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Holomorphic Anosov systems

Étienne Ghys

École Normale Supérieure de Lyon, UMR 128 CNRS, 46 Allée d'Italie, F-69364 Lyon, France

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1. Introduction

The group of holomorphic diffeomorphisms of a compact complex manifold is a finite dimensional Lie group and one could expect a rather complete description of the dynamical behaviour of such diffeomorphisms, at least in low dimensions. For instance, holomorphic diffeomorphisms of compact Kähler manifolds, preserving the cohomology class of the Kähler form, have very simple dynamics, with zero topological entropy [Fu], [Sn]. A theorem of F. Enriques describes *algebraic* surfaces for which the group of holomorphic diffeomorphisms has infinitely many connected components [En], [Ro].

However, very interesting examples do exist and deserve attention. Some K3-surfaces have an infinite discrete group of diffeomorphisms whose dynamical study seems promising [Maz]. More classical examples are provided by matrices A of $\mathrm{GL}(n, \mathbb{C})$ preserving some lattice $\Lambda \subset \mathbb{C}^n$ and, therefore, inducing a holomorphic diffeomorphism $\bar{A}$ of the complex torus $\mathbb{C}^n/\Lambda$. If the spectrum of A is disjoint from the unit circle, this diffeomorphism $\bar{A}$ is of Anosov type, with rich dynamics.

The purpose of this paper is to investigate the structure of holomorphic Anosov diffeomorphisms and flows, especially in low dimensions.

Theorem A. *Let ϕ be a holomorphic Anosov diffeomorphism of a compact complex surface S. Then S is a complex torus $\mathbb{C}^2/\Lambda$ and ϕ is holomorphically conjugate to a linear automorphism of $\mathbb{C}^2/\Lambda$.*

In higher dimensions, we have the following partial result. Recall that a diffeomorphism is called transitive if it has a dense orbit.

Theorem B. *Let ϕ be a transitive holomorphic Anosov diffeomorphism of a compact complex manifold M. Assume that ϕ has complex codimension* 1, *i.e., the unstable foliation of ϕ has real dimension* 2. *Then M is homeomorphic* *to a torus and ϕ is topologically conjugate to a linear automorphism of this torus.*

According to a well-known conjecture, any Anosov diffeomorphism should be topologically conjugate to some automorphism of some infra-nilmanifold [Sm]. In the holomorphic case, it might be true that any Anosov diffeomorphism is *holomorphically conjugate* to some automorphism of a complex infra-nilmanifold.

Next, we study the Anosov flows on compact complex 3-dimensional manifolds. Strictly speaking, a holomorphic vector field never defines an Anosov flow because it defines an action of $\mathbb{C} \simeq \mathbb{R}^2$ and this is of course not compatible with the Anosov property. We shall therefore modify the definition slightly in the following way.

Consider a holomorphic action ϕ of $\mathbb{C}^*$ on a compact complex manifold M denoted by:

$$(T,x) \in \mathbb{C}^* \times M \mapsto \phi(T)(x) \in M .$$

Equip M with a hermitian metric, $\|\cdot\|$. We shall say that ϕ is a *holomorphic Anosov flow* if there exist sub-bundles E^{uu} and E^{ss} of the real tangent bundle $T_{\mathbb{R}}M$ and constants $c > 0$, $\alpha > 0$, such that:

i) $T_{\mathbb{R}}M$ is the direct sum of E^{uu}, E^{ss} and the 2-dimensional bundle tangent to the orbits of the action ϕ.

ii) For all $T \in \mathbb{C}^*$, $v^s \in E^{ss}$ and $v^u \in E^{uu}$, one has:

$$\|d\phi(T)(v^s)\| \leqq c|T|^{-\alpha}\|v^s\|$$

$$\|d\phi(T)(v^u)\| \leqq c|T|^{\alpha}\|v^u\| .$$

If ϕ is an action of $\mathbb{C}^*$ and $k \in \mathbb{Z} - \{0\}$, one can define another action ϕ_k by $\phi_k(T) = \phi(T^k)$. It is clear that if ϕ is Anosov, so is ϕ_k with the same E^{uu}, E^{ss} if $k > 0$ and reversed if $k < 0$. Of course, the k^{th} roots of unity act trivially by ϕ_k so that ϕ_k is not an effective action. Conversely, it is easy to see that any holomorphic Anosov flow is of the form ϕ_k for some k and some effective action ϕ. Therefore, we shall always assume that the holomorphic Anosov flows under consideration are effective.

A *first way* of constructing examples is by *holomorphic suspension.* Let $A \in \mathrm{GL}(2,\mathbb{C})$ preserve a lattice $\Lambda \subset \mathbb{C}^2$ and let $\bar{A}$ be the corresponding diffeomorphism of $\mathbb{C}^2/\Lambda$. Choose ω in $\mathbb{C}^*$ away from the unit circle and consider the following diffeomorphism of $\mathbb{C}^2/\Lambda \times \mathbb{C}^*$:

$$(x,T) \in \mathbb{C}^2/\Lambda \times \mathbb{C}^* \mapsto (\bar{A}(x), \omega T) \in \mathbb{C}^2/\Lambda \times \mathbb{C}^* .$$

It generates a group acting properly discontinuously and freely on $\mathbb{C}^2/\Lambda \times \mathbb{C}^*$, commuting with the obvious $\mathbb{C}^*$-action on the second factor. The quotient manifold M is naturally equipped with a holomorphic Anosov flow as soon as the spectrum of A is disjoint from the unit circle.

This construction can be generalized. Instead of the trivial $\mathbb{C}^*$-bundle over $\mathbb{C}^*/\Lambda$, one can consider any $\mathbb{C}^*$-bundle $E \to \mathbb{C}^2/\Lambda$ over $\mathbb{C}^2/\Lambda$. In some cases, there is an automorphism θ of E over $\bar{A}$ which acts freely and properly discontinuously so that the quotient of E by θ is a compact 3-dimensional manifold equipped with a $\mathbb{C}^*$-action. However, it is not difficult to check (see 7.2) that such a θ can only exist if the bundle E is trivialized by a pullback to a finite cover of $\mathbb{C}^2/\Lambda$. Hence, this generalized construction leads to examples which are the same as the previous ones after lifting to a finite cover. We shall leave the detailed analysis of possible bundles E as an exercise for the reader but we mention that, in other situations, these "twisted holomorphic suspensions" can lead to very different $\mathbb{C}^*$-actions.

A *second kind* of examples of holomorphic Anosov flows comes from the choice of a cocompact discrete subgroup Γ of $SL(2,\mathbb{C})$. The complex manifold $M = SL(2,\mathbb{C})/\Gamma$ is 3-dimensional. The $\mathbb{C}^*$-action on $SL(2,\mathbb{C})$ by left translations by $\begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix}$ commutes with right translations and induces a $\mathbb{C}^*$-action on M which is a holomorphic Anosov flow. The proof of this fact is exactly the same as the proof of the corresponding well known fact in the real domain.

We shall modify these examples by using a construction described in [Gh2]. Let $u : \Gamma \to \mathbb{C}^*$ be a homomorphism. Consider the following right action of Γ on $SL(2,\mathbb{C})$:

$$(x,\gamma) \in SL(2,\mathbb{C}) \times \Gamma \mapsto x \bullet \gamma = \begin{pmatrix} u(\gamma) & 0 \\ 0 & u(\gamma)^{-1} \end{pmatrix} x\gamma \in SL(2,\mathbb{C}).$$

This action still commutes with the action of $\mathbb{C}^*$ on $SL(2,\mathbb{C})$ by left translations by $\begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix}$. If this new action of Γ is free, proper and totally discontinuous, we shall say that u is *admissible* and we shall denote by $SL(2,\mathbb{C})/\!\!/_u\Gamma$ the quotient manifold. This is a compact complex 3-manifold naturally equipped with an action of $\mathbb{C}^*$ which gives rise to a holomorphic Anosov flow. A few remarks are in order (see Sec. 6 for proofs). If u_1 and u_2 are two admissible homomorphisms from Γ to $\mathbb{C}^*$, then $SL(2,\mathbb{C})/\!\!/_{u_1}\Gamma$ and $SL(2,\mathbb{C})/\!\!/_{u_2}\Gamma$ are C^∞-diffeomorphic. One can even choose a diffeomorphism in such a way that it maps the orbits of the $\mathbb{C}^*$-action corresponding to u_1 to those corresponding to u_2. However, the two $\mathbb{C}^*$-actions are *conjugate* by a homeomorphism only if there is an automorphism θ of Γ such that $u_2 = u_1 \circ \theta$. Note that by Mostow's rigidity theorem, the embedding of Γ in $SL(2,\mathbb{C})$ is rigid and the outer automorphism group of Γ is finite—unlike that in the real case.

Let us finally mention an elementary modification of an action ϕ of $\mathbb{C}^*$ on a compact complex manifold M. Choose $k \in \mathbb{Z} - \{0\}$ and assume that the group $\mathbb{Z}/k\mathbb{Z}$ of k^{th} roots of unity acts freely on M. Denote by π_k the projection of M onto the quotient, M_k, of M by $\mathbb{Z}/k\mathbb{Z}$. On M_k one has a natural $\mathbb{C}^*$-action defined by:

$$(T, \pi_k(x)) \in \mathbb{C}^* \times M_k \mapsto {}_k\phi(T)(\pi_k(x)) := \pi_k(\phi(T^{\frac{1}{k}})(x)).$$

It is easy to check that this is indeed well defined and that if ϕ is Anosov so is ${}_k\phi$.

Theorem C. *Let ϕ be a holomorphic Anosov flow on a compact complex 3-dimensional manifold. Then, up to finite covers, ϕ is holomorphically conjugate to one of the examples described above.*

The same construction enables us to describe interesting examples of another phenomenon. Consider a 2-dimensional real foliation on a compact manifold and assume that all leaves are conformally hyperbolic (this does not depend on the choice of a Riemannian metric on the manifold). In [Ca] it is shown, in particular, that under such hypothesis, there is a Riemannian metric on the tangent bundle of the foliation—which is continuous on the manifold and C^∞ along the leaves—for which all leaves have constant negative curvature (see also [Gh3], [Ve]). We show that such a result cannot be extended to the euclidean case.

Theorem D. *There is a holomorphic foliation, $\mathcal{F}$, of complex dimension 1 on a compact complex 3-dimensional manifold with the following properties:*
i) *every leaf of $\mathcal{F}$ is dense and is of polynomial growth,*
ii) *every leaf of $\mathcal{F}$ is conformally equivalent to the complex line $\mathbb{C}$,*
iii) *there is no hermitian metric on the tangent bundle of $\mathcal{F}$, continuous on the manifold and smooth along the leaves, for which all the leaves are flat.*

Let us compare the main results of this paper with their analogues in the real domain [Gh1], [Gh4], [Gh5]. For holomorphic Anosov systems, we make *no assumption* concerning the regularity of the stable and unstable distributions and one of the contributions of this paper is to show that, indeed, these distributions are necessarily holomorphic. Of course, "holomorphic rigidity" helps us a lot, but, in many places, it is also an obstacle. For instance, the non existence of an order in $\mathbb{C}$ prevents us from adapting the proofs of [Gh5] and leads to the discussion of section 4.

2. First properties of the unstable foliation

Let M be a compact complex n-dimensional manifold. We denote by $T_{\mathbb{C}}M$ its complex tangent bundle. This is a holomorphic fibre bundle over M whose underlying real bundle $T_{\mathbb{R}}M$ is the tangent bundle of the underlying real manifold M. We shall always equip $T_{\mathbb{C}}M$ with an auxiliary hermitian metric, $\|\cdot\|$. Let ϕ be a holomorphic Anosov diffeomorphism of M. By definition, there exist complementary sub-bundles E^s and E^u of $T_{\mathbb{R}}M$ and constants $C > 0$, $\lambda > 1$ such that for every $v^s \in E^s$, $v^u \in E^u$ and every $k \geqq 0$, one has:

$$\|d\phi^k(v^s)\| \leqq C\lambda^{-k}\|v^s\|$$

$$\|d\phi^{-k}(v^u)\| \leqq C\lambda^{-k}\|v^u\|.$$

Lemma 2.1. *The sub-bundles E^s, E^u are complex sub-bundles of $T_{\mathbb{C}}M$, i.e., stable under multiplication by $i = \sqrt{-1}$.*

Proof. This is obvious since for $v^s \in E^s$:

$$\|d\phi^k(iv^s)\| = \|i\, d\phi^k(v^s)\| = \|d\phi^k(v^s)\| \leqq C\lambda^{-k}\|iv^s\|,$$

and therefore $iv^s \in E^s$. Same proof for E^u. □

Let $\mathscr{F}^s$ and $\mathscr{F}^u$ be the stable and unstable foliations tangent to E^s and E^u respectively. The leaves of $\mathscr{F}^s$ and $\mathscr{F}^u$ are C^∞-immersed submanifolds of M. It follows from 2.1 that each leaf of $\mathscr{F}^s$ or $\mathscr{F}^u$ is an immersed *holomorphic* submanifold of M.

Remark. For general Anosov diffeomorphisms, $\mathscr{F}^s$ and $\mathscr{F}^u$ are usually only continuous foliations. We know of no example of *holomorphic* Anosov diffeomorphism for which $\mathscr{F}^s$ and $\mathscr{F}^u$ are not holomorphic.

