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On the Arnold conjecture for weakly monotone symplectic manifolds

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Summary. We show the Arnold conjecture concerning symplectic fixed points in the case that the symplectic manifold is weakly-monotone and all the fixed points are non-degenerate. In particular, the conjecture is true in dimension 2, 4, 6, if all the fixed points are non-degenerate.

1 Introduction

A diffeomorphism ϕ on a symplectic manifold (M, ω) is called an exact symplectomorphism, if ϕ is the time 1 map of a time-dependent Hamiltonian vector field. A fixed point p is said to be non-degenerate, if 1 is not an eigenvalue of the differential $d\phi\colon T_pM \to T_pM$. From now on, we assume that M is compact. Arnold conjectured that the number of fixed points of an exact symplectomorphism is estimated below by the sum of the Betti numbers of M, if all the fixed points are non-degenerate. It is well-known that there is a one-to-one correspondence between fixed points of ϕ and 1-periodic solutions of a certain Hamiltonian system ([C-Z]). The periodic Hamiltonian equation is the Euler–Lagrange equation of the action functional on (a certain covering space of) the loop space of M (see Sect. 2). Floer developed an analogue of Morse theory for the action functional, which is now called Floer homology theory.

A symplectic structure ω determines an almost complex structure unique up to homotopy and we denote by $c_1 = c_1(M)$ the first Chern class of TM. A symplectic manifold (M, ω) is called monotone, if there exists $\lambda > 0$ such that $c_1(A) = \lambda\omega(A)$ for any 2-homology class represented by a continuous mapping from the 2-sphere. Floer [F] proved the Arnold conjecture for monotone

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symplectic manifolds. Hofer and Salamon [H-S] refined the argument and proved the Arnold conjecture in the following cases:
(i) (M, ω) is monotone.
(ii) $c_1 = 0$.
(iii) The minimal Chern number is at least $1/2 \cdot \dim M$.
Here the minimal Chern number is the least non-negative integer among $c_1(A)$ for $A \in \mathrm{Im}\{\pi_2(M) \to H_2(M; \mathbf{Z})\}$. A symplectic manifold (M, ω) is called weakly monotone (or semi-positive [MD-2]), if $\omega(A) \leqq 0$ for any $A \in \pi_2(M)$ with $3 - n \leqq c_1(A) < 0$. Actually Hofer and Salamon defined Floer homology groups for periodic Hamiltonian systems on weakly monotone symplectic manifolds. However it is necessary for computation of Floer homology groups that all the connecting orbits of relative index less than 2 should be handled simultaneously. The weak-compactness argument requires an upper bound of the energy functional and they avoid this difficulty by assuming one of the conditions above.

In this note, we introduce a filtration on the Floer complex and define the modified Floer homology group by a certain limit of relative Floer homology groups such that we have an upper bound of the energy functional for each stage, which yields the following

Theorem 1.1 *Let (M, ω) be a weakly monotone symplectic manifold, and ϕ an exact symplectomorphism. If all the fixed points of ϕ are non-degenerate, the number of fixed points of ϕ is bounded below by $\sum b_p(M; \mathbf{Z}/2)$, where $b_p(M; \mathbf{Z}/2)$ denotes the p-th Betti number of M with $\mathbf{Z}/2$-coefficient.*

If $\dim M \leqq 6$, (M, ω) is automatically weakly monotone and the Arnold conjecture holds. We shall show this result by estimating the number of contractible periodic solutions of a periodic Hamiltonian system whose time 1 map is ϕ.

2 Preliminaries

We recall known facts on Floer homology of periodic Hamiltonian systems. Details are found in [F], [H-S], [S-Z]. As for the weak-compactness argument for J-holomorphic curves, details are found in [M-S], [P-W], [Y]. We shall deal with a certain inhomogeneous Cauchy–Riemann equation which is not exactly the Cauchy–Riemann equation. However, it is converted to the Cauchy–Riemann equation for the graph of the mapping with respect to a certain almost complex structure on the product manifold of the domain manifold and the target manifold (see [G]). Hence we can apply the weak-compactness result to our situation. (Note that we cannot apply the transversality argument for J-holomorphic curves to our situation (see Sect. 4)).

Let (M, ω) be a closed symplectic manifold and $H: S^1 \times M \to \mathbf{R}$ a smooth function, called a periodic Hamiltonian function. Denote by $\mathscr{P}(H)$ the set of all contractible loops satisfying

$$\dot{x}(t) + X_H(t, x(t)) = 0\,, \tag{2.1}$$

where X_H is the Hamiltonian vector field of H. If $\langle \omega, \pi_2(M) \rangle = 0$, the equation (2.1) is the Euler–Lagrange equation of the action functional $a_H: \mathscr{L}(H) \to \mathbf{R}$ on the space of contractible loops in M defined as follows:

$$a_H(x) = -\int_{D^2} u^*\omega + \int_0^1 H(t, x(t))\,dt\,, \tag{2.2}$$

where u is the bounding disk of x, i.e. $u|_{\partial D^2} = x$. If $\langle \omega, \pi_2(M) \rangle \neq 0$, the first term of the right-hand-side of (2.2) is not single-valued. However it is well-defined over the covering space $\tilde{\mathscr{L}}(M)$ of $\mathscr{L}(M)$ corresponding to the homomorphism $\phi_\omega: \pi_2(M) \to \mathbf{R}$: $\phi_\omega(A) = \int_A \omega$. After [H-S], we introduce the space $\tilde{\mathscr{L}}(M)$ as follows:

$$\tilde{\mathscr{L}}(M) = \{(x, u) \mid x \in \mathscr{L}(M), u: D^2 \to M \text{ such that } x = u|_{\partial D^2}\}/\sim$$

$$(x, u) \sim (y, v) \Leftrightarrow \begin{cases} x = y \\ \int_{D^2} u^*\omega = \int_{D^2} v^*\omega \\ \int_{D^2} u^*c_1 = \int_{D^2} v^*c_1\,. \end{cases}$$

The covering transformation group of $\tilde{\mathscr{L}}(M) \to \mathscr{L}(M)$ is

$$\Gamma = \frac{\pi_2(M)}{\ker \phi_{c_1} \cap \ker \phi_\omega}\,.$$

Geometrically, $\pi_2(M)$ acts on $\tilde{\mathscr{L}}(M)$ by connected sum of 2-spheres with the bounding disk. Denote by Λ_ω the completion of the group ring of Γ over a field $\mathbf{Z}/2$ with respect to the weight homomorphism $\phi_\omega: \pi_2(M) \to \mathbf{R}$, i.e. the set of all formal sums $\sum_A \lambda_A \cdot \delta_A$, $\lambda_A \in \mathbf{Z}/2$, satisfying that

$$\{A \in \Gamma \mid \lambda_A \neq 0, \phi_\omega(A) < c\} \text{ is finite for all } c \in \mathbf{R}\,.$$

We introduce a grading on Λ_ω by assigning $2c_1(A)$ to δ_A. Fix an almost complex structure J calibrated by ω and consider the space $\mathscr{M}([x^-, u^-], [x^+, u^+])$ of the trajectories of the "(minus) gradient flow" of a_H from $[x^-, u^-]$ to $[x^+, u^+]$, i.e. solutions of the following:

$$\mathscr{F}u = \frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \nabla H(t, u) = 0 \tag{2.3}$$

$$\lim_{s \to -\infty} u(s, t) = x^-(t), \quad \lim_{s \to +\infty} u(s, t) = x^+(t) \tag{2.4}$$

$$(x^+, u^- \# u) \sim (x^+, u^+)\,. \tag{2.5}$$

This equation is invariant under translations in s-variable and $\mathbf{R}$ acts on $\mathcal{M}([x^-,u^-],[x^+,u^+])$ freely unless $[x^-,u^-]=[x^+,u^+]$. The linearized operator of $\mathcal{F}$ at u is

$$F_u\xi = \nabla_s\xi + J(u)\nabla_t\xi + \nabla_\xi J(u)\frac{\partial u}{\partial t} + \nabla_\xi \nabla H(t,u)\,. \tag{2.6}$$

