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The critical order of vanishing of automorphic L-functions with large level

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Introduction

An important property of certain L-functions is the order to which they vanish at their critical points. Let $\mathscr{F}_N$ denote the set of all holomorphic (cuspidal) newforms of weight 2 for $\Gamma_0(N)$ with trivial character. For $f \in \mathscr{F}_N$ let $L_f(s) = \sum_{n \geqq 1} a_f(n) n^{-s}$ (where $a_f(1) = 1$) be the associated automorphic L-function. For any primitive Dirichlet character $\chi \bmod q$ with $(q, N) = 1$ the twisted L-function $L_f(s, \chi) = \sum_{n \geqq 1} \chi(n) a_f(n) n^{-s}$ is entire and satisfies a functional equation for which $s = 1$ is in the center of the critical strip. The first main result of this paper gives the existence of many $f \in \mathscr{F}_N$ with non-vanishing $L_f(1, \chi)$ for χ fixed and N a large prime.

Theorem 1. *Suppose that χ is a fixed primitive Dirichlet character modulo q. Then there is a positive absolute constant C and a constant C_q depending only on q such that for prime $N > C_q$ there are at least $CN \log^{-2} N$ forms $f \in \mathscr{F}_N$ for which $L_f(s, \chi) \neq 0$.*

It is well-known that for $N > 3$ a prime the exact number of forms in $\mathscr{F}_N$ is given by $\# \mathscr{F}_N = \frac{1}{12}(N + \alpha(N))$, where $\alpha(N) = -13, -5, -7$, or 1 according to whether $N \equiv 1, 5, 7$, or $11 \pmod{12}$.

Theorem 1 may be compared with known results giving the non-vanishing of various classes of twists of a fixed L-function (see for instance [12, 13]). The general method used in the proof of Theorem 1, which is based on a comparison of mean values, comes from [8] (see also [9] for a different application of this technique). Higher orders of vanishing of twists are investigated in [5] by a different method. Mazur has kindly pointed out to me that by arithmetic

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means one can show that there are at least $c \log p$ forms in $\mathscr{F}_N$ for which $L_f(1) \neq 0$, where p is the largest prime divisor of the numerator of $(N-1)/12$ (see [10]). Hence for principal χ we may state the following corollary of Theorem 1.

Corollary 1. *There is a positive absolute constant C such that there are at least* $CN \log^{-2} N$ *forms* $f \in \mathscr{F}_N$ *for which* $L_f(1) \neq 0$, *provided* $N = 11$ *or* $N > 13$ *is prime.*

This corollary has an interesting application to the basis problem for weight 3/2 in view of results of Gross and Waldspurger (see [6]) connecting the representability of a cusp form by ternary theta series with the non-vanishing of an associated L-function.[1] To describe this, let M_C^* denote Kohnen's space of those modular forms of weight 3/2 for $\Gamma_0(4N)$ (with trivial character) whose nth Fourier coefficient vanishes unless $-n \equiv 0, 1 \pmod 4$ and $(\frac{-n}{N}) \neq 1$. Also, let Θ_N denote the subspace of M_C^* spanned by ternary theta series (see [6] for a detailed description of these.). Now it is known that $\dim M_C^* = N/24 + O(N^{1/2} \log N)$ for N prime. However, in general $\Theta_N \neq M_C^*$, as the example $N = 389$ where $\dim M_C^* = 22$ while $\dim \Theta_N = 21$ shows, this being a reflection of the nontrivial vanishing of an L-function (see [6, p.181.]). On the other hand, Corollary 1 together with [6, Cor.13.6] imply that Θ_N is not too small.

Corollary 2. *There is a positive absolute constant C such that the dimension of* Θ_N *is at least* $CN \log^{-2} N$ *for* $N = 11$ *or prime* $N > 13$.

Subject to standard conjectures, Corollary 1 also gives information about the Mordell–Weil group of certain Abelian varieties. For example, if A is the factor of the Jacobian of $X_0(N)$ determined by $f \in \mathscr{F}_N$ then $L_f(1)$ is conjectured not to vanish if and only if the rank of the Mordell–Weil group of A over $\mathbf{Q}$ is zero. Thus Corollary 1 gives a lower bound for the frequency of this occurrence for a prime level N. Other similar conditional implications of Theorem 1 may also be formulated.

The second main result of this paper is concerned with the order of vanishing at $s = 1$ of the product

$$P_f(s) = L_f(s, \chi_1) L_f(s, \chi_2)$$

of two such L-functions when χ_1 and χ_2 are both real and distinct. The functional equation implies that $\operatorname{ord}_{s=1} P_f(s) \geq 0$ or 1 according to whether $\chi_1\chi_2(-N) = 1$ or -1. Here it will be shown that for χ_1 and χ_2 fixed and N a large prime many $P_f(s)$ achieve this lower bound.

