NOTE ON THE REDUCING TRANSFORMATION AND SECULAR COUPLING

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1. Introduction

A few years ago, Sessin (1981) and Sessin and Ferraz-Mello (1984) showed that an Hamiltonian approximating the motion of two planets with periods commensurate in the ratio 2:1 is integrable. This is a significant achievement as the problem presents itself as a double resonance problem with two independent critical arguments. Later on, Wisdom (1986) and Henrard et al (1986) showed how Sessin's trick could best be explained by a simple rotation in phase space which converts the two a priori independent critical arguments into a single one. This rotation has been used successfully since then as a first step in the analysis of the 2:1 Jovian resonance (Tsuchida 1985, Henrard et al 1986, Henrard, Lemaitre 1986, 1987) or the 2:1 satellites resonances (Ferraz Mello, 1985a) and we have called it the reducing transformation.

At about the same time Sessin constructed his integrable approximation for the 2:1 commensurability, Pauwels (1983) showed that an approximation of the interaction of two satellites, involving the so-called "secular terms", can be solved by introducing angle-action coordinates by means of a simple explicit canonical transformation. The fact that such a system can be solved at all does not come as a surprise : it is linear (see Hamiltonian 3) in the Poincaré's variables (4). But Pauwels treatment of it depicts vividly its geometry.

Although it is not readily apparent, because Pauwels first use a reduction of his problem to a one degree of freedom intermediary problem, Pauwels' transformation is exactly the same rotation as the one used implicitly by Sessin. Furthermore, S. Ferraz-Mello pointed out to us that this rotation had already been proposed by Poincaré (1899 - Volume II - p. 43) in the frame of the planetary theory.

We thought that this conjunction was worth a note in Celestial Mechanics. We like to think of it as more that the report of a coincidence. It emphasize the usefulness of simple transformations which "disemcumber" (Deprit 1982) a problem and show his underlying geometrical structure.

Note that in the title, we did not follow Pauwels in calling this problem a "secular resonance" problem but rather a "secular coupling" problem. This is because we think that Pauwels' work and this note show that the problem is not really a dynamical resonance problem but rather a geometrical problem. After all a rotation of the coordinate system is all it takes to uncouple the two degrees of freedom.

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2. Secular Coupling Problem

The Hamiltonian of the planetary three-body problem, when truncated at the second order in the masses can be written (see for instance Message 1982)

$$H = -\frac{Gm_0m_1}{2a_1} - \frac{Gm_0m_2}{2a_2} - Gm_1m_2\left\{\frac{1}{|\vec{r}_1 - \vec{r}_2|} - \frac{(\vec{r}_1 \mid \vec{r}_2)}{r_2^3}\right\} \tag{1}$$

where m_0 is the mass of the large body (the Sun or a planet) and $m_i, a_i, \vec{r}_i$ the masses, Jacobian semi-major axes and position vectors of the two smaller bodies (two planets or two satellites).

We are considering here the planar problem and the canonical variables

$$\begin{aligned} \lambda_i &= \text{longitude of } m_i & L_i &= m_i\sqrt{Gm_0\mu_i a_i} \\ -p_i &= \text{longitude of pericenter of } m_i & P_i &= L_i[1 - \sqrt{1 - e_i^2}] \end{aligned} \tag{2}$$

where $\mu_1 = m_0/(m_0 + m_1)$ and $\mu_2 = (m_0 + m_1)/(m_0 + m_1 + m_2)$.

In the non-commensurable case, when the ratio of the mean motions of the satellites are not well approximated by a rational number k_1/k_2 with a small value of $k_1 - k_2$, the Hamiltonian function (1) can be averaged over the "fast" arguments λ_i . The corresponding averaged momenta $\bar{L}_i$ are then constants and the problem is reduced to a two degree of freedom problem in the averaged pericentric variables $(\bar{p}_1, \bar{p}_2, \bar{P}_1, \bar{P}_2)$. From thereon we shall drop the superscript $(^-)$ marking the averaging. No confusion should follow from this as we shall no longer make reference to the osculating variables but deal only with the averaged variables.

When truncated at the second order in the eccentricities the averaged Hamiltonian reads now

$$H = aP_1 + bP_2 + c\sqrt{4P_1P_2}\cos(p_1 - p_2) \tag{3}$$

The coefficients a, b, c are respectively of the order of m_1, m_2 and $\sqrt{m_1, m_2}$ in the pure planetary three body problem. If we further consider that the main body m_0 possesses a significant oblateness (as it is the case most of the time when the main body is a planet) the coefficients a and b can be much larger, having their principal contributions of the order of the oblateness coefficient.

It is obvious that the sum $P_1 + P_2$ is a first integral of the Hamiltonian system described by (3) which can then be reduced to a one degree of freedom system.