Denote by $\tilde{\mathcal{P}}(H)$ the inverse image of $\mathcal{P}(H)$ by the projection $\tilde{\mathcal{L}}(M)\to\mathcal{L}(M)$, then there is the Conley–Zender index $\mu\colon \tilde{\mathcal{P}}(H)\to\mathbf{Z}$ (see: [H-S], [S-Z]), which satisfies

$$\operatorname{index} F_u = \mu([x^-,u^-]) - \mu([x^+,u^+]) \quad \text{for } [x^\pm,u^\pm]\in\tilde{\mathcal{P}}(H)\,.$$

The Sard–Smale theorem [Sm] yields that $\mathcal{M}([x^-,u^-],[x^+,u^+])$ is a manifold of dimension $\mu([x^-,u^-])-\mu([x^+,u^+])$ for a generic pair (J,H). The energy of a solution u of (2.3), (2.4), (2.5) is defined as follows:

$$E(u) = \frac{1}{2}\int_{-\infty}^{\infty}\int_0^1\left(\left|\frac{\partial u}{\partial s}\right|^2 + \left|\frac{\partial u}{\partial t} + X_H(t,u)\right|^2\right)dt\,ds\,. \tag{2.7}$$

For $u\in\mathcal{M}(\tilde{x},\tilde{y})$, we have

$$E(u) = a_H(\tilde{x}) - a_H(\tilde{y})\,.$$

A $2n$-dimensional symplectic manifold (M,ω) is called weakly monotone (or semi-positive) if it satisfies $\omega(A)\leqq 0$ for any $A\in\pi_2(M)$ with $3-n\leqq c_1(A)<0$ [MD-2], [H-S]. This condition yields non-existence of J-holomorphic spheres of negative Chern number for a generic almost complex structure J. Denote by C_k the $\mathbf{Z}/2$-vector space consisting of $\sum_{\mu(\tilde{x})=k}\xi(\tilde{x})\cdot\tilde{x}$, where the coefficients $\xi(\tilde{x})$ satisfy the following finiteness condition.

$$\{\tilde{x}\mid \xi(\tilde{x})\neq 0, \text{ and } a_H(\tilde{x})>c\} \text{ is a finite set for all } c\in\mathbf{R}\,.$$

The boundary operator is defined as follows:

$$\partial\tilde{x} = \sum_{\mu(\tilde{y})=\mu(\tilde{x})-1} n_2(\tilde{x},\tilde{y})\cdot\tilde{y}\,,$$

where $n_2(\tilde{x},\tilde{y})$ is the modulo 2-reduction of the cardinality of $\mathcal{M}(\tilde{x},\tilde{y})/\mathbf{R}$. The complex (C_*,∂) is called the Floer chain complex associated to (H,J). Hofer and Salamon showed $\partial^2=0$ for weakly monotone symplectic manifolds [H-S, Theorem 5.1]. $C_*=\bigoplus_k C_k$ is a graded module over a graded algebra Λ_ω and ∂ is Λ_ω-linear. Hence the homology group $HF_*(H,J)$ of (C_*,∂) is a graded Λ_ω-module. Moreover they proved the following

Theorem 2.8 ([H-S, Theorem 5.2]) *For generic pairs (H^α,J^α) and (H^β,J^β), there exists a natural Λ_ω-module homomorphism*

$$HF^{\beta\alpha}\colon HF_*(H^\alpha,J^\alpha)\to HF_*(H^\beta,J^\beta)$$

which preserves the grading by the Conley–Zehnder index. If (H^{γ}, J^{γ}) is any other such pair then

$$HF^{\gamma\beta} \circ HF^{\beta\alpha} = HF^{\gamma\alpha}, \quad HF^{\alpha\alpha} = id .$$

In particular, $HF^{\beta\alpha}$ is a Λ_{ω}-module isomorphism.

For the proof of this theorem, they considered s-dependent analogue of the equation (2.3). For generic pairs (H^{α}, J^{α}) and (H^{β}, J^{β}), we choose a path $\{(H_s, J_s) \mid s \in \mathbf{R}\}$ which satisfies

$$(H_s, J_s) = (H^{\alpha}, J^{\alpha}) \text{ for } s < -R, \quad (H_s, J_s) = (H^{\beta}, J^{\beta}) \text{ for } s > +R \tag{2.9}$$

for some positive real number R. Let $\tilde{z} = (z, u^-) \in \tilde{\mathscr{P}}(H^{\alpha})$ and $\tilde{w} = (w, u^+) \in \tilde{\mathscr{P}}(H^{\beta})$. $\mathscr{M}(\tilde{z}, \tilde{w}; \{H_s\})$ denotes the space of solutions of the following

$$\frac{\partial u}{\partial s} + J_s(u)\frac{\partial u}{\partial t} + \nabla H_s(t, u) = 0 \tag{2.10}$$

$$\lim_{s \to -\infty} u(s, t) = z(t), \ \lim_{s \to +\infty} u(s, t) = w(t) \tag{2.11}$$

$$(w, u^- \# u) \sim (w, u^+) . \tag{2.12}$$

We define the energy of u: $\mathbf{R} \times S^1 \to M$ as follows:

$$E_{\{H_s\}}(u) = \frac{1}{2} \int_{-\infty}^{+\infty} \int_0^1 \left(\left| \frac{\partial u}{\partial s} \right|^2 + \left| \frac{\partial u}{\partial t} + X_{H_s}(t, u) \right|^2 \right) dt\, ds \tag{2.13}$$

If u is a solution of (2.10), the energy of u is finite if and only if u satisfies the asymptotic condition (2.11) for some $\tilde{z} = [z, u^-]$ and $\tilde{w} = [w, u^+]$. We also have the following estimate of the energy.

$$| E_{\{H_s\}}(u) - \{a_{H^{\alpha}}(\tilde{z}) - a_{H^{\beta}}(\tilde{w})\} | \leqq \int_{-\infty}^{+\infty} \max_{x \in M, t \in S^1} \left| \frac{\partial}{\partial s} H_s(t, x) \right| ds \tag{2.14}$$

Note that the condition (2.9) assures that the last term in the right hand side of (2.14) is finite. The same argument for (2.3) yields that $\mathscr{M}(\tilde{z}, \tilde{w}; \{H_s\})$ is a manifold of dimension $\mu_{H^{\alpha}}(\tilde{z}) - \mu_{H^{\beta}}(\tilde{w})$ for a generic path $\{H_s\}$. Since we have a uniform bound of the energy, the weak compactness holds. In particular, $\mathscr{M}(\tilde{z}, \tilde{w}; \{H_s\})$ is a finite set if $\mu_{H^{\alpha}}(\tilde{z}) = \mu_{H^{\beta}}(\tilde{w})$. We define a Λ_{ω}-module homomorphism $\phi^{\beta\alpha}$: $C_*(H^{\alpha}, J^{\alpha}) \to C_*(H^{\beta}, J^{\beta})$ by

$$\phi^{\beta\alpha}(\tilde{z}) = \sum_{\mu_{H^{\alpha}}(\tilde{z}) = \mu_{H^{\beta}}(\tilde{w})} m_2(\tilde{z}, \tilde{w}) \cdot \tilde{w} ,$$

where $m_2(\tilde{z}, \tilde{w})$ is the modulo 2-reduction of the cardinality of $\mathscr{M}(\tilde{z}, \tilde{w}; \{H_s\})$. Investigating the end of 1-dimensional components of $\mathscr{M}(\tilde{z}, \tilde{w}; \{H_s\})$, we get the fact that $\phi^{\beta\alpha}$ is a Λ_{ω}-linear chain homomorphism. Once we fix a homotopy between two given generic paths in the space of paths satisfying (2.9) for some fixed R, we have a uniform bound of the energy, hence the weak-compactness, of solutions of (2.10). Then we get a chain homotopy between two chain homomorphisms which are obtained from two generic paths satisfying (2.9). Hence the induced homomorphism $HF^{\beta\alpha}$ between homology groups does not depend on the choice of generic paths satisfying (2.9).