However, we shall prove the following:

Proposition 2.2. *If E^u has complex dimension* 1, *then $\mathscr{F}^s$ is a* (*transversely*) *holomorphic foliation.*

Proof. Experts will not be surprised by this result. Indeed, it is well known that if an Anosov diffeomorphism of class C^∞ is such that E^u has (real) dimension 1, then $\mathscr{F}^s$ is a codimension 1 foliation of class C^1. Proposition 2.2. is a "complexification" of this result taking into account the fact that in the complex domain C^1 implies holomorphic. Therefore, the proof that we present is nothing more than an adaptation of this classical fact [HPS].

Let x and y be two points of M lying in the same leaf L of $\mathscr{F}^s$. Let $\mathscr{F}^u_x$ and $\mathscr{F}^u_y$ be the leaves of $\mathscr{F}^u$ going through x and y respectively. Small neighborhoods V_x and V_y of x and y in $\mathscr{F}^u_x$ and $\mathscr{F}^u_y$ can be considered as holomorphic curves which are transversal to $\mathscr{F}^s$. Let h be the holonomy of some path contained in L connecting x and y measured on these transversals. This is a homeomorphism, defined at least on some compact neighborhood W_x of x in V_x, whose image $h(W_x) = W_y$ is a compact neighborhood of y in V_y. Note that, obviously, $h(x) = y$ and that h does not depend on the choice of the path connecting x and y since the leaves of $\mathscr{F}^s$ are simply connected (and even diffeomorphic to some euclidean space). In order to prove the proposition we have to show that h is holomorphic.

Let $W^k_x = \phi^k(W_x)$ and $W^k_y = \phi^k(W_y)$ for $k \geqq 0$. It follows from the definition of Anosov diffeomorphisms that if z is a point in W_x, then $\phi^k(z)$ and $\phi^k(h(z))$ approach each other when k tends to $+\infty$. One knows also that the leaves of $\mathscr{F}^u$ vary continuously in the C^∞-topology (see [HPS] for details). In particular, this implies that there is a sequence of diffeomorphisms π_k, of class C^1, from W^k_x to a compact set $\pi_k(W^k_x)$ in the leaf $\mathscr{F}^u_{y_k}$ of $\mathscr{F}^u$ going through $y_k = \phi^k(y)$ such that:

i) the distance between $\pi_k(z)$ and $\phi^k h \phi^{-k}(z)$ in the leaf $\mathscr{F}^u_{y_k}$ goes uniformly to zero when $z \in W^k_x$ and k goes to $+\infty$.

ii) π_k tends uniformly to the identity in the C^1-topology in the following sense. There is a sequence $\varepsilon_k > 0$ tending to zero when k goes to $+\infty$ such that for all vectors v tangent to W^k_x one has:

$$(1-\varepsilon_k)\|v\| \leqq \|d\pi_k(v)\| \leqq (1+\varepsilon_k)\|v\| .$$

Let us now consider the sequence of diffeomorphisms $h_k = \phi^{-k}\pi_k\phi^k$ defined on W_x with values in $\mathscr{F}^u_y$. As ϕ is holomorphic and acts conformally in each leaf of $\mathscr{F}^u$ the following properties hold:

i) h_k is a $(1+\varepsilon_k)$-quasiconformal diffeomorphism from W_x to $h_k(W_x) \subset \mathscr{F}^u_y$ (see [Ah]).

ii) h_k converges uniformly to h when k tends to infinity.

Therefore, h is 1-quasiconformal, *i.e.*, conformal. In other words h is holomorphic. □

Remark. If M has complex dimension 2, then the proposition can be applied to both stable and unstable foliations so that one deduces that these foliations are not only transversely holomorphic but are actually holomorphic foliations.

3. A complex affine structure along the unstable leaves

In this section, we generalize another property of Anosov diffeomorphisms of real codimension 1: their unstable leaves are naturally equipped with affine structures (see [Su], [Gh4] or [Gh5]).

Recall that a complex affine structure on a holomorphic curve L is an atlas consisting of holomorphic diffeomorphisms f_i from open sets U_i of L to open sets of $\mathbb{C}$ such that the U_i cover L and the $f_i \circ f_j^{-1}$ are restrictions of affine diffeomorphisms of $\mathbb{C}$ to their domains of definition. If x, y and z are three distinct points of L, close enough, then the ratio $\frac{f_i(x)-f_i(y)}{f_i(x)-f_i(z)}$ is well defined and does not depend on the choice of f_i whose domain contains x, y and z. We shall denote this ratio by $\frac{x-y}{x-z}$; this is a holomorphic function of x, y and z. One says that an affine structure on a simply connected Riemann surface is complete if it is isomorphic to $\mathbb{C}$ with its canonical affine structure.

Proposition 3.1. *Let ϕ be a holomorphic Anosov diffeomorphism of a complex compact manifold. Assume that the unstable foliation $\mathscr{F}^u$ has complex dimension* 1. *Then there exists a unique way to equip each leaf of $\mathscr{F}^u$ with a complex affine structure in such a way that*:

i) *ϕ is acting affinely on leaves of $\mathscr{F}^u$,*

ii) *Let γ be a path in a leaf of the stable foliation, parametrized by $t \in [0,1]$, and let $h(t)$ be the corresponding holonomy between open sets of $\mathscr{F}^u_{\gamma(0)}$ and $\mathscr{F}^u_{\gamma(t)}$. If x, y and z are three points of $\mathscr{F}^u_{\gamma(0)}$, close enough to $\gamma(0)$, then the ratio $\frac{h(t)(x)-h(t)(y)}{h(t)(x)-h(t)(z)}$ is a continuous function of t.*

Moreover, for this structure, each leaf of $\mathscr{F}^u$ is complete.

The proof of this proposition is based on the same ideas as in the real case but technical difficulties occur since it is not a priori clear that there is a holomorphic vector field tangent to $\mathcal{F}^u$ or even a continuous vector field which is holomoıphic on the leaves. Before giving the proof, we describe a general construction.

Let L be a holomorphic curve. For $p \geqq 0$ we denote by L_p the fibration over L consisting of p-jets at $0 \in \mathbb{C}$ of local holomorphic diffeomorphisms f from a neighborhood of $0 \in \mathbb{C}$ to some open set of L. Of course L_0 is naturally identified with L, and L_1 with the set of nonzero tangent vectors to L. Moreover, one has a chain of fibrations:

$$\cdots \longrightarrow L_2 \xrightarrow{\rho_2} L_1 \xrightarrow{\rho_1} L_0 = L\,.$$

For $f \in L_1$ and $\omega \in \mathbb{C}^*$, we denote by $\omega \cdot f$ the 1-jet of the map $z \mapsto \bar{f}(\omega z)$ where $\bar{f}$ is any representative of f. This describes ρ_1 explicitly as a $\mathbb{C}^*$-principal fibration. In the same way for $(\omega, \mu) \in \mathbb{C}^* \times \mathbb{C}$ and $f \in L_2$, we denote by $(\omega, \mu) \cdot f$ the 2-jet of $z \mapsto f\left(\frac{\omega z}{1+\mu z}\right)$ and this describes a structure of a principal fibration for $\rho_1 \circ \rho_2 : L_2 \to L$ whose structure group is the affine group Aff, parametrized by $z \mapsto \frac{\omega z}{1+\mu z}$. The fibration $\rho_2 : L_2 \to L_1$ is also principal via the action of $\mathbb{C}$ given by $(1, \mu) \cdot f$.

Lemma 3.2. *There is a natural identification between affine structures on a holomorphic curve L and holomorphic sections $\sigma : L_1 \to L_2$ which are homogeneous in the following sense. For every $\omega \in \mathbb{C}^*$ and $f \in L_1$, one has $\sigma(\omega \cdot f) = (\omega, 0) \cdot \sigma(f)$.*

Proof. If L is equipped with an affine structure and if $f \in L_1$ there is a unique germ of an affine map $\bar{f}$ defined in a neighbourhood of $0 \in \mathbb{C}$, with values in L, having f as its 1-jet at 0. One defines $\sigma(f)$ as the 2-jet of $\bar{f}$ at 0. The homogeneity condition is obviously satisfied.

Conversely, suppose we are given such an equivariant section σ. Let us consider local diffeomorphisms g from open sets of $\mathbb{C}$ to open sets of L such that for every z_0 in the domain of g, the 1 and 2-jets at 0 of the map $z \mapsto g(z - z_0)$ satisfy $j_2(g)(z_0) = \sigma(j_1(g)(z_0))$. It is clear that this second order differential equation has local solutions and the homogeneity condition guarantees that any two local solutions differ by an affine map. In other words, these solutions define an affine structure on L. □

Let us come back to our holomorphic Anosov diffeomorphism ϕ of the compact complex manifold M. For $p \geqq 0$, we denote by M_p the space of p-jets at $0 \in \mathbb{C}$ of holomorphic diffeomorphisms of a neighborhood of 0 into an open set in a leaf of $\mathcal{F}^u$. Just as before, one has a chain of fibrations

$$\cdots \longrightarrow M_2 \xrightarrow{\rho_2} M_1 \xrightarrow{\rho_1} M_0 = M\,.$$

Moreover, ϕ induces natural homeomorphisms ϕ_p of M_p.

In order to prove Proposition 3.1, one has to show the following:

Lemma 3.3. *There is a unique section $\sigma : M_1 \to M_2$ of ρ_2 such that:*
i) *σ is continuous and for every leaf L of $\mathscr{F}^u$, it is a holomorphic map from $L_1 \subset M_1$ to $L_2 \subset M_2$*
ii) *σ is homogeneous, i.e., $\sigma(\omega \cdot f) = (\omega, 0) \cdot \sigma(f)$,*
iii) *$\sigma \circ \phi_1 = \phi_2 \circ \sigma$.*

Proof. Let us consider the space $\mathscr{X}$ of continuous sections $\sigma : M_1 \to M_2$ which are homogeneous (*i.e.*, satisfying ii)). This space is certainly nonempty since it can be naturally identified to the space of sections of a fibration with contractible fibres. If σ and σ' are two elements of $\mathscr{X}$ one can use the structure of the $\mathbb{C}$-principal fibration, ρ_2, to define $\sigma - \sigma'$ as a continuous function from M_1 to $\mathbb{C}$. Moreover, one can use the given hermitian metric on M to define the norm of an element of M_1 (identified with a vector tangent to M). Then we set:

$$d(\sigma, \sigma') = \sup_{x \in M_1} \left\{ \frac{|(\sigma - \sigma')(x)|}{\|x\|} \right\}$$

This is well defined since, by homogeneity, the quotient $\frac{|(\sigma-\sigma')(x)|}{\|x\|}$ only depends on the projection $\rho_1(x)$ which lies in the compact manifold M. It is easy to check that $(\mathscr{X}, d)$ is a complete metric space.

We claim that the bijection $\bar{\phi}^k$ induced by ϕ^k on $\mathscr{X}$ is a dilatation for k large. Indeed:

$$\begin{aligned} d(\bar{\phi}^k(\sigma), \bar{\phi}^k(\sigma')) &= \sup_{x \in M_1} \left\{ \frac{|(\bar{\phi}^k(\sigma) - \bar{\phi}^k(\sigma'))(x)|}{\|x\|} \right\} \\ &= \sup_{x \in M_1} \left\{ \frac{|(\sigma - \sigma')(\phi_1^k(x))|}{\|x\|} \right\} \\ &\geqq C^{-1} \lambda^k d(\sigma, \sigma') \end{aligned}$$

since $\phi_1 : M_1 \to M_1$ is identified with the action of the differential of ϕ on nonzero vectors tangent to $\mathscr{F}^u$ (C and λ are the constants involved in the definition of the Anosov property). Therefore, there is a unique element σ of $\mathscr{X}$ which is invariant under $\bar{\phi}$.

Now, we show that for each leaf L of $\mathscr{F}^u$ the restriction of σ to $L_1 \subset M_1$ is holomorphic. Let U^i ($i = 1, \ldots, N$) be an open covering of M by flowboxes for $\mathscr{F}^u$. This means that U^i is homeomorphic to an open set of $\mathbb{C} \times \mathbb{C}^{n-1}$ by a homeomorphism which maps leaves of $\mathscr{F}^u$ into $\mathbb{C} \times \{\star\}$. This homeomorphism can be assumed to be holomorphic on each leaf of $\mathscr{F}^u$ (restricted to U^i). In other words, one has nonnatural complex affine structures on the leaves of $\mathscr{F}^u$ restricted to U^i.

Let U_1^i and U_2^i be the open sets in M_1 and M_2 corresponding to U^i and let $s_i : U_1^i \to U_2^i$ be the homogeneous section associated with the non canonical affine structures that we have constructed. Let x be any point of M_1 and let

$i(k)$ be a sequence of integers such that $\phi_1^{-k}(x) \in U_1^{i(k)}$. It is clear that the unique invariant section σ satisfies:

$$\sigma(x) = \lim_{k \to +\infty} \phi_1^k s_{i(k)} \phi_1^{-k}(x).$$

Let V be a small open set in a leaf L of $\mathcal{F}^u$ and let V_1 be the corresponding open set of $L_1 \subset M_1$. As ϕ^{-k} is a contraction one can find a sequence $i(k)$ such that *for every* x in V_1, one has $\phi_1^{-k}(x) \in U_1^{i(k)}$. Hence, the above formula shows that the restriction of σ to V_1 is a uniform limit of holomorphic functions. This shows that for every leaf L of $\mathcal{F}^u$ the restriction of σ to $L_1 \subset M_1$ is holomorphic.