[1] The paper [1] contains a different proof of this criterion which also gives the theta series representation explicitly.

Theorem 2. *Suppose that* $\chi_1 \pmod{q_1}$ *and* $\chi_2 \pmod{q_2}$ *are fixed distinct primitive real Dirichlet characters. Then there are positive constants* C_1 *and* C_2 *depending only on* q_1q_2 *such that there are at least* $C_2 N \log^{-10} N$ *forms* $f \in \mathscr{F}_N$ *with*

$$\operatorname{ord}_{s=1} P_f(s) = \begin{cases} 0 & \text{if } \chi_1\chi_2(-N) = 1 \\ 1 & \text{if } \chi_1\chi_2(-N) = -1, \end{cases}$$

provided $N > C_1$ *is prime.*

All constants in these results are effective. With more work it may be possible to improve slightly the lower bounds in Theorems 1 and 2, but as the presence of a factor $\log^{-1} N$ from Proposition 4 below seems unavoidable, it appears hopeless to use the methods here to remove the $\log N$ factors completely and achieve a positive proportion.

Critical values on average

For the proof of Theorems 1 and 2 different averages of critical values are compared, the averaging being done over $\mathscr{F}_N$, the set of all holomorphic newforms of weight 2 for $\Gamma_0(N)$. For $f \in \mathscr{F}_N$ with $N \geqq 1$ let $(f,f) = \int_{\Gamma_0(N)\backslash H} |f(z)|^2 \, dx\, dy$ be the Petersson norm and set

$$\omega_f = \frac{1}{4\pi(f,f)}. \tag{1}$$

If $f(z) = \sum_{n \geqq 1} a_f(n) e(nz)$ is the Fourier expansion at ∞ then $a_f(n)$ are known to generate a totally real number field and to be algebraic integers which satisfy the multiplicativity relation for positive integers m and n

$$a_f(m) a_f(n) = \sum_{\substack{d \mid (m,n) \\ (d,N)=1}} d\, a_f(mn/d^2) \tag{2}$$

and the Ramanujan bound

$$|a_f(n)| \leqq d(n) n^{1/2}, \tag{3}$$

where $d(n)$ is the divisor function. The numbers $a_f(n)/\sqrt{n}$ are also approximately orthogonal in the following sense.

Lemma 1. *For* m *and* n *positive integers and* N *prime we have the inequality*

$$\left| \sum_{f \in \mathscr{F}_N} \omega_f \frac{a_f(m)}{\sqrt{m}} \frac{a_f(n)}{\sqrt{n}} - \delta_{m,n} \right| \leqq 539\, N^{-3/2} (m,n)^{1/2} \sqrt{mn}.$$

Proof. We employ the absolutely convergent "Petersson formula"

$$\sum_{f \in \mathscr{F}_N} \omega_f \frac{a_f(m)}{\sqrt{m}} \frac{a_f(n)}{\sqrt{n}} = \delta_{m,n} - 2\pi \sum_{c \equiv 0 \pmod N} c^{-1} S(m,n;c) J_1\left(\frac{4\pi\sqrt{mn}}{c}\right) \tag{4}$$

where $S(m,n;c) = \sum_{\substack{a \bmod c \\ (a,c)=1}} e(\frac{ma+n\bar{a}}{c})$ is the Kloosterman sum and $J_1(z)$ is the J-Bessel function, which follows from [2, p. 249] together with the fact that for N prime the newforms of weight 2 form an orthogonal basis for the space of all cusp forms. The stated remainder estimate follows easily from Weil's bound

$$|S(m,n;c)| \leqq (m,n,c)^{1/2} d(c) c^{1/2} \tag{5}$$

and the standard bound for $z \geqq 0$

$$|J_1(z)| \leqq z/2 \tag{6}$$

applied in (4). □

Let χ be a primitive Dirichlet character modulo q with $(q,N) = 1$. The L-function $L_f(s,\chi)$ is known to be entire and to satisfy the functional equation

$$(q\sqrt{N}/2\pi)^s \Gamma(s) L_f(s,\chi) = \varepsilon (q\sqrt{N}/2\pi)^{2-s} \Gamma(2-s) L_f(2-s,\bar{\chi}) \tag{7}$$

where $\varepsilon = \varepsilon_f \chi(N) \tau(\chi)^2 q^{-1}$ with $\varepsilon_f = \pm 1$ depending only on f and where $\tau(\chi)$ is the Gauss sum, see [14]. This gives rise to the following standard representation of $L_f(1,\chi)$ as a rapidly convergent series (see [12, p. 411.]).