Let $(H_s^{(1)}, J_s^{(1)})$ and $(H_s^{(2)}, J_s^{(2)})$ be paths satisfying

$$(H_s^{(1)}, J_s^{(1)}) = (H^\alpha, J^\alpha) \text{ for } s < -R, \ (H_s^{(1)}, J_s^{(1)}) = (H^\beta, J^\beta) \text{ for } s > +R$$

$$(H_s^{(2)}, J_s^{(2)}) = (H^\beta, J^\beta) \text{ for } s < -R, \ (H_s^{(2)}, J_s^{(2)}) = (H^\gamma, J^\gamma) \text{ for } s > +R$$

for some R. To show $HF^{\gamma\alpha} = HF^{\gamma\beta} \circ HF^{\beta\alpha}$, we have to consider the following family of paths.

$$H_{s,\lambda} = \begin{cases} H^{(1)}_{s+R+\lambda} & \text{for} \quad s < -\lambda \\ H^\beta & \text{for} \quad -\lambda \leqq s \leqq \lambda \\ H^{(2)}_{s-R-\lambda} & \text{for} \quad s > \lambda \end{cases}$$

For the above family of paths, it is easy to see that the last term of (2.14) is uniformly bounded with respect to $\lambda > 0$. The gluing argument relates $\mathcal{M}(\{H_s^{(1)}\})$ and $\mathcal{M}(\{H_s^{(2)}\})$ with $\mathcal{M}(\{H_{s,\lambda}\})$ for a sufficiently large λ, which yields $HF^{\gamma\beta} \circ HF^{\beta\alpha} = HF^{\gamma\alpha}$.

Hofer and Salamon computed the Floer homology for a generic pair (H, J) under certain conditions.

Theorem 2.15 ([H-S, Theorem 6.1]) *Assume either that (M, ω) is monotone or $c_1(\pi_2(M)) = 0$ or the minimal Chern number is $N \geqq n$. Then for a generic pair (H^α, J^α), there exists a natural isomorphism*

$$HF^\alpha : HF_*(H^\alpha, J^\alpha) \to H_{*+n}(M; \mathbf{Z}/2) \otimes \Lambda_\omega .$$

If (H^β, J^β) is any other such pair, then $HF^\beta \circ HF^{\beta\alpha} = HF^\alpha$.

3 Filtered Floer complex

In this section, we assume that the symplectic form ω has integral periods, i.e. $[\omega] \in \operatorname{Im}\{H^2(M; \mathbf{Z}) \to H^2(M; \mathbf{R})\}$. For a fixed Hamiltonian H, we can choose an increasing sequence of real numbers $\{r_j : j \in \mathbf{Z}\}$ satisfying

(i) $r_j \to \pm\infty$ as $j \to \pm\infty$,

(ii) $\{r_j\}$ does not contain critical values of a_H.

Write $C_{*,j} = \{\sum \xi(\tilde{x}) \cdot \tilde{x} \in C_* \mid \xi(\tilde{x}) = 0 \text{ if } a_H(\tilde{x}) > r_j\}$, which is a subcomplex of the Floer complex C_*. We define the "relative homology" of the pair $(C_{*,j}, C_{*,i})$ $(i < j)$, i.e. the homology group of $C_{*,(i,j)} = C_{*,j}/C_{*,i}$:

$$HF_{*,(i,j)} = H_*(C_{*,j}/C_{*,i}, \partial) .$$

The "relative" Floer homology was introduced in [F-H]. We have the following commutative diagram:

$$
\begin{array}{ccccccccc}
 & & \downarrow & & \downarrow & & \downarrow & & \\
\longrightarrow & & HF_{*,(k-1,l-1)} & \longrightarrow & HF_{*,(k,l-1)} & \longrightarrow & HF_{*,(k+1,l-1)} & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
\longrightarrow & & HF_{*,(k-1,l)} & \longrightarrow & HF_{*,(k,l)} & \xrightarrow{\Phi_{(k,l)}} & HF_{*,(k+1,l)} & \longrightarrow & \\
 & & \downarrow & & \downarrow{\scriptstyle \Psi_{(k,l)}} & & \downarrow & & \\
\longrightarrow & & HF_{*,(k-1,l+1)} & \longrightarrow & HF_{*,(k,l+1)} & \longrightarrow & HF_{*,(k+1,l+1)} & \longrightarrow & \\
 & & \downarrow & & \downarrow & & \downarrow & &
\end{array}
\tag{3.1}
$$

We define the modified Floer homology group as follows:

$$\widehat{HF}_* := \varinjlim_{l \to +\infty} \varprojlim_{k \to -\infty} HF_{*,(k,l)} \, .$$

It is easy to see

Lemma 3.2 *For a generic pair (H, J), $\widehat{HF}_*(H, J)$ does not depend on the choice of $\{r_j\}$.*

As a module, Λ_ω is the completion of the group ring $\mathbf{Z}/2[\Gamma]$ with respect to the following filtration:

$$\mathbf{Z}/2[\Gamma]_{(i,j)} := \left\{ \sum_{A \in \Gamma} \lambda_A \cdot \delta_A \in \mathbf{Z}/2[\Gamma] \mid \lambda_A = 0 \text{ for } \phi_\omega(A) > -i \text{ or } \phi_\omega(A) < -j \right\}$$

$$\Lambda_\omega = \varinjlim_{j \to +\infty} \varprojlim_{i \to -\infty} \mathbf{Z}/2[\Gamma]_{(i,j)} \, .$$

For a generic Hamiltonian function H, $\mathscr{P}(H)$ is a finite set. Since $[\omega]$ is an integral class, we can choose the set $\{r_j\} = \{j + \varepsilon \mid j \in \mathbf{Z}\}$ for some $\varepsilon \geqq 0$.

The Γ-action on $(C_*(H, J), \partial)$ satisfies the following

$$\mathbf{Z}/2[\Gamma]_{(i,j)} \times HF_{*,(k,l)}(H, J) \to HF_{*,(\max(i+l, j+k), j+l)}(H, J) \, ,$$

which induces Λ_ω-action on $\widehat{HF}_*(H, J)$. Thus we get

Lemma 3.3 *$\widehat{HF}_*(H, J)$ has a natural Λ_ω-module structure.*

The following theorem is an analogue of Theorem (2.8).

Theorem 3.4 *For generic pairs* (H^α, J^α) *and* (H^β, J^β), *there exists a* Λ_ω*-module isomorphism*

$$HF^{\beta\alpha}\colon \widehat{HF}_*(H^\alpha, J^\alpha) \to \widehat{HF}_*(H^\beta, J^\beta)$$

preserving the Conley-Zehnder index. If (H^γ, J^γ) *is any other such pair then*

$$HF^{\gamma\beta} \circ HF^{\beta\alpha} = HF^{\gamma\alpha}, \quad HF^{\alpha\alpha} = id\,.$$

Proof. Let $\{(H^{(\sigma)}, J^{(\sigma)}) \mid \sigma \in [0,1]\}$ be a path connecting (H^α, J^α) and (H^β, J^β). Subdivide $[0, 1]$ into $[s_k, s_{k+1}]$ such that there exists $\{r_j^{(k)}\}$ and $\varepsilon > 0$ satisfying

(i) $r_j^{(k)} \to \pm\infty$ as $j \to \pm\infty$.

(ii) ε-neighborhood of $\{r_j^{(k)}\}$ contains no critical values of a_{H_s} for $s = s_k, s_{k+1}$.

(iii) $\displaystyle\int_{s_k}^{s_{k+1}} \max_{x \in M, t \in S^1} \left|\frac{\partial}{\partial\sigma} H^{(\sigma)}(t,x)\right| d\sigma < \varepsilon/2\,.$

This is possible, since $[\omega]$ is an integral class and Lemma (3.5) below assures that $H^{(\sigma)}$ can be chosen such that $\mathscr{P}(H^{(\sigma)})$ is finite for $\sigma \in [0, 1]$. Then we can choose a generic path $\{(H_s, J_s) \mid s \in \mathbf{R}\}$ satisfying (2.9) with $(H^\alpha, J^\alpha) = (H_{s_k}, J_{s_k})$ and $(H^\beta, J^\beta) = (H_{s_{k+1}}, J_{s_{k+1}})$ and

(iii′) $\displaystyle\int_{-\infty}^{\infty} \max_{x \in M, t \in S^1} \left|\frac{\partial}{\partial s} H_s(t,x)\right| ds < \varepsilon\,.$

We shall consider the equation (2.10) with limits $\tilde{z} = [z, u^-]$, $\tilde{w} = [w, u^+]$ which satisfy $a_{H^\alpha}(\tilde{z})$, $a_{H^\beta}(\tilde{w}) \in [r_i^{(k)}, r_j^{(k)}]$. Let $\{u_l\}$ be a sequence of solutions. Since we have a uniform upper bound for the energy functional (2.14), $\{u_l\}$ contains a subsequence which converges to a solution of (2.10) with $\tilde{z}' \in \tilde{\mathscr{P}}(H^\alpha)$ and $\tilde{w}' \in \tilde{\mathscr{P}}(H^\beta)$, and possibly solutions of (2.3) with $H = H^\alpha$, $\tilde{z}'$, $\tilde{z}'' \in \tilde{\mathscr{P}}(H^\alpha)$ or $H = H^\beta$, $\tilde{w}'$, $\tilde{w}'' \in \tilde{\mathscr{P}}(H^\beta)$. The conditions (ii) and (iii′) imply that $a_{H^\alpha}(\tilde{z}')$, $a_{H^\alpha}(\tilde{z}'')$, $a_{H^\beta}(\tilde{w}')$, $a_{H^\beta}(\tilde{w}'') \in [r_i^{(k)}, r_j^{(k)}]$. Hence the proof of Theorem (2.8) yields that $\phi^{\beta\alpha}$ induces a chain homomorphism $C_{*,(i,j)}(H^\alpha, J^\alpha) \to C_{*,(i,j)}(H^\beta, J^\beta)$. In a similar way, we can show that $\phi^{\beta\alpha}$ does not depend on the choice of $\{H_s, J_s\}$ and that $\phi^{\beta\alpha}$ induces an isomorphism between "relative" Floer homology groups. It is also easy to see that homomorphisms $\Phi_{(i,j)}$, $\Psi_{(i,j)}$ in the diagram (3.1) are compatible with the isomorphism obtained above. Moreover, the actions of Λ_ω are preserved under the induced isomorphism between modified Floer homology groups $\widehat{HF}_*(H^\alpha, J^\alpha)$ and $\widehat{HF}_*(H^\beta, J^\beta)$. □

Lemma 3.5 *For a generic Hamiltonian functions* H^α *and* H^β, *there is a path* $\{H(s)\}$ *connecting them such that* $\mathscr{P}(H(s))$ *is finite for all s.*

Proof. Let $\mathscr{H}$ be the Banach space of periodic Hamiltonian functions (see [H-S]) and $\{H_s\}$ a generic path in $\mathscr{H}$ connecting H^α and H^β. The implicit function theorem and the Sard–Smale theorem [Sm] yield that $\mathscr{X}(\{H_s\}) = \{(x, s) \in \mathscr{L}(M) \times [0,1] \mid x \in \mathscr{P}(H_s)\}$ is a 1-dimensional manifold with boundary $\mathscr{P}(H_0) \times \{0\} \cup \mathscr{P}(H_1) \times \{1\}$.

More precisely, let $\mathscr{E}$ denote the Banach space bundle over $W^{1,2}$-completion of $\mathscr{L}(M)$, which we shall also denote by $\mathscr{L}(M)$, with fiber $\mathscr{E}_x = L^2\Gamma(x^*TM)$. We define a Fredholm mapping $F: \mathscr{L}(M) \times \mathbf{R} \to \mathscr{E}$ by

$$F(x, s) = \dot{x} + X_{H_s}(t, x(t)) .$$

Then the linearization DF of F is

$$DF(\xi, \sigma) = \nabla_{\frac{\partial}{\partial t}} \xi + \nabla_\xi X_{H_s}(t, x(t)) + \sigma X_{\frac{\partial}{\partial s} H_s}(t, x(t))$$

for $(\xi, \sigma) \in T_{(x,s)}(\mathscr{L}(M) \times \mathbf{R})$. For a generic path $\{H_s\}$, DF is surjective, this fact and the index computation imply that $\mathscr{X}(\{H_s\})$ is a 1-dimensional manifold.

Let p: $\mathscr{X} \to [0, 1]$ be the projection to the second factor. To prove Lemma (3.5), it suffices to show that dp: $T\mathscr{X} \to T([0, 1])$ is transversal to the zero section of $T([0, 1])$ outside of the zero section of $T\mathscr{X}$, i.e. the differential of the projection p_4 to the fourth factor $p_4(x, s, \xi, \sigma) = \sigma$ is surjective if $\xi \neq 0$. Namely we have to get the transversality on 1-jets. We define $\mathscr{F}: \mathscr{L}(M) \times \mathbf{R} \times \mathscr{H} \to \mathscr{E}$ by

$$\mathscr{F}(x, s, H) = \dot{x} + X_{H_s + H}(t, x(t)) .$$

Restricting the linearization of $\mathscr{F}$ to $T(\mathscr{L}(M) \times \mathbf{R}) \times \mathscr{H}$, we get $\mathscr{F}'$: $T(\mathscr{L}(M) \times \mathbf{R}) \times \mathscr{H} \to T\mathscr{E}$ as follows:

$$\mathscr{F}'(x, s, \xi, \sigma, H) = (\mathscr{F}'_{(1)}(x, s, \xi, \sigma, H), \mathscr{F}'_{(2)}(x, s, \xi, \sigma, H)) ,$$

where

$$\mathscr{F}'_{(1)}(x, s, \xi, \sigma, H) = \dot{x} + X_{H_s + H}(t, x(t)) ,$$

$$\mathscr{F}'_{(2)}(x, s, \xi, \sigma, H) = \nabla_{\frac{\partial}{\partial t}} \xi + \nabla_\xi X_{H_s + H}(t, x(t)) + \sigma X_{\frac{\partial}{\partial s} H_s}(t, x(t))$$

for $(\xi, \sigma) \in T_{(x,s)}(\mathscr{L}(M) \times \mathbf{R})$, $H \in \mathscr{H}$. The linearization $D\mathscr{F}'$ of $\mathscr{F}'$ is given by

$$D\mathscr{F}'(a, b, c, \tau, h) = (D\mathscr{F}'_{(1)}(a, b, c, \tau, h), D\mathscr{F}'_{(2)}(a, b, c, \tau, h))$$

where

$$D\mathscr{F}'_{(1)}(a, b, c, \tau, h) = \nabla_{\frac{\partial}{\partial t}} a + \nabla_a X_{H_s + H}(t, x(t)) + b X_{\frac{\partial}{\partial s} H_s}(t, x(t)) + X_h(t, x(t)) ,$$

and

$$\begin{aligned} D\mathscr{F}'_{(2)}(a, b, c, \tau, h) = {} & \nabla_{\frac{\partial}{\partial t}} c + \nabla_c X_{H_s + H}(t, x(t)) + \tau X_{\frac{\partial}{\partial s} H_s}(t, x(t)) \\ & + \nabla_\xi X_h(t, x(t)) + b \nabla_\xi X_{\frac{\partial}{\partial s} H_s}(t, x(t)) \\ & + b\sigma X_{\frac{\partial^2}{\partial s^2} H_s}(t, x(t)) + \nabla_a \nabla_\xi X_{H_s + H}(t, x(t)) \\ & + \sigma \nabla_a X_{\frac{\partial}{\partial s} H_s}(t, x(t)). \end{aligned}$$

$\bar{p}_4$: $T(\mathscr{L}(M) \times \mathbf{R}) \times \mathscr{H} \to \mathbf{R}$ denotes the projection to the fourth factor.