It would seems at first that this one degree of freedom system is very similar to one of the "resonance" problems which have been investigated by several authors (see for instance Henrard-Lemaitre 1983, Lemaitre 1984, Ferraz-Mello 1985b, Henrard-Murigande 1987 etc.).

Indeed such a problem has been investigated by Greenberg (1975, 1977) in the frame of the restricted problem (at the limit $m_1 = 0$) to explain the Rhea-Titan interaction. Greenberg is very careful in his use of the term "secular resonance" and prefers to use the word "libration". More recently, Pauwels (1983), noting that Rhea is not that much smaller than Titan (a factor of 60 in the masses), reopened the investigation in the frame of the full planetary three body problem (as we have stated it above in *eq.*1). There the problem is called a "secular resonance" without restriction but at the same time, it is shown that this "resonance" has some strange peculiarities. All orbits are periodics. No unstable equilibrium and associated separatrices are present like in other resonances.

Actually the careful and well documented analysis of Pauwels (1983) shows that the effects of the "resonance" is purely geometrical. The orbits reduce to circle on a sphere, but the traditional phase space diagrams show projections of this sphere on a plane in such a way that the basic simplicity of the motion is lost.

Pauwels proceeds to show that a particular explicit canonical transformation transforms the one degree of freedom Hamiltonian deduced from (3) by using the integral $P_1 + P_2$ into a linear function of the new momentum. The transformed problem is thus trivial.

Pauwels canonical transformation is actually related to a rotation in phase space which has already shown itself so be quite useful in other contexts.

3. The reducing rotation

Consider the cartesian coordinates

$$Y_i = \sqrt{2P_i}\sin p_i \quad X_i = \sqrt{2P_i}\cos p_i \tag{4}$$

and the rotation in phase space

$$\begin{aligned} Z_1 &= Y_1\cos\alpha + Y_2\sin\alpha & W_1 &= X_1\cos\alpha + X_2\sin\alpha \\ Z_2 &= -Y_1\sin\alpha + Y_2\cos\alpha & W_2 &= -X_1\sin\alpha + X_2\cos\alpha \end{aligned} \tag{5}$$

which can easily be shown to be a canonical transformation.

Coming back to polar coordinates by means of

$$Z_i = \sqrt{2R_i}\sin r_i \quad W_i = \sqrt{2R_i}\cos r_i \tag{6}$$

the Hamiltonian function (3) is transformed into

$$\begin{aligned} K = {} & (s + d\cos 2\alpha + c\sin 2\alpha)R_1 + (s - d\cos 2\alpha - c\sin 2\alpha)R_2 \\ & +(c\cos 2\alpha - d\sin 2\alpha)\sqrt{4R_1R_2}\cos(r_1 - r_2) \end{aligned} \tag{7}$$

where

$$s = \frac{a+b}{2} \ ; \ d = \frac{a-b}{2}. \tag{8}$$

If the rotation angle α is choosen in such a way that

$$c\cos 2\alpha - d\sin 2\alpha = 0. \tag{9}$$

The Hamiltonian function reduces to

$$K = (s + \sqrt{d^2+c^2})R_1 + (s - \sqrt{d^2+c^2})R_2. \tag{10}$$

The momenta R_1 and R_2 are constants and the angular variables r_1 and r_2 precess with constant angular velocities. The polar coordinates R_i, r_i are angle-action variables of the problem.

4. Conclusions

The use of a rotation to uncouple the secular two planet problem can be generalized to uncouple the secular n-planet problem and indeed it is what is done implicitely or explicitely by the solution of the "Lagrange secular problem" in planetary theories (see Message 1982 for instance). The fact that already in the two planet problem the simplicity of the motion is hidden when a projection onto the phase space of each individual planet is used (see Pauwels for a good illustration of this fact) should tell us something. In the general planetary problem also such projections are likely to hide the real geometry of the problem. Projections unto the phase space of the normal modes as used by Laskar (1988) are likely to provide more readable information.

If we have emphasized the fact that the problem we have investigated is not a real "resonance" problem, this does not mean that "secular resonance" problems do not exist. When both the difference $(a - b)$ and the ratio m_1/m_2 are small, the eccentricity of m_1 can become large. This is what was pointed out by Greenberg (1977). In this case the term in P_1^2 neglected in the Hamiltonian (3) becomes important and should be included in the analysis. Such is the frame in which are investigated the secular resonance problem of asteroids (see for instance Nakai and Kinoshita 1985, Scholl and Froeschlé 1986, Froeschlé and Scholl 1986, Yoshikawa 1987) or the problem of resonance between normal modes as investigated by Laskar (1988).

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