Finally, we show that the affine complex structures that we have constructed on leaves of $\mathcal{F}^u$ are complete. Recall that for every affine structure on a simply connected space L there is a *developing map* $D : L \to \mathbb{C}$ which is a local diffeomorphism and such that the given structure is the pull-back by D of the canonical structure (see [Th] for instance). This map D is unique up to post-composition by an affine map and the structure is complete if and only if D is a diffeomorphism.

By compactness of M, one knows that there is some $\varepsilon > 0$ such that if two points are in the same leaf L of $\mathcal{F}^u$ and are at a distance less than ε, then the images of these points under the developing map are distinct. Since ϕ acts affinely on leaves of $\mathcal{F}^u$ and is expanding these leaves, it follows that the developing map is indeed injective for every leaf L of $\mathcal{F}^u$.

In the same way, there is an $\varepsilon > 0$ with the following property. Let $p \in M$ and choose a developing map D from the leaf L through p to $\mathbb{C}$ such that the norm of the differential of D at p is 1. This fixes D up to post-composition by a rotation. Then the image of D contains at least the disc of radius ε and center 0. Since ϕ is affine on leaves of $\mathcal{F}^u$ and expands these leaves, one deduces that these developing maps D have images which cover arbitrarily large discs, *i.e.*, they are onto. Therefore, the affine structures that we have constructed on leaves of $\mathcal{F}^u$ are complete. □

4. A transversely projective structure for the stable foliation

In this section, we establish the following proposition.

Proposition 4.1. *Let ϕ be a holomorphic Anosov diffeomorphism of a compact complex manifold. Assume that the unstable foliation $\mathcal{F}^u$ has complex dimension* 1. *Then the holonomy pseudogroup of the stable foliation $\mathcal{F}^s$ acts projectively with respect to the complex affine structures that we introduced on leaves of $\mathcal{F}^u$.*

One could be tempted to adapt the arguments of [Gh5]. However, we shall have to modify them a lot since an order structure is lacking in the complex line!

Let us fix a point x_0 in the ambient manifold M and a vector v tangent to $\mathscr{F}^u$ at x_0 and let us consider the leaf L of $\mathscr{F}^s$ through the point x_0. If x is another point of L, there is a well defined germ of a holomorphic diffeomorphism $h_{x_0,x}$ from a neighborhood of x_0 in the leaf $\mathscr{F}^u_{x_0}$ through x_0 to a neighborhood of x in the leaf $\mathscr{F}^u_x$ through x: this is the holonomy of any path in a leaf of $\mathscr{F}^s$ connecting x_0 and x. Since leaves of $\mathscr{F}^u$ are affine, one can compute the Schwarzian derivative of $h_{x_0,x}$: this is a quadratic differential in a neighborhood of x_0 in $\mathscr{F}^u_{x_0}$. At x_0 we get a complex quadratic form $q_{x_0,x}$ on the 1-dimensional complex vector space tangent to $\mathscr{F}^u$ at x_0. Evaluating $q_{x_0,x}$ on the vector v we obtain a function:

$$x \in L \mapsto q_{x_0,x}(v) \in \mathbb{C}.$$

A priori, this is only a continuous function of $x \in L$ since we only know that the affine structures along the leaves of $\mathscr{F}^u$ vary continuously. In order to prove the proposition, we have to show that all germs $h_{x_0,x}$ are projective, which is equivalent to showing that their Schwarzian derivatives vanish. In other words, we want to show that $q_{x_0,x}(v) = 0$.

Before going through the proof, let us note that if x'_0 is another point of L and if $v' = dh_{x_0,x'_0}(v)$ is the corresponding vector tangent to $\mathscr{F}^u$ at x_0, then the composition formula for Schwarzian derivatives yields:

$$q_{x'_0,x}(v') = q_{x'_0,x_0}(v') + q_{x_0,x}(v).$$

Therefore, if Ω is a domain contained in L, the diameter of the image of the map

$$x \in \Omega \mapsto q_{x_0,x}(v) \in \mathbb{C}$$

does not depend on the choice of the base point x_0 (if one changes the vector v accordingly). This will be used later.

We shall also need a Markov partition for ϕ. Let us recall some definitions and introduce some notations. For every $\varepsilon > 0$, there is a $\eta > 0$ such that if two points x and y of M are at a distance less than η then the discs in the leaves $\mathscr{F}^s_x$ and $\mathscr{F}^u_y$ of centers x and y and radii ε intersect in a single point denoted by $[x, y]$.

A *rectangle* is a compact connected set R in M, which is the closure of its interior, of diameter less than η, and such that if x and y are in R so does $[x, y]$.

If R is a rectangle and $x \in R$, we set:

$$R_u(x) = \{[x, y], y \in R\}$$

$$R_s(x) = \{[y, x], y \in R\}$$

A Markov partition for ϕ is a finite collection of rectangles $R^1, \dots, R^N$ covering M, with disjoint interiors, satisfying the following property. If x belongs to the interior of R^i and $\phi(x)$ belongs to the interior of R^j then:

$$\phi(R_u^i(x)) \supset R_u^j(\phi(x))$$
$$R_s^i(x) \subset \phi^{-1}(R_s^j(\phi(x)).$$

For the existence of Markov partitions, see [Bo], [F-J] or [Si].

For each rectangle R^i, we choose a base point x_i in the interior of R^i and we shall simply write R_u^i and R_s^i instead of $R_u^i(x_i)$ and $R_s^i(x_i)$ respectively. Sometimes, when we do not want to specify to which rectangle R^i a point x belongs, we write $R_u(x)$ or $R_s(x)$ instead of $R_u^i(x)$ and $R_s^i(x)$ respectively.

Let $\mathscr{U}$ be the disjoint union of the R_u^i $(i = 1,\ldots,N)$. Let $x \in \mathscr{U}$. If $\phi(x)$ belongs to the interior of R^j, we set $\Phi(x) = [x_j, \phi(x)]$. In this way, we get a partially defined expanding map Φ from an open dense subset of $\mathscr{U}$ to $\mathscr{U}$. We shall not try to define $\Phi(x)$ for those points x such that $\phi(x)$ belongs to two rectangles. However, we note the following. For each $i = 1,\ldots,N$, there is a finite collection of holomorphic diffeomorphisms $\Psi_1^i,\ldots,\Psi_{N_i}^i$ from R_u^i to $\mathscr{U}$ which are the "branches of the inverse of Φ", *i.e.*, such that if $\Phi(x) = y$ is defined and is in R_u^i, then x is one of the points $\Psi_1^i(y),\ldots,\Psi_{N_i}^i(y)$. If the Markov partition is thin enough, one can always assume that for each i, the images of the $\Psi_1^i,\ldots,\Psi_{N_i}^i$ are contained in different R_u^j.

After these preliminaries, we can begin the proof of proposition 4.1.

Let v be a vector tangent to $\mathscr{U}$, thought of as being a vector tangent to $\mathscr{F}^u$ at some point x in some R_u^i. As explained before, one can consider the continuous map:

$$y \in R_s^i(x) \mapsto q_{x,y}(v) \in \mathbb{C}.$$

Let $\delta(v)$ be the diameter of the image of this map. Clearly, if λ is any complex number, one has:

$$\delta(\lambda v) = |\lambda|^2 \delta(v).$$

Hence δ has the tensorial character of an area form *i.e.*, a 2-form of type $(1,1)$. More precisely, let z be a holomorphic coordinate in a leaf $\mathscr{F}_x^u$, in a neighborhood of x. Then the measure $\delta\left(\frac{\partial}{\partial z}\right) dz\, d\bar{z}$ does not depend on the choice of this parameter. In this way, we have constructed a natural measure δ on $\mathscr{U}$.

Recall that we can always assume that the boundaries of R_u^i have zero (2-dimensional) Lebesgue measure (see [Man]).

Lemma 4.2. *The measure δ is invariant under Φ.*

Proof. Let v be a vector tangent to $\mathscr{U}$ at a point x belonging to R_u^i. Consider the ϕ^{-1}-image of $R_s^i(x)$. It is the union of $R_s(\Psi_1^i(x)),\ldots,R_s(\Psi_{N_i}^i(x))$.

Taking into account that:

i) ϕ acts affinely on leaves of $\mathscr{F}^u$,

ii) $R_s^i(x)$ is connected,

iii) $q_{x,y}(v) = q_{\phi^{-1}(x),\phi^{-1}(y)}(d\phi^{-1}(v))$,

one gets:

$$\begin{aligned}\delta(v) &= \operatorname{diam}\{q_{x,y}(v) \mid y \in R^i_s(x)\}\\ &= \operatorname{diam}\{q_{\phi^{-1}(x),z}(d\phi^{-1}(v)) \mid z \in \phi^{-1}(R^i_s(x))\}\\ &\leqq \sum_{k=1}^{N_i} \operatorname{diam}\{q_{\phi^{-1}(x),z}(d\phi^{-1}(v)) \mid z \in R^i_s(\Psi^i_k(x))\}\\ &\leqq \sum_{k=1}^{N_i} \delta(d\Psi^i_k(v)).\end{aligned}$$

In other words, δ is sub-invariant, *i.e.*, for every Borel set $\mathscr{B} \subset \mathscr{U}$, one has:

$$\delta(\mathscr{B}) \leqq \delta(\Phi^{-1}(\mathscr{B})).$$

It is now easy to deduce that δ is invariant. If $\mathscr{B}$ is a Borel set in $\mathscr{U}$, one has:

$$\delta(\mathscr{U}) = \delta(\mathscr{B})+\delta(\mathscr{U}-\mathscr{B}) \leqq \delta(\Phi^{-1}(\mathscr{B}))+\delta(\Phi^{-1}(\mathscr{U}-\mathscr{B})) = \delta(\Phi^{-1}(\mathscr{U})) = \delta(\mathscr{U}).$$

Hence, all inequalities are equalities and δ is an invariant measure. □

Lemma 4.3. *The measure δ is the zero measure.*

Proof. Assume by contradiction that δ is nontrivial. Then Φ admits an absolutely continuous invariant measure and is therefore topologically transitive. This implies that ϕ is also topologically transitive and that δ is everywhere nonzero.

We know that all inequalities in Lemma 4.2 are equalities. This means that:

$$\begin{aligned}&\operatorname{diam}\{q_{\phi^{-1}(x),y}(d\phi^{-1}(v)) \mid y \in \phi^{-1}(R^i_s(x)\}\\ &\qquad = \sum_{k=1}^{N_i} \operatorname{diam}\{q_{\phi^{-1}(x),y}(d\phi^{-1}(v)) \mid y \in R_s(\Psi^i_k(x))\}.\end{aligned}$$

We leave the proof of the following easy fact for the reader:

Fact 1. *Let K be a connected compact set in $\mathbb{C}$ which is the union of finitely many nonempty compact sets K_j. If the diameter of K is the sum of the diameters of K_j, then one can find closed discs D_j containing K_j with disjoint interiors and whose centers are on the same line. In particular, if no K_j is reduced to a point then no point of K belongs to three distinct K_j.*

Since by hypothesis δ is nowhere zero, no point of $C = \phi^{-1}(R^i_s(x))$ can belong to three distinct compact sets among the $R_s(\Psi^i_1(x)), \ldots, R_s(\Psi^i_{N_i}(x))$. Of course, one can apply the same argument for iterates of ϕ and we get arbitrarily thin covers of C by finitely many compact sets C_j such that no point of C belongs to three distinct C_j. However, this is in contradiction with the following topological fact.

Fact 2. *A compact set C in $\mathbb{R}^p$ ($p \geqq 2$) with nonempty interior cannot be covered by finitely many compact sets C_j with arbitrarily small diameters and such that no point of C belongs to three distinct C_j.*

Indeed, let us denote by C^ε and C_j^ε the ε-neighborhoods of C and C_j in $\mathbb{R}^p$. By the classical definition of topological dimension, if the C_j have sufficiently small diameters and ε is sufficiently small, there is at least one point in C^ε belonging to three distinct C_j^ε. Letting ε go to zero and taking a limit, one deduces the assertion.

This contradiction proves the lemma. □

We can now finish the proof of Proposition 4.1. Let x_0 and x be two points in the same leaf L of $\mathscr{F}^s$, $h_{x_0,x}$ the corresponding holonomy and $q_{x_0,x}$ the Schwarzian derivative of $h_{x_0,x}$. Of course h_{x_0,x_0} is the identity so that $q_{x_0,x_0} = 0$. Since we have shown that δ vanishes, we know that q_{x,x_0} does not depend on x and therefore vanishes identically. In other words, the holonomy pseudogroup acts projectively with respect to the affine structures on the leaves of $\mathscr{F}^u$.

5. Proofs of Theorems A and B

Let ϕ be a holomorphic Anosov diffeomorphism of a compact complex manifold M. Assume that the unstable foliation $\mathscr{F}^u$ has complex dimension 1. We have shown that the stable foliation $\mathscr{F}^s$ admits a transversely projective structure. This implies that the lifted foliation $\widetilde{\mathscr{F}}^s$ in the universal covering space $\widetilde{M}$ of M is defined by a global submersion:

$$D : \widetilde{M} \to \mathbb{CP}^1 .$$

Moreover, there exists a (holonomy) homomorphism

$$H : \pi_1(M) \to \mathrm{PSL}(2, \mathbb{C})$$

such that for every $\tilde{x} \in \widetilde{M}$ and $\gamma \in \pi_1(M)$ one has the following property of the developing map:

$$D(\gamma \cdot \tilde{x}) = H(\gamma)(D(\tilde{x})) .$$

(See for instance [Th]).

Let $\widetilde{\mathscr{F}}^u$ be the lift of $\mathscr{F}^u$ to $\widetilde{M}$ and $\widetilde{\mathscr{F}}^u_{\tilde{x}}$ be the leaf of $\widetilde{\mathscr{F}}^u$ through a point $\tilde{x}$. We know that $\widetilde{\mathscr{F}}^u_{\tilde{x}}$ is naturally equipped with a complete affine structure. In other words, the restriction of D to $\widetilde{\mathscr{F}}^u_{\tilde{x}}$ is a diffeomorphism from $\widetilde{\mathscr{F}}^u_{\tilde{x}}$ to $\mathbb{CP}^1 - \{\omega(\tilde{x})\}$ where $\omega(\tilde{x})$ is some point of $\mathbb{CP}^1$.

Let us begin with the proof of theorem B. Assume that ϕ is topologically transitive so that $\mathscr{F}^s$ admits a transverse invariant measure μ with full support (see [Pl1]).

Lemma 5.1. *Let x and y be two points of M in the same leaf of $\mathscr{F}^s$ and γ be a path connecting them and contained in that leaf. Then the holonomy of γ is a global diffeomorphism of the leaf $\mathscr{F}^u_x$ of $\mathscr{F}^u$ through x to the leaf $\mathscr{F}^u_y$ through y.*

Proof. Each leaf of $\mathscr{F}^u$ is equipped with a measure μ, which is finite on compact sets. Each leaf of $\mathscr{F}^u$ is also equipped with a complete affine structure so that closed discs in $\mathcal{F}^u$ have an intrinsic meaning. Since each leaf of $\mathcal{F}^u$ is dense and μ has full support, for each $K > 0$ the set of closed discs contained in some leaf of $\mathcal{F}^u$ and with measure less than or equal to K is a compact set.

Let us consider the path, $\gamma : [0,1] \to M$, the holonomy of which we want to study, and let B be a closed disc in $\mathcal{F}^u_x$, where $x = \gamma(0)$. Let t_0 be the upper bound of the set of $t \in [0,1]$ such that the holonomy h_t of $\mathcal{F}^s$ along the restriction of γ to $[0,t]$ is defined on all the disc B with values in $\mathcal{F}^u_{\gamma(t)}$. Since $\mathcal{F}^s$ is transversely projective, we know that, for $t < t_0$, $h_t(B)$ is a closed disc in $\mathcal{F}^u_{\gamma(t)}$. Of course, all discs $h_t(B)$ have the same measure so that by the remark above, this family of discs $h_t(B)$ stay in a compact set. In particular, the diameters of $h_t(B)$ in $\mathcal{F}^u_{h(t)}$ (measured with the auxiliary hermitian metric) are bounded, hence $h_{t_0}(B)$ is defined. If $t_0 < 1$, it would be easy to extend h_t beyond t_0, contradicting the definition of t_0. This shows that the holonomy of γ can be defined on all B. As the choice of B was arbitrary, this establishes the lemma. □

Proof of theorem B. The proof of the preceding lemma shows that two leaves of $\widetilde{\mathcal{F}}^u$ meet exactly the same leaves of $\widetilde{\mathcal{F}}^s$. In other words, all leaves $\widetilde{\mathcal{F}}^u_{\tilde{x}}$ of $\widetilde{\mathcal{F}}^u$ have the same image $\mathbb{CP}^1 - \{a\}$ under the developing map D. Choosing a as a point at infinity, one sees that $\widetilde{\mathcal{F}}^s$ is defined by a submersion $D : \widetilde{M} \to \mathbb{C}$ so that $\mathcal{F}^s$ is actually a transversely affine foliation. Moreover, since each leaf of $\widetilde{\mathcal{F}}^u$ is mapped diffeomorphically onto $\mathbb{C}$, it follows that D is a (trivial) fibration. Since leaves of $\mathcal{F}^s$ are simply connected, the global holonomy group $H(\pi_1(M))$ is a group of affine transformations of $\mathbb{C}$ with no fixed point in $\mathbb{C}$. Hence H is an injection of $\pi_1(M)$ into the group of translations of $\mathbb{C}$. In particular, $\pi_1(M)$ is abelian and M has the homotopy type of a torus. If M has complex dimension 2 the fact that M is covered by $\mathbb{R}^4$ implies that M is actually homeomorphic to a 4-torus. In higher dimension, it is known that a smooth manifold having the homotopy type of a torus is homeomorphic to a torus [H-S](I thank A. Verjovsky for this argument). Theorem B now follows from Franks-Manning's theorem: ϕ is topologically conjugate to an automorphism of a torus, [Fr], [Mann]. □

The proof of theorem A is more delicate. Observe that we do not assume that ϕ is topologically transitive so that we cannot argue that there is a transversal invariant measure with full support for $\mathcal{F}^s$. Moreover, the proof that follows will be used again later.

Since we now assume that $\mathcal{F}^s$ and $\mathcal{F}^u$ are one dimensional, we can apply the result of section 4 for both foliations. Hence, M admits local coordinates in $\mathbb{CP}^1 \times \mathbb{CP}^1$ such that changes of coordinates are in $\mathrm{PSL}(2,\mathbb{C}) \times \mathrm{PSL}(2,\mathbb{C})$. Let:

$$D : \widetilde{M} \to \mathbb{CP}^1 \times \mathbb{CP}^1$$

and

$$H : \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C}) \times \mathrm{PSL}(2,\mathbb{C})$$

be the developing map and holonomy corresponding to this structure on M respectively.

Lemma 5.2. *D is injective.*

Proof. Let $\tilde{L}$ be a leaf of $\tilde{\mathscr{F}}^s$ and let $U(\tilde{L})$ be the union of all leaves of $\tilde{\mathscr{F}}^u$ intersecting $\tilde{L}$. Choose a sequence of leaves $(\tilde{L}_i)_{i\in\mathbb{N}}$ such that the open connected sets $U_i = U(\tilde{L}_i)$ satisfy:

i) The U_i cover $\tilde{M}$.

ii) For each $k \geqq 1$, the union $\Omega_k = \bigcup_{i=1}^k U_i$ is connected.

We shall prove the following assertion, stronger than Lemma 5.2, by induction on k.

Assertion. *The restriction of D to Ω_k is injective and the image $D(\Omega_k)$ is the complement in $\mathbb{CP}^1 \times \mathbb{CP}^1$ of the graph of a continuous map from $\mathbb{CP}^1$ to $\mathbb{CP}^1$, or, of the union of vertical $\{\star\}\times\mathbb{CP}^1$ and of the graph of a continuous map from $\mathbb{CP}^1 - \{\star\}$ to $\mathbb{CP}^1$.*

Let us prove this assertion for $k = 1$. We know that the restriction of D to a leaf $\tilde{L}_1$ of $\tilde{\mathscr{F}}^s$ is a diffeomorphism onto a set of the form $(\mathbb{CP}^1 - \{a\}) \times \{b\}$. Each point x of $\mathbb{CP}^1 - \{a\}$ corresponds to a point of $\tilde{L}_1$, still denoted by x, and the leaf containing x of $\tilde{\mathscr{F}}^u$, $\tilde{\mathscr{F}}^u_x$, is mapped diffeomorphically by D onto a set of the form $\{x\} \times (\mathbb{CP}^1 - \{\omega(x)\})$. The map $\omega : \mathbb{CP}^1 - \{a\} \to \mathbb{CP}^1$ is continuous since the complement of its graph is the open set $D(U_1)$. Hence D restricted to U_1 is injective and its image has the announced form.

Assume the assertion has been proven up to k. Since we already know that D is injective on Ω_k and $D(\Omega_k)$, is the complement in $\mathbb{CP}^1 \times \mathbb{CP}^1$ of the graph of some continuous map $u_k : \mathbb{CP}^1 \to \mathbb{CP}^1$, or, of the union of a vertical $\{a_k\} \times \mathbb{CP}^1$ and of the graph of a continuous map $u_k : \mathbb{CP}^1 - \{a_k\} \to \mathbb{CP}^1$.

Moreover we know that D is injective on U_{k+1} and that $D(U_{k+1})$ is the complement of the union of $\{b_{k+1}\} \times \mathbb{CP}^1$ and of the graph of $v_{k+1} : \mathbb{CP}^1 - \{b_{k+1}\} \to \mathbb{CP}^1$ (this is the case $k = 1$).

It follows that $D(\Omega_k) \cap D(U_{k+1})$ is connected since it is the complement in $\mathbb{CP}^1 \times \mathbb{CP}^1$ of a finite union of real codimension 2 topological submanifolds.

Let x and y be two points in $\Omega_{k+1} = \Omega_k \cup U_{k+1}$ such that $D(x) = D(y)$. Since we already know that D is injective on Ω_k and U_{k+1}, we can assume that x is in Ω_k and y in U_{k+1}. Let z be a point in $U_{k+1} \cap \Omega_k$ and γ be a path in $D(\Omega_k) \cap D(U_{k+1})$ connecting $D(z)$ and $D(x) = D(y)$. This path lifts in a unique way in Ω_k from z to x. In the same way, γ lifts in a unique way as a path in U_{k+1} from z to y. Since D is a local diffeomorphism, two lifts of γ in $\tilde{M}$ with the same origin are equal. In particular, the endpoints of these lifts are the same, *i.e.*, $x = y$. Therefore the restriction of D to Ω_{k+1} is injective.

We still have to describe the image $D(\Omega_{k+1})$. Let p be a point of $\mathbb{CP}^1$ distinct from b_{k+1} and a_k (in case a_k is defined). We claim that $u_k(p) = v_{k+1}(p)$. Otherwise $D(\Omega_{k+1}) = D(\Omega_k) \cup D(U_{k+1})$ would contain a complete vertical $\{p\} \times \mathbb{CP}^1$. This is of course not possible since we would have an embedding of $\mathbb{CP}^1$ in $D^{-1}(\Omega_{k+1}) \subset \widetilde{M}$ whose image would be in a leaf of $\mathscr{F}^u$ and we know that all leaves of $\mathscr{F}^u$ are copies of $\mathbb{C}$. This implies that the open set $D(\Omega_{k+1})$ has the required form.

Proof of Theorem A. The previous lemma shows that D is injective and its image is the complement of the graph of a continuous map $u : \mathbb{CP}^1 \to \mathbb{CP}^1$, or, of the union of a vertical $\{c\} \times \mathbb{CP}^1$ and of the graph of $u : \mathbb{CP}^1 - \{a\} \to \mathbb{CP}^1$. Reversing the roles of $\mathscr{F}^s$ and $\mathscr{F}^u$, one sees that only two possibilities can occur:

i) $D(\widetilde{M})$ is the complement of the graph of a homeomorphism $u : \mathbb{CP}^1 \to \mathbb{CP}^1$,

ii) $D(\widetilde{M})$ has the form $(\mathbb{CP}^1 - \{a\}) \times (\mathbb{CP}^1 - \{b\})$.

Let us first consider case i) and let us show that it is not possible (even though it will be important for the study of Anosov flows; see section 7). Since leaves of $\mathscr{F}^u$ and $\mathscr{F}^s$ are simply connected, a nontrivial element γ of $\pi_1(M)$ is such that $H(\gamma)$ does not fix a leaf of $\widetilde{\mathscr{F}}^s$ or $\widetilde{\mathscr{F}}^u$. This excludes case i) since a nontrivial element of $\mathrm{PSL}(2,\mathbb{C})$ has at least one fixed point in $\mathbb{CP}^1$ and, in case i), this would give rise to a fixed leaf.