$$\bar{p}_4(x, s, \xi, \sigma, H) = \sigma$$

For $H \in \mathscr{H}$ sufficiently close to 0, the operator

$$(a, b) \mapsto \nabla_{\frac{\partial}{\partial t}} a + \nabla_a X_{H_s + H}(t, x(t)) + bX_{\frac{\partial}{\partial s} H_s}(t, x(t))$$

is surjective. Because of the term $\nabla_\xi X_h(t, x(t))$ and the unique continuation theorem, $(c, h) \mapsto \nabla_{\frac{\partial}{\partial t}} c + \nabla_c X_{H_s + H}(t, x(t)) + \nabla_\xi X_h(t, x(t))$ is surjective, if $\xi \neq 0$. Hence, $\mathscr{F}' \times \bar{p}_4$ is transversal to the zero section at points $(x, s, \xi, 0, H) \in T(\mathscr{L}(M) \times \mathbf{R}) \times \mathscr{H}$ satisfying $\xi \neq 0$. By the Sard-Smale theorem, $\{(x, s, \xi, 0) \in T(\mathscr{L}(M) \times \mathbf{R}) \mid x \in \mathscr{P}(H_s + H), \xi \in T_{(x,s)}\mathscr{X}(\{H_s + H\}), \xi \neq 0\}$ is a 1-dimensional manifold for a generic H. In particular, $\mathscr{C} = \{(x, s) \in \mathscr{L}(M) \times \mathbf{R} \mid x \in \mathscr{P}(H_s + H), p_{4*}(\xi) = 0$ for all $\xi \in T_{(x,s)}\mathscr{X}(\{H_s + H\})\}$ is a 0-dimensional submanifold of $\mathscr{X}(H_s + H)$. By the assumption on H^α and H^β, $\mathscr{C}$ does not intersect the boundary of $\mathscr{X}(\{H_s + H\})$ for H sufficiently close to 0. Moreover, there is a path $\gamma\colon [0, 1] \to \mathscr{H}$ such that $\gamma(s) = 0$ near $s = 0$, $\gamma(s) = H$ near $s = 1$, $\mathscr{X}(\{\gamma + H^{\alpha \operatorname{or} \beta}\}) = \{(x, s) \mid x \in \mathscr{P}(\gamma(s) + H^{\alpha \operatorname{or} \beta})\}$ is a manifold, and the projection to the second factor $\mathscr{X}(\gamma + \mathscr{H}^{\alpha \operatorname{or} \beta})\} \to [0, 1]$ is a submersion. Then we define $H(s) = H^\alpha + \gamma(\varepsilon^{-1} \cdot s)$ for $0 \leqq s \leqq \varepsilon$, $H(s) = H_{(1-2\varepsilon)^{-1}(s-\varepsilon)} + H$ for $\varepsilon \leqq s \leqq 1 - \varepsilon$, and $H(s) = H^\beta + \gamma(\varepsilon^{-1}(1 - s))$ for $1 - \varepsilon \leqq s \leqq 1$. This satisfies the property of Lemma (3.5). □

4 Computation of the modified Floer homology group

In [F], [H-S], they compare the Floer complex of a generic pair (H, J) with the Morse complex of a C^2-small Morse function. An almost complex structure J calibrated by ω determines a Riemannian metric on M. For a Morse function $f\colon M \to \mathbf{R}$ whose gradient flow is of Morse–Smale type, we denote by $C_*(f)$ the Morse complex associated to f [Sa].

Under the assumption that (M, ω) is monotone or $c_1(M)(\pi_2(M)) = 0$, or the minimal Chern number $N \geqq n$, they proved that $HF_*(f, J)$ is isomorphic to $H_{*+n}(C_*(f)) \otimes \Lambda_\omega \cong H_{*+n}(M; \mathbf{Z}/2) \otimes \Lambda_\omega$ are graded Λ_ω-modules. Here $H_*(M; \mathbf{Z}/2) \otimes \Lambda_\omega$ is the tensor product of graded modules. This result and Theorem (2.8) yield Theorem (2.15).

In this section, we compute the modified Floer homology group of (H, J) on a weakly monotone symplectic manifold (M, ω) without the assumption concerning the minimal Chern number.

First of all, we show the following

Lemma 4.1 *For a fixed $C > 0$, there exists a positive integer $j_0(C)$ such that for $\tilde{x}, \tilde{y} \in \tilde{\mathscr{P}}(f)$ satisfying $a_f(\tilde{x}) - a_f(\tilde{y}) < C$, $\mu_f(\tilde{x}) - \mu_f(\tilde{y}) \leqq 1$, all solutions of the following equation with $\varepsilon = 1/j$ are independent of t-variable for $j > j_0(C)$ and a generic almost complex structure J.*

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \varepsilon \cdot (\nabla f)(u) = 0 \tag{4.2}$$

$$\lim_{s \to -\infty} u(s, t) = \tilde{x}(t), \quad \lim_{s \to +\infty} u(s, t) = \tilde{y}(t) \tag{4.3}$$

Proof. First of all, we assume that $|f| < 1, f$ is sufficiently C^2-small such that all the 1-periodic solutions of the equation

$$\dot{x}(t) + X_{\varepsilon f}(x(t)) = 0$$

are constant solutions at critical points of f and $\mu_{\varepsilon \cdot f}(\tilde{x}) = \mu_f(\tilde{x})$ for $0 < \varepsilon \leqq 1$. If the statement is false, we can choose a sequence of integers j_l diverging to $+\infty$ and a sequence of solutions u_l of (4.2), (4.3) with $\varepsilon = 1/j_l$.

Firstly, we consider the case that $c_1(u_l) < 0$. (Note that $c_1(u_l)$ does not depend on l as long as u_l is a connecting orbit from $\tilde{x}$ to $\tilde{y}$.) Since the energy of u_l is uniformly bounded by $C + 2$, after choosing a subsequence if necessary, u_l converges, up to J-holomorphic bubbles, to a solution u_∞ of (4.2) with $\varepsilon = 0$, i.e. a J-holomorphic mapping and u_∞ extends to a J-holomorphic mapping from the Riemann sphere to M. The limit u_∞ may be a constant mapping. (If $\varepsilon \neq 0$, only constant solutions of (4.2) are constant mappings to critical points of f. On the other hand, every constant mapping is a solution of (4.2)(4.3) with $\varepsilon = 0$.) For real numbers σ_l, ψ_l denotes the reparametrization $(s, t) \to (s + \sigma_l, t)$ of the infinite cylinder $\mathbf{R} \times S^1$. The above argument yields that, after choosing a subsequence if necessary, $u_l \circ \psi_l$ also converges to a J-holomorphic sphere up to J-holomorphic bubbles. Let $\{S_j\}$ be the set of all possible J-holomorphic spheres appearing as a limit of $u_l \circ \psi_l$ or bubbles.

Claim 1 *$\{S_j\}$ is not an empty set.*

Proof of Claim 1 Let δ be the injectivity radius of M and $v: \mathbf{R} \times S^1 \to M$ a smooth mapping satisfying the following asymptotic condition.

$$\lim_{s \to -\infty} v(s, t) = x, \quad \lim_{s \to +\infty} v(s, t) = y .$$

Denote by v the extension of v to $S^2 \to M$. If v satisfies $|v_*(s, t)(\frac{\partial}{\partial t})| \leqq \delta$ for all $(s, t) \in \mathbf{R} \times S^1$, v is homologous to zero. The condition that $c_1(u_l) < 0$ implies that u_l is not homologous to zero, hence there exists (s_l, t_l) such that $|u_{l*}(s_l, t_l)(\frac{\partial}{\partial t})| > \delta$. We reparametrize u_l by $u_l'(s, t) = u_l(s + s_l, t + t_l)$, then u_l' is still a solution of (4.2) with $\varepsilon = 1/j_l$, and satisfies

$$\left| u'_{l*}(0, 0)\left(\frac{\partial}{\partial t}\right)\right| > \delta \tag{4.4}$$

Since we have a uniform energy bound, the weak-compactness argument tells us that u_l' converges to a J-holomorphic sphere possibly with J-holomorphic bubbles, which are also J-holomorphic spheres. The condition (4.4) assures that at least one of the J-holomorphic spheres above is not a constant mapping.

