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Representations of quantum algebras

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Dedicated to Prof. Cao Xihua on his 70th birthday

Introduction

Let $\mathscr{A}$ denote the local ring $\mathbb{Z}[v]_{\mathfrak{m}}$ where v is an indeterminate and $\mathfrak{m}$ is the maximal ideal in $\mathbb{Z}[v]$ generated by $v - 1$ and a fixed *odd* prime p. The residue field $\mathscr{A}/\mathfrak{m} = \mathbf{F}_p$ is denoted by k.

To each Cartan matrix $(a_{ij})_{i,j=1}^n$ Drinfeld [Dr] and Jimbo [Ji] have associated a so-called quantum group U', which is a Hopf algebra over $\mathbb{Q}(v)$ defined by certain generators and relations. Following Lusztig [L 5, L 6] we consider an $\mathscr{A}$-lattice U of U' which is a Hopf algebra over $\mathscr{A}$, and also the "specializations" $U_\Gamma = U \otimes \Gamma$ for various $\mathscr{A}$-algebras Γ.

Firstly we introduce the coordinate algebra $\mathscr{A}[U]$ as a suitable dual of U. Our first main result says that $\mathscr{A}[U]$ is a free $\mathscr{A}$-module (Theorem 1.33). This relies on the connection, established in [loc. cit.], betweeen U_k and the hyperalgebra of the semi-simple algebraic group G_k corresponding to (a_{ij}). Here k is made into an $\mathscr{A}$-algebra by sending v to 1. The point is—and this will be used repeatedly throughout the paper—that this connection allows us to carry over information from the representation theory of G_k to that of U_k.

Next we use the coordinate algebra to set up a general theory of induction. A crucial result here is that induction from the trivial subalgebra as well as from U^0 (see Section 0 for notations) is exact, see Theorem 1.31 and Proposition 2.11. Also, we emphasize the study of induction from "generalized parabolic subalgebras". We check that our induction functors have the standard properties, e.g. Frobenius reciprocity, transitivity and the tensor identity (Section 2). Moreover, we study their behaviour under base change, thereby getting explicit connections to the analogous functors in the representation theory of G_k and $G_{\mathbb{Q}}$, see Section 3.

The above results together with a detailed examination of the rank 1 case (Section 4) then enable us to obtain some deeper results about induction from a "Borel subalgebra". These include analogues of Serre's theorem, Grothendieck's theorem, Kempf's vanishing theorem for dominant characters and Demazure's character formula. Moreover, we show that the concepts and results about good, respectively excellent filtrations carry over to the quantum case, see Section 5.

Consider now a specialization of $\mathscr{A}$ into a field Γ. We develop a Borel-Weil-Bott theory for U_Γ, see Section 6. If the image ζ of v is not a root of 1 then the theory is completely analogous to the classical theory for $G_{\mathbb{Q}}$ (regardless of the characteristic of Γ) whereas if $\mathrm{char}(\Gamma) = 0$ and ζ is a root of 1 then we have a situation resembling the modular representation theory for G_k.

This latter situation is explored further in Section 8 where we prove a linkage principle and a translation principle for U_Γ. An important ingredient in the arguments there is Serre duality (Theorem 7.3) which in turn requires a special case of Bott's theorem.

Everything has now been set up in a way which invites us to define a "Jantzen type" filtration and prove a sum formula. In fact, we obtain several such filtrations and corresponding sum formulas (see Section 10). Working over k this gives filtrations of the classical Weyl modules and it is an interesting question to compare these with the ordinary Jantzen filtration. We conjecture that if the highest weight in question is in the lowest p^2-alcove then the filtrations coincide. As we point out a positive answer to this conjecture would settle Lusztig's conjecture relating the irreducible characters in the quantum and modular case ([L 3]). At least in rank 2 and also for type A_3 the conjecture is true. In fact, in these cases the sum formula together with the translation principle and the Steinberg-Lusztig tensor product theorem give all the irreducible characters, see Section 11.

So far most of our results concern the so-called integrable modules of type 1 (see Section 1). In Section 9 we prove that finite dimensional U_Γ modules are integrable. If v is not specialized to a root of unity, we just reproduce the argument given by Rosso ([R 1]), whereas in the root of unity case we have to work somewhat harder and use both results of Lusztig ([L 3]) and some properties of the Steinberg module. In the course of the proof, we obtain the somewhat surprising result that the category of *finite dimensional* U_Γ-modules has enough projectives (injectives).

Also, in an appendix by the second author it is proved that for type A_n the quantum coordinate algebra, defined in Section 1, coincides with the one studied by Parshall and Wang ([PW 1–2]). The appendix is independent of the results in Sections 2–11.

Some of the results in this paper are contained in the first author's preprint [A 5]. However, the proof of the exactness of induction from U^0 given in [A 5, 1.12] is not correct and also some of the steps in section 4 are incomplete. In this paper we have overcome these difficulties by relying on the relation between U_k and the hyperalgebra for G.

Finally, we acknowledge our debt to G. Lusztig, whose preprints [L 1–6] have both aroused our interest in quantum algebras and provided the start for our work.

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Contents

0. Notation

Throughout the paper we use the following notation, mostly following Lusztig [L 1–6].

$(a_{ij})_{i,j=1}^n$ is a Cartan matrix

$d_1, \ldots, d_n \in \{1, 2, 3\}$ such that $(d_i a_{ij})$ is symmetric

$\mathscr{A}$ $= \mathbb{Z}[v]_{m}$ where v is an indeterminate and m is the ideal in $\mathbb{Z}[v]$ generated by $v - 1$ and an odd prime p. We assume $p > 3$ if (a_{ij}) has a component of type G_2

$\mathscr{A}'$ $= \mathbb{Q}(v)$ the fraction field of $\mathscr{A}$

k $= \mathbf{F}_p$ the residue field of $\mathscr{A}$

Γ an $\mathscr{A}$-algebra

$[m]_d$ $= \dfrac{v^{dm} - v^{-dm}}{v^d - v^{-d}} \in \mathscr{A}$ where $m, d \in \mathbb{N}$

$[m]_d^!$ $= \displaystyle\prod_{j=1}^{m} \frac{v^{dj} - v^{-dj}}{v^d - v^{-d}} = \prod_{j=1}^{m} [j]_d \in \mathscr{A}$ where $m, d \in \mathbb{N}$

$\begin{bmatrix} m \\ t \end{bmatrix}_d$ $= \displaystyle\prod_{j=1}^{t} \frac{v^{d(m-j+1)} - v^{-d(m-j+1)}}{v^{dj} - v^{-dj}} \in \mathscr{A}$ where $m \in \mathbb{Z}$, $t, d \in \mathbb{N}$

(We omit the subscript d if $d = 1$)

ϕ_l $= \dfrac{v^l - 1}{v - l} \in \mathscr{A}$ where $l \in \mathbb{N}$

U' is the quantum algebra over $\mathscr{A}'$ associated to $(a_{i,j})$, i.e. the $\mathscr{A}'$-algebra with generators, $E_i, F_i, K_i, K_i^{-1}, i = 1, \ldots, n$ and relations

$$K_i K_j = K_j K_i, \; K_i K_i^{-1} = 1 = K_i^{-1} K_i$$

$$K_i E_j = v^{d_i a_{ij}} E_j K_i, \; K_i F_j = v^{-d_i a_{ij}} F_j K_i$$

$$E_i F_j - F_j E_i = \delta_{ij} \frac{K_i - K_i^{-1}}{v^{d_i} - v^{-d_i}}$$

$$\sum_{r+s=1-a_{ij}} (-1)^s \begin{bmatrix} 1 - a_{ij} \\ s \end{bmatrix}_{d_i} E_i^r E_j E_i^s = 0 \quad \text{if } i \neq j$$

$$\sum_{r+s=1-a_{ij}} (-1)^s \begin{bmatrix} 1-a_{ij} \\ s \end{bmatrix}_{d_i} F_i^r F_j F_i^s = 0 \quad \text{if } i \neq j$$

$E_i^{(m)}$ $= \dfrac{E_i^m}{[m]_{d_i}^!}$ for $m \geqq 0$

$F_i^{(m)}$ $= \dfrac{F_i^m}{[m]_{d_i}^!}$ for $m \geqq 0$

$\begin{bmatrix} K_i; c \\ t \end{bmatrix}$ $= \prod_{s=1}^{t} \dfrac{K_i v^{d_i(c-s+1)} - K_i^{-1} v^{-d_i(c-s+1)}}{v^{sd_i} - v^{-sd_i}}$

$\begin{bmatrix} K_i \\ t \end{bmatrix}$ $= \begin{bmatrix} K_i; 0 \\ t \end{bmatrix}$

U the quantum algebra over $\mathscr{A}$ introduced in [L 5, L 6], i.e. the $\mathscr{A}$-subalgebra of U' generated by $E_i^{(N)}, F_i^{(N)}, K_i, K_i^{-1}, i = 1, \ldots, n, N \geqq 0$

U^+ (resp. U^-) the $\mathscr{A}$-subalgebra of U generated by $E_i^{(N)}$ (resp. $F_i^{(N)}$), $i = 1, \ldots, n, N \geqq 0$

U^0 the $\mathscr{A}$-subalgebra of U generated by

$$K_i, K_i^{-1}, \begin{bmatrix} K_i; c \\ t \end{bmatrix}, i = 1, \ldots, n, t \geqq 0$$

$U^\flat$ $= U^- U^0$

$U^\natural$ $= U^0 U^+$

U_I resp. $U(I)$ the subalgebra of U generated by $\{E_i^{(r)}, F_i^{(s)}, K_i^{\pm 1} | i \in I, r, s \geqq 0\}$ resp. by $U^\flat$ and $\{E_i^{(r)} | i \in I, r \geqq 0\}$ where $I \subset \{1, \ldots, n\}$. When $I = \{i\}$ we simply write U_i resp. $U(i)$ instead of U_I resp. $U(I)$

U_Γ $= U \otimes \Gamma$ for any $\mathscr{A}$-algebra Γ. Same definition of $U_\Gamma^+, U_\Gamma^-, U_\Gamma^0, U_\Gamma^\flat$ and $U_\Gamma^\natural$. By [L 5], $U_{\mathscr{A}'}$ identifies with U'

U' is a Hopf algebra with comultiplication Δ, antipode S and counit ε defined by

$$\Delta(E_i) = E_i \otimes 1 + K_i \otimes E_i, \quad \Delta(F_i) = F_i \otimes K_i^{-1} + 1 \otimes F_i, \quad \Delta(K_i) = K_i \otimes K_i$$

$$S(E_i) = -K_i^{-1} E_i, \qquad S(F_i) = -F_i K_i, \qquad S(K_i) = K_i^{-1}$$

$$\varepsilon(E_i) = 0 = \varepsilon(F_i), \qquad \varepsilon(K_i) = 1$$

and U is a sub-Hopf-algebra of U' (see [L5])

$\alpha_1, \ldots, \alpha_n$ a set of simple roots associated to (a_{ij}), i.e. $\langle \alpha_i, \alpha_j^\vee \rangle = a_{ij}$

R (resp. R^+) the corresponding root system (resp. positive roots). We set $N = |R^+|$

X the set of weights, i.e. $X = \mathbb{Z}^n$. If $\lambda = (\lambda_1, \ldots, \lambda_n) \in X$, we write $\lambda_i = \langle \lambda, \alpha_i^\vee \rangle$, $i = 1, \ldots, n$

X^+ the set of dominant weights, i.e. $X^+ = \{\lambda \in X | \langle \lambda, \alpha_i^\vee \rangle \geqq 0, i = 1, \ldots, n\}$

W — the Weyl group corresponding to R. There are two actions of W on X. The first one is the natural one, given by $s_\alpha(\lambda) = \lambda - \langle \lambda, \alpha^\vee \rangle \alpha$, for any $\alpha \in R$, $\lambda \in X$. The second is the dot action given by $w \cdot \lambda = w(\lambda + \rho) - \rho$, for any $w \in W, \lambda \in X$. Here $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$. For $i = 1, \ldots, n$ we set $s_i = s_{\alpha_i}$

W_l — the affine Weyl group corresponding to W and a positive integer l. It is generated by the reflections $s_{\alpha,r}: X \to X, \alpha \in R^+, r \in \mathbb{Z}$, where $s_{\alpha,r} \cdot \lambda = s_\alpha \cdot \lambda + rl\alpha, \lambda \in X$

1. The quantum coordinate algebra

The aim of this section is to define the quantum coordinate algebra as a suitable dual of the quantum algebra.

We start with some generalities on characters. Since U^0 is a commutative Hopf algebra, the set of characters of U^0 is a group, where multiplication and inverse are defined as follows. If χ, χ' are characters, then $\chi\chi' = (\chi \otimes \chi') \circ \Delta$ and $\chi^{-1} = \chi \circ S$.

Let X be the weight lattice, Σ the group $\{\pm 1\}^n$, and $\tilde{X}$ the direct product $\Sigma \times X$. Then:

Lemma 1.1. (*i*) *For each* $(\sigma, \lambda) \in \tilde{X}$, *there exists a unique algebra homomorphism* $\chi_{\sigma,\lambda}$: $U^0 \to \mathscr{A}$ *such that:*

$$\chi_{\sigma,\lambda}(K_i) = \sigma_i v^{d_i\lambda_i} \quad \textit{and} \quad \chi_{\sigma,\lambda}\left(\begin{bmatrix} K_i \\ t \end{bmatrix}\right) = (\sigma_i)^t \begin{bmatrix} \lambda_i \\ t \end{bmatrix}_{d_i}$$

Moreover, $\chi_{\sigma,\lambda}$ *satisfies:* $\chi_{\sigma,\lambda}\left(\begin{bmatrix} K_i; c \\ t \end{bmatrix}\right) = (\sigma_i)^t \begin{bmatrix} \lambda_i + c \\ t \end{bmatrix}_{d_i}$ *for all* $c \in \mathbb{Z}$.

(*ii*) *If* $(\sigma, \lambda), (\tau, \mu) \in \tilde{X}$, *then* $\chi_{\sigma,\lambda}\chi_{\tau,\mu} = \chi_{\sigma\tau,\lambda+\mu}$. *Therefore, the map:* $(\sigma, \lambda) \mapsto \chi_{\sigma,\lambda}$ *is a group homomorphism.*

Proof. (*i*) U^0 is a subalgebra of $U^0_{\mathscr{A}'}$, and the latter is a Laurent polynomial ring over $\mathscr{A}'$, in the variables $K_i^{\pm 1}$. Therefore, there exists an algebra homomorphism $\chi_{\sigma,\lambda}: U^0_{\mathscr{A}'} \to \mathscr{A}'$, such that: $\chi_{\sigma,\lambda}(K_i) = \sigma_i v^{d_i\lambda_i}$. Since:

$$\begin{bmatrix} K_i; c \\ t \end{bmatrix} = \prod_{s=1}^{t} \frac{K_i v^{d_i(c-s+1)} - K_i^{-1} v^{-d_i(c-s+1)}}{v^{d_i s} - v^{-d_i s}}$$

we obtain that:

$$\chi_{\sigma,\lambda}\left(\begin{bmatrix} K_i; c \\ t \end{bmatrix}\right) = (\sigma_i)^t \prod_{s=1}^{t} \frac{v^{d_i(\lambda_i+c-s+1)} - v^{-d_i(\lambda_i+c-s+1)}}{v^{d_i s} - v^{-d_i s}} = (\sigma_i)^t \begin{bmatrix} \lambda_i + c \\ t \end{bmatrix}_{d_i}$$

which belongs to $\mathscr{A}$. Hence, by restriction, $\chi_{\sigma,\lambda}$ induces an algebra homomorphism: $U^0 \to \mathscr{A}$ with the required properties. Uniqueness follows from the fact that the monomials:

$\prod_{s=1}^{n} K_i^{\delta_i} \begin{bmatrix} K_i \\ t_i \end{bmatrix}$, where $t_i \in \mathbb{N}$, $\delta_i \in \{0, 1\}$ form an $\mathscr{A}$-basis of U^0. See [L 6] (*ii*) We have: $\Delta(K_i) = K_i \otimes K_i$ and:

$$\Delta\left(\begin{bmatrix} K_i \\ t \end{bmatrix}\right) = \sum_{s=0}^{t} \begin{bmatrix} K_i \\ t-s \end{bmatrix} K_i^{-s} \otimes K_i^{t-s} \begin{bmatrix} K_i \\ s \end{bmatrix}.$$

Therefore:

$$\chi_{\sigma,\lambda}\chi_{\tau,\mu}\left(\begin{bmatrix} K_i \\ t \end{bmatrix}\right) = (\sigma_i\tau_i)^t \sum_{s=0}^{t} \begin{bmatrix} \lambda_i \\ t-s \end{bmatrix}_{d_i} v^{-sd_i\lambda_i} v^{(t-s)d_i\mu_i} \begin{bmatrix} \mu_i \\ s \end{bmatrix}_{d_i}$$

Now, by [L 5, 2.3 (g10)], the R.H.S. is equal to $\chi_{\sigma\tau,\lambda}\left(\begin{bmatrix} K_i, \mu_i \\ t \end{bmatrix}\right)$, which is:

$$(\sigma_i\tau_i)^t \begin{bmatrix} \lambda_i + \mu_i \\ t \end{bmatrix}_{d_i} = \chi_{\sigma\tau,\lambda+\mu}\left(\begin{bmatrix} K_i \\ t \end{bmatrix}\right).$$

Hence Lemma 1.1 is proved. □

We will sometimes denote $\chi_{\sigma,\lambda}$ simply by: λ_σ.

Remark. Lemma 1.1 is implicit in Lusztig's work.

1.2. If M is a U^0-module, and χ a character of U^0, the χ-weight space of M is: $M_\chi = \{x \in M \mid ux = \chi(u)x \text{ for all } u \in U^0\}$. If M, N are U^0-modules, then $M \otimes N$ is a U^0-module, and $M_\chi \otimes N_{\chi'} \subseteq (M \otimes N)_{\chi+\chi'}$.

1.3. Remark. The emphasis on the characters λ_σ comes from the following fact: If $\mathscr{A} \to \Gamma$ is a specialization of $\mathscr{A}$ into a field Γ, then any finite dimensional U_Γ-module is the (direct) sum of its weight-spaces M_{λ_σ}. This will be proved in Section 9.

1.4. Let M be a U-module. Then $E_i^{(r)} M_{\sigma,\lambda} \subseteq M_{\sigma,\lambda+r\alpha_i}$, and $F_j^{(s)} M_{\sigma,\lambda} \subseteq M_{\sigma,\lambda-s\alpha_j}$. Therefore, if we define:

$$\mathcal{O}_\sigma(M) = \bigoplus_{\lambda \in X} M_{\sigma,\lambda} \text{ then we have the:}$$

Lemma. *Each $\mathcal{O}_\sigma(M)$ is a U-submodule of M.*

1.5. Now, we define: $F_\sigma(M) = \{x \in \mathcal{O}_\sigma(M) \mid E_i^{(r)}x = F_i^{(r)}x = 0, 1 \leqq i \leqq n, r \gg 0\}$. Then, we also have the:

Lemma. *Each $F_\sigma(M)$ is a U-submodule of $\mathcal{O}_\sigma(M)$.*

Proof. Let $x \in F_\sigma(M)$. We want to prove that: $E_j^{(s)}x, F_j^{(s)}x \in \mathcal{O}_\sigma(M)$ for all j, s. For this, we have to check that these elements are killed by all $E_i^{(r)}$ and $F_i^{(r)}$ when $r \gg 0$. But this follows from the commutation relations given in [L 6, Section 5, and 6.5]. □

1.6. For each $\sigma \in \Sigma$, we introduce the category $\mathscr{C}_\sigma$ of those U-modules M such that $M = F_\sigma(M)$. These are called integrable U-modules of type σ. When $\sigma = 1$, we denote the corresponding category simply by $\mathscr{C}$, and we omit the subscript 1 in the notation elsewhere as well.

We claim that the categories $\mathscr{C}_\sigma$ are all isomorphic to $\mathscr{C}$. In fact, for each $\sigma \in \Sigma$ the character $\chi_{\sigma,0}$ of U^0 extends to a character of U, which we denote by ε_σ. Observe that ε_1 is nothing but ε, the co-unit of U. Let $\mathscr{A}_\sigma$ denote the U-module $\mathscr{A}$, on which U acts by the character ε_σ. Clearly, tensoring by $\mathscr{A}_\sigma$ gives an isomorphism of categories: $\mathscr{C}_\sigma \simeq \mathscr{C}$.

Therefore, we can concentrate without loss of generality on the category $\mathscr{C}$.

1.7. Let $\mathscr{A} \to \Gamma$ be a specialization of $\mathscr{A}$ into a field Γ. As we shall see in Section 9 the characters $\chi_{\sigma,\lambda} \otimes 1$, $(\sigma, \lambda) \in \tilde{X}$, of U_Γ^0 are pairwise distinct. If M is a U_Γ^0-module, the weight spaces $M_{\sigma,\lambda}$ are defined in the obvious way and their sum is a direct sum. If M is a U_Γ-module then $F(M)$ is defined as in 1.5, and the category $\mathscr{C}_\Gamma$ consists of those M such that $M = F(M)$. For $M \in \mathscr{C}_\Gamma$ such that all weight spaces are finite dimensional, we set as usual:

$$\mathrm{ch}(M) = \sum_{\lambda \in X} \dim_\Gamma(M_\lambda) e^\lambda .$$

Also if $M \in \mathscr{C}$ is such that each M_λ is a finite free $\mathscr{A}$-module, we set:

$$\mathrm{ch}(M) = \sum_{\lambda \in X} \mathrm{rank}_{\mathscr{A}}(M_\lambda) e^\lambda .$$

1.8. We now define an induction functor $H: \{\mathscr{A}\text{-modules}\} \to \mathscr{C}$. Firstly, let $\mathscr{I}$ be the set of two-sided ideals I of U which satisfy the following conditions:

(1) U/I is a finite $\mathscr{A}$-module

(2) $I \cap U^0$ contains a finite intersection of ideals $\mathrm{Ker}(\chi_\lambda)$, $\lambda \in X$.

We shall define below a functor H, called induction from $\mathscr{A}$ to U, such that, for an $\mathscr{A}$-module M, $H(M)$ will coincide with

$$(*) \qquad \{f \in \mathrm{Hom}_{\mathscr{A}}(U, M) \mid f(I) = 0 \text{ for some } I \in \mathscr{I}\} .$$

and we shall define the *quantum coordinate algebra* $\mathscr{A}[U]$ to be $H(\mathscr{A})$.

Our aim is to prove that induction from $\mathscr{A}$ to U is an exact functor, and that $\mathscr{A}[U]$ is a free $\mathscr{A}$-module. The definition used in (*) above has the aesthetic advantage of being intrinsic, and making no use of a particular U-module structure on $\mathrm{Hom}_{\mathscr{A}}(U, M)$. But in order to investigate the properties of H, we have to work with a more explicit definition, which will be shown to be equivalent to the first one.

1.9. The $\mathscr{A}$-module $\mathscr{H}(M) = \mathrm{Hom}_{\mathscr{A}}(U, M)$ carries two structures of (left) U-modules, γ and δ, defined as follows:

$$\text{if } u \in U, \theta \in \mathscr{H}(M), x \in U \text{ then } (\gamma(u)\theta)(x) = \theta(S(u)x) \text{ and } (\delta(u)\theta)(x) = \theta(xu) .$$

Clearly, the subset considered in 1.8(*) is both a γ and δ-submodule of $\mathscr{H}(M)$. Now, assume that $\theta \in \mathscr{H}(M)$ satisfy $\theta(I) = 0$ for some $I \in \mathscr{I}$, see 1.8. Then the $\delta(U)$-submodule N generated by θ is a finite $\mathscr{A}$-module, and is the direct sum of weight spaces N_λ, where $\lambda \in X$. From this, it follows that: $\delta(E_i^{(r)})\theta = \delta(F_i^{(r)})\theta = 0$ for all i, and $r \gg 0$. Hence, we obtain: $\theta \in F_\delta(\mathscr{H}(M))$. (Here, F is the functor defined in 1.5, and the subscript δ means that $\mathscr{H}(M)$ is considered as a $\delta(U)$-module).

Remark. Of course, we also obtain $\theta \in F_\gamma(\mathscr{H}(M))$.

1.10. Now, we take as a definition:

Definition. $H(M) = F_\delta(\mathscr{H}(M))$.

(In fact, we shall see later (Corollary 1.30) that this definition coincides with the one proposed in 1.8 (*)).

Lemma 1.11. *Let* $M \in \mathscr{C}$, $\lambda \in X$, $0 \neq x \in M_\lambda$. *Set* $r_i = \mathrm{Max}\{r | E_i^{(r)} x \neq 0\}$ *and* $s_i = \mathrm{Max}\{s | F_i^{(s)} x \neq 0\}$. *Then, for each i, we have:* $s_i - r_i = \lambda_i$.

Proof. We will use the following commutation relations (see [L 6, 6.5 (a2)]):

$$E_i^{(r)} F_i^{(s)} = \sum_{0 \leq t \leq r,s} F_i^{(s-t)} \begin{bmatrix} K_i; 2t-r-s \\ t \end{bmatrix} E_i^{(r-t)} \tag{1}$$

$$F_i^{(s)} E_i^{(r)} = \sum_{0 \leq t \leq r,s} E_i^{(r-t)} \begin{bmatrix} K_i^{-1}; 2t-r-s \\ t \end{bmatrix} F_i^{(s-t)} \tag{2}$$

Set $y = E_i^{(r_i)} x$. Then y has weight $\lambda + r_i \alpha_i$, and $E_i^{(s)} y = 0$ for $s > 0$. Let s be large enough so that $F_i^{(s)} y = 0$. Then, by (1), we have:

$$0 = E_i^{(s)} F_i^{(s)} y = \begin{bmatrix} K_i \\ s \end{bmatrix} y = \begin{bmatrix} \lambda_i + 2r_i \\ s \end{bmatrix}_{d_i} y$$

From this we deduce that $\lambda_i + 2r_i \geq 0$. Now, let $z = F_i^{(\lambda_i + 2r_i)} y$. We claim that $z \neq 0$. In fact, by (1) we have: $E_i^{(\lambda_i + 2r_i)} z = y \neq 0$. On the other hand, by (2) we have:

$$0 \neq z = \sum_t E_i^{(r_i - t)} \begin{bmatrix} K_i^{-1}; 2t - \lambda_i - 3r_i \\ t \end{bmatrix} F_i^{(\lambda_i + 2r_i - t)} x .$$

Hence $\lambda_i + 2r_i - t \leq s_i$ for some $t \leq r_i$, and therefore $\lambda_i + r_i \leq s_i$.

Now, consider $y' = F_i^{(s_i)} x$. Then y' has weight $\lambda - s_i \alpha_i$, and $F_i^{(t)} = 0$ for all $t > 0$. As before we obtain that $\lambda_i - 2s_i \leq 0$, and $z' = E_i^{(2s_i - \lambda_i)} y' \neq 0$, and $s_i - \lambda_i \leq r_i$. Therefore, we conclude that $s_i - r_i = \lambda_i$. □

1.12. Remark. Keep the notations of the lemma, and let U^i be the subalgebra of U generated by $U^0, E_i^{(r)}, F_i^{(s)}, r, s \geq 0$. From the commutation formulas 1.11 (1)–(2), we deduce that the $\mathscr{A}$-span of the elements $\{F_i^{(s)} E_i^{(r)} x | 0 \leq r \leq r_i, 0 \leq s \leq \lambda_i + 2r\}$ is a U^i-submodule of M. This is the U^i-submodule generated by x, and it is a finite $\mathscr{A}$-module. This will be used in the proof of the next:

Lemma 1.13. *Let* $M \in \mathscr{C}$. *If* λ *is a weight of* M, *then so is* $w\lambda$, *for any* $w \in W$.

Proof. We can reduce to the case where $w = s_i$, a simple reflection. Let $0 \neq x \in M_\lambda$. By the preceding remark, the U^i-submodule N generated by x is a finite $\mathscr{A}$-module. Let $\bar{N} = N \otimes k$. Then $\bar{N}$ is a finite dimensional U_k^i-module. Moreover, $\bar{K}_i = K_i \otimes 1$ acts as the identity on $\bar{N}$, hence $\bar{N}$ is a $U_k^i/(\bar{K}_i - 1)$-module. By [L 5, 6.7], the latter algebra identifies with the hyperalgebra $\bar{U}_k(SL_2)$. Moreover, since each $\begin{bmatrix} K_i \\ t \end{bmatrix}$ corresponds under this isomorphism to the usual element $\binom{H_i}{t}$ of $\bar{U}_k^0$ (hyperalgebra of the torus), we obtain that the decomposition: $N = \oplus_\mu N_\mu$ induces the decomposition $\bar{N} = \oplus_\mu \bar{N}_\mu$ of $\bar{N}$ into SL_2-weight spaces. Now, $N_\lambda \neq 0$, hence $\bar{N}_\lambda \neq 0$ by Nakayama. Then, by SL_2-theory, $\bar{N}_{s_i\lambda} \neq 0$, and therefore $N_{s_i\lambda} \neq 0$. □

1.14. In order to investigate the properties of the "universal $\mathscr{A}$-finite highest weight modules", we need to develop the machinery of Joseph's induction functors. Define a category $\mathscr{C}^\natural$ as follows.

If M is a $U^\natural$-module, set $F(M) = \{x \in \Sigma_{\lambda \in X} M_\lambda | E_i^{(r)} x = 0 \text{ for all } i \text{ and } r \gg 0\}$. We say that $M \in \mathscr{C}^\natural$ if $M = F(M)$. We denote by $\mathscr{C}_f^\natural$, resp. $\mathscr{C}_f$ the category of $\mathscr{A}$-finite objects in $\mathscr{C}^\natural$, resp. $\mathscr{C}$. Following Joseph [Jo] (see also [Do, section 12.3]), we define a functor $D: \mathscr{C}_f^\natural \to \mathscr{C}_f$ as follows:

Proposition. *Let $N \in \mathscr{C}_f^\natural$. Set $M = U \otimes_{U^\natural} N$ and let $\mathscr{S}$ be the set of U-submodules K of M such that M/K is a finite $\mathscr{A}$-module. Then $\mathscr{S}$ has a unique minimal element K_0. We define: $D(N) = M/K_0$.*

Proof. Since $N \in \mathscr{C}_f^\natural$, then the weights of N form a finite set $\Omega \subseteq X$. There is a U^--isomorphism $M \simeq U^- \otimes N$, and therefore all weights of M belong to $\Omega' = \Omega + \mathbb{N}R^-$, and all weight spaces are finite $\mathscr{A}$-modules. Observe that Ω' only contains finitely many dominant weights, and therefore the set $\Omega'' = W(X^+ \cap \Omega')$ is also finite.

Now, let $K \in \mathscr{S}$. By the previous lemma, the set of weights of M/K is W-stable, and is therefore contained in Ω''. It follows that K contains the $\mathscr{A}$-submodule $M' = \oplus_{\mu \notin \Omega''} M_\mu$. Conversely, let K_0 be the U-submodule of M generated by M'. Then the set of weights of M/K_0 is contained in Ω'', and is therefore finite. Since all weight spaces in M are finite $\mathscr{A}$-modules, we conclude that M/K_0 is a finite $\mathscr{A}$-module. It follows that K_0 is the unique minimal element of $\mathscr{S}$. □

1.15. Remark. For $\lambda \in X$, we denote by $\mathscr{A}_\lambda$ the $U^\natural$-module $\mathscr{A}$ on which $U^\natural$ acts by the character χ_λ. We simply write $D(\lambda)$ for $D(\mathscr{A}_\lambda)$. Observe that if $\lambda \notin X^+$ then the dominant conjugate of λ does not belong to $\lambda + \mathbb{N}R^-$. Hence, with the notations of the above proof, we have $\lambda \notin \Omega''$. Therefore we conclude: if $\lambda \notin X^+$ then $D(\lambda) = 0$.

1.16. We leave it to the reader to check that D is a right exact covariant functor.

1.17. For each $N \in \mathscr{C}_f^\natural$, there is a natural $U^\natural$-homomorphism $\sigma: N \to D(N)$. Then, we have the:

Proposition. (Frobenius reciprocity.) *Let $N \in \mathscr{C}_f^\natural$, $E \in \mathscr{C}_f$. For any $\varphi \in \mathrm{Hom}_{U^\natural}(N, E)$ there exists a unique $\tilde{\varphi} \in \mathrm{Hom}_U(D(N), E)$ such that $\tilde{\varphi} \circ \sigma = \varphi$. Moreover,* $\mathrm{Im}(\tilde{\varphi})$ *is the U-submodule of E generated by* $\mathrm{Im}(\varphi)$.

Proof. Clear. □

1.18. Let E be a U-module, and let γ be an anti-endomorphism of U. Then $\mathrm{Hom}_{\mathscr{A}}(E, \mathscr{A})$ is made into a U-module as follows:

$$\text{if } f \in \mathrm{Hom}_{\mathscr{A}}(E, \mathscr{A}),\, u \in U,\, x \in E, \text{ then } (u.f)(x) = f(\gamma(u)x)\,.$$

If $\gamma = S$, the antipode of U, then the resulting U-module is denoted by E^*. But, since S is bijective (see [L 6, 1.1 (c1)]), we can also take $\gamma = S^{-1}$, and then the resulting U-module is denoted by E^t.

Now, assume that as an $\mathscr{A}$-module E is free of finite rank. Then we have U-isomorphisms: $(E^*)^t \simeq E \simeq (E^t)^*$, and also $\mathscr{A}$-isomorphisms: $E \otimes E^* \simeq \mathrm{End}_{\mathscr{A}}(E) \simeq E^t \otimes E$. These, composed with the injection: $\mathscr{A} \hookrightarrow \mathrm{End}_{\mathscr{A}}(E)$, $a \mapsto a\, id_E$, give injections: $\mathscr{A} \overset{\tau}{\hookrightarrow} E \otimes E^*$ and $\mathscr{A} \overset{\tau'}{\hookrightarrow} E^t \otimes E$. Then, regarding $\mathscr{A}$ as a U-module via the co-unit ε, we have the:

Proposition. *(i) The maps τ and τ' are U-homomorphisms.*

(ii) The contraction maps: $E^* \otimes E \xrightarrow{c} \mathscr{A}$ and $E \otimes E^t \xrightarrow{c'} \mathscr{A}$ are U-homomorphisms.

(ii) For any U-modules M, N we have isomorphisms:

$$\mathrm{Hom}_U(M, N \otimes E) \simeq \mathrm{Hom}_U(M \otimes E^t, N)$$

$$\text{and } \mathrm{Hom}_U(E^* \otimes M, N) \simeq \mathrm{Hom}_U(M, E \otimes N).$$

Proof. (iii) follows from (i) and (ii), which are easily checked. □

1.19. Proposition. (Tensor identities.) *Let $N \in \mathscr{C}_f^\natural$, $E \in \mathscr{C}_f$. Assume that E is a finite free $\mathscr{A}$-module. Then, there are U-isomorphisms: $D(E \otimes N) \simeq E \otimes D(N)$ and $D(N \otimes E) \simeq D(N) \otimes E$.*

Proof. Denote by f and g the natural $U^\natural$-homomorphisms $N \to D(N)$ and $E \otimes N \to D(E \otimes N)$. By 1.17, there exist U-homomorphisms $\varphi: D(E \otimes N) \to E \otimes D(N)$ and $\psi': D(N) \to E^t \otimes D(E \otimes N)$ such that $\varphi \circ g = 1 \otimes f$ and $\psi' \circ f = (1 \otimes g) \circ (\tau' \otimes 1)$. By 1.18, ψ' corresponds to some ψ: $E \otimes D(N) \to D(E \otimes N)$ such that $\psi \circ (1 \otimes f) = g$. Then we have: $\psi \circ \varphi \circ g = g$, hence by 1.17 $\psi \circ \varphi$ is the identity on $D(E \otimes N)$. Now, by 1.18 and 1.17 we have isomorphisms:

$$\begin{aligned} \mathrm{Hom}_U(E \otimes D(N), E \otimes D(N)) &\simeq \mathrm{Hom}_U(D(N), E^t \otimes E \otimes D(N)) \\ &\simeq \mathrm{Hom}_{U^\natural}(N, E^t \otimes E \otimes D(N)) \\ &\simeq \mathrm{Hom}_{U^\natural}(E \otimes N, E \otimes D(N)) . \end{aligned}$$

Therefore, the equality: $\varphi \circ \psi \circ (1 \otimes f) = \varphi \circ g = 1 \otimes f$ shows that $\varphi \circ \psi$ is the identity on $E \otimes D(N)$. Hence, φ and ψ are reciprocal isomorphisms.

The second isomorphism is proved similarly, using E^* instead of E^t. □

1.20. For each $\lambda \in X$, let $M(\lambda) = U \otimes_{U^\natural} \mathscr{A}_\lambda$ be the Verma module with highest weight λ. Observe that $M(\lambda) \simeq U/I(\lambda)$, where $I(\lambda)$ is the left ideal of U generated by $\mathrm{Ker}(\chi_\lambda)$ when we regard here χ_λ as a character of $U^\natural$.

A U-module M is said to be a module of highest weight λ if it is generated by an element x of weight λ such that $E_i^{(r)} x = 0$ for all i and $r > 0$. Clearly, any such M is a quotient of $M(\lambda)$. Moreover, if M is a finite $\mathscr{A}$-module then it is a quotient of $D(\lambda)$, and necessarily $\lambda \in X^+$ (see 1.15). Therefore, for $\lambda \in X^+$, $D(\lambda)$ is the universal $\mathscr{A}$-finite U-module of highest weight λ.

We have the following description of $D(\lambda)$. Let $J^-(\lambda)$ be the left ideal of U^- generated by all $F_i^{(s_i)}$, where $s_i > \lambda_i$ (recall $\lambda \in X^+$), take $x_\lambda \in M(\lambda)$ of weight λ and let $N(\lambda)$ be the U^--submodule $J^-(\lambda) x_\lambda$ of $M(\lambda)$. Then:

Proposition. *(i) $N(\lambda)$ is a U-submodule of $M(\lambda)$. Equivalently, $J(\lambda) = J^-(\lambda) + I(\lambda)$ is a left ideal of U.*

(ii) $M(\lambda)/N(\lambda)$ is the largest $\mathscr{A}$-finite quotient module of $M(\lambda)$. In other words:

$$D(\lambda) = M(\lambda)/N(\lambda) = U/J(\lambda) .$$

Remark. It follows from (ii) that $D(\lambda)$ is the universal $\mathscr{A}$-finite highest weight module with highest weight λ.

Proof. (i) Let $s > \lambda_i$ and $r > 0$. By the commutation formula 1.11(1), and since x_λ has weight λ, we have:

$$E_i^{(r)} F_i^{(s)} x_\lambda = F_i^{(s-r)} \begin{bmatrix} \lambda_i + r - s \\ r \end{bmatrix}_{d_i} x_\lambda (\text{this being } 0 \text{ if } r > s) .$$

Assume: $0 \leqq s - r \leqq \lambda_i$. Then $0 \leqq \lambda_i - s + r < r$ and therefore

$$\begin{bmatrix} \lambda_i + r - s \\ r \end{bmatrix}_{d_i} = 0.$$

It follows that the U^--submodule of $M(\lambda)$ generated by all $\{F_i^{(s_i)} x_\lambda | s_i > \lambda_i\}$ is actually a U-submodule.

(ii) If K is a U-submodule of $M(\lambda)$ such that $M(\lambda)/K$ is $\mathscr{A}$-finite then, by Lemma 1.11, $F_i^{(s)} x_\lambda \in K$ whenever $s > \lambda_i$, and therefore K contains $N(\lambda)$. Conversely, let $Q = M(\lambda)/N(\lambda)$. By Lemma 1.5, $F(Q)$ is a U-submodule of Q, and since it contains the generator x_λ we conclude that $F(Q) = Q$. Hence, by Lemma 1.13, the set of weights of Q is W-stable. Then, as in the proof of 1.14, we conclude that Q has only finitely many weight spaces, and is therefore $\mathscr{A}$-finite. □

1.21. The following criterion of freeness will be useful.

Lemma. *Let A be a local domain, k the residue field and K the fraction field. Let M be a finite A-module such that* $\dim_K(M \otimes K) = \dim_k(M \otimes k)$. *Then M is a free A-module.*

Proof. Let $x_1, \ldots, x_m \in M$ such that their images form a basis of $M \otimes k$. By Nakayama's lemma, the x_i's generate M and therefore the $x_i \otimes 1$'s generate $M \otimes K$. But $\dim_K(M \otimes K) = \dim_k(M \otimes k)$, hence the $x_i \otimes 1$'s are linearly independent. It follows that $\{x_1, \ldots, x_m\}$ is a free A-basis of M. □

Proposition 1.22. *Let $\lambda \in X^+$. Then $D(\lambda)$ is a free $\mathscr{A}$-module, and its character is given by Weyl's formula.*

Proof. We compare the dimension of $D(\lambda) \otimes \mathscr{A}' = D(\lambda)_{\mathscr{A}'}$ and $D(\lambda) \otimes k = D(\lambda)_k$. Firstly, $D(\lambda)_{\mathscr{A}'}$ is a finite dimensional quotient of $M(\lambda)_{\mathscr{A}'}$. After ([L 2]), $M(\lambda)_{\mathscr{A}'}$ has a unique such quotient, and its character is given by Weyl's formula.

On the other hand, $D(\lambda)_k$ is a finite dimensional U_k-module. Moreover, since v is specialized to 1, then each K_i acts as the identity on $D(\lambda)_k$. By [L 6 8.15], the algebra $U_k/(K_i - 1)$ identifies with the hyperalgebra $\bar{U}_k$ of G_k, hence we obtain that $D(\lambda)_k$ is a finite dimensional $\bar{U}_k$-module, generated by a highest weight vector of weight x_λ. As observed in ([Ja 1, Satz 1]), this implies that $D(\lambda)_k$ is a quotient of the Weyl module $E(\lambda)_k$. (This uses Kempf's vanishing theorem!). Therefore, we obtain:

$$\dim_k(D(\lambda) \otimes k) \leqq \dim_{\mathscr{A}'}(D(\lambda) \otimes \mathscr{A}') .$$

Then lemma 1.21 gives that $D(\lambda)$ is a finite free $\mathscr{A}$-module. It follows that $\operatorname{ch} D(\lambda) = \operatorname{ch} D(\lambda)_{\mathscr{A}'} = \operatorname{ch} D(\lambda)_k$ is given by Weyl's character formula. □

The previous results are needed in order to prove that the quantum coordinate algebra is a free $\mathscr{A}$-module. Let us make a disgression in order to derive some byproducts of our analysis.

1.23. Recall that Lusztig has defined an action of the braid group on U ([L 6, Section 3]). Let $\lambda \in X^+$. For any $w \in W$, Let $J_w(\lambda) = T_w(J(\lambda))$. This is a left ideal of U.

Proposition. *(i) For any* $w \in W$, $D(\lambda)$ *is generated as a* $T_w(U^-)$*-module by its* $w\lambda$*-weight space.*

(ii) For any $w \in W$, *there is a U-isomorphism* $\varphi_w : D(\lambda) \simeq U/J_w(\lambda)$.

Proof. (i) By induction on $l(w)$, we reduce to the case where $w = s_i$, a simple reflection. We prove that $D(\lambda)$ is generated as a $T_i(U^-)$-module by the element $x_{s_i\lambda} = F_i^{(\lambda_i)} x_\lambda$.

By [DL], the monomials $F_{\beta_1}^{(r_1)} \dots F_{\beta_N}^{(r_N)}$ form an $\mathscr{A}$-basis of U^-, where $\{\beta_1, \dots, \beta_N\}$ is the ordering of R^+ corresponding to an arbitrary reduced expression of the longest element w_0. We arrange that $\beta_N = \alpha_i$. Let $U^{(i)}$ be the subalgebra of U^- generated by all $F_\beta^{(r)}$, where $\beta \neq \alpha_i$. Then $D(\lambda)$ is generated as a $U^{(i)}$-module by the elements $\{F_i^{(s)} x_\lambda | 0 \leqq s \leqq \lambda_i\}$.

Clearly, T_i maps $U^{(i)}$ onto itself, and $T_i(F_i^{(s)}) = E_i^{(s)}$ for all s. Since $E_i^{(s)} F_i^{(\lambda_i)} x_\lambda = F_i^{(\lambda_i - s)} x_\lambda$, we obtain that $D(\lambda)$ is generated as a $T_i(U^-)$-module by $F_i^{(\lambda_i)} x_\lambda$.

(ii) By (i) we know that $D(\lambda)$ is generated by its $w\lambda$-weight space, a free rank one $\mathscr{A}$-module which is annihilated by $T_w(U^+)$. (This follows e.g. from Weyl's character formula). Since $U/J_w(\lambda)$ has the obvious universal property, we obtain a surjective U-homomorphism $U/J_w(\lambda) \overset{\varphi}{\twoheadrightarrow} D(\lambda)$. But the automorphism T_w of U induces an $\mathscr{A}$-isomorphism from $D(\lambda) = U/J(\lambda)$ onto $U/J_w(\lambda)$. Therefore, the latter is free, of the same rank as $D(\lambda)$. Then, by Nakayama, φ is injective, and is therefore a U-isomorphism. □

Corollary 1.24. *Let* $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{N}^n$ *and let* J^+ *(resp.* J^-*) be the left ideal of* U^+ *(resp.* U^-*) generated by* $\{E_i^{(r_i)} | r_i > \lambda_i\}$ *(resp.* $\{F_i^{(s_i)} | s_i > \lambda_i\}$*. Then* U^+/J^+ *and* U^-/J^- *are finite free* $\mathscr{A}$*-modules.*

Proof. By 1.20 and 1.23 we have isomorphisms: $U^-/J^- \simeq D(\lambda)$ and $U^+/J^+ \simeq D(-w_0\lambda)$. □

1.25. Remark. Following [Jo], we have defined the functor D as induction from $U^\natural$ to U. This agrees with the tradition of privileging the dominant weights and considering the Weyl modules as generated over U^- by their highest weight vector. But the tradition for algebraic groups (where induction replaces co-induction) is still to work with dominant weights, but induce from the negative Borel subgroup. Thus, we also introduce the functor D', which is defined in the same way as D (see 1.14) with $U^\natural$ replaced by $U^\flat$. Of course, all properties of D have their analogue for D'. Observe that $D'(\mu) \neq 0$ if and only if $\mu \in X^-$. In fact, for $\lambda \in X^+$ we have by 1.24:

$$D'(-\lambda) \simeq D(-w_0\lambda) .$$

Proposition 1.26. *For all* $\lambda, \mu \in X^+$, *the U-modules* $D(\lambda) \otimes D(\mu)$ *and* $D(\mu) \otimes D(\lambda)$ *are isomorphic.*

Proof. Let I be the left ideal of U^+ generated by $\{E_i^{(r_i)} | r_i > \langle -w_0\mu, \alpha_i^\vee \rangle\}$. By 1.24, we have a U^+-isomorphism $U^+/I \simeq D(\mu)$. Let J be the left ideal of $U^\natural$ generated by I and $\mathrm{Ker}(\chi_\nu)$, where $\nu = \lambda + w_0\mu$. Then we have a U^+-isomorphism: $U^+/I \simeq U^\natural/J$. Now, as $U^\natural$-modules both $\mathscr{A}_\lambda \otimes D(\mu)$ and $D(\mu) \otimes \mathscr{A}_\lambda$ are generated by an element of weight ν, which is killed by J. Therefore, we have surjective $U^\natural$-homomorphisms:

$$\varphi : U^\natural/J \twoheadrightarrow \mathscr{A}_\lambda \otimes D(\mu) \text{ and } \psi : U^\natural/J \twoheadrightarrow D(\mu) \otimes \mathscr{A}_\lambda .$$

But all three are free $\mathscr{A}$-modules of the same rank, and therefore both φ and ψ are isomorphisms. Hence we obtain a $U^{\natural}$-isomorphism: $\mathscr{A}_\lambda \otimes D(\mu) \simeq D(\mu) \otimes \mathscr{A}_\lambda$. Since $D(\mu)$ is free as $\mathscr{A}$-module, we can apply the tensor identities 1.19 and get:

$$D(\lambda) \otimes D(\mu) \simeq D(\mathscr{A}_\lambda \otimes D(\mu)) \simeq D(D(\mu) \otimes \mathscr{A}_\lambda) \simeq D(\mu) \otimes D(\lambda). \quad \square$$

Remark. Proposition 1.26 answers a question raised in [PW 2, 3.4.3]. (We shall see that $D'(-\lambda)^* \simeq H^0(\lambda)$, see Proposition 3.3).

1.27. We now compute the annihilator in U of the element $x = x_{-\lambda} \otimes x_\mu \in D'(-\lambda) \otimes D(\mu)$. Set $\nu = \mu - \lambda$ and let $J(\lambda, \mu)$ be the left ideal of U generated by $\mathrm{Ker}(\chi_\nu)$, $J^+(-\lambda) = \mathrm{Ann}_{U^+}(x_{-\lambda})$, $J^-(\mu) = \mathrm{Ann}_{U^-}(x_\mu)$. Then, we have the:

Proposition. *(i) The U-module $D'(-\lambda) \otimes D(\mu)$ is generated by the element x.*
(ii) $\mathrm{Ann}_U(x) = J(\lambda, \mu)$.
(iii) $U/J(\lambda, \mu)$ is a finite free $\mathscr{A}$-module.

Proof. Of course, (iii) follows from (i) and (ii). We prove (i). The U^+-submodule of $M = D'(-\lambda) \otimes D(\mu)$ generated by x is equal to $D'(-\lambda) \otimes x_\mu$. By 1.19 we have $M \simeq D(D'(-\lambda) \otimes \mu)$, and therefore M is generated as a U^--module by $D'(-\lambda) \otimes x_\mu$. Hence, M is generated as a U-module by x.

Now, we prove (ii). From the definition of comultiplication, we obtain that $J \subseteq \mathrm{Ann}(x)$, and moreover:

$$(\Delta(u^+) - u^+ \otimes 1) \cdot x = 0 = (\Delta(u^-) - 1 \otimes u^-) \cdot x \text{ for any } u^\pm \in U^\pm .$$

From this it follows that:

$$\mathrm{Ann}_{U^+}(x) = \mathrm{Ann}_{U^+}(x_{-\lambda}) = J^+(-\lambda), \text{ and } \mathrm{Ann}_{U^-}(x) = \mathrm{Ann}_{U^-}(x_\mu) = J^-(\mu) .$$

Set $P = U/J(\lambda, \mu)$. There is a surjective U-homomorphism $P \overset{\pi}{\twoheadrightarrow} M$. We prove firstly that P is a finite $\mathscr{A}$-module. By Lemma 1.5, $F(P)$ is a U-submodule of P. Since it contains the generator $y = \bar{1}$, we conclude that $F(P) = P$. Hence, by Lemma 1.13, the set of weights of P is W-stable. Now, let N be the $U^\flat$-submodule of P generated by y. Since:

$$J^-(\mu) \subseteq \mathrm{Ann}_{U^-}(y) \subseteq \mathrm{Ann}_{U^-}(x) \subseteq J^-(\mu)$$

we obtain that: $N \simeq \mathscr{A}_{-\lambda} \otimes D(\mu)$.

Now, since P is generated as a U^+-module by N, it follows that all weights of P are bigger than $-\lambda + w_0\mu$, and all weight spaces are finite $\mathscr{A}$-modules. Finally, since any weight of P is conjugate to some antidominant weight bigger than $-\lambda + w_0\mu$, and since there are finitely many of these, we conclude that P is a finite $\mathscr{A}$-module.

Hence, by 1.17, the inclusion $N \subseteq P$ induces a surjective U-homomorphism $D'(N) \overset{\varphi}{\twoheadrightarrow} P$. Also, $D'(N) \simeq M$, by 1.19. Hence, we have two surjective U-homomorphisms:

$$M \overset{\varphi}{\twoheadrightarrow} P \overset{\pi}{\twoheadrightarrow} M .$$

Since M is a finite free $\mathscr{A}$-module, it follows by Nakayama that $\pi \circ \varphi$ is injective. Hence both φ and π are isomorphisms. This proves that $J(\lambda, \mu) = \mathrm{Ann}(x)$. □

Corollary 1.28. *Let $M \in \mathscr{C}$. Assume that M is a finite U-module. Then M is also a finite $\mathscr{A}$-module.*

Proof. We may assume that M is generated by an element x of weight $v \in X$. For each $i \in \{1, \ldots, n\}$ let r_i and s_i be the largest integers such that $E_i^{(r_i)} x \neq 0$ and $F_i^{(s_i)} x \neq 0$. Define λ, $\mu \in X^+$ by: $\lambda_i = r_i$ and $\mu_i = s_i$ for all i. Then $\mu - \lambda = v$ by Lemma 1.11, and $J(\lambda, \mu)$ annihilates x. Hence $M = Ux$ is a quotient of $U/J(\lambda, \mu)$ and is therefore $\mathscr{A}$-finite by Proposition 1.27 (iii). □

1.29. We can now derive the main properties of the induction functor H. Let M be an $\mathscr{A}$-module. Then the $\mathscr{A}$-module $\mathscr{H}(M) = \mathrm{Hom}_{\mathscr{A}}(U, M)$ carries two structures of U-module γ and δ (see 1.9). Recall that $H(M) = F_\delta(\mathscr{H}(M))$ (see 1.10). As a $\delta(U^0)$-module, $H(M)$ is the direct sum of weight spaces $H(M)_v$, $v \in X$, and as we shall see in the proposition below these are $\gamma(U)$-submodules of $H(M)$.

For each $v \in X$, let $I(v)$ be the left ideal of U generated by the ideal $\mathrm{Ker}(\chi_v)$ of U^0, and let $U(v) = U/I(v)$. Then $\mathrm{Hom}_{\mathscr{A}}(U(v), M)$ is made into a U-module as follows: $(u.\theta)(x) = \theta(S(u)x)$. Let $\Omega(v) = \{(\lambda, \mu) \in X^+ \times X^+ \mid \mu - \lambda = v\}$.

For $(\lambda, \mu) \in \Omega(v)$ recall that $J(\lambda, \mu)$ has been defined in 1.27. Note that $I(v) \subseteq J(\lambda, \mu)$. Set $D(\lambda, \mu) = U/J(\lambda, \mu)$. Then $\mathrm{Hom}_{\mathscr{A}}(D(\lambda, \mu), M)$ is a U-submodule of $\mathrm{Hom}_{\mathscr{A}}(U(v), M)$. Let $H_v(M)$ be the union of these submodules, for all $(\lambda, \mu) \in \Omega(v)$. Then, we have the:

Proposition. *For each $v \in X$, there are isomorphisms of $\gamma(U)$-modules: $\mathrm{Hom}_{\mathscr{A}}(U, M)_v \simeq \mathrm{Hom}_{\mathscr{A}}(U(v), M)$ and $H(M)_v \simeq H_v(M)$.*

Proof. The first isomorphism is clear, and identifies $H(M)_v$ with a $\gamma(U)$-submodule of $\mathrm{Hom}_{\mathscr{A}}(U(v), M)$. Let $\varphi \in H(M)_v$. For each i, let r_i and s_i be the largest integers such that $\delta(E_i^{(r_i)}) \cdot \varphi \neq 0$ and $\delta(F_i^{(s_i)}) \cdot \varphi \neq 0$. Define λ, $\mu \in X^+$ by: $\lambda_i = r_i$ and $\mu_i = s_i$. Then $\mu - \lambda = v$ by Lemma 1.11, and we obtain that φ is zero on the left ideal $J(\lambda, \mu)$. Hence, φ belongs to $\mathrm{Hom}_{\mathscr{A}}(D(\lambda, \mu), M)$. □

1.30. Keep the notations of 1.29

Corollary. *Let $\varphi \in \mathscr{H}(M)$.*

(*i*) *The following are equivalent:*

(*a*) *There exists a two-sided ideal $I \in \mathscr{I}$ (see 1.8) such that $\varphi(I) = 0$.*

(*b*) $\varphi \in F_\delta(\mathscr{H}(M))$

(*c*) $\varphi \in F_\gamma(\mathscr{H}(M))$

(*ii*) *If these conditions are satisfied and if moreover φ has weight v for the action of $\delta(U^0)$, then there exist λ, $\mu \in X^+$ with $\mu - \lambda = v$ such that $\varphi \in \mathrm{Hom}_{\mathscr{A}}(D(\lambda, \mu), M)$.*

1.31. We can now derive the:

Theorem. (*i*) *H is an exact functor.*

(*ii*) *H commutes with direct sum (possibly infinite).*

(*iii*) *For any $\mathscr{A}$-module M, the natural map $\theta_M : H(\mathscr{A}) \otimes M \to H(M)$ is an $\mathscr{A}$-isomorphism.*

Proof. (i) Since H is left exact, we only have to prove that, if $M \twoheadrightarrow Q$ is a surjective $\mathscr{A}$-homomorphism, then $H(M) \overset{\pi}{\to} H(Q)$ is onto. So, let $\varphi \in H(Q)$. We can assume that φ has weight ν. Then $\varphi \in \mathrm{Hom}_{\mathscr{A}}(D(\lambda, \mu), Q)$ for some λ, μ. Since $D(\lambda, \mu)$ is a free $\mathscr{A}$-module, then φ can be lifted to M. This proves that π is surjective.

(ii) Let $M = \bigoplus M_i$ be a direct sum of $\mathscr{A}$-modules. Then $\bigoplus H(M_i) \subseteq H(M) \subseteq \Pi H(M_i)$. Let $\varphi \in H(M)$. Again, we can assume that φ belongs to some $\mathrm{Hom}_{\mathscr{A}}(D(\lambda, \mu), M)$. Since $D(\lambda, \mu)$ is a finite $\mathscr{A}$-module, then $\mathrm{Im}(\varphi)$ is contained in a finite direct sum of the M_i's. Hence $\varphi \in \bigoplus H(M_i)$.

As for (iii), it follows from (ii) that $H(\mathscr{A}^{(I)}) \simeq H(\mathscr{A})^{(I)} \simeq H(\mathscr{A}) \otimes \mathscr{A}^{(I)}$ for any free $\mathscr{A}$-module $\mathscr{A}^{(I)}$. Now, consider an exact sequence $0 \to K \to F \to M \to 0$ of $\mathscr{A}$-modules, where F is free. By naturality of θ, we get a commutative diagram:

$$\begin{array}{ccccccccc} & & H(\mathscr{A}) \otimes K & \to & H(\mathscr{A}) \otimes F & \to & H(\mathscr{A}) \otimes M & \to & 0 \\ & & \downarrow \theta_K & & \downarrow \theta_F & & \downarrow \theta_M & & \\ 0 & \to & H(K) & \to & H(F) & \to & H(M) & \to & 0 \end{array}$$

The bottom row is exact by (i), and θ_F is an isomorphism since F is free. Hence, θ_M is surjective. The same argument applies to K instead of M, and therefore θ_K is also surjective. From this it follows that θ_M is bijective. Hence the Theorem is proved.

We shall see below that $H(\mathscr{A})$ is a free $\mathscr{A}$-module. Note that we have already obtained that $H(\mathscr{A})$ is flat. Indeed, since θ_K is also bijective, then the top row is also exact, and therefore $H(\mathscr{A})$ is a flat $\mathscr{A}$-module. □

Remark. θ_M is both a $\gamma(U)$ and $\delta(U)$ isomorphism.

1.32. For future use, we record here the following lemma.

Lemma. (Kaplansky). *Over a local ring, any projective module is free.*

Proof. See [Mu, Theorem 2.5 p. 9].

Theorem 1.33. *$H(\mathscr{A})$ is a free $\mathscr{A}$-module.*

Proof. Since $H(\mathscr{A}) = \bigoplus_{\nu \in X} H(\mathscr{A})_\nu$, it is enough to prove that each $H(\mathscr{A})_\nu$ is free. So, let $\nu \in X$ be fixed. Let (λ_0, μ_0) be the element of $\Omega(\nu)$ defined by the conditions:

$$\lambda_{0,i} = \mathrm{Max}\{0, \nu_i\} \quad \text{and} \quad \mu_{0,i} = \mathrm{Max}\{0, -\nu_i\}\ .$$

For $m \geqq 0$, set $\lambda_m = \lambda_0 + m\rho$, $\mu_m = \mu_0 + m\rho$, and $J(m) = J(\lambda_m, \mu_m)$.

If $(\lambda, \mu) \in \Omega(\mu)$, then $J(\lambda, \mu)$ contains some $J(m)$ and therefore $(U/J(\lambda, \mu))^*$ is contained in $(U/J(m))^*$. Hence we obtain

$$H(\mathscr{A})_\nu = \bigcup_{m \geqq 0} (U/J(m))^*\ .$$

Let $m \geqq 0$. Since $U/J(m)$ and $U/J(m+1)$ are free, there is an $\mathscr{A}$-isomorphism: $U/J(m+1) \simeq U/J(m) \oplus J(m)/J(m+1)$. Hence $J(m)/J(m+1)$ is a projective $\mathscr{A}$-module, and is therefore free, by lemma 1.32. From this we deduce that

$$H(\mathscr{A})_\nu \simeq \bigoplus_{m \geqq 0} (J(m)/J(m+1))^* \text{ is a free } \mathscr{A}\text{-module. } \square$$

We call H the induction functor from $\mathscr{A}$ to U, and set $H(\mathscr{A}) = \mathscr{A}[U]$, the *quantum coordinate algebra.*

1.34. The Hopf algebra structure of $\mathscr{A}[U]$. When H is a Hopf algebra over a field K, it is well-known (and easy to prove) that the restricted dual of H (also called space of representative functions) is again a Hopf algebra. The argument also applies to our Hopf algebra U over the (base) ring $\mathscr{A}$, except that we may have difficulties in checking the isomorphism: $(M \otimes N)^* \simeq M^* \otimes N^*$, when M, N are $\mathscr{A}$-finite U-modules, and M^* denotes $\mathrm{Hom}_{\mathscr{A}}(M, \mathscr{A})$. But this difficulty is overcome by Corollary 1.30 (ii) (taking $M = \mathscr{A}$) by which we deduce that we only have to consider U-modules which are finite free $\mathscr{A}$-modules.

Hence, we obtain that $\mathscr{A}[U]$ is a Hopf algebra. It consists of the coefficient spaces of all $\mathscr{A}$-finite (free) U-modules.

1.35. Tensor identity. Let M be any $\mathscr{A}$-module. Then Theorem 1.31 gives an $\mathscr{A}$-isomorphism $H(M) \simeq H(\mathscr{A}) \otimes M$. The action of U is defined as follows. If $\theta \in \mathrm{Hom}_{\mathscr{A}}(U, M)$, u, $x \in U$ then: $(u \cdot \theta)(x) = \theta(xu)$. Equivalently, if $\varphi \otimes m \in \mathscr{A}[U] \otimes M$, then $u \cdot (\varphi \otimes m)(x) = \varphi(xu)m$. In other words, $u \cdot (\varphi \otimes m) = (u \cdot \varphi) \otimes m$. Now, if M is already a U-module, then $\mathscr{A}[U] \otimes M$ is a $U \otimes U$-module, and it becomes a U-module via the comultiplication $\Delta : U \to U \otimes U$. In fact, we prove that these two structures of U-module are equivalent, when M is an integrable U-module.

We need some notations. Let $\mathscr{M}$ denote the $\mathscr{A}$-module $M \otimes \mathscr{A}[U]$. We shall use the fact that $\mathscr{M} \simeq H(M)$ can be identified with an $\mathscr{A}$-submodule of $\mathrm{Hom}_{\mathscr{A}}(U, M)$. There are two actions of U on $\mathscr{M}$, defined as follows. Let $m \otimes \varphi \in \mathscr{M}$, $u \in U$. Write $\Delta(u) = \sum_i u_i \otimes u_i'$.

One action is defined by: $u \cdot (m \otimes \varphi) = m \otimes (u\varphi)$. The resulting U-module is called $\mathscr{M}_1$.

The other action is defined by: $u \cdot (m \otimes \varphi) = \sum_i u_i m \otimes u_i' \varphi$. The resulting U-module is called $\mathscr{M}_2$.

We shall define an $\mathscr{A}$-automorphism of $\mathscr{M}$, which will be a U-isomorphism from $\mathscr{M}_1$ onto $\mathscr{M}_2$. Let $m \otimes \varphi \in \mathscr{M}$. We define $\alpha(m \otimes \varphi) \in \mathrm{Hom}_{\mathscr{A}}(U, M)$ as follows. If $u \in U$, $\Delta(u) = \sum_i u_i \otimes u_i'$ then:

$$\alpha(m \otimes \varphi)(u) = \sum_i u_i \varphi(u_i') m \ .$$

We claim that $\alpha(m \otimes \varphi) \in \mathscr{M}$. To see this observe that since M is integrable then the map $\psi_m : U \to M, x \mapsto x \cdot m$ belongs to $\mathscr{M}$. Then, $\alpha(m \otimes \varphi)$ is nothing but $\psi_m \varphi$, which again belongs to $\mathscr{M}$.

We leave it to the reader to check that α is a U-homomorphism from $\mathscr{M}_1$ into $\mathscr{M}_2$. In order to prove that α is an isomorphism, we construct an inverse. Using the same notations as above, if $m \otimes \varphi \in \mathscr{M}$ we define $\beta(m \otimes \varphi) \in \mathrm{Hom}_{\mathscr{A}}(U, M)$ as follows:

$$\beta(m \otimes \varphi)(u) = \sum_i S(u_i) \varphi(u_i') m, \text{ where } S \text{ is the antipode of } U \ .$$

Since the antipode S' of $\mathscr{A}[U]$ is defined by $S'(\psi) = \psi \circ S$ ($\psi \in \mathscr{A}[U]$), we see that $\beta(m \otimes \varphi)$ is equal to $S'(\psi_m)\varphi$, which again belongs to $\mathscr{M}$. Hence, β takes $\mathscr{M}$ into itself. Now, it is a formal manipulation to check that α and β are reciprocal bijections. Hence, $\alpha : \mathscr{M}_1 \xrightarrow{\sim} \mathscr{M}_2$ is a U-isomorphism.

Therefore, if we denote $\mathscr{M}_2$ by $M \otimes \mathscr{A}[U]$, whereas $\mathscr{M}_1$ is denoted by $\mathscr{M}_t \otimes \mathscr{A}[U]$ (here, t stands for: trivial action) and also by $H(M)$ then the result reads as (i) of the proposition below. For the sake of completeness, we observe that the $\mathscr{A}$-module $\mathscr{A}[U] \otimes M$ also carries two different structures of U-module, and since the antipode S of U is bijective, these two structures can also be intertwined. This gives (ii) in the proposition below.

Proposition. *Let M be a U-module. Then we have U-isomorphisms:*
(*i*) $H(M) \simeq M \otimes \mathscr{A}[U]$
(*ii*) $H(M) \simeq \mathscr{A}[U] \otimes M$

2. Induction

In this section we study induction functors for quantum algebras. The functor H considered in Section 1 corresponds to induction from the trivial subalgebra $\mathscr{A}$ to the whole algebra U. Most of the results here are deduced via standard arguments from the key properties of H developed in Section 1.

2.1. For two subalgebras $U^2 \subseteq U^1$ of U, we will define an induction functor from U^2 to U^1. Firstly, we only want to consider subalgebras of the following type. Let I, J be subsets of $\{1, \ldots, n\}$. We denote by $U(I, J)$ the subalgebra of U generated by U^0 and $\{E_i^{(r)}, F_j^{(s)}, | i \in I, j \in J, r, s \geqq 0\}$, and we simply write U_I for $U(I, I)$.

Note that U_I is isomorphic (as an algebra) to the tensor product of the subalgebra of U^0 generated by $\{K_j^{\pm 1}, \left[{K_j \atop t}\right] | j \notin I, t \geqq 0\}$ with the quantum algebra associated to the Cartan submatrix $(a_{i,j})_{i,j \in I}$, so that all the results from Section 1 apply to U_I.

Secondly, in order to have good properties of induction, we need induction from $\mathscr{A}$ to the given subalgebra $U(I, J)$ to be exact. We do not know whether this is true for arbitrary I, J, but we prove it when $I \subseteq J$ or $J \subseteq I$.

2.2. Let I, J as above. If V is a $U(I, J)$-module, we set:

$$F^{I,J}(V) = \left\{ x \in \bigoplus_{\lambda \in X} V_\lambda \,|\, E_i^{(r)} x = 0 = F_j^{(r)} x, \text{ for all } i \in I, j \in J, r \gg 0 \right\}$$

Let $\mathscr{C}^{I,J}$ be the category of those $U(I, J)$-modules V such that $V = F^{I,J}(V)$. The modules in $\mathscr{C}^{I,J}$ are called integrable $U(I, J)$-modules (of type 1, see 1.6). Observe that $\mathscr{C}^{I,J}$ is an abelian category, see the *Note added in proof.*

When $I = J = \{1, \ldots, n\}$, we have $\mathscr{C}^{I,J} = \mathscr{C}$ (see 1.6).

When $I = \varnothing$ and $J = \{1, \ldots, n\}$, we have $U(I, J) = U^\flat$ and we write $\mathscr{C}^\flat$ for the corresponding category.

When $I = J = \varnothing$, then $U(I, J) = U^0$ and the corresponding category is denoted by $\mathscr{C}^0$.

2.3. If $\mathscr{A} \to \Gamma$ is a specialization of $\mathscr{A}$ into a field Γ, the categories $\mathscr{C}_\Gamma^0$, $\mathscr{C}_\Gamma^\flat$, etc. are defined similarly (see 1.7).

2.4. We define a functor $H^{I,J}$:$\{\mathscr{A}\text{-modules}\} \to \mathscr{C}^{I,J}$ as follows. For an $\mathscr{A}$-module M, we set (see 1.9–1.10):

$$H^{I,J}(M) = F_\delta^{I,J}(\mathrm{Hom}_{\mathscr{A}}(U(I, J), M)).$$

Also, we set: $H^{I,J}(\mathscr{A}) = \mathscr{A}[U(I, J)]$.

2.5. Assume that $I, J \subseteq \{1, \ldots, n\}$ satisfy: $I \subseteq J$ or $J \subseteq I$. We denote by $U^\flat(I)$, respectively $U^\natural(J)$ the subalgebra generated by $U^\flat$ and $\{E_i^{(r)} | i \in I, r \geqq 0\}$, respectively $U^\natural$ and $\{F_j^{(s)} | j \in J, s \geqq 0\}$, and call them: parabolic subalgebras. Essentially, these subalgebras encompass all the subalgebras that we want to consider. Indeed, if $I \subseteq J$ (resp. $J \subseteq I$) then $U(I, J)$ is the parabolic subalgebra $U_J^\flat(I)$ (resp. $U_I^\natural(J)$) of U_J (resp. U_I), and might therefore be called a generalized parabolic subalgebra.

So, we see that there is no loss of generality in assuming that $J = \{1, \ldots, n\}$. We shall do this in the rest of this section, and omit the letter J everywhere in the notation. Hence, $U(I, J)$ becomes $U^\flat(I)$, and $\mathscr{C}^{I,J}$, $H^{I,J}$ are simply denoted $\mathscr{C}^I$, H^I, etc.

2.6. Let $\lambda \in X^+$. As in 1.14 we define $D'_I(-\lambda)$ as the largest $\mathscr{A}$-finite quotient U_I-module of $U_I \otimes_{U_I^\flat} \mathscr{A}_{-\lambda}$. Note that there is a U_I-isomorphism: $U_I \otimes_{U_I^\flat} \mathscr{A}_{-\lambda} \simeq U^\flat(I) \otimes_{U^\flat} \mathscr{A}_{-\lambda}$. From this we deduce that the U_I-module structure on $D'_I(\lambda)$ extends to a $U^\flat(I)$-module structure.

Now, let also $\mu \in X^+$ and as in 1.27 denote by $J(\lambda, \mu)$ the left ideal of $U^\flat(I)$ generated by $\mathrm{Ker}(\chi_{\mu-\lambda})$ and $\{E_i^{(r_i)}, F_j^{(s_j)} | i \in I, 1 \leqq j \leqq n, r_i > \lambda_i, s_j > \mu_j\}$. Then, as in 1.27, we have the:

Proposition. (*i*) *The* $U^\flat(I)$*-module* $D'_I(-\lambda) \otimes D(\mu)$ *is generated by* $x = x_{-\lambda} \otimes x_\mu$.

(*ii*) *The annihilator of* x *is equal to* $J(\lambda, \mu)$.

(*iii*) $U^\flat(I)/J(\lambda, \mu)$ *is a finite free* $\mathscr{A}$*-module.*

Remark. For this proposition, it is crucial for $U^\flat(I)$ to be a parabolic algebra, in order to be able to apply the tensor identity 1.19, see the proof of 1.27.

2.7. Then, by 2.5–2.6 we obtain, as in 1.31–1.33–1.35, the:

Proposition. (*i*) H^I *is an exact functor. It takes free* $\mathscr{A}$*-modules to* $\mathscr{A}$*-free modules in* $\mathscr{C}^I$.

(*ii*) $\mathscr{A}[U^\flat(I)]$ *is a direct sum of weight spaces, and each of these is a free* $\mathscr{A}$*-module.*

(*iii*) *The restriction map*: $\mathscr{A}[U] \to \mathscr{A}[U^\flat(I)]$ *is surjective.*

(*iv*) *For any* $\mathscr{A}$*-module* M *the natural map*: $\mathscr{A}[U^\flat(I)] \otimes M \to H^I(M)$ *is an* $\mathscr{A}$*-module isomorphism.*

(*v*) *Tensor identities. If* M *is already a* $U^\flat(I)$*-module (considered as an* $\mathscr{A}$*-module by restriction), then we have* $U^\flat(I)$*-isomorphisms*:

$$H^I(M) \simeq M \otimes \mathscr{A}[U^\flat(I)] \text{ and } H^I(M) \simeq \mathscr{A}[U^\flat(I)] \otimes M$$

where the terms on the R.H.S. are regarded as $U^\flat(I)$*-modules for the "diagonal" action.*

Proof. We have seen all of this already, except for the fact that $\mathscr{A}[U] \to \mathscr{A}[U^\flat(I)]$ is surjective. So, let $\varphi \in \mathscr{A}[U^\flat(I)]$. We can assume that φ has weight ν. Then there exist $\lambda, \mu \in X^+$ with $\mu - \lambda = \nu$ such that $\varphi \in (D'_I(-\lambda) \otimes D(\mu))^*$. Now, the sum N of all weight spaces $D(-\lambda)_\eta$, for $\eta \in -\lambda + \mathbb{N}R_I^+$, is clearly a $U^\flat(I)$-submodule of $D'(-\lambda)$, generated by the $(-\lambda)$-weight space. Moreover, it is well known that N and $D'_I(-\lambda)$ have the same character, see e.g. [Ja 3, II 5.21]. Therefore $D'_I(-\lambda)$ identifies with a direct summand (as $\mathscr{A}$-module) of $D'(-\lambda)$. It follows that φ is the restriction of some $\psi \in (D'(-\lambda) \otimes D(\mu))^* \subseteq \mathscr{A}[U]$. □

2.8. Now, we define induction from subalgebras. Let $I' \subseteq I \subseteq \{1, \ldots, n\}$. We set $U^1 = U^b(I)$, and take U^2 to be either $U^b(I')$ or U^0, and call $\mathscr{C}^1$ and $\mathscr{C}^2$ the corresponding categories.

Induction from $\mathscr{C}^2$ to $\mathscr{C}^1$ is defined as follows. Let $M \in \mathscr{C}^2$. Then $\mathrm{Hom}_{U^2}(U^1, M)$ is a $\delta(U^{-1})$-submodule of $\mathrm{Hom}_{\mathscr{A}}(U^1, M))$ (see 1.9) and we set:

$$H^0(U^1/U^2, M) = F^I_\delta(\mathrm{Hom}_{U^2}(U^1, M)) .$$

Clearly, this is a left exact covariant functor. There is a natural U^2-homomorphism $\mathscr{E}v: H^0(U^1/U^2, M) \to M$, defined by: $\mathscr{E}v(\varphi) = \varphi(1)$.

Note that $H^0(U^1/U^2, M)$ is the same as $\{f \in H^I(M) | f \text{ is a } U^2\text{-homomorphism}\}$. We shall elaborate on this in the next subsection.

2.9. Homomorphisms and invariants for Hopf algebras. In order to push further the analogy with algebraic groups, we recall some facts about Hopf algebras. In this subsection, we denote by $\mathscr{A}$ a commutative ring, and by U a Hopf algebra over $\mathscr{A}$, with comultiplication Δ, co-unit ε and antipode S.

If M is a U-module, we set: $M^U = \{x \in M | u \cdot x = \varepsilon(u)x \text{ for all } u \in U\}$. This is called the space of U-invariants in M.

Let M, N be U-modules. Then $\mathrm{Hom}_{\mathscr{A}}(M, N)$ is made into a U-module as follows: let $\theta \in \mathrm{Hom}_{\mathscr{A}}(M, N)$, $x \in M$, $u \in U$. Write $\Delta(u) = \sum_i u_i \otimes u_i'$. Then:

$$(u\theta)(x) = \sum_i u_i \theta(S(u_i')x) .$$

If $N = \mathscr{A}$, with trivial action, i.e. $u \,.\, y = \varepsilon(u)y$ for all $u \in U$, $y \in \mathscr{A}$, we obtain: $(u\theta)(x) = \theta(S(u)x)$. This is the usual action on M^*. Now, for general N, the natural $\mathscr{A}$-homomorphism $N \otimes M^* \to \mathrm{Hom}_{\mathscr{A}}(M, N)$ is a U-homomorphism.

Proposition. *Let M, N be U-modules. Set $Y = \mathrm{Hom}_{\mathscr{A}}(M, N)$. Then:*

$$\mathrm{Hom}_U(M, N) \subseteq Y^U \subseteq \{\theta \in Y | \theta(S(u)x) = S(u)\theta(x) \textit{ for all } u \in U, x \in M\} .$$

Therefore, if S is surjective then $\mathrm{Hom}_U(M, N) = \mathrm{Hom}_{\mathscr{A}}(M, N)^U$.

Proof. Let $\theta \in \mathrm{Hom}_U(M, N), x \in M$, $u \in U$. Write $\Delta(u) = \sum_i u_i \otimes u_i'$. Then $\sum_i u_i\theta(S(u_i')x) = \sum_i u_i S(u_i')\theta(x) = \varepsilon(u)\theta(x)$. Hence $\theta \in \mathrm{Hom}_{\mathscr{A}}(M, N)^U$.

The proof of the reverse inclusion is a little harder. We make $\mathrm{Hom}_{\mathscr{A}}(M, N)$ into a $U \otimes U$-module as follows:

$$((u \otimes v)\theta)(x) = u\theta(S(v)x) \text{ for all } u, v \in U, \theta \in \mathrm{Hom}_{\mathscr{A}}(M, N), x \in M .$$

Let $\theta \in \mathrm{Hom}_{\mathscr{A}}(M, N)^U$. Then we have: $(z \otimes \varepsilon(t))\theta = \sum_i (zt_i \otimes t_i')\theta$, for any $z, t \in U$, with $\Delta(t) = \sum_i t_i \otimes t_i'$.

Now, let $u \in U$, $\Delta(u) = \sum_i u_i \otimes u_i'$. Since $u = \sum_i u_i \varepsilon(u_i')$, we have $S(u) = \sum_i S(u_i)\varepsilon(u_i')$. Therefore:

$$\begin{aligned}(S(u) \otimes 1)\theta = \sum_i (S(u_i) \otimes \varepsilon(u_i'))\theta &= (m \otimes \mathrm{id})(S \otimes \mathrm{id} \otimes \mathrm{id})(\mathrm{id} \otimes \Delta)\Delta(u)\theta \\ &= (m \otimes \mathrm{id})(S \otimes \mathrm{id} \otimes \mathrm{id})(\Delta \otimes \mathrm{id})\Delta(u)\theta \\ &= ((m(S \otimes \mathrm{id})\Delta) \otimes \mathrm{id})\Delta(u)\theta \\ &= (\varepsilon \otimes \mathrm{id})\Delta(u)\theta = (1 \otimes u)\theta\end{aligned}$$

The proposition follows. □

2.10. Let $M \in \mathscr{C}^2$. In analogy with induction for algebraic groups, we obtain from 2.8–2.9 that:

$$H^0(U^1/U^2, M) = (M \otimes A[U^1])^{U^2}$$

2.11. The formula in 2.10 is of particular interest when $U^2 = U^0$. In that case we obtain:

$$H^0(U^1/U^0, M) = (M \otimes A[U^1])^{U^0} = \bigoplus_{\lambda \in X} (M_\lambda \otimes A[U^1]_{-\lambda})$$

From this we deduce the:

Proposition. *Induction from $\mathscr{C}^0$ to $\mathscr{C}^1$ is an exact functor. It takes $\mathscr{A}$-free modules in $\mathscr{C}^0$ to $\mathscr{A}$-free modules in $\mathscr{C}^1$.*

Proof. Let $0 \to N \to M \to P \to 0$ be an exact sequence of modules in $\mathscr{C}^0$. Then, for each $\lambda \in X$, the sequence $0 \to N_\lambda \to M_\lambda \to P_\lambda \to 0$ is exact, and remains so after tensoring by $\mathscr{A}[U^1]_{-\lambda}$ which is a free $\mathscr{A}$-module by 2.7 (ii). This proves that $H^0(U^1/U^0, -)$ is exact.

Now, assume that $M \in \mathscr{C}^0$ is a free $\mathscr{A}$-module. Then each weight space M_λ, being a direct summand, is projective and therefore free, since $\mathscr{A}$ is a local ring (see 1.32). It follows that $H^0(U^1/U^0, M)$ is a free $\mathscr{A}$-module. □

2.12. We shall prove that induction satisfies Frobenius reciprocity, i.e. is right adjoint to restriction. Let the notation be as in 2.8.

Proposition. (Frobenius reciprocity.) *Let $M \in \mathscr{C}^2$ and $V \in \mathscr{C}^1$. Then, the map Φ: $f \mapsto \mathscr{E}v \circ f$ is an isomorphism of $\mathscr{A}$-modules:*

$$\mathrm{Hom}_{U^1}(V, H^0(U^1/U^2, M)) \xrightarrow{\sim} \mathrm{Hom}_{U^2}(V, M)$$

Proof. To each $h \in \mathrm{Hom}_{U^2}(V, M)$ one can associate $\Psi(h) \in \mathrm{Hom}_{U^1}(V, H^0(U^1/U^2, M))$ defined as follows. For $x \in V$, $\Psi(h)(x)$ is the map sending $u \in U^1$ to $h(ux) \in M$. It is easy to check that Φ and Ψ are reciprocal isomorphisms. □

Corollary 2.13. (*i*) *The induction functor: $\mathscr{C}^2 \to \mathscr{C}^1$ takes injective objects to injective objects.*

(*ii*) *The category $\mathscr{C}^1$ has enough injective objects.*

Proof. Assertion (i) is a standard consequence of Proposition 2.12. As for assertion (ii), we consider first the category $\mathscr{C}^0$. Let $M \in \mathscr{C}^0$. For each $\lambda \in X$, let Q_λ be the injective hull of the $\mathscr{A}$-module M_λ. We let U^0 act on Q_λ by the character χ_λ. Then, $M = \oplus M_\lambda$ is a U^0-submodule of $Q = \oplus Q_\lambda$, and the latter is an injective object in $\mathscr{C}^0$.

Consider now the category $\mathscr{C}^1$. Let $M \in \mathscr{C}^1$. As a U^0-module, M belongs to $\mathscr{C}^0$ and is therefore a U^0-submodule of some injective object $Q \in \mathscr{C}^0$. By Frobenius reciprocity, we obtain an injective U^1-homomorphism: $M \hookrightarrow H^0(U^1/U^0, M)$; and the latter is an injective object in $\mathscr{C}^1$, by assertion (i). Hence, (ii) is proved. □

2.14. Since the category $\mathscr{C}^2$ has enough injectives, we can define the right derived functors of induction. We denote them by: $H^i(U^1/U^2, -)$.

2.15. Suppose we have three subsets $I'' \subseteq I' \subseteq I \subseteq \{1, \ldots, n\}$. Take U^3 to be one of the following subalgebras: $\mathscr{A}$, U^0, $U(I'')$, and let $\mathscr{C}^3$ be the corresponding category. Then we have the:

Corollary. (Transitivity of induction.) *Let* $M \in \mathscr{C}^3$.

(*i*) *There is a natural isomorphism of* U^1-*modules*:

$$H^0(U^1/U^3, M) \simeq H^0(U^1/U^2, H^0(U^2/U^3, M))$$

(*ii*) *There is a spectral sequence*:

$$H^i(U^1/U^2, H^j(U^2/U^3, M)) \Rightarrow H^{i+j}(U^1/U^3, M)$$

Proof. Again (i) is a standard consequence of Proposition 2.12. The spectral sequence is the usual one associated to the composite of two functors, the first of which takes injective objects to objects which are acyclic for the second functor. This property is satisfied here, thanks to Corollary 2.13 (i). □

2.16. We shall now prove that induction satisfies the "tensor identity". The key to this result is Proposition 1.35 which might be called the tensor identity for induction from the trivial subalgebra. Let the notations be as in 2.8.

Proposition. (Tensor identity.) *Let* $V \in \mathscr{C}^1$.

(*i*) *For all* $M \in \mathscr{C}^0$ *there is a natural* U^1-*isomorphism*: $V \otimes H^0(U^1/U^0, M) \simeq H^0(U^1/U^0, V \otimes M)$

(*ii*) *Assume that* V *is flat as an* $\mathscr{A}$-*module. Then, for all* $M \in \mathscr{C}^2$ *there is a natural* U^1-*isomorphism*: $V \otimes H^0(U^1/U^2, M) \simeq H^0(U^1/U^2, V \otimes M)$.

Proof. We will prove both assertions simultaneously. So in the following U^2 denotes either U^0 or U^2. First, we consider the case where $M = H^{I'}(M')$ for some $\mathscr{A}$-module M'. In this case we get via Propositions 2.15 (i) and 2.7 (v):

$$V \otimes H^0(U^1/U^2, M) = V \otimes H^0(U^1/U^2, H^{I'}(M')) \simeq V \otimes H^I(M') \simeq H^I(V \otimes M') \simeq$$
$$\simeq H^0(U^1/U^2, H^{I'}(V \otimes M')) \simeq H^0(U^1/U^2, V \otimes H^{I'}(M')) \simeq H^0(U^1/U^2, V \otimes M)\,.$$

This proves the proposition for such M. For a general M we note that M is a U^2-submodule and a direct $\mathscr{A}$-summand of $H^{I'}(M) = M \otimes \mathscr{A}[U^2]$. Set $R = H^{I'}(M)/M$. From the short exact sequence: $0 \to M \to H^{I'}(M) \to R \to 0$ we obtain the commutative diagram:

$$\begin{array}{ccccccc} 0 \to & V \otimes H^0(U^1/U^2, M) & \to & V \otimes H^0(U^1/U^2, H^{I'}(M)) & \to & V \otimes H^0(U^1/U^2, R) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \to & H^0(U^1/U^2, V \otimes M) & \to & H^0(U^1/U^2, V \otimes H^{I'}(M)) & \to & H^0(U^1/U^2, V \otimes R) \end{array}$$

Here the bottom row is clearly exact and we claim that so is the top row. In case (i) this is because $H^0(U^1/U^0, M)$, being the zero-weight space of $H^I(M) = H^0(U^1/U^2, H^{I'}(M))$, is a direct summand. In case (ii), this follows from the flatness assumption on V. Now, we have seen that the middle vertical map is an isomorphism, and the proposition follows. □

Remark. Just as for the tensor identity in the case of algebraic groups, assertion (ii) is false without the flatness assumption on V (in spite of [Ja 3, Proposition I.3.6]).

2.17. Let $M \in \mathscr{C}^1$. Then M is a U^1-submodule and U^0-summand of $Q_0 = H^0(U^1/U^0, M)$. The same is true for Q_0/M and $Q_1 = H^0(U^1/U^0, Q_0/M)$, etc. It follows that we obtain a resolution in $\mathscr{C}^1$:

$$0 \to M \to Q_0 \to Q_1 \to \dots$$

which is $\mathscr{A}$-split (and even U^0-split) and such that each Q_i equals $H^0(U^1/U^0, Q_i')$ for some $Q_i' \in \mathscr{C}^0$. We shall call this the *standard resolution of M*.

Lemma 2.18. *If M is a free $\mathscr{A}$-module then the standard resolution of M consists of free $\mathscr{A}$-modules.*

Proof. Since M is free, then Q_0 is free, by Proposition 2.11. Then, Q_0/M is a direct summand of the free $\mathscr{A}$-module Q_0, and is therefore projective, hence free, since $\mathscr{A}$ is a local ring (see 1.32). Repeating this argument, we obtain that each Q_i is a free $\mathscr{A}$-module. □

Proposition 2.19. *Keep the notations of 2.8 and let $M \in \mathscr{C}^2$. Then:*

(i) The standard resolution of M (in $\mathscr{C}^2$) consists of modules which are acyclic for $H^0(U^1/U^2, -)$.

(ii) If $V \in \mathscr{C}^1$ is a flat $\mathscr{A}$-module then there is for each $i \geqq 0$ a natural U^1-isomorphism: $V \otimes H^i(U^1/U^2, M) \simeq H^i(U^1/U^2, V \otimes M)$.

Proof. Let $Q_i = H^0(U^1/U^2, Q_i')$ be the ith term in the standard resolution of M. By Corollary 2.15 (ii) and Proposition 2.11 we get:

$$H^j(U^1/U^2, Q_i) \simeq H^j(U^1/U^0, Q_i') = 0 \text{ for } j > 0$$

This proves (i). Moreover, $V \otimes Q_i \simeq H^0(U^1/U^0, V \otimes Q_i')$ for all i and hence $V \otimes Q_\bullet$ identifies with the standard resolution of $V \otimes M$. This together with the flatness of V gives (ii). □

3. Base change

In this section we study the relations between U-modules and U_Γ-modules, where $U_\Gamma = U \otimes \Gamma$ for some $\mathscr{A}$-algebra Γ.

If V is a U-module we write V_Γ for the U_Γ-module $V \otimes \Gamma$. Also, $U_\Gamma^0 = U^0 \otimes \Gamma$, $U_\Gamma^\flat = U^\flat \otimes \Gamma$, etc.

Although our results remain true for more general inductions, we shall state most of our results only in the case of induction from $U^\flat$ to U. We write $H^i(V)$ instead of $H^i(U/U^\flat, V)$, resp. $H_\Gamma^i(V)$, instead of $H^i(U_\Gamma/U_\Gamma^\flat, V)$, when V is a $U^\flat$-module, resp. $U_\Gamma^\flat$-module.

Lemma 3.1. *For any $M \in \mathscr{C}^0$, there is a natural isomorphism of U_Γ-modules: $H^0(U/U^0, M)_\Gamma \simeq H^0(U_\Gamma/U_\Gamma^0, M_\Gamma)$.*

Proof. This is the special case $V = \Gamma$ (with trivial action) in Proposition 2.16 (i). □

3.2. Induction from $U^\flat$ to U has the special property that it takes $\mathscr{A}$-finite modules to $\mathscr{A}$-finite modules. Let us denote by $\mathscr{C}_f^\flat$ (resp. $\mathscr{C}_f$) the subcategory of objects in $\mathscr{C}^\flat$ (resp. $\mathscr{C}$) which are finite $\mathscr{A}$-modules. Then we have the:

Proposition. *Let $M \in \mathscr{C}_f^\flat$. Then $H^0(M) \in \mathscr{C}_f$.*

Proof. Let $\varphi \in H^0(M)$. By Corollary 1.30 we know that $\varphi(J) = 0$ for some right ideal J such that U/J is a finite $\mathscr{A}$-module. We prove that all $\varphi \in H^0(M)$ are zero on some *fixed* such ideal J. Consider the larger $\mathscr{A}$-module $\mathscr{H}(M) = \operatorname{Hom}_{\mathscr{A}}(U, M)$. This is a $\gamma(U) \times \delta(U)$-module (see 1.9), and $H^0(M)$ is contained in $F_\delta(\mathscr{H}(M)) = F_\gamma(\mathscr{H}(M))$ (see 1.10–1.30).

Since $M \in \mathscr{C}_f^\flat$, then the set Ω of weights of M is finite and there exists $s_0 \gg 0$ such that $F_i^{(s)} M = 0$, hence $\gamma(F_i^{(s)}) M = 0$, for all i, and $s \geqq s_0$. Assume now that $\varphi \in H^0(M)$ has weight v for the action of $\gamma(U^0)$. This means that:

$$\chi_v(u)\varphi(x) = (\gamma(u)\varphi)(x) = \varphi(S(u)x) \text{ for all } u \in U^0, x \in U .$$

Then, any $\varphi(x) \neq 0$ is an element of M of weight $-v$. Hence the weights of the $\gamma(U^0)$-module $H^0(M)$ are contained in the (finite) set $-\Omega$. Together with Lemma 1.11 and the fact that $\gamma(F_i^{(s)})H^0(M) = 0$ for all i, and $s \geqq s_0$, this implies that there exists r_0 such that $\gamma(E_i^{(r)})M = 0$ for all i, and $r \geqq r_0$.

We conclude that all $\varphi \in H^0(M)$ are zero on the right ideal J generated by $\bigcap_{v \in \Omega} \mathrm{Ker}(\chi_{-v})$ and $\{E_i^{(r)}, F_i^{(s)} | r \geqq r_0, \ s \geqq s_0\}$. It follows that $H^0(M) \subseteq \mathrm{Hom}_{\mathscr{A}}(U/J, M)$. Since U/J is a finite $\mathscr{A}$-module and since $\mathscr{A}$ is noetherian, we conclude that $H^0(M)$ is a finite $\mathscr{A}$-module. □

Remark. The proposition remains valid for induction from $U(\varnothing, J)$ to $U(I, J)$, for any $I \subseteq J$; and in particular for induction from $U^\flat$ to any $U^\flat(I)$.

3.3. Let $\lambda \in X^+$, let J be the right ideal of U generated by $\mathrm{Ker}(\chi_\lambda)$ and $\{F_i^{(s)}, E_i^{(r_i)} | s > 0, r_i > \lambda_i\}$. Then the proof of Proposition 3.2 shows that $H^0(\lambda) \subseteq (U/J)^*$. The reverse inclusion is easily checked, and therefore we get: $H^0(\lambda) = (U/J)^*$. Now, U/J is a right U-module. We leave it to the reader to check that there exists an anti-automorphism Ψ of $U_{\mathscr{A}'}$ defined by the conditions:

$$\Psi(E_i) = -E_i \quad \Psi(F_i) = -F_i \quad \Psi(K_i) = K_i^{-1} \quad (1 \leqq i \leqq n)$$

and that Ψ restricts to an anti-automorphism of U. This allows us to make U/J into a left U-module. Then it identifies with $D'(-\lambda)$. Therefore, we obtain the:

Proposition. *Let $\lambda \in X^+$. Then $H^0(\lambda) \simeq D'(-\lambda)^* \simeq D(-w_0\lambda)^*$.*

Corollary. (*i*) *$H^0(\lambda)$ is a free $\mathscr{A}$-module, and its character is given by Weyl's formula.*
(*ii*) *If M is a $U^\flat$* (resp. $U^\natural$) *submodule of $H^0(\lambda)$ then $M_{w_0\lambda} \neq 0$* (resp. $M_\lambda \neq 0$).
(*iii*) $H^0(\lambda) \otimes k \simeq H_k^0(\lambda)$.

Proof. (i) follows from the proposition above together with Proposition 1.22; and (ii) obtains since $D(-w_0\lambda) \simeq D'(-\lambda)$ is generated as a $U^\flat$ (resp. $U^\natural$) module by its $-w_0\lambda$ (resp. $-\lambda$) weight space. As for (iii), the $U_k^\flat$-homomorphism $H^0(\lambda) \otimes k \to k_\lambda$ induces by Frobenius reciprocity a U_k-homomorphism $\phi: H^0(\lambda) \otimes k \to H_k^0(\lambda)$ which is injective on the λ-weight space. Since $H^0(\lambda) \otimes k$ is isomorphic to the k-dual of $D'(-\lambda) \otimes k$ and since the latter is generated by its $(-\lambda)$ weight space, we conclude that ϕ is injective. Since the two modules have the same character we obtain that ϕ is an isomorphism. □

3.4. Let Γ be an $\mathscr{A}$-algebra. Since $\mathscr{A}$ is a regular local ring (of dimension 2), then it has finite global homological dimension (equal to 2) and therefore there exists a finite resolution:

$$0 \to P^2 \to P^1 \to P^0 \to \Gamma \to 0$$

where the P^i are free $\mathscr{A}$-modules (and $P^i = 0$ for $i > 2$). We regard this resolution as an exact sequence of trivial U-modules.

Let now $M \in \mathscr{C}^\flat$ and consider the standard resolution of M (see 2.17):

$$0 \to M \to Q_0 \to Q_1 \to \dots$$

Assuming that M is free as $\mathscr{A}$-module, each Q_j is also free, by Lemma 2.18. Hence, for each j, we have a free resolution $Q_j \otimes P^{\bullet}$ of $Q_j \otimes \Gamma$. Moreover, for each i the resolution $Q_\bullet \otimes P^i$ of $M \otimes P^i$ is H^0-acyclic. Since the complex $P^{\bullet}$ is finite, then the double complex $Q_\bullet \otimes P^{\bullet}$ gives rise to a spectral sequence:

$$E_2^{i,-j} = \mathrm{Tor}_j^{\mathscr{A}}(H^i(M), \Gamma) \Rightarrow H_\Gamma^{i-j}(M_\Gamma)$$

3.5. Instead of the spectral sequence in 3.4 we can formulate the relations between the functors in question as a six-term exact sequence.

Keep the notations in 3.4. Set $M^i = H^0(Q_i)$ and let $d^i : M^i \to M^{i+1}$ be the differential in the complex $M^{\bullet}$. Setting $B^i = \mathrm{Im}(d^i)$ and $R^i = \mathrm{Coker}(d^i)$ we obtain the exact sequences below, where $i \geqq 0$. (Note that (3) is the special case $i = 0$ of (2)).

$$0 \to B^i \to M^{i+1} \to R^i \to 0 \tag{1}$$

$$0 \to H^i(M) \to R^{i-1} \to B^i \to 0 \tag{2}$$

$$0 \to H^0(M) \to M^0 \to B^0 \to 0 \tag{3}$$

Note that M^i is a free $\mathscr{A}$-module. In fact, $Q_i = H^0(U^\flat/U^0, Q_i')$ for some $\mathscr{A}$-free U^0-module Q_i' so that $M^i = H^0(U/U^0, Q_i')$ is free by Proposition 2.11. Therefore (1) and (3) respectively give:

$$\mathrm{Tor}_j^{\mathscr{A}}(B^i, \Gamma) \simeq \mathrm{Tor}_{j+1}^{\mathscr{A}}(R^i, \Gamma) \text{ and } \mathrm{Tor}_j^{\mathscr{A}}(H^0(M), \Gamma) \simeq \mathrm{Tor}_{j+1}^{\mathscr{A}}(B^0, \Gamma) \tag{4}$$

$$i \geqq 0, j \geqq 1.$$

Since $\mathrm{gldim}(\mathscr{A}) = 2$ we get $\mathrm{Tor}_j^{\mathscr{A}}(B^i, \Gamma) = 0$ for all $j \geqq 2$, $i \geqq 0$, hence $\mathrm{Tor}_j^{\mathscr{A}}(H^0(M), \Gamma) = 0$ for all $j \geqq 1$. