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On the resummation of the $\alpha \ln^2 z$ terms for QED corrections to deep-inelastic ep scattering and e^+e^- annihilation

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Abstract. The resummation of the $\alpha \ln^2(z)$ non-singlet contributions is performed for initial state QED corrections. As examples, the effect of the resummation on neutral-current deep-inelastic scattering and the $e^+e^- \to \mu^+\mu^-$ scattering cross section near the Z^0-peak is investigated.

1 Introduction

The non-singlet splitting functions of QCD are known to behave as $\alpha_s^{l+1} \ln^{2l}(z)$ [1] for small values of z, the momentum fraction determining the corresponding radiator function. A similar behaviour is observed also in QED[1] [3, 4]. These terms may potentially yield large contributions to the radiative corrections. In an approach based on the systematic evaluation of the Feynman diagrams at a fixed order in the coupling constant, the contributions of $O[\alpha^{l+1} \ln^{2l}(z)]$ emerge from a wide class of terms, see for example [3,5]. Therefore the all–order resummation of these terms cannot be carried out by direct diagram calculations but is performed by solving so-called infrared evolution equations [1]. It is interesting to note that double logarithmic terms emerge also in other physical situations. As an example we mention the Sudakov form factor, which is mostly studied for the region $z \sim 1$. For a fixed coupling it behaves $\propto \alpha \ln^2(Q^2/m^2)$, see [6]. Contributions to this quantity $\propto \ln^2 z$ may become important as $z \to 0$.

In the present paper we calculate the contribution of the small-z resummed terms to the initial state radiative corrections for deep-inelastic ep scattering (DIS). We compare these corrections with those resummed by the non–singlet Altarelli–Parisi equation in QED, $\propto \alpha^l \ln^l(Q^2/m_e^2)$. We also evaluate the contribution of these terms to the initial state corrections to $e^+e^- \to \mu^+\mu^-$ at the Z^0-peak.

2 Basic relations

The evolution of the non-singlet electron structure function $D(z, Q^2)$ is governed by

$$\frac{\partial D(z,Q^2)}{\partial \ln Q^2} = P\left[z, \alpha(Q^2)\right] \otimes D(z,Q^2), \qquad (1)$$

[1] A first application to QED was discussed in [2], considering forward $e^+e^- \to \mu^+\mu^-$ annihilation in the high energy limit

where $\otimes$ denotes the Mellin convolution

$$A(z) \otimes B(z) \equiv \int_0^1 \int_0^1 dz_1 dz_2 A(z_1) B(z_2) \delta(z - z_1 z_2). \qquad (2)$$

Equation (1) is a flavor non-singlet evolution equation, which results from the renormalization group equation [7] ruling the mass factorization [8]. As well-known this equation can be equivalently derived using the operator product expansion [9], which is valid for $z \in\,]0,1]$, if $P(z,\alpha)$ and $D(z,Q^2)$ are interpreted as distributions [10]. This we will do in the following.

The splitting function $P\left[z, \alpha(Q^2)\right]$ can be represented by the series

$$P\left[z, \alpha(Q^2)\right] = \sum_{k=1}^{\infty} a^k(Q^2) P_k(z), \qquad (3)$$

with $a(Q^2) = \alpha(Q^2)/(4\pi)$. In leading order, the evolution of the QED coupling constant $a(Q^2)$ is described by

$$\frac{\partial a(Q^2)}{\partial \ln Q^2} = \frac{4}{3} a^2(Q^2), \qquad (4)$$

yielding

$$a(Q^2) = \frac{a(m_e^2)}{1 - \frac{4}{3} a(m_e^2) \ln\left(\frac{Q^2}{m_e^2}\right)}. \qquad (5)$$

Here we have considered only the electron threshold in the evolution. For the solution of (1), we use the first-order splitting function[2]

$$P_1(z) = 2\left(\frac{1+z^2}{1-z}\right)_+ . \qquad (6)$$

[2] This solution of the evolution equation is well-known from QCD [11]. For the non-singlet case the $O(\alpha \ln(Q^2/m^2))$ result for the initial state QED corrections to e^+e^- annihilation was obtained in [12] already. Later applications were given, e.g., in [13]

For the higher order contributions in $a(Q^2)$, we account for the leading terms at small z, denoted by $P_{z\to 0}$, which are $\propto a^{l+1}\ln^{2l}(z)$. The latter terms are obtained in resummed form in Mellin space by

$$\mathcal{M}[P_{z\to 0}](N,a) \equiv \int_0^1 dz\, z^{N-1} P_{z\to 0}(z,a) \equiv -\frac{1}{2}\Gamma^{-}_{z\to 0}(N,a) = \frac{1}{8\pi^2} f_0^-(N,a). \quad (7)$$

$f_0^-(N,a)$ is the solution to the equation [1]

$$f_0^-(N,a) = 16\pi^2\frac{a}{N} + 8\frac{a}{N^2} f_V^+(N,a) + \frac{1}{8\pi^2}\frac{1}{N}\left[f_0^-(N,a)\right]^2, \quad (8)$$

and $f_V^+(N,a)$ obeys

$$f_V^+(N,a) = 16\pi^2\frac{a}{N} + \frac{1}{8\pi^2}\frac{1}{N}\left[f_V^+(N,a)\right]^2 . \quad (9)$$

Here the coefficients in (8, 9), originally given for $SU(N)$ in [1], were adjusted to the case of QED, see [4].

For the resummed anomalous dimension, one finally obtains

$$\boldsymbol{\Gamma}^{-,\mathrm{QED}}_{z\to 0}(N,a) = -N\left\{1-\sqrt{1+\frac{8a}{N^2}\left[1-2\sqrt{1-\frac{8a}{N^2}}\right]}\right\}. \quad (10)$$

$\mathcal{M}[P_{z\to 0}](N,a)$ can be represented in terms of a Taylor series in a by

$$\mathcal{M}[P_{z\to 0}](N,a) = \sum_{k=0}^{\infty} a^{k+1}\frac{p_k}{N^{2k+1}} = \frac{2a}{N} - \frac{12a^2}{N^3} - \frac{80a^3}{N^5} - \frac{304a^4}{N^7} + \dots . \quad (11)$$

The resummed small-z part of the splitting function $P(z,a)$ is obtained transforming (11) back to z-space,

$$P_{z\to 0}(z,a) = \sum_{k=0}^{\infty} c_k a^{k+1}\ln^{2k}(z), \quad c_k = \frac{p_k}{(2k)!} . \quad (12)$$

The numerical values of the first coefficients c_k are listed in Table 1. Evidently the first term in (11) agrees with the limit $z \to 0$ of the lowest order (LO) splitting function (6). In [4], (23, 26), we showed that the NLO contribution to (11) agrees both with the limit of the splitting function derived in [5] in the $\overline{\mathrm{MS}}$ scheme[3] and with the result of [3] in the on-mass-shell scheme.

We use

$$D(z, Q_0^2 = m_e^2) = \delta(1-z) \quad (13)$$

[3] The QED result is obtained setting the color factors in [5] to $C_F = T_F = 1$ and $C_G = 0$, see [4]

Table 1. Coefficients of the expansion of the small-z resummation $P_{z\to 0}(z,a) = \sum_{k=0}^{\infty} c_k a^{k+1}\ln^{2k}(z)$

k	c_k
0	$2.0000E+0$
1	$-6.0000E+0$
2	$-3.3333E+0$
3	$-0.4222E+0$
4	$1.5873E-3$
5	$2.8571E-3$
6	$1.4000E-4$
7	$-3.8468E-7$
8	$-2.0649E-7$
9	$-6.1484E-9$

as the initial condition for the solution of (1). For the splitting functions $P_k(z)$ in (3),

$$\int_0^1 dz P_k(z) = 0 \quad (14)$$

holds due to fermion number conservation. For the resummed kernel $P_{z\to 0}(z,a)$, (12), the integral condition (14) is not obeyed *a priori* but has to be restored. In the subsequent treatment we will subtract the term $p_{k-1}\delta(1-z)$ in $O(a^k)$.