For any J-holomorphic sphere S in $\{S_j\}$, we can find a subsequence $\{l_p\}$ of $\{l\}$ and $\{\sigma_p \in \mathbf{R}\}$ such that S is the limit of $u_{l_p} \circ \psi_{l_p}$ or a bubble attached to it. Choose subsequences successively and denote the subsequence by the same symbol $\{l\}$.

Suppose that each J-holomorphic spheres S_j appears as the limit or a bubble of (possible reparametrized) solutions u_l of (4.2).

Claim 2 *$\{S_j\}$ is a finite set.*

Proof of Claim 2. Let $\{T_i\}$ be J-holomorphic spheres obtained as limits of (reparametrized) solutions u_l of (4.2) with $\varepsilon = 1/j_l$ except finitely many points in $\mathbf{R} \times S^1$ and $\{T_{i,h} \mid 1 \leqq h \leqq d(i)\}$ J-holomorphic bubbles attached to T_i. Clearly, we have $\{S_j\} = \{T_i, T_{i,h}\}$. For any $\varepsilon > 0$, we can take l large enough such that there exist mutually disjoint intervals $[R_i, L_i]$ $(i = 1, \ldots, k)$ and $u_l([R_i, L_i] \times S^1)$ is close to T_i and possibly some bubbles $T_{i,l}$ enough to satisfy

$$\frac{1}{2}\int_{R_i}^{L_i}\int_0^1 \left|\frac{\partial u_l}{\partial s}\right|^2 + \left|\frac{\partial u_l}{\partial t} + \frac{1}{j_l}\cdot X_f\right|^2 dt\, ds \geqq E(T_i) + \sum_{h=1}^{d(i)} E(T_{i,h}) - \varepsilon .$$

Hence

$$E(u_l) \geqq \sum_{i=1}^{k} \left\{E(T_i) + \sum_{h=1}^{d(i)} E(T_{i,h})\right\} - k\varepsilon ,$$

if $\{T_i\}$ contains at least k J-holomorphic spheres. On the other hand, we have $E(u_l) \leqq C + 2$ and $E(S) = \int_S \omega \geqq 1$ for any J-holomorphic sphere S, because $[\omega]$ is an integral class. Since ε is arbitrary, the cardinality of $\{S_j\} = \{T_i, T_{i,l}\}$ is bounded by $C + 2$.

Remark. Hofer and Salamon showed the estimate $E(S) > \hbar$ for some positive constant $\hbar$ without assuming $[\omega]$ is an integral class.

Claim 3 $c_1(u_l) = \sum_j c_1(S_j)$.

Proof of Claim 3. Let U be a regular neighborhood of $\bigcup\{T_i \cup (\cup T_{i,l})\}$. For a fixed $\varepsilon > 0$, there exists l and sequence of real numbers $-\infty = L_0 < R_1 < L_1 < R_2 < L_2 < \cdots < R_k < L_k < R_{k+1} = +\infty$, such that

$$\operatorname{Im} u_l([R_i, L_i] \times S^1) \subset U ,$$

and

$$\left|\frac{\partial}{\partial t} u_l(s, t)\right| < \varepsilon \text{ if } s < [L_i, R_{i+1}] \text{ for some } i = 0, \ldots, k . \tag{4.5}$$

We choose $\varepsilon < \delta$, then $u_l|_{R_i}$, $u_l|_{L_i}$ bound disks D_i^-, D_i^+ in δ-balls, which are unique up to homotopy. It is easy to see that $C_i = D_i^- \cup u_l([R_i, L_i] \times S^1) \cup D_i^+$ is homologous to $T_i \cup (\cup T_{i,l})$. The condition (4.5) assures that $D_{i-1}^+ \cup u_l([L_{i-1}, R_i] \times S^1) \cup D_i^-$ is homologous to zero. Therefore we get

$$c_1(u_l) = \sum c_1(C_i) = \sum (c_1(T_i) + \sum c_1(T_{i,l})) .$$

Since $c_1(u_l) < 0$, one of the J-holomorphic spheres S_j has negative Chern number. However the weak monotonicity excludes this possibility for a generic almost complex structure J. This is a contradiction.

From now on, we assume that the Chern number of $u = u_l$ is non-negative.

Case 1 *rank* $du = 0$ *on an open subset* U *of* $\mathbf{R} \times S^1$.

The unique continuation theorem [A], [J, Lemma 2.6.1] yields that u is a constant mapping. However we assumed that u is t-dependent, and this is a contradiction.

Case 2 *rank* $du = 1$ *on an open subset* U *of* $\mathbf{R} \times S^1$.

In this case, the image of U by the mapping u is an immersed curve $\gamma: (a, b) \to M$. Since $\dot{\gamma}$ and $J\dot{\gamma}$ are linearly independent, we have

$$\frac{\partial u}{\partial s}(s, t) = \frac{\partial u}{\partial s}(s', t')$$

and

$$\frac{\partial u}{\partial t}(s, t) = \frac{\partial u}{\partial t}(s', t')$$

for $(s, t), (s', t') \in U$ satisfying $u(s, t) = u(s', t')$. This fact implies that $u(s, t) = \gamma(\lambda s + \mu t)$ after a reparametrization of γ and λ and μ are some constants in $\mathbf{R}$. In fact, we can show that the foliation given by the level sets of u is invariant under translations in the s-direction and the t-direction, which implies that the level sets are segments of parallel straight lines. Since $\gamma: (a, b) \to M$ satisfies the equation

$$\dot{\gamma} = -\varepsilon \frac{\lambda - J\mu}{\lambda^2 + \mu^2} \nabla f,$$

we can extend it to $\gamma: \mathbf{R} \to M$ as the integral curve of the equation above which coincide with the original γ on (a, b). We define a mapping $u_\gamma: \mathbf{R} \times \mathbf{R} \to M$ by $u_\gamma(s, t) := \gamma(\lambda s + \mu t)$. Then u_γ is a solution of (4.2). Apply the unique continuation theorem to u_γ and the composition of the projection $p: \mathbf{R} \times \mathbf{R} \to \mathbf{R} \times S^1$ with u, we get $u_\gamma = u \circ p$ on $\mathbf{R} \times \mathbf{R}$.

If $\mu = 0$, γ is a gradient trajectory of f and u degenerates to γ on the whole domain $\mathbf{R} \times S^1$. Since u is a t-dependent solution, we get $\mu \neq 0$.

If $\lambda = 0$, $\tilde{\gamma}(t) = \gamma(\mu t)$ is a 1-periodic solution for $\dot{x}(t) = X_{\varepsilon f}(x(t))$. However, there are no non-trivial 1-periodic solutions by our assumption, hence $\lambda \neq 0$.

We can assume that $\lambda = 1$ and $u(s, t) = \gamma(s + \mu t)$ with $\mu \neq 0$. Since u is 1-periodic in t-variable, $\gamma(s) = \gamma(s + \mu)$. On the other hand, $\langle \dot{\gamma}(s), \nabla f \rangle = -\varepsilon\{1/(1 + \mu^2)\} \| \nabla f \|^2 < 0$, hence $f(\gamma(s)) \neq f(\gamma(s + \mu))$, which is a contradiction.

A solution u is called multiple, if there exist an integer $k \geqq 2$ and $v: \mathbf{R} \times S^1 \to M$ such that $u(s, t) = v(ks, kt)$. If u is not multiple, we call u a simple solution.

Case 3 *u is simple.*

If u is somewhere injective, we can apply the same argument in [MD-1] to prove that $\mathscr{M}(\tilde{x}, \tilde{y})$ is a manifold of dimension $\mu_f(\tilde{x}) - \mu_f(\tilde{y})$ around u for a generic almost complex structure J. On the other hand, there is a 2-parameter family of solutions $u_{\sigma,\rho}(s, t) = u(s + \sigma, r + \rho)$ in $\mathscr{M}(\tilde{x}, \tilde{y})$. Hence $\mu_f(\tilde{x}) - \mu_f(\tilde{y}) = \dim \mathscr{M}(\tilde{x}, \tilde{y}) \geqq 2$, which contradicts our assumption that $\mu_f(\tilde{x}) - \mu_f(\tilde{y}) \leqq 1$.