In case ii), we can always assume that $a = b = \infty$ and the same argument shows that $H(\pi_1(M))$ should consist of affine maps fixing no leaf of $\mathscr{F}^s$ or $\mathscr{F}^u$, *i.e.*, of translations. Hence M is the quotient of $\mathbb{C}^2$ by a lattice; it is a complex torus. A lift of ϕ to $\widetilde{M}$ has to be an affine transformation of $\mathbb{C}^2$ preserving vertical and horizontal lines and normalizing the lattice. Theorem A is proved. □

6. Some examples of holomorphic Anosov flows and of nonuniformizable foliations

We first explain why we have chosen to define a holomorphic Anosov flow as being an action of $\mathbb{C}^*$. We could have considered holomorphic vector fields X on a compact complex manifold M generating a flow $\phi^\tau (\tau \in \mathbb{C})$ for which there is a $d\phi^\tau$ invariant splitting:

$$T_{\mathbb{C}}M = E^{ss} \oplus E^{uu} \oplus \mathbb{C}X$$

and constants $C > 0$, $\lambda > 0$ such that:

$$\|d\phi^\tau(v^s)\| \leqq C\exp(-\lambda\Re(\tau))\|v^s\|$$

$$\|d\phi^\tau(v^u)\| \geqq C\exp(\lambda\Re(\tau))\|v^u\|$$

for all $v^s \in E^{ss}$, $v^u \in E^{uu}$ and $\tau \in \mathbb{C}$ (where $\Re$ is the real part). However, this is not more general than $\mathbb{C}^*$-actions. Indeed, the real flow $\phi^{it}(t \in \mathbb{R})$ is obviously uniformly equicontinuous so that its closure in the diffeomorphism group of M is a torus T^l for some $l \geqq 1$ commuting with the flow ϕ^τ. The above hyperbolicity assumptions imply that a vector tangent ot the orbits of T^l has a zero component in $E^{ss} \oplus E^{uu}$ and it easily follows that $l = 1$. In other words, there is a real number $\bar{\omega}$ such that $\phi^{i\bar{\omega}} = id$. Normalizing, we can always assume that $\bar{\omega} = 2\pi$ so that $\phi^{2i\pi} = id$ and the flow ϕ^{τ} corresponds to a $\mathbb{C}^*$-action of $\exp(\tau) \in \mathbb{C}^*$.

Closed orbits of $\mathbb{C}^*$-actions have a richer structure than that in the real case; their "period" is an element ω in $\mathbb{C}^*$ and the closed orbit is the elliptic curve $\mathbb{C}^*/\{\omega^k\}$. In case of a holomorphic suspension, as described in the introduction, periods are precisely the powers of the complex number ω used in the construction.

We now describe with more details the examples mentioned in the introduction. Let Γ be a discrete cocompact subgroup in $\mathrm{SL}(2,\mathbb{C})$. Observe that, up to a $\mathbb{Z}/2\mathbb{Z}$-extension, $\mathrm{SL}(2,\mathbb{C})$ is the isometry group of the real hyperbolic 3-dimensional space $\mathbb{H}^3$ so that Γ is the fundamental group of a hyperbolic 3-dimensional orbifold. Many examples have nonvanishing first Betti number, *i.e.*, are such that there exist nontrivial homomorphisms $u : \Gamma \to \mathbb{C}^*$ (see [Th]).

If u is such a homomorphism, we have considered the right action of Γ on $\mathrm{SL}(2,\mathbb{C})$ defined by:

$$(x,\gamma) \in \mathrm{SL}(2,\mathbb{C}) \times \Gamma \mapsto x \bullet \gamma = \begin{pmatrix} u(\gamma) & 0 \\ 0 & u(\gamma)^{-1} \end{pmatrix} x\gamma .$$

If this action is free, proper and totally discontinuous, we denote by $\mathrm{SL}(2,\mathbb{C})/\!\!/_u\Gamma$ the quotient, and we say that u is admissible. We noted that there is a natural $\mathbb{C}^*$-action on this quotient, coming from left translations by matrices $\begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix}$.

Let H^+ and H^- be the right invariant holomorphic vector fields in $\mathrm{SL}(2,\mathbb{C})$ corresponding to the elements $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ and $\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ of the Lie algebra of $\mathrm{SL}(2,\mathbb{C})$ and denote by $\mathscr{H}^+$ and $\mathscr{H}^-$ the one-dimensional holomorphic foliations generated by H^+ and H^-. It is easy to check that the differential of the right action by γ in $\mathrm{SL}(2,\mathbb{C})$ maps H^+ and H^- to $u(\gamma)^2H^+$ and $u(\gamma)^{-2}H^-$ so that H^+ and H^- are not invariant (unless u^2 is trivial) but $\mathscr{H}^+$ and $\mathscr{H}^-$ are invariant. In other words, on the compact manifold $\mathrm{SL}(2,\mathbb{C})/\!\!/_u\Gamma$, we have two natural foliations $\mathscr{H}^+$ and $\mathscr{H}^-$ which are invariant under the $\mathbb{C}^*$-action. When u^2 is trivial, $\mathscr{H}^+$ and $\mathscr{H}^-$ are parametrized by vector fields H^+ and H^- which are expanded and contracted respectively by the action so that, at least in that case, the $\mathbb{C}^*$-action is a holomorphic Anosov flow.

In order to simplify our description of these examples, we shall assume that Γ is torsion-free (this can always be achieved by replacing Γ by a finite index subgroup by a theorem of Selberg). In particular, Γ injects into $\mathrm{PSL}(2,\mathbb{C}) = \mathrm{SL}(2,\mathbb{C})/\{\pm id\}$.

Note that if $\varepsilon : \Gamma \to \{\pm 1\}$ is a homomorphism, the map $\tau : \gamma \in \Gamma \mapsto \varepsilon(\gamma)\gamma \in \mathrm{SL}(2,\mathbb{C})$ is an injective homomorphism whose image is another discrete subgroup Γ' of $\mathrm{SL}(2,\mathbb{C})$. In such a situation, we shall write $\Gamma = \pm\Gamma'$. This happens precisely when Γ and Γ' have the same projection in $\mathrm{PSL}(2,\mathbb{C})$. Of course, $u : \Gamma \to \mathrm{SL}(2,\mathbb{C})$ is admissible if and only if $\varepsilon.u \circ \tau^{-1} : \Gamma' \to \mathbb{C}^*$ is admissible and the corresponding actions of $\mathbb{C}^*$ are conjugate.

Proposition 6.1. *Let Γ be a discrete torsion-free cocompact subgroup of* $\mathrm{SL}(2,\mathbb{C})$. *Then:*

i) *homomorphisms $u : \Gamma \to \mathbb{C}^*$ which are close enough to the trivial homomorphism are admissible,*

ii) *if $u : \Gamma \to \mathbb{C}^*$ is admissible, the corresponding $\mathbb{C}^*$-action on $\mathrm{SL}(2,\mathbb{C})/\!\!/_u\Gamma$ is a holomorphic Anosov flow.*

Let Γ_1 and Γ_2 be two discrete torsion-free cocompact subgroups of $\mathrm{SL}(2,\mathbb{C})$. *Then* $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_1}\Gamma_1$ *and* $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_2}\Gamma_2$ *are homeomorphic if and only if there is a continuous automorphism θ of* $\mathrm{SL}(2,\mathbb{C})$ *such that $\theta(\Gamma_1) = \pm\Gamma_2$. In such a case, there is a C^∞-diffeomorphism between* $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_1}\Gamma_1$ *and* $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_2}\Gamma_2$ *sending orbits of the first $\mathbb{C}^*$-action to orbits of the second (without necessarily commuting with the actions).*

Proof. Property i) is a complex version of the analogous property in the case of $\mathrm{SL}(2,\mathbb{R})$ and can be proven exactly in the same way (see [Gh2] or [Gol] for details).

Assume $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_1}\Gamma_1$ and $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_2}\Gamma_2$ are homeomorphic. Then Γ_1 and Γ_2 are isomorphic as abstract groups and it follows from Mostow's rigidity theorem that there is a continuous automorphism θ of $\mathrm{SL}(2,\mathbb{C})$ such that $\theta(\Gamma_1) = \pm\Gamma_2$. Note that, up to conjugacy, the only nontrivial continuous automorphism of $\mathrm{SL}(2,\mathbb{C})$ is given by $\theta(x) = \bar{x}$.

We now show that if $\Gamma_2 = \pm\theta(\Gamma_1)$ then $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_1}\Gamma_1$ and $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_2}\Gamma_2$ are diffeomorphic. We can of course assume that $\theta = id$, and that $\Gamma_1 = \Gamma_2 = \Gamma$. Let us consider first of all the quotients $M_i = \mathrm{U}(1)\backslash\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_i}\Gamma_i$ $(i = 1,2)$. These are manifolds since we assumed that Γ is torsion free. Note that if u_i is trivial, then $\mathrm{SL}(2,\mathbb{C})/\Gamma_i$ is the 2-fold (spin)-cover of the orthonormal frame bundle of the 3-manifold V which is the quotient of the hyperbolic 3-space by the action of Γ and M_i is the unit tangent bundle of V.

On M_i, we have a real one-parameter flow f_i^t coming from the complex one-parameter flow on $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_i}\Gamma$. Of course when u_i is trivial the flow f_i^t is nothing but the geodesic flow of V.

The quotient $\mathbb{C}^*\backslash\mathrm{SL}(2,\mathbb{C})$ of $\mathrm{SL}(2,\mathbb{C})$ by the diagonal subgroup is isomorphic to the complement of the diagonal in $\mathbb{CP}^1 \times \mathbb{CP}^1$. The universal cover $\widetilde{M}_i$ of M_i, naturally identified with $\mathrm{U}(1)\backslash\mathrm{SL}(2,\mathbb{C})$, fibres over the complement of the diagonal Δ in $\mathbb{CP}^1 \times \mathbb{CP}^1$:

$$D_i : \widetilde{M}_i \to \mathbb{CP}^1 \times \mathbb{CP}^1 - \Delta$$

and this fibration is equivariant under the diagonal embedding:

$$H : \gamma \in \Gamma \mapsto (\gamma,\gamma) \in \mathrm{PSL}(2,\mathbb{C}) \times \mathrm{PSL}(2,\mathbb{C}).$$

The fibres of D_i are the orbits of the lifted flow $\tilde{f}_i^t$. We therefore observe that both flows f_1^t and f_2^t have the same transverse structure, *i.e.*, equivalent holonomy pseudogroups. It follows from [Ha] (see also [Gr], [Ba]) that *there* *is a C^∞-diffeomorphism between M_1 and M_2 sending orbits of f_1^t to orbits of f_2^t and, in particular, that M_1 and M_2 are diffeomorphic.*

We claim that the circle fibrations $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_i}\Gamma \to M_i$ are trivial fibrations. This follows from the fact that orientable closed 3-manifolds are parallelizable and from the fact that the space of homomorphisms from Γ to $\mathbb{C}^*$ is connected. Indeed, choose a path u_t ($t \in [0,1]$) connecting the trivial homomorphism to u_1 and consider the right action of Γ on $\mathrm{SL}(2,\mathbb{C}) \times \mathbb{H}^3$ (where $\mathbb{H}^3$ is the hyperbolic 3-space) given by:

$$(x,p) \bullet_t \gamma = \left(\begin{pmatrix} u_t(\gamma) & 0 \\ 0 & u_t(\gamma)^{-1} \end{pmatrix} x\gamma, \gamma^{-1}(p) \right) \in \mathrm{SL}(2,\mathbb{C}) \times \mathbb{H}^3 .$$

The second factor has been introduced in such a way that the action is free, proper, and totally discontinuous for each $t \in [0,1]$. The quotient spaces are homotopy equivalent to $\mathrm{SL}(2,\mathbb{C})/\Gamma$ and $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_1}\Gamma$ for $t=0$ and $t=1$. Moreover, for each t, the right-action of Γ commutes with left translations by $\mathrm{U}(1)$ so that each quotient space is the total space of circle bundle. Since we noticed that this circle bundle is trivial for $t=0$, we deduce that it is also trivial for $t=1$. Hence the circle bundles $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_i}\Gamma \to M_i$ are trivial and the diffeomorphism between M_1 and M_2 sending orbits of f_1^t to orbits of f_2^t can be lifted to a diffeomorphism between $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_1}\Gamma$ and $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_2}\Gamma$ sending orbits of the first $\mathbb{C}^*$-action to orbits of the second one.