As outlined in [4,14] for the resummation of the small-x terms for different processes in QCD, less singular terms can be as important as the leading singular terms. In QED, the $O[\alpha^2\ln(z)\ln(Q^2/m_e^2)]$ terms are known for e^+e^- annihilation [3]. From the different contributions, all terms but the well-known term due to the vacuum polarization function cancel. In $O(\alpha^2)$ the respective correction is

$$-12\frac{a^2}{N^3}\left(1-\frac{2}{9}N\right).$$

In this order, the coefficient of the term being one order less singular in $1/N$ is much smaller than that of the leading term.

3 Non-singlet QED radiative corrections to deeply inelastic ep scattering

The Born cross section for neutral-current deep-inelastic ep scattering is given by

$$\frac{d^2\sigma^B_{NC}}{dx\,dy} = \frac{2\pi\alpha^2 S}{Q^4}\left[Y_+\mathcal{F}_2(x,Q^2) + Y_- x\mathcal{F}_3(x,Q^2)\right], \quad (15)$$

with $Y_\pm = 1 \pm (1\pm y)^2$, x and y are the Bjorken variables, S is the cm energy squared, $Q^2 = xyS$ and

$$\mathcal{F}_2(x,Q^2) = F_2(x,Q^2) + 2|Q_e|(v_e+\lambda a_e)\chi(Q^2)G_2(x,Q^2)$$

$$+4(v_e^2 + a_e^2 + 2\lambda v_e a_e)\chi^2(Q^2)H_2(x,Q^2) \quad (16)$$

$$x\mathcal{F}_3(x,Q^2) = -2\,\mathrm{sign}(Q_e)\Big\{|Q_e|(a_e + \lambda v_e)\chi(Q^2)xG_3(x,Q^2) \quad (17)$$
$$+[2v_e a_e + \lambda(v_e^2 + a_e^2)]\chi^2(Q^2)xH_3(x,Q^2)\Big\},$$

with $Q_e = -1$ for electron and $Q_e = 1$ for positron scattering. $\lambda = \xi\ \mathrm{sign}(Q_e)$ denotes the lepton polarization, $v_e = 1 - 4\sin^2\theta_W, a_e = 1$, θ_W the weak mixing angle, and

$$\chi(Q^2) = \frac{G_F}{\sqrt{2}}\frac{M_Z^2}{8\pi\alpha^2}\frac{Q^2}{Q^2 + M_Z^2}. \quad (18)$$

G_F is the Fermi constant and M_Z the mass of the Z^0-boson. The neutral-current structure functions in (16, 17) are described in the parton model by

$$F_2(x,Q^2) = x\sum_{i=1}^{N_f} e_i^2[q_i(x,Q^2) + \overline{q}_i(x,Q^2)], \quad (19)$$

$$G_2(x,Q^2) = x\sum_{i=1}^{N_f} |e_i|v_i[q_i(x,Q^2) + \overline{q}_i(x,Q^2)], \quad (20)$$

$$H_2(x,Q^2) = \frac{x}{4}\sum_{i=1}^{N_f}(v_i^2 + a_i^2)[q_i(x,Q^2) + \overline{q}_i(x,Q^2)], \quad (21)$$

$$xG_3(x,Q^2) = x\sum_{i=1}^{N_f} |e_i|a_i[q_i(x,Q^2) - \overline{q}_i(x,Q^2)], \quad (22)$$

$$xH_3(x,Q^2) = \frac{x}{2}\sum_{i=1}^{N_f} v_i a_i[q_i(x,Q^2) - \overline{q}_i(x,Q^2)], \quad (23)$$

with $v_i = 1 - 4|e_i|\sin^2\theta_W$, $a_i = 1$, N_f the number of flavors, and q_i, $\overline{q}_i$ denote the quark and antiquark densities, respectively.

The QED radiative corrections due to initial state electron radiation can be expressed by

$$\frac{d^2\sigma_{NC}^{isr}}{dx\,dy} = \frac{d^2\sigma_{NC}^{B}}{dx\,dy} + \int_0^1 dz D(z,Q^2) \quad (24)$$
$$\times\left\{\theta(z-z_0)\mathcal{J}(x,y,z)\frac{d^2\sigma_{NC}^B}{dx\,dy}\Big|_{x=\hat{x},y=\hat{y},s=\hat{s}} - \frac{d^2\sigma_{NC}^B}{dx\,dy}\right\},$$

with $D(z,Q^2)$ the solution of (1) for $z < 1$, and

$$\mathcal{J}(z,y,z) = \begin{vmatrix} \partial\hat{x}/\partial x\ \partial\hat{y}/\partial x \\ \partial\hat{x}/\partial y\ \partial\hat{y}/\partial y \end{vmatrix}. \quad (25)$$

$D(z,Q^2)$ receives contributions from the iteration of the non-singlet kernel $R_1(z) = P_1(z)|_{z<1}$, which are obtained by

$$D_{AP}(z,Q^2) = \sum_{k=1}^{\infty}\frac{1}{k!}\zeta^k(Q^2)\otimes_{l=1}^k R_1(z), \quad (26)$$

where

$$\zeta(Q^2) = -\frac{3}{2}\ln\left[1 - \frac{4}{3}a_0\ln\left(\frac{Q^2}{m_e^2}\right)\right]. \quad (27)$$

In the subsequent numerical calculation, we evaluate the initial state radiative corrections for the case of leptonic variables [15,16]. Here the shifted quantities $\hat{x},\hat{y},\hat{s}$ and the threshold z_0 are

$$\hat{x} = \frac{xyz}{z+y-1}, \quad \hat{y} = \frac{z+y-1}{z},$$
$$\hat{s} = zS, \quad z_0 = \frac{1-y}{1-xy}. \quad (28)$$

The contributions to (26) are taken into account up to $k = 3$ completely[4]. For the higher-order terms we add the solution of (1) in the soft limit

$$D_{AP}^{\mathrm{soft}}(z,Q^2)|_{(4)} \quad (29)$$
$$= 2\zeta(1-z)^{2\zeta-1}\frac{\exp[\zeta(\frac{3}{2} - 2\gamma_E)]}{\Gamma(1+2\zeta)} - \frac{2\zeta}{1-z}$$
$$- [3 + 4\ln(1-z)]\frac{\zeta^2}{1-z}$$
$$- \left[4\ln^2(1-z) + 6\ln(1-z) + \frac{9}{4} - \frac{2\pi^2}{3}\right]\frac{\zeta^3}{1-z}$$

for $z < 1$. The regularization is inherent in (24).
The contribution of the small-z resummed terms to $D(z,Q^2)$ is

$$D_{z\to 0}(z,Q^2) = \sum_{k=1}^{\infty} c_k \int_{m_e^2}^{Q^2}\frac{dq^2}{q^2}a^{k+1}(q^2)\ln^{2k}(z). \quad (30)$$

In Fig. 1a–c, we show the contributions to the initial state QED corrections to $d^2\sigma_{NC}^B/dxdy$ in the kinematic range of HERA starting with the terms in $O(\alpha^2)$ to allow for a better comparison. The first order corrections are well-known, see [18,15]. We compare the small-z resummed terms to those obtained by iterating the kernel $R_1(z)$. The small-z resummed terms are negative and contribute only for large values of y. There they diminish the positive leading order corrections $O[\alpha^l\ln^l(Q^2/m_e^2)]$ significantly. These corrections are therefore relevant and need to be considered in the case of high y measurements, such as the determination of the structure function $F_L(x,Q^2)$ in the small-x range. For larger values of x, the small-z resummed corrections contribute only for the highest values of y.

4 $\alpha\ln^2(z)$ QED corrections to the Z^0 peak

A second important application of the small-z resummation concerns its possible effect upon the $e^+e^- \to \mu^+\mu^-$ cross section near the Z^0-peak. The implications would be quite profound were the resummation-improved cross section to have a measurable impact on the total cross

[4] Analytic results for the convolutions of $R_1(z)$ are easily obtained, see [16] and [17] for explicit expressions

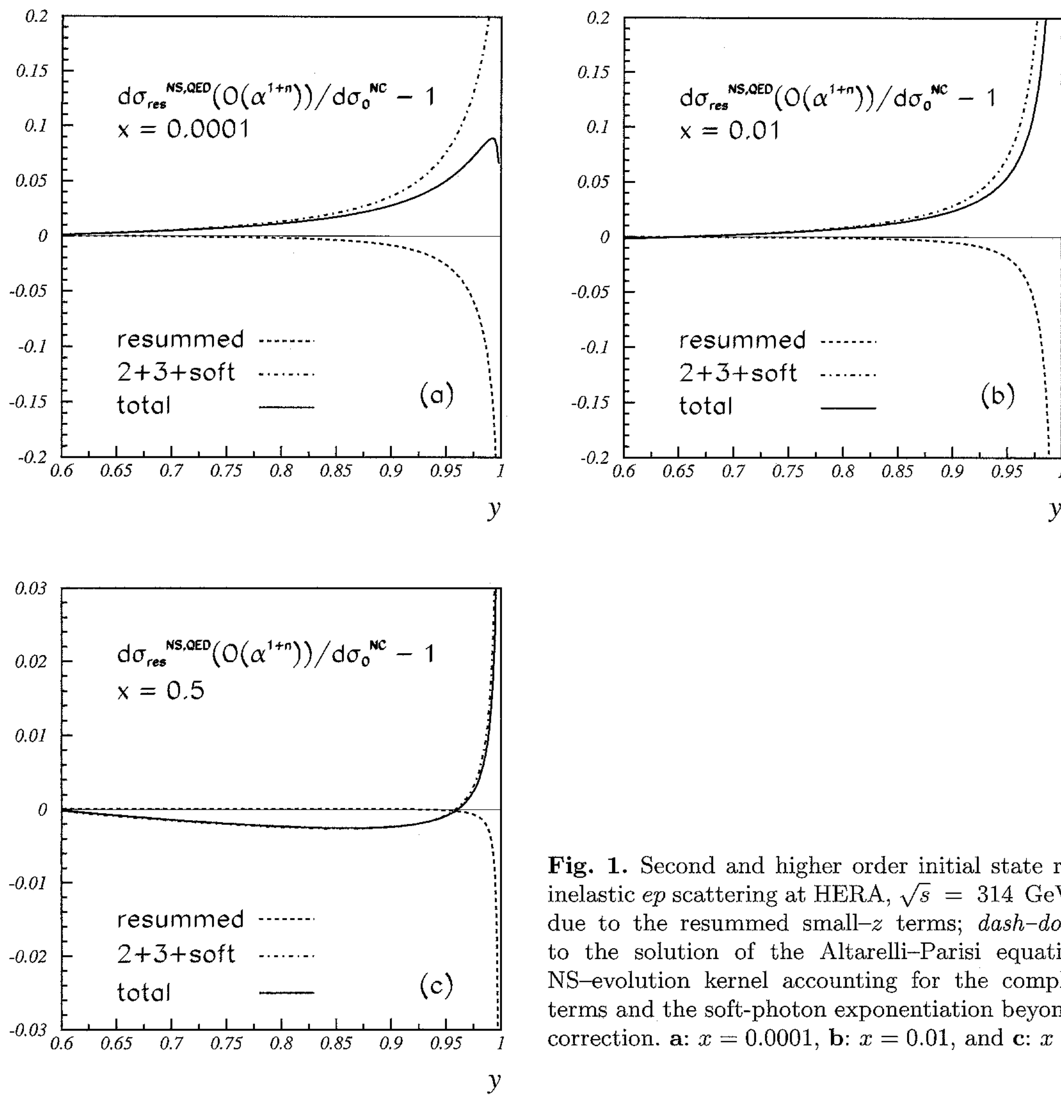

Fig. 1. Second and higher order initial state radiative corrections to deep inelastic ep scattering at HERA, $\sqrt{s} = 314$ GeV. *Dashed lines*: contribution due to the resummed small-z terms; *dash-dotted lines*: contribution due to the solution of the Altarelli–Parisi equation with the leading order NS–evolution kernel accounting for the complete $O(\alpha^2 L^2)$ and $O(\alpha^3 L^3)$ terms and the soft-photon exponentiation beyond $O(\alpha^3)$; *full lines*: resulting correction. **a**: $x = 0.0001$, **b**: $x = 0.01$, and **c**: $x = 0.5$

section or upon the position of the Z^0-peak or width on the order of an MeV.

The QED corrections up to $O(\alpha^2)$ were calculated in [3]. We consider the initial state corrections which are calculated accounting for the contributions to $O(\alpha^2)$ and soft-photon exponentiated terms using the code `ZFITTER` [19]. The small-z resummed terms (12) are taken into account for the contributions higher than second order by

$$\sigma_{z\to 0} = 2\int_0^1 dz\,[\Theta(z-z_0)\sigma_B(zs) - \sigma_B(s)]\,R^-_{z\to 0}(z,s)\,, \tag{31}$$

where $z_0 = s'/s$, s' being the cm energy entering the annihilation, and $\sigma_B(s)$ the Born cross section. The radiator $R^-_{z\to 0}(z,s)$ is given by

$$R^-_{z\to 0}(z,s) = \int_{m_e^2}^{s} \frac{ds'}{s'} \sum_{k=3}^{\infty} c_k a^{k+1}(s') \ln^{2k}(z)\,. \tag{32}$$

The factor of 2 enters in (31) because of the initial state radiation from both the electron and the positron line. A series of cuts on s' has been made and the results are listed in Table 2. The parameters of the calculation are $M_Z = 91.1887$ GeV, $\Gamma_Z = 2.4974$ GeV, and $\sin^2\theta_W = 0.2319$. The small-z contribution is six orders of magnitude down from the cross section containing the standard QED corrections. A measurement of this effect is clearly out of the question.

We also have compared the maximum of the cross section of `ZFITTER` as a function of the cm energy with and without the small-z resummation. We find the difference to be smaller than 40 eV, widely independent of the z-cut. Here the effects of the small-z contributions beyond second order are much smaller than the experimental resolution.

5 Conclusions

The resummation of the $O[\alpha \ln^2(z)]$ non-singlet contributions was performed for initial state QED corrections. As examples, we investigated the effects of the resummation

Table 2. Dependence of the cross section $\sigma(e^+e^- \to \mu^+\mu^-)$ at the Z^0-peak on the minimum energy of the final state muons. σ_R : scattering cross section including the initial state QED corrections to $O(\alpha^2)$ and soft-photon exponentiation; $\sigma_{z\to 0}$: small-z resummed contributions beyond $O(\alpha^2)$, (31)

E^{μ}_{min} (GeV)	5	10	20	40
σ_R (nb)	1.4723	1.4713	1.4702	1.4674
$\sigma_{z\to 0}$ (nb)	1.05341 10^{-6}	1.13476 10^{-6}	1.11480 10^{-6}	1.11465 10^{-6}

for two processes: neutral-current deep-inelastic scattering and $e^+e^- \to \mu^+\mu^-$ scattering near the Z^0-peak. The influence upon the DIS results is particularly strong in the low-x, high-y region. In this region, the small-z corrections negate a sizeable portion of the $O[\alpha^2 \ln^2(Q^2/m_e^2)]$ and higher-order contributions. The effect diminishes as x becomes larger but still remains important near $y \approx 1$. The incorporation of these corrections is therefore important in analyses of deep-inelastic data in the high y range.

The small-z resummation, on the other hand, has no visible effect upon the $e^+e^- \to \mu^+\mu^-$ cross section near the Z^0 peak. It contributes to the cross section at a level of 10^{-6} only. Correspondingly the shift in the peak cross section is negligibly small.

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