$ be such that $d := \mathrm{dist}_{n+1}(\boldsymbol{p}, \boldsymbol{q}) = \mathrm{diam}(\mathscr{K})$.

We have $\boldsymbol{p}, \boldsymbol{q} \in \mathscr{K}_t \quad \forall t$, and as $V(\mathscr{K}) = 0$ there is a (two dimensional) plane E through $\boldsymbol{p}, \boldsymbol{q}$ such that the area of the convex subset $C_t := \mathscr{K}_t \cap E \subset \mathbb{R}^2$ of E fulfils

$$(*) \qquad A(C_t) \overset{t \to T_C}{\longrightarrow} 0 .$$

As $\mathscr{K}_{t_2} \subset \mathscr{K}_{t_1}$ whenever $t_2 \geq t_1$, we have

$$(**) \qquad C_{t_2} \subset C_{t_1} \quad \forall\, t_2 \geq t_1 .$$

The convex curve $\gamma_t := \partial C_t$ can be decomposed into $\gamma_t = \gamma_t' \cup \gamma_t''$ such that $\gamma_t' \subset M_t$, $\gamma_t'' \subset \Sigma$ (hereby $\gamma_t'' = \emptyset$ being allowed) and (from $(**)$):

$$(***) \qquad \gamma_{t_2}'' \subset \gamma_{t_1}'' \quad \forall\, t_2 \geq t_1 .$$

This means that as t increases, no "new" boundary points of Σ can occur on γ_t. As γ_t is convex, we conclude from $(*)$ that there must exist points on γ_t with smaller and smaller curvature as t increases. From $(***)$ it follows that these points must belong to $\gamma_t' \subset M_t$ which is a contradiction to the uniform convexity equation 5.1. Thus $\mathscr{K}$ must be a point, and the result follows. □

Proof of Proposition 1.4. In the case $n = 1$ strict convexity of the M_t is equivalent to $H \geq C_0 > 0$, and by Theorem 3.2 this condition is preserved for arbitrary convex hypersurfaces Σ. □

5.2 Convergence to a hemisphere

Now we want to study the *shape* of the singularity; hereby, we follow an idea due to Hamilton [9] and rescale the entire flow. A similar rescaling for curves was carried out by Altschuler in [1]. In addition to Assumption 5.1, we will require $n \geq 2$ in the sequel. Note that by Theorem 4.8 and Corollary 4.9, a pinching condition

$$h_{ij} \geq \varepsilon H\, g_{ij}$$

will hold for $t \in [0, T_C)$.

Definition 5.3 *Let (x_k, t_k) be a sequence in $M^n \times [0, T_C)$ such that $t_k \nearrow T_C$. If*

$$H(x_k, t_k) = \max\{H(x, t_k) : x \in M^n\} \quad \forall k\ ,$$

(x_k, t_k) is called a blowup-sequence.

In the sequel we will rescale the mean curvature flow equation 1.2 along a given blowup-sequence. To this end, let

$$\begin{aligned} \lambda_k &:= H(x_k, t_k) = \max_{M^n} H(., t_k)\ , \\ \alpha_k &:= -\lambda_k^2\, t_k\ , \\ \omega_k &:= \lambda_k^2\,(T_C - t_k)\ , \\ \tau &:= \lambda_k^2\,(t - t_k)\ , \\ \tilde{\boldsymbol{F}}_k(., \tau) &:= \lambda_k\,(A_k \cdot \boldsymbol{F}(., t) + \boldsymbol{b}_k)\ ,\ A_k \in SO_{n+1}\mathbb{R}\ ,\ \boldsymbol{b}_k \in \mathbb{R}^{n+1}\ , \\ \tilde{M}_\tau^{(k)} &:= \tilde{\boldsymbol{F}}_k(M^n, \tau)\ . \end{aligned}$$

Thereby let A_k and $\boldsymbol{b}_k$ be such that $\tilde{\boldsymbol{F}}_k(x_k, 0) = \boldsymbol{0}$ and $\tilde{\boldsymbol{\nu}}_k(x_k, 0) = \boldsymbol{e}_{n+1}$ for any k. A straightforward calculation then yields that each $\tilde{\boldsymbol{F}}_k$ is itself a mean curvature flow, and $\tilde{H}_k(x_k, 0) = 1 \quad \forall k$. As usual, a singularity will be referred to as of *type I* if $\max_{M^n} |A|^2(., t) \leq K\,(T_C - t)^{-1}$ as $t \to T_C$ with a given constant $K > 0$; otherwise it will be called a *type-II*-singularity. Now, given a blowup-sequence, $\alpha_k \searrow -\infty$ as $t \to T_C$. Moreover, by taking a subsequence, we can obtain $\omega_k \nearrow \omega < \infty$ in the case of a type-*I*-singularity, and $\omega_k \nearrow \infty$, otherwise. Finally, as $\lambda_k \to \infty$, we can assume $\lambda_k > 1\ \forall k$. Now the crucial step is to control the curvature of the rescaled hypersurfaces:

Proposition 5.4 *(Curvature estimates for the rescaled hypersurfaces)*

$$\begin{aligned}
\textit{Type I:}\quad & \forall\delta > 0\ \exists k_0 = k_0(\delta)\ \text{ and }\ C = C(\delta)\ \text{ such that} \\
& \tilde{H}_k \le C\ \text{ on }\ M^n \times [\alpha_{k_0}, \omega - \delta]\,,\ \ \forall k \ge k_0\,. \\
\textit{Type II:}\quad & \forall\,\bar{\omega} > 0,\ \forall\varepsilon > 0\ \exists k_0 = k_0(\bar{\omega}, \varepsilon)\ \text{ such that} \\
& \tilde{H}_k \le 1 + \varepsilon\ \text{ on }\ M^n \times [\alpha_{k_0}, \bar{\omega}]\,,\ \ \forall k \ge k_0\,.
\end{aligned}$$

Hence on compact domains of spacetime, the curvature of the rescaled flows is bounded uniformly in k.

Proof.

I: As $\omega_k \to \omega$, we can choose k_0 such that $[\alpha_{k_0}, \omega - \delta] \subset [\alpha_k, \omega_k - \frac{\delta}{2}]\ \forall k \ge k_0$. Now suppose the statement does not hold. Then there is a sequence $(y_k, \tau_k) \in M^n \times [\alpha_{k_0}, \omega - \delta]$ such that $\tilde{H}_k(y_k, \tau_k) \to \infty$. As for $k \ge k_0$, $\omega_k \ge \omega - \frac{\delta}{2}$ we can choose a subsequence with

$$\tilde{H}_{\hat{k}}(y_{\hat{k}}, \tau_{\hat{k}}) \ge C_{\hat{k}}\,, \quad y_{\hat{k}} \in M^n\,, \quad \tau_{\hat{k}} \le \omega_{\hat{k}} - \tfrac{\delta}{2}\,, \quad C_{\hat{k}} \nearrow \infty$$

and thereby, defining $\tilde{t}_{\hat{k}} := \lambda_{\hat{k}}^{-2}\tau_{\hat{k}} + t_{\hat{k}}$:

$$(*) \qquad H(y_{\hat{k}}, \tilde{t}_{\hat{k}}) \ge \lambda_{\hat{k}}\, C_{\hat{k}}\,, \quad y_{\hat{k}} \in M^n\,, \quad 0 \le \tilde{t}_{\hat{k}} \le T_C - \lambda_{\hat{k}}^{-2}\tfrac{\delta}{2}\,.$$

But the singularity being of type I implies

$$H(y_{\hat{k}}, \tilde{t}_{\hat{k}}) \le \sqrt{\tfrac{2K}{\delta}}\,\lambda_{\hat{k}}\,;$$

hence $\hat{k} \to \infty$ yields a contradiction to $(*)$.

II: This has been proved by Hamilton [9] as follows: Given the sequence (x_k, t_k), choose $(\tilde{x}_k, \tilde{t}_k)$ such that

$$(*) \qquad (t_k - \tilde{t}_k) H^2(\tilde{x}_k, \tilde{t}_k) = \max_{M^n \times [0, t_k]} (t_k - t) H^2(.,t)\,.$$

Defining $\tilde{\omega}_k := \lambda_k^2(t_k - \tilde{t}_k)$, we obtain $\tilde{H}_k(\tilde{x}_k, 0) = 1\ \ \forall k$ and $\tilde{\omega}_k \to +\infty$ as $k \to \infty$. Furthermore, $(*)$ yields $\tilde{H}_k^2(.,\tau) \le \frac{\omega_k}{\omega_k - \tau}\ \ \forall \tau \in [\alpha_k, \omega_k)$ and thereby the result. □

Now the curvature bounds immediately imply local gradient bounds which are *uniform in k*: Let $\boldsymbol{p}_0^{(k)} := \tilde{\boldsymbol{F}}_k(x_0, \tau_0)$, $(x_0, \tau_0) \in M^n \times [\alpha_k, \omega_k]$ and $\boldsymbol{\omega}_k := \tilde{\boldsymbol{\nu}}_k(x_0, \tau_0)$. Furthermore, let

$$\tilde{v}_k(x, \tau) := \langle\, \boldsymbol{\omega}_k\,,\, \tilde{\boldsymbol{\nu}}_k(x, \tau)\,\rangle^{-1}\,.$$

Theorems 6.7 and 6.13 of [15] then yield local gradient bounds for the rescaled flows $\tilde{\boldsymbol{F}}_k$ on domains $\mathcal{U}_{r_k} \times [\tau_0, \tau_0 + \Delta_k]$, where r_k and Δ_k depend only on the maximum of the curvatures $|\tilde{A}_k|, |{}^{\Sigma}\tilde{A}_k|$ of the rescaled hypersurfaces $\tilde{M}_\tau$, $\tilde{\Sigma}_k$. But as the $\tilde{M}_\tau$ are convex, the above Proposition yields a bound on $|\tilde{A}_k|$ which is uniform in k. Moreover, as we assumed $\lambda_k \ge 1$, the curvatures of the $\tilde{\Sigma}_k$ will be uniformly bounded in k as well. Hence we obtain a local gradient estimate which is *uniform in k*. Now we can state the first Theorem concerning a limit flow:

Theorem 5.5 *A subsequence of the rescaled flows $\tilde{F}_k$ converges to a limit flow $\tilde{F}_\infty : M^n \times (\alpha, \omega) \longrightarrow \mathbb{R}^{n+1}$ ($\omega = \infty$ allowed) such that*

(i) *$\tilde{F}$ is a mean curvature flow;*
(ii) *If the M_τ^∞ possess a boundary, then $\partial M_\tau^\infty \subset \tilde{\Sigma}_\infty$, $\tilde{\Sigma}_\infty$ is a hyperplane and $\langle \tilde{\nu}_\infty, \tilde{\mu}_\infty \rangle = 0$;*
(iii) *The hypersurfaces $\tilde{M}_\tau^\infty$ are pinched as the original hypersurfaces, i.e. $\varepsilon H\, g_{ij} \le h_{ij} \le \kappa H\, g_{ij}$ implies $\varepsilon \tilde{H}_\infty \tilde{g}_{ij}^\infty \le \tilde{h}_{ij}^\infty \le \kappa \tilde{H}_\infty \tilde{g}_{ij}^\infty$.*

Proof. In view of the above considerations, the only thing to check is the convergence of a subsequence. This can be done by standard methods involving the Arzelà-Ascoli Theorem: We first let $A := B_R^{n+1}(\mathbf{0}) \times [\tau_-, \tau_+]$. Using the above k-uniform local gradient estimate , we can cover A by finitely many spacetime-cylinders $C_{r,\Delta}(\boldsymbol{p}_m, \tau_m)$ such that for each $(\boldsymbol{p}_m, \tau_m)$ *and every* k the intersection of the hypersurfaces $\tilde{M}_\tau^k$ with $C_{r,\Delta}(\boldsymbol{p}_m, \tau_m)$ can be written as a graph with bounded gradient. Hence the Arzelà-Ascoli Theorem yields the convergence of a subsequence on A with the desired properties. Doing this for a whole *sequence* $A_{R^l,[\tau_-^l,\tau_+^l]}$ and choosing a diagonal sequence completes the proof. □

Remark 5.6 $\mathscr{H} := \tilde{\Sigma}_\infty$ is a hyperplane if the hypersurfaces of the limit flow do have a boundary. Thus, in this case we can by reflection at $\mathscr{H}$ extend the hypersurfaces symmetrically to closed, convex and pinched hypersurfaces without boundary which again move under the mean curvature flow. Thus, in any case we can in the sequel assume the $\tilde{M}_\tau^\infty$ to be complete, strictly convex, pinched hypersurfaces.

The limit flow can now be described by means of a result due to Hamilton ([8], Main Theorem):

Theorem 5.7 *Let M be a smooth, strictly convex, complete hypersurface bounding a region in $\mathbb{R}^{n+1}$, $n \ge 2$. Let M be ε-pinched, i.e. let $h_{ij} \ge \varepsilon H\, g_{ij}$. Then M is compact.*

We remark that this is the point where we need $n \ge 2$. An immediate consequence of this Theorem is

Theorem 5.8 *Under Assumption 5.1 the singularity at $t = T_C$ is a type-I-singularity.*

Proof. By Theorem 5.5, $\tilde{F}_\infty$ moves under the mean curvature flow. Thus the singularity being of Type *II* would imply that the solution would be defined for $\tau \in (-\infty, \infty)$. But then all the $\tilde{M}_\tau$ would have to be noncompact, as compact hypersurfaces are contained in a ball $B_R \subset \mathbb{R}^{n+1}$ and therefore cannot "live" eternally. Now the $\tilde{M}_\tau$ are pinched by Theorem 5.5, and thus we run into a contradiction to Theorem 5.7. □

Hence the boundary of the $\tilde{M}_k$ cannot be "shot away" to infinity as $k \to \infty$. But, up to now, it could shrink away such that the $\tilde{M}_k$ would be compact, *closed* hypersurfaces. In order to rule out this possibility we observe that the image of the Gauss map of the "original" hypersurfaces M_t is contained in a hemisphere. Hence this holds for each of the rescaled flows, the various hemispheres depending on k. Again by the Arzelà-Ascoli Theorem, we obtain the C^∞-convergence of a subsequence to a limit flow $\tilde{F}_\infty$ such that in addition to the above properties the images of all $\tilde{M}_\tau^\infty$ are contained in a *unique* hemisphere. According to Theorem 5.5 the boundary of these hypersurfaces will therefore lie on a hyperplane. From there, the proof of Theorem 1.3 is an immediate consequence of Remark 5.6 and a Theorem of Huisken [10] which states that closed, strictly convex hypersurfaces moving by mean curvature shrink to a single point within finite time, the (appropriately) rescaled hypersurfaces thereby converging to a sphere in the C^∞-topology.

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