EQUATIONS OF MOTION OF ELLIPTIC RESTRICTED PROBLEM OF THREE BODIES WITH VARIABLE MASS

R. K. DAS
R. K. Mission Vidyapith, B. Deoghar (Bihar), India.
A. K. SHRIVASTAVA
Dept. of Mathematics, S. P. College, Dumka (Bihar), India.
and
B. ISHWAR
Department of Mathematics, University of Bihar, Muzaffarpur (Bihar), India.

(Received: 5 January 1987; accepted: 23 March 1988)

Abstract. The differential equations of motion of the elliptic restricted problem of three bodies with decreasing mass are derived. The mass of the infinitesimal body varies with time. We have applied Jeans' law and the space-time transformation of Meshcherskii. In this problem the space-time transformation is applicable only in the special case when $n = 1, k = 0, q = \frac{1}{2}$. We have applied Nechvile's transformation for the elliptic problem. We find that the equations of motion of our problem differ from that of constant mass only by a small perturbing force.

1. Introduction

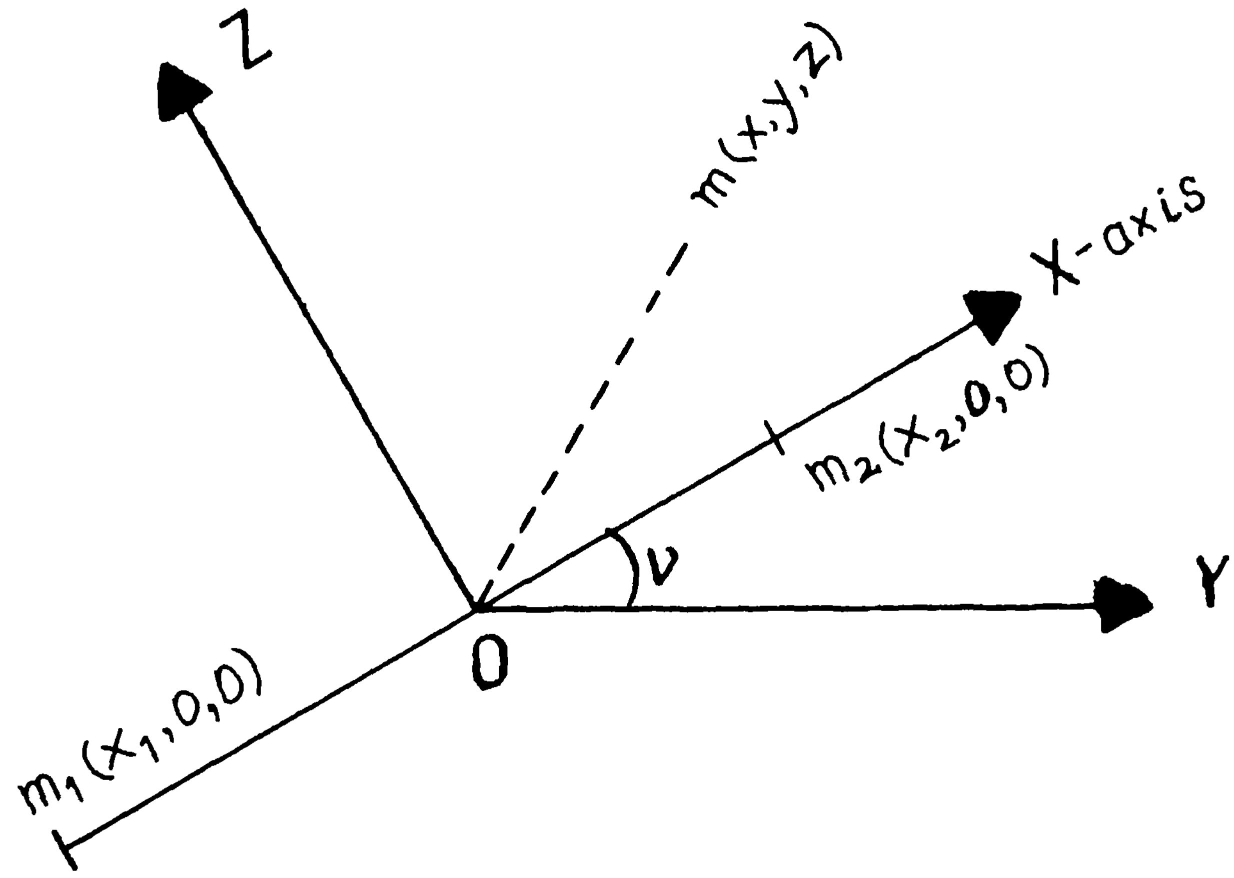
The phenomenon of isotropic radiation or absorption in stars led scientists to formulate the restricted problem of three bodies with variable mass. The two-body problem with variable mass was studied by Jeans (1928) regarding the evolution of a binary system. The restricted problem of perturbed motion of two bodies with variable mass was considered by Omarov (1963). Meshcherskii (1949) assumed that the mass is ejected isotropically from the two-body system at very high velocities and is lost to the system. He examined the change in orbits, the variation in angular momentum and the energy of the system. Verhulst (1972) has discussed the two-body problem with slowly decreasing mass according to Jeans, by a non-linear, non-autonomous system of differential equations.

Shrivastava and Ishwar (1983) established the equations of motion of the circular restricted problem of three bodies with variable mass under the assumption that the mass of the infinitesimal body varies with respect to time. Singh and Ishwar (1984) considered the effect of perturbations on the locations of the stability of the triangular points in the above problem.

2. Equations of Motion

Let us suppose that the mass of the primaries, m_1 and m_2 is constant and that m, the mass of the infinitesimal body, varies with time. The primaries are moving on elliptic orbits. We consider a barycentric rotating coordinate system $OXYZ$ rotating relative to inertial space with angular velocity $\dot{v}$.

Celestial Mechanics **45**: 387–393, 1988.



We have taken the line joining m_1 and m_2 as the x-axis. The coordinates of m_1 and m_2 are $(x_1, 0, 0)$ and $(x_2, 0, 0)$ and the coordinates of m are (x, y, z). Let the radius vector from m to m_1 be ρ_1 and m to m_2 be ρ_2 and the distance between m_1 and m_2 be r. As the motion of the primaries is known we have only to find the motion of m. The kinetic energy in the rotating frame of reference $OXYZ$ is give by

$$T = \tfrac{1}{2}m\,(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + m\dot{v}(x\dot{y} - y\dot{x}) + \tfrac{1}{2}m(x^2 + y^2)\dot{v}^2 \tag{1}$$

and the potential energy ω is given by

$$\omega = -Km\left(\frac{m_1}{\rho_1} + \frac{m_2}{\rho_2}\right) \tag{2}$$

where K is the gravitational constant and

$$\rho_1^2 = (x - x_1)^2 + y^2 + z^2,$$

$$\rho_2^2 = (x - x_2)^2 + y^2 + z^2.$$

Now $r = P/(1 + e\cos v)$, where P = semi latus rectum for the elliptic orbit of m_2 relative to m_1, e = eccentricity, and v = true anomaly.

The Lagrangian equations of motion are given by

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{y}}\right) - \frac{\partial L}{\partial y} = 0, \tag{3}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{z}}\right) - \frac{\partial L}{\partial z} = 0,$$

where $L = T - \omega$, or

$$L = \tfrac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) + m\dot{v}(x\dot{y} - y\dot{x}) + \tfrac{1}{2}m\dot{v}^2(x^2 + y^2) + Km\left(\frac{m_1}{\rho_1} + \frac{m_2}{\rho_2}\right).$$

The equations of motion can be written in the form

$$\begin{aligned}
&\frac{\dot{m}}{m}(\dot{x} - y\dot{v}) + \ddot{x} - 2\dot{v}\dot{y} - y\frac{d\dot{v}}{dv}\dot{v} - \dot{v}^2 x = -\frac{1}{m}\frac{\partial\omega}{\partial x},\\
&\frac{\dot{m}}{m}(\dot{y} + x\dot{v}) + \ddot{y} + 2\dot{v}\dot{x} + x\frac{d\dot{v}}{dv}\dot{v} - \dot{v}^2 y = -\frac{1}{m}\frac{\partial\omega}{\partial y},\\
&\frac{\dot{m}}{m}\dot{z} + \ddot{z} = -\frac{1}{m}\frac{\partial\omega}{\partial x},
\end{aligned}\tag{4}$$

where from (2)

$$\begin{aligned}
&\frac{\partial\omega}{\partial x} = -Km\left[\frac{m_1}{\rho_1^3}(x - x_1) + \frac{m_2}{\rho_2^3}(x - x_2)\right],\\
&\frac{\partial\omega}{\partial y} = -Km\left[\frac{m_1}{\rho_1^3}y + \frac{m_2}{\rho_2^3}y\right],\\
&\frac{\partial\omega}{\partial z} = -Km\left[\frac{m_1}{\rho_1^3}z + \frac{m_2}{\rho_2^3}z\right],
\end{aligned}\tag{5}$$

and we have

$$\begin{aligned}
&(m_1 + m_2)x_1 = -m_2 r,\\
&(m_1 + m_2)x_2 = m_1 r.
\end{aligned}\tag{6}$$

From Jeans' law

$$\frac{dm}{dt} = -\alpha m^n \tag{7}$$

where α is a constant coefficient and n is within the limits $0.4 \leqslant n \leqslant 4.4$ for stars of the main sequence.

Now we introduce the following space-time transformation of Meshcherskii which preserve the dimensions of the space and time

$$\begin{aligned}
&\xi = \gamma^q x, \quad \eta = \gamma^q y, \quad \zeta = \gamma^q z, \quad d\Gamma = \gamma^k dt,\\
&r_1 = \gamma^q \rho_1, \quad r_2 = \gamma^q \rho_2,
\end{aligned}\tag{8}$$

where $\gamma = m/m_0$, and m_0 is the mass of the satellite when $t = 0$. Therefore $d\gamma/dt = -\beta\gamma^n$, where $\beta = \alpha m_0^{n-1}$.

Now since $\xi = \gamma^q x$, we have

$$x = \xi\gamma^{-q}, \qquad \frac{dt}{d\Gamma} = \gamma^{-k},$$

$$\xi' = \gamma^{q-k}\dot{x} - q\beta\gamma^{n-k-1}\xi,$$

then $\dot{x} = \gamma^{k-q}\xi' + q\beta\gamma^{n-q-1}\xi$. Therefore

$$\frac{d\dot{x}}{d\Gamma} = \xi''\gamma^{k-q} + \xi'(k-q)\gamma^{k-q-1}\frac{d\gamma}{dt}\frac{dt}{d\Gamma} + q\beta\gamma^{n-q-1}\xi' +$$

$$+ q\beta\xi(n-q-1)\gamma^{n-q-2}\frac{d\gamma}{dt}\frac{dt}{d\Gamma}$$

i.e.

$$\ddot{x}\gamma^{-k} = \xi''\gamma^{k-q} - \beta\xi'(k-q)\gamma^{n-q-1} + q\beta\xi'\gamma^{n-q-1} -$$

$$- q\beta^2(n-q-1)\xi\gamma^{2n-k-q-2}$$

which implies that

$$\ddot{x} = \xi''\gamma^{2k-q} + (2q-k)\beta\xi'\gamma^{n+k-q-1} - q\beta^2(n-q-1)\xi\gamma^{2n-q-2}$$

Similarly

$$\dot{y} = \eta'\gamma^{k-q} + q\beta\gamma^{n-q-1}\eta$$

$$\ddot{y} = \eta''\gamma^{2k-q} + (2q-k)\beta\eta'\gamma^{n+k-q-1} - q\beta^2(n-q-1)\eta\gamma^{2n-q-2}$$

$$\dot{z} = \zeta'\gamma^{k-q} + q\beta\gamma^{n-q-1}\zeta$$

$$\ddot{z} = \zeta''\gamma^{2k-q} + (2q-k)\beta\zeta'\gamma^{n+k-q-1} - q\beta^2(n-q-1)\zeta\gamma^{2n-q-2}$$

Putting these values of $\dot{x}, \ddot{x}, \dot{y}, \ddot{y}, \dot{z}$ and $\ddot{z}$ into the equations of motion (4), we have

$$\xi'' + (2q-k-1)\beta\xi'\gamma^{n-k-1} - q\beta^2(n-q)\xi\gamma^{2(n-k-1)} - \dot{v}^2\gamma^{-2k}\xi - 2\dot{v}\eta'\gamma^{-k} -$$

$$- \dot{v}\beta(2q-1)\gamma^{n-2k-1}\eta - \frac{d\dot{v}}{dv}\dot{v}\gamma^{-2k}\eta = -\frac{1}{m_0}\gamma^{2q-2k-1}\frac{\partial\omega}{\partial\xi}$$

Similarly,

$$\eta'' + (2q-k-1)\beta\eta'\gamma^{n-k-1} - q\beta^2(n-q)\eta\gamma^{2(n-k-1)} - \dot{v}^2\gamma^{-2k}\eta +$$

$$+ 2\dot{v}\xi'\gamma^{-k} + \dot{v}\beta(2q-1)\gamma^{n-2k-1}\xi + \frac{d\dot{v}}{dv}\dot{v}\gamma^{-2k}\xi = -\frac{1}{m_0}\gamma^{2q-2k-1}\frac{\partial\omega}{\partial\eta} \quad (9)$$

$$\zeta'' + (2q-k-1)\beta\zeta'\gamma^{n-k-1} - q\beta^2(n-q)\zeta\gamma^{2(n-k-1)} = -\frac{1}{m_0}\gamma^{2q-2k-1}\frac{\partial\omega}{\partial\zeta}$$

Here the prime denotes differentiation with respect to Γ.

In order to free the equations (9) from the factor which depends on the variation of mass, it is sufficient to set

$$n-k-1=0,$$

$$2q-2k-1=0 \quad \text{and} \quad n=1,$$

i.e., $k=0$ and $q=\frac{1}{2}$.
Hence the equations (9) can be written in the form:

$$\begin{aligned}
&\xi''-\dot{v}^2\xi-2\dot{v}\eta'-\dot{v}\frac{\mathrm{d}\dot{v}}{\mathrm{d}v}\eta=-\frac{1}{m_0}\frac{\partial\omega}{\partial\xi}+\tfrac{1}{4}\beta^2\xi,\\
&\eta''-\dot{v}^2\eta+2\dot{v}\xi'+\dot{v}\frac{\mathrm{d}\dot{v}}{\mathrm{d}v}\xi=-\frac{1}{m_0}\frac{\partial\omega}{\partial\eta}+\tfrac{1}{4}\beta^2\eta,\\
&\zeta''=-\frac{1}{m_0}\frac{\partial\omega}{\partial\zeta}+\tfrac{1}{4}\beta^2\zeta.
\end{aligned}\tag{10}$$

Next let us introduce Nechvile's variables (Szebehely, 1967) by writing

$$\xi=\bar{r}\bar{\xi},\quad \eta=\bar{r}\bar{\eta},\quad \zeta=\bar{r}\bar{\zeta}$$

where

$$\bar{r}=\frac{r}{p}=\frac{1}{1+e\cos v}.$$

Also we know from the integral of area that

$$\begin{aligned}
&r^2\dot{v}=\bar{c}\quad\text{or}\\
&\dot{v}=\frac{\bar{c}}{r^2}=\frac{\bar{c}}{p^2}\frac{1}{\bar{r}^2}.
\end{aligned}\tag{11}$$

Differentiating $\xi=\bar{r}\bar{\xi}$ w.r.t. Γ, we have

$$\xi'=\frac{\mathrm{d}\bar{r}}{\mathrm{d}v}\bar{\xi}\dot{v}+\bar{r}\frac{\mathrm{d}\bar{\xi}}{\mathrm{d}v}\dot{v}, \text{ and}$$

$$\xi''=\left(\frac{\mathrm{d}^2\bar{r}}{\mathrm{d}v^2}\bar{\xi}+2\frac{\mathrm{d}\bar{r}}{\mathrm{d}v}\frac{\mathrm{d}\bar{\xi}}{\mathrm{d}v}+\bar{r}\frac{\mathrm{d}^2\bar{\xi}}{\mathrm{d}v^2}\right)\dot{v}^2+\left(\frac{\mathrm{d}\bar{r}}{\mathrm{d}v}\bar{\xi}+\bar{r}\frac{\mathrm{d}\bar{\xi}}{\mathrm{d}v}\right)\frac{\mathrm{d}\dot{v}}{\mathrm{d}v}\dot{v}$$

Similarly

$$\eta'=\frac{\mathrm{d}\bar{r}}{\mathrm{d}v}\bar{\eta}\dot{v}+\bar{r}\frac{\mathrm{d}\bar{\eta}}{\mathrm{d}v}\dot{v}$$

$$\eta''=\left(\frac{\mathrm{d}^2\bar{r}}{\mathrm{d}v^2}\bar{\eta}+2\frac{\mathrm{d}\bar{r}}{\mathrm{d}v}\frac{\mathrm{d}\bar{\eta}}{\mathrm{d}v}+\bar{r}\frac{\mathrm{d}^2\bar{\eta}}{\mathrm{d}v^2}\right)\dot{v}^2+\left(\frac{\mathrm{d}\bar{r}}{\mathrm{d}v}\bar{\eta}+\bar{r}\frac{\mathrm{d}\bar{\eta}}{\mathrm{d}v}\right)\frac{\mathrm{d}\dot{v}}{\mathrm{d}v}\dot{v}$$

$$\zeta' = \frac{d\bar{r}}{dv}\bar{\zeta}\dot{v} + \bar{r}\frac{d\bar{\zeta}}{dv}\dot{v}$$

$$\zeta'' = \left(\frac{d^2\bar{r}}{dv^2}\bar{\zeta} + 2\frac{d\bar{r}}{dv}\frac{d\bar{\zeta}}{dv} + \bar{r}\frac{d^2\bar{\zeta}}{dv^2}\right)\dot{v}^2 + \left(\frac{d\bar{r}}{dv}\bar{\zeta} + \bar{r}\frac{d\bar{\zeta}}{dv}\right)\frac{d\dot{v}}{dv}\dot{v}$$

Putting these values in the equations of motion (10), we get

$$\begin{aligned}
&\bar{r}\frac{d^2\bar{\xi}}{dv^2}\dot{v}^2 + \frac{d\bar{\xi}}{dv}\left(2\frac{d\bar{r}}{dv}\dot{v}^2 + \bar{r}\dot{v}\frac{d\dot{v}}{dv}\right) - 2\dot{v}^2\bar{r}\frac{d\bar{\eta}}{d\dot{v}} - \\
&-\bar{\eta}\dot{v}\left(2\frac{d\bar{r}}{dv}\dot{v} + \bar{r}\frac{d\dot{v}}{dv}\right) + \dot{v}\bar{\xi}\left(\frac{d^2\bar{r}}{dv^2}\dot{v} + \frac{d\bar{r}}{dv}\frac{d\dot{v}}{dv} - \bar{r}\dot{v}\right) = -\frac{1}{m_0}\frac{\partial\omega}{\partial\xi} + \tfrac{1}{4}\beta^2\bar{r}\bar{\xi} \\
&\bar{r}\frac{d^2\bar{\eta}}{dv^2}\dot{v}^2 + \frac{d\bar{\eta}}{dv}\left(2\frac{d\bar{r}}{dv}\dot{v}^2 + \bar{r}\dot{v}\frac{d\dot{v}}{dv}\right) + 2\dot{v}^2\bar{r}\frac{d\bar{\xi}}{dv} \\
&+\bar{\xi}\dot{v}\left(2\frac{d\bar{r}}{dv}\dot{v} + \bar{r}\frac{d\dot{v}}{dv}\right) + \dot{v}\bar{\eta}\left(\frac{d^2\bar{r}}{dv^2}\dot{v} + \frac{d\bar{r}}{dv}\frac{d\dot{v}}{dv} - \bar{r}\dot{v}\right) = -\frac{1}{m_0}\frac{\partial\omega}{\partial\eta} + \tfrac{1}{4}\beta^2\bar{r}\bar{\eta} \\
&\bar{r}\frac{d^2\bar{\zeta}}{dv^2}\dot{v}^2 + \frac{d\bar{\zeta}}{dv}\left(2\frac{d\bar{r}}{dv}\dot{v}^2 + \bar{r}\dot{v}\frac{d\dot{v}}{dv}\right) + \dot{v}\bar{\zeta}\left(\frac{d^2\bar{r}}{dv^2}\dot{v} + \frac{d\bar{r}}{dv}\frac{d\dot{v}}{dv}\right) \\
&= -\frac{1}{m_0}\frac{\partial\omega}{\partial\zeta} + \tfrac{1}{4}\beta^2\bar{r}\bar{\zeta}.
\end{aligned} \tag{12}$$

From the fact that $\bar{r}^2\dot{v} = \bar{c}/p^2$, we find by differentiation that

$$2\frac{d\bar{r}}{dv}\dot{v} + \bar{r}\frac{d\dot{v}}{dv} = 0 \tag{13}$$

and

$$\frac{d^2\bar{r}}{dv^2}\dot{v} + \frac{d\bar{r}}{dv}\frac{d\dot{v}}{dv} - \bar{r}\dot{v} = \frac{\bar{c}}{p^2} \tag{14}$$

Now using identities (13) and (14), the equations of motion given by (12) can be simplified to the form given as

$$\begin{aligned}
&\frac{d^2\bar{\xi}}{dv^2} - 2\frac{d\bar{\eta}}{dv} = \frac{\partial\bar{\Omega}}{\partial\bar{\xi}} + \frac{\bar{r}^4p^4}{\bar{c}^2}\frac{\beta^2}{4}\bar{\xi}, \\
&\frac{d^2\bar{\eta}}{dv^2} + 2\frac{d\bar{\xi}}{dv} = \frac{\partial\bar{\Omega}}{\partial\bar{\eta}} + \frac{\bar{r}^4p^4}{\bar{c}^2}\frac{\beta^2}{4}\bar{\eta}, \\
&\frac{d^2\bar{\zeta}}{dv^2} = \frac{\partial\bar{\Omega}}{\partial\bar{\zeta}} + \frac{\bar{r}^4p^4}{\bar{c}^2}\frac{\beta^2}{4}\bar{\zeta},
\end{aligned} \tag{15}$$

where

$$\bar{\Omega} = \bar{r}\left[\tfrac{1}{2}(\bar{\xi}^2 + \bar{\eta}^2) - \tfrac{1}{2}e\cos v\bar{\zeta}^2 + \frac{p^4}{\bar{c}^2}\bar{\omega}\right], \tag{16}$$

$$\bar{\omega} = K\gamma^{3/2}\left(\frac{m_1}{\bar{r}_1} + \frac{m_2}{\bar{r}_2}\right), \tag{17}$$

$$\begin{aligned} \bar{r}_1^2 &= (\bar{\xi} - \bar{\xi}_1)^2 + \bar{\eta}^2 + \bar{\zeta}^2, \\ \bar{r}_2^2 &= (\bar{\xi} - \bar{\xi}_2)^2 + \bar{\eta}^2 + \bar{\zeta}^2, \end{aligned} \tag{18}$$

$$\bar{\xi}_1 = -\frac{m_2 p\gamma^{1/2}}{m_1 + m_2}, \quad \bar{\xi}_2 = \frac{m_1 p\gamma^{1/2}}{m_1 + m_2} \tag{19}$$

The set of equation (15) is known as the equations of motion of the elliptic restricted three-body problem with variable mass.

Equations (15) differ from the equation of motion of restricted elliptical problem of three bodies with constant mass only by a factor $\frac{1}{4}\beta^2$.

Acknowledgements

We are extremely grateful to Prof. K. B. Bhatnagar, Zakir Hussain College, Delhi (India) who reviewed this paper and made many valuable suggestions. We will like to express our heartiest thanks to Dr. R. K. Choudhry, University Professor and Head of the Department of Mathematics, Bhagalpur University, Bhagalpur, who inspired us to do the work.

We are also indebted to Prof. V. Szebehely, University of Texas, Austin, U.S.A. for his valuable suggestions for improvement of this paper.

References

1. Jeans, J. H.: 1924, *Monthly Notices Roy. Astro. Sec.* 85, 2.
2. Jeans, J. H.: 1928, *Astronomy and Cosmogomy*, Cambridge.
3. Markeev, A. P.: 1971, *Soviet Astronomy*, **15**, 682.
4. Meshcherskii, I. V.: 1949, *Studies on the Mechanics of Bodies of Variable Mass*, Gostekhizdat, Moscow, Leningrad.
5. Omarov, T. B.: 1963, *Sov. Astron.*, **8**, 127.
6. Shrivastava, A. K. and Ishwar, B.: 1983, *Celest. Mech.* **30**, 323.
7. Singh, J. and Ishwar, B.: 1984, *Celest. Mech.* **32**, 297.
8. Singh, J. and Ishwar, B: 1985, *Celest. Mech.* **35**, 201.
9. Szebehely, V.: 1967, *Theory of Orbits*, Academic Press.
10. Verhulst, F.: 1972, *Celest. Mech.* **5**, 27.