Lemma 2. *For any $x > 0$ let $A(x) = \sum_{n \geqq 1} \chi(n) a_f(n) n^{-1} e^{-2\pi n/x}$. Then we have*

$$L_f(1,\chi) = A(x) + \varepsilon \bar{A}(Nq^2/x).$$

When combined with Lemma 1, Lemma 2 yields the following asymptotic formula.

Proposition 1. *Let χ be a fixed primitive character modulo q. Then we have*

$$\sum_{f \in \mathcal{F}_N} \omega_f L_f(1,\chi) = 1 + O(N^{-1/2} \log N)$$

for N prime, the implied constant depending only on q.

Proof. Choosing $x = q^2 N \log N$ in Lemma 2 gives

$$L_f(1,\chi) = \sum_{n \geqq 1} \chi(n) a_f(n) n^{-1} e^{-2\pi n/q^2 N \log N} + O(N^{-6})$$

and applying Lemma 1 with $m = 1$ easily yields the result. □

It may be worth remarking that the apparently inefficient choice of $x = q^2 N \log N$ in Lemma 2 (the smaller choice $x = q\sqrt{N}$ equalizes the two terms there) is made above to avoid the variation of ε_f as f runs over $\mathcal{F}_N$. Since the sign in the functional equation for $P_f(s)$ does not so vary we are still able to obtain corresponding results for $P_f(1)$ and $P_f'(1)$ even though these require approximations which are, in effect, twice as long.

Turning to these let, for $x > 0$,

$$g_0(x) = 4\pi\sqrt{x} K_1(4\pi\sqrt{x}) \tag{8}$$

and

$$g_1(x) = 2K_0(4\pi\sqrt{x}), \tag{9}$$

where K_v is the K-Bessel function. For any primitive characters $\chi_1 \pmod{q_1}$ and $\chi_2 \pmod{q_2}$ let $P_f(s) = L_f(s,\chi_1)L_f(s,\chi_2) = \sum_{\ell \geqq 1} \mathrm{b}_f(\ell)\ell^{-s}$ so that

$$b_f(\ell) = \sum_{mn=\ell} \chi_1(m)\chi_2(n)a_f(m)a_f(n). \tag{10}$$

Define the sums for $x > 0$ and $i = 0, 1$

$$B_i(x) = \sum_{\ell \geqq 1} \mathrm{b}_f(\ell)\ell^{-1} g_i(\ell/x). \tag{11}$$

These are absolutely convergent since from (3) and (10)

$$b_f(\ell) \ll_\varepsilon \ell^{1/2+\varepsilon} \tag{12}$$

while we also have the standard estimates

$$g_0(x) \ll \begin{cases} 1 & \text{for } x \leqq 1 \\ x^{1/4} e^{-4\pi\sqrt{x}} & \text{for } x > 1 \end{cases} \tag{13}$$

and

$$g_1(x) \ll \begin{cases} \log(2/x) & \text{for } x \leqq 1 \\ x^{-1/4} e^{-4\pi\sqrt{x}} & \text{for } x > 1. \end{cases} \tag{14}$$

Lemma 3. *Let $f \in \mathscr{F}_N$ for $N \geqq 1$ and suppose that χ_1 and χ_2 are primitive with $(q_1 q_2, N) = 1$. For any $x > 0$ we have*

$$P_f(1) = B_0(x) + \hat{\varepsilon}\bar{B}_0((Nq_1 q_2)^2/x)$$

while if $P_f(1) = 0$ then for any $x > 0$ we have

$$P'_f(1) = B_1(x) - \hat{\varepsilon}\bar{B}_1((Nq_1 q_2)^2/x)$$

where

$$\hat{\varepsilon} = \chi_1\chi_2(N)(\tau(\chi_1)\tau(\chi_2))^2(q_1 q_2)^{-1}.$$

Proof. We have the integral representations for $i = 0, 1$

$$g_i(x) = \frac{1}{2\pi i} \int_{Re(s)=3/4} (2\pi)^{-2s}\Gamma(s)\Gamma(s-i+1)x^{-s}\,ds. \tag{15}$$

To prove the first statement in Lemma 3, consider that by (15) and (11)

$$B_0(x) = \frac{1}{2\pi i} \int_{(3/4)} x^s(2\pi)^{-2s}\cdot\Gamma(s+1)^2 P_f(s+1)s^{-1}\,ds$$

and this is

$$= P_f(1) + \frac{\hat{\varepsilon}}{2\pi i} \int_{(-3/4)} ((Nq_1q_2)^2/x)^{-s}(2\pi)^{2s}\Gamma(-s+1)^2 \bar{P}_f(-\bar{s}+1)s^{-1}\,ds$$

upon moving the contour and using the functional equation for $P_f(s)$ which follows from (7). Changing variables $s \mapsto -s$ yields the first statement. Similarly,

$$B_1(x) = \frac{1}{2\pi i} \int_{(3/4)} x^s(2\pi)^{-2s}\Gamma(s+1)^2 P_f(s+1)s^{-1}\,ds$$

which, if $P_f(1) = 0$, is

$$= P'_f(1) + \hat{\varepsilon}\bar{B}_1((Nq_1q_2)^2/x),$$

giving the second statement. □

We come now to the main result of this section.

Proposition 2. *Let $\chi_1 \pmod{q_1}$ and $\chi_2 \pmod{q_2}$ be primitive Dirichlet characters such that either $\chi_1 = \bar{\chi}_2$ or χ_1 and χ_2 are real and distinct. In the first case we have*

$$\sum_{f \in \mathscr{F}_N} \omega_f P_f(1) = \prod_{p \mid q_1} (1 - p^{-1}) \log N + c_1 + O(N^{-1/2}\log N)$$

for N prime with $(q_1, N) = 1$, where c_1 and the implied constant depend only on q_1. Otherwise

$$\sum_{f \in \mathscr{F}_N} \omega_f P_f(1) = 2L(1, \chi_1\chi_2) + O(N^{-1/2}\log N)$$

for N prime with $\chi_1\chi_2(-N) = 1$ while

$$\sum_{f \in \mathscr{F}_N} \omega_f P'_f(1) = 2L(1, \chi_1\chi_2)\log N + c_2 + O(N^{-1/2}\log N)$$

for N prime with $\chi_1\chi_2(-N) = -1$, where c_2 and the implied constants depend only on q_1q_2.

Proof. Under our assumptions $B_i = \bar{B}_i$ and $\chi_1\chi_2(N)(\tau(\chi_1)\tau(\chi_2))^2(q_1q_2)^{-1} = \chi_1\chi_2(-N)$. Thus by Lemma 3 with $x = Nq_1q_2$ and (11) we have for prime N with $\chi_1\chi_2(-N) = 1$

$$\sum_{f \in \mathscr{F}_N} \omega_f P_f(1) = 2\sum_f \omega_f \sum_{\ell \geq 1} b_f(\ell)\ell^{-1} g_0(l/Nq_1q_2)$$

and by (10) this is

$$= 2 \sum_{m,n \geq 1} \chi_1(m)\chi_2(n) g_0(mn/Nq_1q_2) \sum_f \omega_f \frac{a_f(m)}{m}\frac{a_f(n)}{n}.$$

By Lemma 1 we get

$$\sum_{f \in \mathcal{F}_N} \omega_f P_f(1) = 2 \sum_{n \geq 1} \chi_1\chi_2(n) g_0(n^2/Nq_1q_2) n^{-1} + R, \tag{16}$$

where

$$R \ll N^{-3/2} \sum_{m,n \geq 1} g_0\left(\frac{mn}{Nq_1q_2}\right)(m,n)^{1/2}. \tag{17}$$

Now the first term on the right hand side of (16) is evaluated using (15) as

$$\frac{1}{\pi i} \int_{(3/4)} L(2s+1, \chi_1\chi_2)(2\pi)^{-2s}\Gamma(s)\Gamma(s+1)(Nq_1q_2)^s \, ds.$$

In case $\chi_1 = \bar{\chi}_2$ this is

$$\prod_{p|q_1} (1-p^{-1})\log N + c_1 + O(N^{-1/2}). \tag{18}$$

Otherwise it is

$$2L(1, \chi_1\chi_2) + O(N^{-1/2}) \tag{19}$$

for N prime with $\chi_1\chi_2(-N) = 1$. The remainder term R in (16) is estimated by the following standard lemma in case $i = 0$.

Lemma 4. *For $i = 0, 1$ we have*

$$\sum_{m,n \geq 1} (m,n)^{1/2} g_i(mn/x) \sim \kappa_i x \log x$$

as $x \to \infty$ for some positive constant κ_i.

Proof. This follows easily from the identity

$$\sum_{m,n \geq 1} (m,n)^{1/2}(mn)^{-s} = \frac{\zeta(2s-1/2)\zeta(s)^2}{\zeta(2s)}$$

and (15). □

Thus $R \ll N^{-1/2}\log N$ by (17) and Lemma 4 so by (16), (18) and (19) we deduce the first two asymptotic formulas in Proposition 2. The last one is proved similarly using the second part of Lemma 3 and Lemma 4 with $i = 1$. □

Non-vanishing critical values

The object of this section is to establish the next Proposition.

Proposition 3. *Let χ be a primitive Dirichlet character modulo q. Then there is a constant C_q depending only on q* such that for prime $N > C_q$

$$\sum_{f \in \mathscr{F}_N : L_f(1,\chi) \neq 0} \omega_f \gg \log^{-1} N,$$

the implied constant being absolute. Let χ_1 and χ_2 be distinct real primitive Dirichlet characters modulo q_1 and q_2, respectively. Then, for $i = 0$ or 1,

$$\sum_{f \in \mathscr{F}_N : P_f^{(i)}(1) > 0} \omega_f \gg \log^{-9} N$$

for N a sufficiently large prime with $\chi_1\chi_2(-N) = (-1)^i$, the implied constants depending only on $q_1 q_2$.

Proof. By Cauchy's inequality we have

$$\left| \sum_{f \in \mathscr{F}_N} \omega_f L_f(1,\chi) \right|^2 \leqq \left(\sum_{f : L_f(1,\chi) \neq 0} \omega_f \right) \left(\sum_{f \in \mathscr{F}_N} \omega_f |L_f(1,\chi)|^2 \right). \tag{20}$$

Thus the first statement of Proposition 3 follows from Proposition 1 and the first statement of Proposition 2 since here $P_f(1) = |L_f(1,\chi)|^2$.

For the second statement we need the next Lemma.

Lemma 5. *Under the assumptions of Proposition 2 we have the estimates*

$$\sum_{f \in \mathscr{F}_N} \omega_f |P_1(1)|^2 \ll \log^9 N$$

for N prime with $\chi_1\chi_2(-N) = 1$ and

$$\sum_{f \in \mathscr{F}_N} \omega_f |P_f'(1)|^2 \ll \log^{11} N$$

for N prime with $\chi_1\chi_2(-N) = -1$. The implied constants depend only on $q_1 q_2$.

Proof. By Lemma 3, (12) and (13) we have for $\chi_1\chi_2(-N) = 1$ that

$$P_f(1) = 2 \sum_{\ell \leqq X} b_f(\ell)\ell^{-1} g_0(\ell/Nq_1q_2) + O(N^{-12}) \tag{21}$$

where $X = Nq_1q_2 \log^2 N$. By using (2), (10) and (13) we can write (21) as

$$P_f(1) = \sum_{\ell \leqq X} c_\ell a_f(\ell) + O(N^{-12}) \tag{22}$$

where $c_\ell \ll d(\ell)\ell^{-1} \log N$. We now employ the following mean value result, which is an immediate consequence of [3, Theorem 1].

Lemma 6. *For N prime and any complex numbers c_n we have*

$$\sum_{f \in \mathscr{F}_N} \omega_f \left| \sum_{l \leqq X} c_l a_f(l) \right|^2 = (1 + O(N^{-1} X \log X)) \sum_{l \leqq X} l |c_l|^2$$

with an absolute implied constant.

Thus by (22), Lemma 6 and the bound $\sum_{\ell \leq X} d^2(\ell)\ell^{-1} \ll \log^4 N$ we get the first estimate of Lemma 5. The second one is similar using (14) in place of (13). □

The second part of Proposition 3 now follows as did the first from Cauchy's inequality, Lemma 5 and the last two statements of Proposition 2 together with the nonvanishing of $L(1, \chi_1\chi_2)$ when $\chi_1 \neq \chi_2$. □

The function ω_f

In order to derive Theorems 1 and 2 it is necessary to estimate ω_f defined in (1) from above. Now ω_f is approximately a density function on $\mathscr{F}_N$ as is shown by the asymptotic formula from Lemma 1 when $m = n = 1$:

$$\sum_{f \in \mathscr{F}_N} \omega_f = 1 + O(N^{-3/2})$$

for N prime. In fact, ω_f is not far from being uniform. We apply a recent important result from [7, 4] which, together with Proposition 3, proves Theorems 1 and 2.

Proposition 4. *For N prime we have the estimate*

$$\omega_f \ll N^{-1} \cdot \log N$$

with an absolute implied constant.

Proof. This follows the extension of the Main Theorem of [4] to holomorphic cusp forms, together with the fact that for prime N no $f \in \mathscr{F}_N$ is a lift from $GL(1)$, see [4, Remark and paragraph following the Main Theorem]. □

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