If u is not somewhere injective, there are open subsets U_1 and U_2 in $\mathbf{R} \times S^1$ such that $u(U_1) = u(U_2)$. We may assume that rank $du = 2$ on U_1 and U_2.

If u satisfies the following condition:

$$\frac{\partial u}{\partial s}(s_1, t_1) = \frac{\partial u}{\partial s}(s_2, t_2)$$

and

$$\frac{\partial u}{\partial t}(s_1, t_1) = \frac{\partial u}{\partial t}(s_2, t_2)$$

for all $(s_1, t_1) \in U_1$ and $(s_2, t_2) \in U_2$ satisfying $u(s_1, t_1) = u(s_2, t_2)$, then the unique continuation theorem yields that $u(s, t) = u(s + \sigma, t + \tau)$ for some $\sigma, \tau \in \mathbf{R}$. If $\sigma \neq 0$, the unique continuation theorem implies that the energy $E(u)$ is infinite, which contradicts our hypothesis. Therefore $\sigma = 0$, and u is a multiple solution or degenerates to a gradient trajectories of f according to cases that τ is a rational number or not. However we assumed that u is a simple solution. Hence these cases never happen.

We assume that u does not satisfy the above condition. Since u is not somewhere injective, there is a diffeomorphism $\phi: U_1 \to U_2$ between open subsets of $\mathbf{R} \times S^1$ such that $u \circ \phi|_{U_1} = u|_{U_1}$. Since $u \circ \phi$ and u have the same image on U_1, there are four functions a, b, c, d on U_1 such that

$$(a, b, c, d) \neq (1, 0, 0, 1)\,,$$

$$\frac{\partial u}{\partial s}(\phi(s, t)) = a \cdot \frac{\partial u}{\partial s}(s, t) + b \cdot \frac{\partial u}{\partial t}(s, t)$$

and

$$\frac{\partial u}{\partial t}(\phi(s, t)) = c \cdot \frac{\partial u}{\partial s}(s, t) + d \cdot \frac{\partial u}{\partial t}(s, t)\,.$$

From the equation (4.2) for u on U_1 and U_2, we get

$$\{(a - 1) + cJ\} \frac{\partial u}{\partial s} + \{b + (d - 1)J\} \frac{\partial u}{\partial t} = 0 \text{ on } U_1\,.$$

Since $(a, b, c, d) \neq (1, 0, 0, 1)$, $\partial u/\partial s$ and $\partial u/\partial t$ are linearly dependent over complex numbers. Hence the gradient vector field ∇f is tangent to a 2-dimensional immersed surface $u(U_1) = u(U_2)$. By the discussion in Case 1 and 2, we can assume that u is an immersion on an open dense subset of $\mathbf{R} \times S^1$. Hence the image of u is swept by gradient trajectories of f. More precisely, the gradient vector field ∇f is tangent to the image of u on an open dense subset.

If the statement of Lemma 4.1 is false, we can choose a sequence of integers j_l diverging to $+\infty$ and a sequence of t-dependent solutions u_l of (4.2), (4.3) with $\varepsilon = 1/j_l$. If $c_1(u_l) > 0$ or $\omega(u_l) \neq 0$, u_l is not homologous to zero. Hence the argument in the proof of Claim 1 and 2 implies that at least one of the J-holomorphic spheres appearing in the limit of u_l is nontrivial. Since u_l is swept by the gradient trajectories of f, i.e. the gradient vector field ∇f is tangent to the image of u_l almost everywhere, the C^1-limit of (reparametrized) connecting orbits u_l should also be swept by the gradient trajectories of f. This also holds for J-holomorphic bubbles, since they are the C^1-limit of rescaled mapping of u_l, to which the gradient vector field ∇f is tangent. Since a J-holomorphic sphere has only finitely many singular points, i.e. the points where the differential of the J-holomorphic mapping is not injective, only finitely many gradient trajectories of f contain singular points of S. On the other hand, if a part of a gradient trajectory $\gamma: \mathbf{R} \to M$ of f lies on S but the whole image of γ is not contained in S, γ should pass one of the critical points of S. Hence, there are gradient trajectories contained in S completely, so S should contain at least two critical points of f.

The Sard–Smale transversality argument [Sm] tells us that for given two points p and q, we can choose a generic almost complex structure J such that there are no J-holomorphic spheres containing p and q with the Chern number less than $n + 1$. Since S contains at least two of the critical points of f, we have $c_1(S) \geqq n + 1$. Then the proof of Claim 3 implies $c_1(u_l) \geqq n + 1$, because there are no J-holomorphic spheres of negative Chern number for a generic almost complex structure J.

Recall that $\mu_{\varepsilon \cdot f}(\tilde{x}) - \mu_{\varepsilon \cdot f}(\tilde{y}) = \mu_f(\tilde{x}) - \mu_f(\tilde{y}) = \mathrm{ind}_f(x) - \mathrm{ind}_f(y) + 2c_1(u)$, where $\mathrm{ind}_f(x)$ denote the index of the Hessian of f at a critical point x. Then we get $\mu_f(\tilde{x}) - \mu_f(\tilde{y}) \geqq 2$, which contradicts our assumption.

If $c_1(u_l) = \omega(u_l) = 0$, the argument in [S-Z, Theorem 7.3(2)] tells us that there are no t-dependent solution of (4.2), (4.3) for a sufficiently small $\varepsilon = 1/j$.

Case 4 *u is a multiple solution.*

In this case, $u(s, t) = v(ks, kt)$ and v is a simple solution of the following

$$\frac{\partial v}{\partial s} + J(v)\frac{\partial v}{\partial t} + \frac{\varepsilon}{k} \cdot (\nabla f)(v) = 0 .$$

Note that the argument in [S-Z, Theorem 7.3(2)] does not require that u is a simple solution. Hence in the case that $c_1(u) = \omega(u) = 0$, we have no

t-dependent solutions for a sufficiently large j. We assume that $c_1(u) \neq 0$ or $\omega(u) \neq 0$. Since $c_1(M)$ and $[\omega]$ are integral cohomology classes, the multiplicity k of u is uniformly bounded as long as u is a connecting orbit from $\tilde{x}$ to $\tilde{y}$. If $c_1(u) = 0$, the argument in case 3 implies that there are no simple solutions for a sufficiently small ε, which implies that such a u does not exist.

If $c_1(u) > 0$, the index of the linearization at the simple solution v is at most -1. Hence such a v does not exist for a generic almost complex structure J, which implies that such a u does not exist either. □

Remark. If a symplectic manifold (M, ω) satisfies the condition that $c_1(A) = \lambda \cdot \omega(A)$ for any spherical 2-homology class A with $\lambda \neq 0$, the condition that $a_f(\tilde{x}) - a_f(\tilde{y}) < C$ follows automatically from $\mu_f(\tilde{x}) - \mu_f(\tilde{y}) = 0, 1$. Therefore the conclusion of Lemma (4.1) holds for such symplectic manifolds without assuming the energy bound. If $\lambda < 0$, the weak monotonicity is equivalent to the condition that the minimal Chern number $N > n - 3$ or $\dim M = 6$. If $c_1(A) = 0$ for any spherical 2-homology class A, the first part of the proof yields that the same conclusion holds. Namely we get the following

Corollary 4.6 *Let (M, ω) be a closed symplectic manifold such that $c_1(A) = \lambda \cdot \omega(A)$ for any $A \in \pi_2(M)$ and a real constant λ. Suppose that the minimal Chern number $N > n - 3$ or $\dim M = 6$ if $\lambda < 0$. Then there exists a positive integer j_0 such that for $\tilde{x}, \tilde{y} \in \tilde{\mathscr{P}}(f)$ satisfying $\mu_f(\tilde{x}) - \mu_f(\tilde{y}) \leqq 1$, all solutions of* (4.2)(4.3) *with $\varepsilon = 1/j$ are independent of t-variable for $j > j_0$ and a generic almost complex structure J.*

We choose a C^2-small Morse function f satisfying $-1/8 < f(x) < 1/8$ and $\{r_j\} = \{j + 1/2 \mid j \in \mathbf{Z}\}$. Let $\{(H_s, J_s) \mid 0 \leqq s \leqq 1\}$ be a generic path from (f, J) to a generic pair (H_1, J_1), which is sufficiently small in C^1-sense. More precisely, the set of critical values of a_{H_s} is disjoint from $\{j + 1/2 + \delta \mid j \in \mathbf{Z}, -1/16 < \delta < 1/16\}$ and

$$\left| \frac{\partial}{\partial s} H_s(t, x) \right| < \frac{1}{16}$$

for all $x \in M$ and $s \in [0, 1]$. We prove the following

Theorem 4.7 $\widehat{HF}_*(H_1, J_1) \cong H_{*+n}(M; \mathbf{Z}/2) \otimes \Lambda_\omega$.