It is well known that the Anosov property is invariant under time reparametrization, *i.e.*, if there is a C^∞-diffeomorphism sending orbits of a (real) flow f_1^t to orbits of another flow f_2^t and if f_1^t is Anosov, so is f_2^t (see, for instance [Gh4]). The same fact applies (with the same proof) in the holomorphic case. Since we have already observed that when $u : \Gamma \to \mathbb{C}^*$ is trivial the $\mathbb{C}^*$-action on $\mathrm{SL}(2,\mathbb{C})/\Gamma$ is Anosov, it follows that $\mathbb{C}^*$-actions on $\mathrm{SL}(2,\mathbb{C})/\!\!/_u\Gamma$ are Anosov for admissible u. This completes the proof of proposition 6.1. □

Proposition 6.2. *If $u : \Gamma \to \mathbb{C}^*$ is an admissible homomorphism such that u^2 is non trivial, then the space of holomorphic vector fields on $\mathrm{SL}(2,\mathbb{C})/\!\!/_u\Gamma$ has complex dimension 1 and is generated by the vector field corresponding to the $\mathbb{C}^*$-action.*

Proof. We have already noticed that there are two holomorphic one dimensional foliations $\mathscr{H}^+$ and $\mathscr{H}^-$ on $V = \mathrm{SL}(2,\mathbb{C})/\!\!/_u\Gamma$ which are invariant under the $\mathbb{C}^*$-action and which provide, together with the tangent bundle to the orbits of $\mathbb{C}^*$, a splitting of $T_{\mathbb{C}}V$ as a sum of three line-bundles. In order to show the proposition, it is enough to show that there is no nonzero holomorphic vector field in V tangent to $\mathscr{H}^+$ (or to $\mathscr{H}^-$) if u^2 is nontrivial. Assume there is such a vector field ξ. Using the fact that the $\mathbb{C}^*$-action preserves $\mathscr{H}^+$ and that the space of holomorphic vector fields is finite dimensional, one can choose ξ such that the $\mathbb{C}^*$- action $\phi(T)$ ($T \in \mathbb{C}^*$) satisfies, for some $k \in \mathbb{Z}$:

$$d\phi(T)(\xi) = T^k\xi \text{ for all } T \in \mathbb{C}^* .$$

If one lifts ξ to $\mathrm{SL}(2,\mathbb{C})$, one gets a vector field $\tilde{\xi}$ which is of the form $f \cdot H^+$ where f is holomorphic on $\mathrm{SL}(2,\mathbb{C})$. Taking into account the invariance of $\tilde{\xi}$ under the action of Γ and the non-invariance of H^+ already observed, we get:

$$(1) \qquad f(x \bullet \gamma) = u(\gamma)^{-2} f(x) \text{ for } \gamma \in \Gamma \text{ and } x \in \mathrm{SL}(2,\mathbb{C}).$$

Moreover, we have:

$$(2) \quad f\left(\begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix} \cdot x\right) = T^k f(x) \text{ for all } T \in \mathbb{C}^* \text{ and } x \in \mathrm{SL}(2,\mathbb{C})$$

Assume first that $k = 0$ so that f actually defines a function $\bar{f}$ on

$$\mathbb{C}^* \backslash \mathrm{SL}(2,\mathbb{C}) \cong \mathbb{CP}^1 \times \mathbb{CP}^1 - \Delta .$$

Then, by (1), $\bar{f}$ is invariant under the action of the first commutator group Γ' of Γ (on which u is obviously trivial). Now this action of Γ' on $\mathbb{CP}^1 \times \mathbb{CP}^1$ is topologically transitive. This is equivalent to the fact that the geodesic flow of the homology cover of a compact hyperbolic manifold is topologically transitive. Indeed all non trivial normal subgroups of a discrete group of isometries of a hyperbolic space have the same limit set and all non elementary groups act topologically transitively on the square of their limit set (see [Th] and [G-H] page 123). Therefore f is constant—but this is impossible if u^2 and f are not trivial.

Now, assume that $k \neq 0$. Consider the function $f : V' = \mathrm{SL}(2,\mathbb{C})/\Gamma' \to \mathbb{C}$. According to (2), f has to vanish on periodic orbits of the $\mathbb{C}^*$-action on V'. But, on any compact hyperbolic manifold the union of closed geodesics homologous to zero is dense (as follows also from [G-H]). This shows that the union of closed orbits of the $\mathbb{C}^*$-action on V' is dense in V'. It follows that f is zero. □

Corollary 6.3. *Let Γ be a discrete torsion free cocompact subgroup of* $\mathrm{SL}(2,\mathbb{C})$ *and $u_1, u_2 : \Gamma \to C^*$ be two admissible homomorphisms. Then the compact complex manifolds* $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_i}\Gamma$ *($i = 1,2$) are holomorphically diffeomorphic only if there is an automorphism θ of Γ such that $u_2^{\pm 1} = u_1 \circ \theta$.*

Proof. If u_1^2 is trivial, then $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_1}\Gamma$ is a homogeneous space of $\mathrm{SL}(2,\mathbb{C})$ and therefore admits three linearly independent holomorphic vector fields. According to 6.2, on deduces that u_2^2 is also trivial if $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_2}\Gamma$ is holomorphically diffeomorphic to $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_1}\Gamma$. The corresponding complex manifolds are therefore of the form $\mathrm{SL}(2,\mathbb{C})/\Gamma_i$ ($i = 1,2$) and $\Gamma_1 = \pm\Gamma_2$. Any holomorphic diffeomorphism between these two homogeneous spaces induces an isomorphism between the Lie algebras of holomorphic vector fields which are themselves isomorphic to the Lie algebra of $\mathrm{SL}(2,\mathbb{C})$. The corollary follows in this special case.

Now, assume that u_1^2 and u_2^2 are nontrivial and that there is a holomorphic diffeomorphism F between the corresponding compact complex manifolds.

Proposition 6.2 implies that F conjugates the $\mathbb{C}^*$-actions or one with the inverse of the other. Let γ be a nontrivial element of Γ and denote by $\lambda(\gamma)$, $\lambda(\gamma)^{-1}$ its two eigenvalues. The $\mathbb{C}^*$-action on $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_i}\Gamma$ contains precisely two closed orbits containing a loop freely homotopic to $\gamma^{\pm 1}$, whose "periods" are $\lambda(\gamma)u_i(\gamma)$ and $\lambda^{-1}(\gamma)u_i(\gamma)$. Note that periods of closed orbits related under F should be equal or inverse. If θ denotes the automorphism of Γ (defined up to conjugacy) induced by F, it follows that either $u_2 = u_1 \circ \theta$ or $u_2^{-1} = u_1 \circ \theta$. □

Corollary 6.4. *Let Γ be a discrete torsion free cocompact subgroup of $\mathrm{SL}(2,\mathbb{C})$ and $u_1, u_2 : \Gamma \to \mathbb{C}^*$ be two admissible homomorphisms. Then the $\mathbb{C}^*$-actions on $\mathrm{SL}(2,\mathbb{C})/\!\!/_{u_i}\Gamma$ are conjugate by a homeomorphism only if there is an automorphism θ of Γ such that $u_2 = u_1 \circ \theta$.*

Proof. This is the same proof as that of Corollary 6.3 since we only used preservation of periods of closed orbits. □

It would not be difficult to give a sufficient condition in 6.3. and 6.4.

We can now prove theorem D.

Theorem 6.5. *Let Γ be a discrete torsion-free cocompact subgroup of $\mathrm{SL}(2,\mathbb{C})$ and $u : \Gamma \to \mathbb{C}^*$ be an admissible homomorphism such that $|u| : \Gamma \to \mathbb{R}_+^*$ is nontrivial. Then the holomorphic one-dimensional foliation $\mathcal{H}^+$ on $\mathrm{SL}(2,\mathbb{C})/\!\!/_u\Gamma$ has the following properties:*

i) *all leaves of $\mathcal{H}^+$ are dense with polynomial growth,*

ii) *every leaf of $\mathcal{H}^+$ is conformally equivalent to $\mathbb{C}$,*

iii) *there is no hermitian metric on the tangent bundle of $\mathcal{H}^+$, continuous on the manifold and smooth along the leaves, for which all the leaves are flat.*

Proof. Property i) follows from the fact that the holomorphic Anosov flow is not a suspension (the proof of the density of stable leaves of transitive Anosov flows which are not suspension, as given in [Pl1], immediately generalizes to the complex case).

Property ii) also follows from the Anosov property. Indeed, by compactness of the ambient manifold, one can find $\varepsilon > 0$ in such a way that for every point x, there is a holomorphic map f_x from the disc of center 0 and radius ε in $\mathbb{C}$ to the leaf of $\mathcal{H}^+$ through x sending 0 to x and such that the norm of the derivative of f_x at 0 is 1. Using the $\mathbb{C}^*$-action $\phi(T)$, we get holomorphic maps $\phi(T) \circ f_{\phi(T^{-1}(x))}$ from a disc of radius ε to the leaf of $\mathcal{H}^+$ through x with an arbitrarily large derivative at 0. This implies that the leaf of $\mathcal{H}^+$ through x is parabolic, *i.e.*, conformally equivalent to $\mathbb{C}$.

We now prove the crucial property iii). Assume by contradiction that there is such a hermitian metric g and let $\tilde{g}$ be its lift to $\mathrm{SL}(2,\mathbb{C})$. Consider the function $\rho : \mathrm{SL}(2,\mathbb{C}) \to \mathbb{R}_+^*$ which is the $\tilde{g}$-length of the vector field H^+. By invariance of $\tilde{g}$ and non-invariance of H^+ we get for every γ in Γ and x in $\mathrm{SL}(2,\mathbb{C})$:

$$\rho(x \bullet \gamma) = \rho(x)|u(\gamma)|^2 .$$

Now, consider, an orbit of H^+ in $\mathrm{SL}(2,\mathbb{C})$, *i.e.*, $\left\{\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\cdot x\right\}$ parametrized by $s \in \mathbb{C}$. It is equipped with the complete flat metric $\rho\left(\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\cdot x\right)|ds|^2$ (completeness follows from the fact that it is isometric to a leaf of a compact manifold). Hence ρ is constant on orbits of H^+ and defines a function $\bar\rho$ on the left quotient of $\mathrm{SL}(2,\mathbb{C})$ by $\left\{\begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix}\right\}$, naturally identified to $\mathbb{C}^2 - \{(0,0)\}$ (via the natural linear action of $\mathrm{SL}(2,\mathbb{C})$). This continuous function $\bar\rho : \mathbb{C}^2 - \{(0,0)\} \to \mathbb{R}_+^*$ satisfies:

$$\bar\rho(i(\gamma)(p)) = |u(\gamma)|^2(\bar\rho(p)) \tag{3}$$

for every γ in Γ and p in $\mathbb{C}^2 - \{(0,0)\}$ where $i(\gamma) = \mu(\gamma^{-1})\cdot {}^t\gamma \in \mathrm{GL}(2,\mathbb{C})$. Fix $\omega \in \mathbb{C}^*$ and consider the following function on $\mathbb{C}^2 - \{(0,0)\}$:

$$p \in \mathbb{C}^2 - \{(0,0)\} \mapsto F_\omega(p) = \frac{\bar\rho(\omega\cdot p)}{\bar\rho(p)}$$

This function F_ω is invariant under the action of $i(\Gamma)$ on $\mathbb{C}^2 - \{(0,0)\}$ which has dense orbits (this is a way of expressing minimality of the horocyclic foliation $\mathscr{H}^+$). So, there is a constant c_ω such that $F_\omega(p) = c_\omega$. Of course, $c_{\omega_1\omega_2} = c_{\omega_1}c_{\omega_2}$ so that there is an $\alpha \in \mathbb{R}$ such that for every $\omega \in \mathbb{C}^*$ and $p \in \mathbb{C}^2 - \{(0,0)\}$ one has:

$$\bar\rho(\omega\cdot p) = |\omega|^\alpha \bar\rho(p).$$

Assume that $\alpha \neq 0$. Then the equation $\bar\rho \leqq 1$ would define a domain in $\mathbb{C}^2 - \{(0,0)\}$ bounded by a topological sphere transversal to real straight lines going through the origin. This domain should be invariant by elements γ in the first commutator group Γ' of Γ. This is impossible since a non trivial element of Γ' is a hyperbolic matrix in $\mathrm{SL}(2,\mathbb{C})$ and its action on $\mathbb{C}^2$ cannot preserve a topological sphere.

Now, if $\alpha = 0$, the function $\bar\rho$ is constant on rays and, being continuous, is bounded. This is obviously in contradiction with (3) since we assumed that $|u|$ is a nontrivial homomorphism in $\mathbb{R}_+^*$. This contradiction shows that there is no flat hermitian metric on $\mathscr{H}^+$. □

It is probable that theorem 6.4 can be strengthened in the following way: there should not exist a measurable Riemannian metric on the tangent bundle of $\mathscr{H}^+$, smooth along the leaves, such that almost all leaves are flat.

7. Classification of holomorphic Anosov flows on compact complex 3-manifolds

In this section, we prove theorem C. Let $\phi(T)$ be a holomorphic Anosov flow on a compact complex 3-dimensional manifold M and let X be the corresponding holomorphic vector field whose flow is $\phi(\exp(s))$, ($s \in \mathbb{C}$). By definition, there is a splitting:

$$T_{\mathbb{C}}M = E^{ss} \oplus E^{uu} \oplus \mathbb{C} \cdot X$$

as a sum of three line bundles. Since the action of $\phi(T)$ is hyperbolic in the normal direction to the (complex) one-dimensional orbits, it follows, as in the usual real case, that $E^s = E^{ss} \oplus \mathbb{C}X$ and $E^u = E^{uu} \oplus \mathbb{C}X$ are integrable and generate foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ called the *central stable and unstable* foliations respectively.

Note that the action of the circle group $S^1 \subset \mathbb{C}^*$ is locally free on M so that the quotient of M by this action is a 5-dimensional real orbifold M' equipped with a real flow f^t coming from the action of $\mathbb{C}^*$ on M. This real flow f^t is Anosov (if one generalizes appropriately the classical definition to orbifolds, which is not a problem). As a matter of fact, most of what follows could be stated for Anosov flows on 5-manifolds generating a transversely holomorphic 1-dimensional foliation.

Many results from sections 2 to 5 generalize to flows:

Proposition 7.1. *The central foliations $\mathcal{F}^s$, $\mathcal{F}^u$ are transversely projective. More precisely, there is a holomorphic fibration D from the universal covering space $\widetilde{M}$ of M to some open set in $\mathbb{CP}^1 \times \mathbb{CP}^1$ and a homomorphism H from the fundamental group $\pi_1(M)$ of M to* $\mathrm{PSL}(2,\mathbb{C}) \times \mathrm{PSL}(2,\mathbb{C})$ *such that:*

i) *the lifts of $\mathcal{F}^s$ and $\mathcal{F}^u$ to $\widetilde{M}$ are the inverse images of horizontal and vertical fibrations of $\mathbb{CP}^1 \times \mathbb{CP}^1$ by D respectively. The fibres of D are lifted orbits of the Anosov flow,*

ii) *for $\widetilde{m} \in \widetilde{M}$ and $\gamma \in \pi_1(M)$, one has $D(\gamma \cdot \widetilde{m}) = H(\gamma)D(\widetilde{m})$ where, of course, $H(\gamma)D(\widetilde{m})$ denotes the action of the pair $H(\gamma)$ of elements of* $\mathrm{PSL}(2,\mathbb{C})$ *on the pair $D(\widetilde{m})$ of points of $\mathbb{CP}^1$.*

Proof. The existence of a transverse invariant projective structure for $\mathcal{F}^s$ and $\mathcal{F}^u$ is obtained exactly in the same way as in sections 2 through 4: one simply constructs affine structures on strongly stable and unstable leaves and uses a Markov partition for the associated Anosov flow f^t. So, we get the existence of a global holomorphic submersion D from $\widetilde{M}$ to some open set in $\mathbb{CP}^1 \times \mathbb{CP}^1$ and a holonomy homomorphism $H : \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C}) \times \mathrm{PSL}(2,\mathbb{C})$.

The method of proof of theorem A generalizes in this case and shows that D is a locally trivial fibration over its image. Moreover the image of D is either:

I) $(\mathbb{CP}^1 - \{a\}) \times (\mathbb{CP}^1 - \{b\})$

II) the complement of the graph of some homeomorphism of $\mathbb{CP}^1$.

In case I, one can of course assume that a and b are points at infinity so that the image of D is $\mathbb{C} \times \mathbb{C}$. In this case, the holonomy H has its image contained in $\mathrm{Aff} \times \mathrm{Aff}$ where Aff is the complex affine group of $\mathbb{C}$. This will eventually correspond to the suspension case.

Let H_1 and H_2 denote the two components of H. We will show that in case II we can assume that $H_1 = H_2$ and that the image of $H_1 : \pi_1(M) \to \mathrm{PSL}(2,\mathbb{C})$ is a discrete cocompact subgroup Γ. Indeed, let $\gamma \in \pi_1(M)$ and let

$\lambda(H_1(\gamma))$, $(\lambda(H_1(\gamma)))^{-1}$ be the two eigenvalues of $H_1(\gamma)$ (defined up to ± 1). If $\lambda(H_1(\gamma)) \neq \pm 1$, one produces a closed orbit of the associated Anosov flow f^t for which the eigenvalue of the corresponding Poincaré map on the stable manifold is $\lambda(H_1(\gamma))^{\pm 2}$. From the obvious fact that an Anosov flow has a finite number of periodic orbits of period less than a given bound, one deduces that, for every $K > 0$, there is a finite number of conjugacy classes of elements γ in $\pi_1(M)$ such that $\frac{1}{K} \leqq |\lambda(H_1(\gamma))| \leqq K$ and $\lambda(H_1(\gamma)) \neq \pm 1$. This easily implies that the identity component of the closure of $H_1(\pi_1(M))$ in $\mathrm{PSL}(2, \mathbb{C})$ contains only parabolic elements. It follows that $H(\pi_1(M))$ is discrete unless it is conjugate to a subgroup of Aff. The latter case is impossible since the inverse image by D of $\{\infty\} \times \mathbb{CP}^1$ (which is nonempty in case II) would be a leaf of the lift of $\mathscr{F}^u$ to $\widetilde{M}$, invariant by all $\pi_1(M)$, and would therefore produce a closed leaf for $\mathscr{F}^u$ in M. Hence $H_1(\pi_1(M))$ is discrete in $\mathrm{PSL}(2, \mathbb{C})$ and, of course, the same arguments show that $H_2(\pi_1(M))$ is also discrete.

If these discrete subgroups of $\mathrm{PSL}(2, \mathbb{C})$ were not cocompact, they would be of virtual cohomological dimension less than or equal to 2, *i.e.*, they would contain finite index subgroups of cohomological dimension less than or equal to 2. This follows from the fact that the quotient of hyperbolic 3-space by a torsion-free not cocompact discrete subgroup of $\mathrm{PSL}(2, \mathbb{C})$ retracts to its 2-skeleton and it is an Eilenberg-MacLane space. Consider now the 5-dimensional real orbifold M' that we described above. We know that its universal covering space $\widetilde{M}'$ is a trivial $\mathbb{R}_+$ fibration over the complement of the graph of some homeomorphism of $\mathbb{CP}^1$ so that it has the homotopy type of the 2-sphere. This is a contradiction since a group of virtual cohomological dimension less than or equal to 2 cannot act cocompactly on a 5-manifold homotopy equivalent to a 2-sphere (by an elementary spectral sequence argument). Hence $H_1(\pi_1(M))$ and $H_2(\pi_1(M))$ are both discrete cocompact subgroups of $\mathrm{PSL}(2, \mathbb{C})$.

Let us consider the homeomorphism h of $\mathbb{CP}^1$ whose graph is not in the image of D. Clearly h conjugates H_1 and H_2 so that it follows from Mostow's rigidity theorem that h is actually an element of $\mathrm{PSL}(2, \mathbb{C})$. *Finally, conjugating D by* id $\times$ *h, one sees that $H_1 = H_2$ and that the image $H_1(\pi_1(M)) = H_2(\pi_1(M)) = \Gamma$ is a discrete cocompact subgroup of* $\mathrm{PSL}(2, \mathbb{C})$.

We shall now prove theorem C in case I, where $D(\widetilde{M}) = \mathbb{C} \times \mathbb{C}$.

Proposition 7.2. *In case I, up to a finite cover, the holomorphic Anosov flow is holomorphically conjugate to a holomorphic suspension of some linear automorphism of a torus $\mathbb{C}^2/\Lambda$.*

Proof. Passing to a finite cover, we can assume that the holonomy group $\Gamma = H(\pi_1(M))$ is torsion-free. It follows that the S^1-action is a free action so that the associated real Anosov flow f^t is a flow on a manifold M' (not only an orbifold). According to [Pl2], any Anosov flow admitting a transverse affine structure has a global transversal section (note however that in our special case it is much easier to prove this than to use the full strength of [Pl2]). This global section inherits a complex affine structure and the first return map is Anosov.

So, it is a complex torus $\mathbb{C}^2/\Lambda$ and the first return map is a $\mathbb{C}$-linear mapping $A \in \mathrm{GL}(2,\mathbb{C})$.

In other words, there is a cyclic covering $\widehat{M}$ of M which is a $\mathbb{C}^*$-fibration over $\mathbb{C}^2/\Lambda$ and the Galois group of this fibration is generated by an automorphism inducing A on $\mathbb{C}^2/\Lambda$. Recall that $\mathbb{C}^*$-fibrations over tori $\mathbb{C}^2/\Lambda$ are topologically classified by their first Chern class in $H^2(\mathbb{C}^2/\Lambda,\mathbb{Z})$, which is described by the imaginary part of a hermitian form on $\mathbb{C}^2$, taking integral values on Λ. Note that the hyperbolic matrix A does not preserve any nontrivial hermitian form on $\mathbb{C}^2$ so that the $\mathbb{C}^*$-fibration $\widehat{M} \to \mathbb{C}^2/\Lambda$ is topologically trivial. These topologically trivial $\mathbb{C}^*$-fibrations are characterized by a homomorphism $u : \Lambda \to \mathbb{C}^*$ which should be invariant by A. Consider $\log|u|$ and note that A being hyperbolic, no nontrivial homomorphism from Λ to $\mathbb{R}$ is invariant under A. It follows that u takes its values in S^1, *i.e.*, defines an element of the dual torus—quotient of the dual of $\mathbb{C}^2$ by the dual lattice. Once again the hyperbolicity of A implies that its transpose has finitely many fixed points on the dual torus, *i.e.*, that u is trivial on a finite index sublattice of Λ. Hence, passing once again to a finite cover of M, one can always assume that the $\mathbb{C}^*$-fibration $\widehat{M} \to \mathbb{C}^2/\Lambda$ is holomorphically trivial, *i.e.*, isomorphic to $\mathbb{C}^2/\Lambda \times \mathbb{C}^*$.

The generator of the Galois group should act by

$$(x,T) \in \mathbb{C}^2/\Lambda \times \mathbb{C}^* \mapsto (A(x),\omega(x)\cdot T) \in \mathbb{C}^2/\Lambda \times \mathbb{C}^*$$

where $\omega : \mathbb{C}^2/\Lambda \to \mathbb{C}^*$ is holomorphic, *hence does not depend on x*. We recover the definition of a holomorphic suspension. $\square$

We now study holomorphic Anosov flows of type II for which the image of the developing map D is the complement of the diagonal Δ in $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Recall that $\mathbb{CP}^1 \times \mathbb{CP}^1 - \Delta$ is a Stein manifold and can be identified with the affine quadric $Q = \{(a,b,c) \in \mathbb{C}^3 \mid b^2 - 4ac = 1\}$ via the map:

$$(\alpha,\beta) \in \mathbb{CP}^1 \times \mathbb{CP}^1 - \Delta \mapsto \left(\frac{1}{\alpha-\beta},\frac{\alpha+\beta}{\alpha-\beta},\frac{\alpha\beta}{\alpha-\beta}\right) \in Q.$$

This identification is such that the action of $\mathrm{PSL}(2,\mathbb{C})$ on $\mathbb{CP}^1 \times \mathbb{CP}^1 - \Delta$ corresponds to the restriction to Q of the natural linear irreducible representation of $\mathrm{PSL}(2,\mathbb{C})$ on $\mathbb{C}^3$, considered as the space of homogeneous polynomials of degree 2, $aX^2 + bXY + cY^2$.

Let $\mathscr{H}$ be the space of holomorphic functions on $\mathbb{CP}^1\times\mathbb{CP}^1-\Delta$, isomorphic to the space of holomorphic functions on $\mathbb{C}^3$ modulo the ideal generated by $b^2 - 4ac - 1$. One can find irreducible finite dimensional subspaces $\mathscr{H}_k \subset \mathscr{H}$, ($k \geqq 0$) invariant under the action of $\mathrm{PSL}(2,\mathbb{C})$ such that:

i) $\mathscr{H}_0 = \mathbb{C}$ consists of constant functions,

ii) $\bigoplus_{k=0}^{\infty}\mathscr{H}_k$ consists of polynomial functions on $Q \subset \mathbb{C}^3$ and is therefore dense in $\mathscr{H}$ (in the topology of uniform convergence on compact sets in Q).

Note also that the embeddings $\mathscr{H}_k \hookrightarrow \mathscr{H}$ split in a $\mathrm{PSL}(2,\mathbb{C})$-equivariant way. Indeed, one can project the space of holomorphic functions on $\mathbb{C}^3$ to the space of polynomials by considering finite jets at the origin; this is $\mathrm{PSL}(2,\mathbb{C})$-equivariant and it suffices then to use the complete reducibility of finite dimensional representations of $\mathrm{PSL}(2,\mathbb{C})$.

Let $\ell \in \mathbb{Z} - \{0\}$ and denote by E_ℓ the quotient $(\mathbb{Z}/\ell\mathbb{Z})\backslash \mathrm{SL}(2,\mathbb{C})$ where $\mathbb{Z}/\ell\mathbb{Z}$ consists of diagonal matrices of order $|\ell|$. On E_ℓ, one has a natural free $\mathbb{C}^*$-action:

$$(\omega, x) \in \mathbb{C}^* \times E_\ell \mapsto \omega \cdot x = \begin{pmatrix} \omega^{\frac{\varepsilon}{\ell}} & 0 \\ 0 & \omega^{-\frac{\varepsilon}{\ell}} \end{pmatrix} x \in E_\ell$$

where $\varepsilon = \mathrm{sign}\ (\ell)$ and the right hand side does not depend on the choice of the ℓ^{th} root. Of course, the orbit space is the same as the quotient of $\mathrm{SL}(2,\mathbb{C})$ by the left action by diagonal matrices, *i.e.*, $\mathbb{CP}^1 \times \mathbb{CP}^1 - \Delta$.

We shall denote by E_0 the product $(\mathbb{CP}^1 \times \mathbb{CP}^1 - \Delta) \times \mathbb{C}^*$ considered as a trivial $\mathbb{C}^*$-fibration. Note that on each E_ℓ, there is a natural action of $\mathrm{SL}(2,\mathbb{C})$ commuting with the $\mathbb{C}^*$-action.

Lemma 7.3. *Any $\mathbb{C}^*$-fibration over $\mathbb{CP}^1 \times \mathbb{CP}^1 - \Delta$ is isomorphic to E_ℓ for some $\ell \in \mathbb{Z}$.*

Proof. Since $\mathbb{CP}^1 \times \mathbb{CP}^1 - \Delta$ is a Stein manifold, any $\mathbb{C}^*$-fibration is characterized by its first Chern class in $H^2(\mathbb{CP}^1 \times \mathbb{CP}^1 - \Delta, \mathbb{Z}) \cong \mathbb{Z}$. The lemma follows from the fact that the first Chern class of E_ℓ is precisely ℓ. □

After these preliminaries, we consider a holomorphic Anosov flow of type II on M. We assume that the $\mathbb{C}^*$-action is effective and we consider the holonomy cover $\widehat{M}$ of M, with Galois group $\Gamma \subset \mathrm{PSL}(2,\mathbb{C})$ and the fibration:

$$D : \widehat{M} \to \mathbb{CP}^1 \times \mathbb{CP}^1 - \Delta\,.$$

The $\mathbb{C}^*$-action on M lifts to a free action on $\widehat{M}$. Hence by Lemma 7.3, there is an $\ell \in \mathbb{Z}$ such that the two $\mathbb{C}^*$-fibrations E_ℓ and $\widehat{M}$ are isomorphic. Choose such an isomorphism. By going to a finite covering space, we can assume that Γ is torsion-free so that the embedding of Γ in $\mathrm{PSL}(2,\mathbb{C})$ lifts to $\mathrm{SL}(2,\mathbb{C})$ (this follows from the fact that orientable closed 3-manifolds have a trivial second Stiefel-Whitney class [Mi]). Choose such a lift and consider Γ as a subgroup of $\mathrm{SL}(2,\mathbb{C})$.

We know that Γ acts in two ways on $E_\ell \cong \widehat{M}$. The first action comes from the action of $\mathrm{SL}(2,\mathbb{C})$ on E_ℓ and the second from the action of Γ on $\widehat{M}$. Both actions commute with the $\mathbb{C}^*$-action so that, comparing the two actions of Γ, we get for each $\gamma \in \Gamma$ a holomorphic map

$$f_\gamma : \mathbb{CP}^1 \times \mathbb{CP}^1 - \Delta \to \mathbb{C}^*\,.$$

Note that these maps satisfy a cocycle condition, *i.e.*, for $\gamma_1, \gamma_2 \in \Gamma$ and $x \in \mathbb{CP}^1 \times \mathbb{CP}^1 - \Delta$, one has:

$$f_{\gamma_1\gamma_2}(x) = f_{\gamma_1}(\gamma_2 \cdot x) f_{\gamma_2}(x)\,.$$

If one denotes by $\mathscr{H}^*$ the multiplicative group of holomorphic functions from $\mathbb{CP}^1 \times \mathbb{CP}^1 - \Delta$ to $\mathbb{C}^*$ on which Γ acts naturally we obtain an element of $H^1(\Gamma, \mathscr{H}^*)$. Clearly, changing the identification between $\widehat{M}$ and E_ℓ amounts to the choice of some element f in $\mathscr{H}^*$ and f_γ changes accordingly by the coboundary of f.

Lemma 7.4. *Any* 1*-cocycle f_γ of Γ with values in $\mathscr{H}^*$ is cohomologous to a* 1*-cocycle with values in the subgroup $\mathbb{C}^* \subset \mathscr{H}^*$ of constant functions.*

Before we prove the lemma, let us show how it settles theorem C easily.

We first show that the case $\ell = 0$ cannot occur. A 1-cocycle of Γ with values in $\mathbb{C}^*$ is nothing but a homomorphism $u : \Gamma \to \mathbb{C}^*$ and, in particular, vanishes on the first commutator subgroup $\Gamma' \subset \Gamma$. Therefore, if $\ell = 0$, there would be an identification between $\widehat{M}$ and $(\mathbb{CP}^1 \times \mathbb{CP}^1 - \Delta) \times \mathbb{C}^*$ such that Γ' acts trivially on the second factor. If Γ is a discrete cocompact subgroup of $\mathrm{SL}(2, \mathbb{C})$, the action of Γ' on $\mathbb{CP}^1 \times \mathbb{CP}^1 - \Delta$ is not free. Since the action of Γ on $\widehat{M}$ is of course free this shows that ℓ cannot be zero.

Therefore, Lemma 7.4 shows that there is an identification between $\widehat{M}$ and $E_\ell \cong (\mathbb{Z}/\ell\mathbb{Z})\backslash \mathrm{SL}(2, \mathbb{C})$ as $\mathbb{C}^*$-fibrations and a homomorphism $u : \Gamma \to \mathbb{C}^*$ such that Γ acts on $\widehat{M}$ in the following way:

$$(x, \gamma) \in \widehat{M} \times \Gamma \mapsto \begin{pmatrix} u(\gamma)^{\frac{1}{\ell}} & 0 \\ 0 & u(\gamma)^{-\frac{1}{\ell}} \end{pmatrix} x\gamma.$$

This is precisely the description of the examples of flows given in the introduction. This proves theorem C. $\square$

We now prove Lemma 7.4. We begin by two elementary remarks.

The first is that it suffices to show that the embedding $\mathbb{C} \hookrightarrow \mathscr{H}$ induces an isomorphism $H^1(\Gamma, \mathbb{C}) \to H^1(\Gamma, \mathscr{H})$. Indeed, noting that $\mathbb{CP}^1 \times \mathbb{CP}^1 - \Delta$ is simply connected, the exact sequences:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathbb{C} & \longrightarrow & \mathbb{C}^* & \longrightarrow & 1 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \mathbb{Z} & \longrightarrow & \mathscr{H} & \longrightarrow & \mathscr{H}^* & \longrightarrow & 1 \end{array}$$

give:

$$\begin{array}{ccccccc} H^1(\Gamma, \mathbb{C}) & \longrightarrow & H^1(\Gamma, \mathbb{C}^*) & \longrightarrow & H^2(\Gamma, \mathbb{Z}) & \longrightarrow & H^2(\Gamma, \mathbb{C}) \\ \downarrow & & \downarrow & & \downarrow{\cong} & & \downarrow \\ H^1(\Gamma, \mathscr{H}) & \longrightarrow & H^1(\Gamma, \mathscr{H}^*) & \longrightarrow & H^2(\Gamma, \mathbb{Z}) & \longrightarrow & H^2(\Gamma, \mathscr{H}). \end{array}$$

The last vertical map is injective since the injection $\mathbb{C} \to \mathscr{H}$ splits as a map of Γ-modules. So, by a classical diagram chasing argument, we can deduce the surjectivity of the second vertical map from the surjectivity of the first.

The second preliminary remark is that $H^1(\Gamma, \mathscr{H})$ is finite dimensional. Indeed, let us consider the sheaf Ω on M of germs of holomorphic functions, constant on the orbits of the $\mathbb{C}^*$-action, and let $\widehat{\Omega}$ be its inverse image in $\widehat{M}$. Since $\widehat{\Omega}$ is also the inverse image by D of the sheaf of germs of holomorphic functions on the Stein manifold $\mathbb{CP}^1 \times \mathbb{CP}^1 - \Delta$ we have $H^1(\widehat{M}, \widehat{\Omega}) = 0$. It easily follows (see [Mu, p. 22]) that the natural map:

$$H^1(\Gamma, H^0(\widehat{M}, \widehat{\Omega})) \to H^1(M, \Omega)$$

is an isomorphism. Of course, the left hand side is precisely $H^1(\Gamma, \mathscr{H})$. Consider the sheaf $\mathcal{O}$ of germs of holomorphic functions on M. The Lie derivative along the $\mathbb{C}^*$-action gives an exact sequence of sheaves:

$$0 \to \Omega \to \mathcal{O} \to \mathcal{O} \to 0,$$

so that the finite dimensionality of $H^1(M, \Omega)$ follows from that of $H^1(M, \mathcal{O})$.

Recall the following theorem of Raghunathan [Ra].

Theorem. *Let Γ be a discrete cocompact subgroup of a simple noncompact Lie group G and let $\rho : G \to \mathrm{GL}(N, \mathbb{R})$ be a nontrivial irreducible representation. Then, the cohomology group $H^1(\Gamma, \mathbb{R}^N)$ of the Γ-module $\mathbb{R}^N$ (defined by ρ) vanishes unless G is locally isomorphic to $\mathrm{SO}(n, 1)$ or $\mathrm{SU}(n, 1)$ and the highest weight of ρ is a multiple of the highest weight of the canonical representation.*

In our case, $\mathrm{PSL}(2, \mathbb{C})$ is isomorphic to the identity component $\mathrm{SO}_0(3, 1)$ of $\mathrm{SO}(3, 1)$. If one considers an irreducible finite dimensional complex representation $\rho : \mathrm{PSL}(2, \mathbb{C}) \to \mathrm{GL}(N, \mathbb{C})$ considered as a real representation, $\mathrm{SO}_0(3, 1) \to \mathrm{GL}(2N, \mathbb{R})$, then the highest weight has the form $k(e_1 + e_2)$, whereas the highest weight of the natural representation is e_1. Here, e_1 and e_2 denote the standard basis of the dual of the standard Cartan subalgebra of the complexification of $\mathfrak{sl}(2, \mathbb{C})$. Therefore, one deduces from Raghunathan's theorem that for any nontrivial irreducible representation $\rho : \mathrm{PSL}(2, \mathbb{C}) \to \mathrm{GL}(N, \mathbb{C})$, the cohomology $H^1(\Gamma; \mathbb{R}^{2N})$ vanishes and hence $H^1(\Gamma; \mathbb{C}^N)$ also vanishes.

Consider now a cocycle f_γ in $H^1(\Gamma, \mathscr{H})$. Using the Γ-equivariant projection $\mathscr{H} \to \mathbb{C}$, we get a cocycle c_γ in $H^1(\Gamma, \mathbb{C})$. Let us show that $g_\gamma = f_\gamma - c_\gamma$ is cohomologous to 0. Indeed, projecting on each irreducible finite dimensional submodule $\mathscr{H}_k \subset \mathscr{H}$, and using Raghunathan's result, one gets that g_γ is cohomologous to a cocycle whose projections on arbitrarily many $\mathscr{H}_k$ vanish. This implies that g_γ is cohomologous to arbitrarily small 1-cocycles. The finite dimensionality of $H^1(\Gamma, \mathscr{H})$ implies that g_γ is cohomologous to zero.

This proves Lemma 7.4, and therefore, theorem D. □

We mention two corollaries of theorem D.

Corollary 7.5. *For any holomorphic Anosov flow on a compact complex 3-manifold the stable and unstable bundles are holomorphic sub-bundles of the complex tangent bundle of the ambient manifold.*

Proof. We know that this is true for the lift in some finite covering space of the ambient manifold. The result follows from the fact that the stable and unstable bundle of the lifted flow is the lifted stable and unstable bundle of the flow. □

Corollary 7.6. *A holomorphic Anosov flow on a compact complex 3-manifold preserves a volume form. If it is of type II, it actually preserves a nonvanishing holomorphic 3-form.*

Proof. Holomorphic suspensions and flows on $\mathrm{SL}(2,\mathbb{C})/\!\!/_u\Gamma$ are obviously volume preserving. It follows that this is also the case for flows which are finite quotients of these. This proves the first part of the corollary.

Consider the holomorphic 1-form η on $\mathrm{SL}(2,\mathbb{C})$ which vanishes on H^+ and H^- and equals 1 on the vector field corresponding to left translations by $\begin{pmatrix} T & 0 \\ 0 & T^{-1} \end{pmatrix}$. It is clear that η naturally defines a 1-form, still denoted by η, in $\mathrm{SL}(2,\mathbb{C})/\!\!/_u\Gamma$ (for any admissible $u : \Gamma \to \mathbb{C}^*$) which is invariant under the corresponding Anosov flow $\phi(T)$. One easily checks that $\eta \wedge d\eta$ is a nonsingular holomorphic 3-form. Any holomorphic diffeomorphism of $\mathrm{SL}(2,\mathbb{C})/\!\!/_u\Gamma$ commuting with $\phi(T)$ preserves both η and $\eta \wedge d\eta$ so that any finite quotient of $\phi(T)$ also preserves a nonsingular holomorphic 3-form. □

Let us remark that the second part of corollary 7.6 cannot be generalized to type I Anosov flows, *i.e.*, to suspensions. Indeed, let $B \in \mathrm{SL}(4,\mathbb{Z})$ be a matrix whose spectrum is $\{\rho e^{i\theta}, \rho e^{-i\theta}, \frac{1}{\rho}e^{i\theta'}, \frac{1}{\rho}e^{-i\theta'}\}$ with $\rho < 1$. This B is conjugate in $\mathrm{GL}(4,\mathbb{R})$ to $A = \begin{pmatrix} \rho e^{i\theta} & 0 \\ 0 & \frac{1}{\rho}e^{i\theta'} \end{pmatrix} \in \mathrm{GL}(2,\mathbb{C}) \subset \mathrm{GL}(4,\mathbb{R})$. Hence A preserves a lattice $\Lambda \subset \mathbb{C}^2$. Since $\det A = e^{i(\theta+\theta')}$, if $e^{i(\theta+\theta')} \neq 1$, the diffeomorphism $\bar{A}$ of $\mathbb{C}^2/\Lambda$ does not preserve any holomorphic 2-form. If $e^{i(\theta+\theta')}$ is not a root of unity, the same is true for any lift of $\bar{A}$ to any finite covering space. By considering the holomorphic suspension of A, one produces examples of holomorphic Anosov flows which do not preserve any nontrivial holomorphic 3-form, even after lifting to finite covering spaces. The canonical bundles of these 3-manifolds are nontrivial. Therefore these 3-manifolds are not complex homogeneous spaces (see [G-V] for more details).

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