Proof. Let $\tilde{x}, \tilde{y} \in \tilde{\mathscr{P}}(f)$ satisfy $k + 1/2 < a_f(\tilde{x}), a_f(\tilde{y}) < l + 1/2$ and $\mu_f(\tilde{x}) - \mu_f(\tilde{y}) \leqq 1$. By Lemma (4.1), there exists a positive integer $j_0(k, l)$ such that all solutions of the equation (4.2) with $\varepsilon = 1/j$ are independent of t-variable if $j > j_0(k, l)$. In particular, the equation (4.2) has no non-trivial solutions if $\mu_f(\tilde{x}) = \mu_f(\tilde{y})$, and the chain complex $C_{*,(k,l)}(1/j \cdot f, J)$ is isomorphic to $C_{*+n}(f) \otimes \mathbf{Z}/2[\Gamma]_{(k,l)}$. The argument in the proof of Theorem (3.4) yields a chain homomorphism $\phi_{(k,l)}: C_{*,(k,l)}(H, J) \to C_{*+n,(k,l)}(1/j \cdot f, J) \cong C_{*+n}(f) \otimes \mathbf{Z}/2[\Gamma]_{(k,l)}$, which induces an isomorphism between homology groups. Moreover the argument in the proof of Lemma (4.1) yields that there exists an integer $j_1(k, l) \geqq j_0(k, l)$ such that all the solutions of the equation below

$$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} + \varphi(s)(\nabla f)(u) = 0$$

$$\lim_{s \to -\infty} u(s, t) = x \in \mathscr{P}(f), \quad \lim_{s \to \infty} u(s, t) = y \in \mathscr{P}(f)$$

$$(y, u^- \# u) \sim (y, u^+)$$

are t-independent, if $\mu(\tilde{x}) = \mu(\tilde{y})$, where $\tilde{x} = (x, u^-)$, $\tilde{y} = (y, u^+)$, and a positive function $\varphi(s)$ on $\mathbf{R}$ satisfies

$$|\varphi(s)| \leqq \frac{1}{j_1(k, l)},$$

and

$$\varphi(s) = \frac{1}{j_1} \text{ for } s < -R, \quad \varphi(s) = \frac{1}{j_2} \text{ for } s > R,$$

for some $R > 0$. t-independent solutions of the above equation are reparametrized paths of the gradient trajectory of f. Since we assume that the gradient flow is of Morse–Smale type, gradient trajectories are constant paths at critical points of f, if the Morse indices at end points coincide. This observation implies that the induced homomorphism $\phi_{(k,l)_*}$ between homology groups does not depend on the choice of $j > j_1(k, l)$. Therefore homomorphisms $\{\phi_{(k,l)_*}\}$ commute with homomorphisms $\Phi_{(k,l)}$, $\Psi_{(k,l)}$ in the diagram (3.1). Thus $\{\phi_{(k,l)_*}\}$ induces a homomorphism $\phi\colon \widehat{HF}_*(H, J) \to H_{*+n}(M; \mathbf{Z}/2) \otimes \Lambda_\omega$. Since $\{\phi_{(k,l)_*}\}$ are isomorphisms, ϕ is an isomorphism. By the construction, it is easy to see that ϕ is Λ_ω-linear. □

Theorem (3.4) and Theorem (4.7) yield

Theorem 4.8 *Let (M, ω) be a weakly monotone symplectic manifold such that $[\omega] \in Im(H^2(M; \mathbf{Z}) \to H^2(M, \mathbf{R}))$. Then for a generic pair (H, J),*

$$\widehat{HF}_*(H, J) \cong H_{*+n}(M; \mathbf{Z}/2) \otimes \Lambda_\omega .$$

Theorem (1.1) is equivalent to the following

Corollary 4.9 *Let (M, ω) be a weakly monotone symplectic manifold and H a Hamiltonian function such that all periodic solutions of (2.1) are nondegenerate. The number of periodic solutions of (2.1) is bounded below by $\sum_p b_p(M; \mathbf{Z}/2)$, where $b_p(M; \mathbf{Z}/2)$ is the p-th $\mathbf{Z}/2$-Betti number of M.*

Proof. If $[\omega]$ is an integral class, the conclusion is a direct consequence of Theorem (4.8). The same conclusion holds if $[\omega]$ is in $H^2(M; \mathbf{Q})$. We shall show that the general case is reduced to this case. Let $\{x_i\}$ be all periodic solutions of (2.1), $N_i(\varepsilon)$ an ε-neighborhood of the orbit of x_i, and ϕ_i a cut off function, i.e. $\phi_i = 1$ on $N_i(\varepsilon/2)$ and $\phi_i = 0$ outside of $N_i(3\varepsilon/4)$. $\eta_1, \ldots, \eta_q$ denote closed 2-forms on M representing generators of $H^1(M; \mathbf{R})$. For a sufficiently small ε, $N_i(\varepsilon)$ has the same homotopy type as the orbit of x_i, hence $H^2(N_i(\varepsilon); \mathbf{R}) = 0$. Thus $\eta_j|_{N_i(\varepsilon)} = dg_{j,i}$ for some function $g_{j,i}$ on $N_i(\varepsilon)$. $\eta'_j = \eta_j - d(\sum_i \phi_i \cdot g_{j,i})$ is cohomologous to η_j, with support in $M - \cup_i N_i(\varepsilon/2)$. It is easy to see that there exists $\sigma > 0$, such that the equation (2.1) has exactly same number of solutions for symplectic forms $\omega' = \omega + \sum a_k \cdot \eta'_k$ if $|a_k| < \sigma$. Since $H^2(M; \mathbf{Q})$ is dense in $H^2(M; \mathbf{R})$, there exist real numbers a_k such that $\omega' \in H^2(M; \mathbf{Q})$ and $|a_k| < \sigma$. Moreover, there are no J-holomorphic spheres for a generic J calibrated by ω' and tamed by ω. This condition is the only one we use in the proof of Theorem (4.8). Therefore we get the desired estimate. □

Since the weak monotonicity is automatic in dimension 2, 4 and 6, we get

Corollary 4.10 *Let (M, ω) be a closed symplectic manifold of dimension* 2, 4 *or* 6. *If all periodic solutions of* (2.1) *are non-degenerate, the number of periodic solutions is at least* $\sum b_p(M; \mathbf{Z}/2)$.

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Note added in proof

After finishing this paper, we showed that a natural homomorphism $HF_*(H,J) \to \widehat{HF}_*(H,J)$ is surjective. We also noticed that abuse of notation (H^α, J^α), (H^β, J^β) may cause misunderstanding in the proof of Theorem 3.4. Precisely, the isomorphism $\widehat{HF}_*(H_\alpha, J_\alpha) \to \widehat{HF}_*(H_\beta, J_\beta)$ is obtained by composition of $\widehat{HF}_*(H_{s_k}, J_{s_k}) \to \widehat{HF}_*(H_{s_{k+1}}, J_{s_{k+1}})$.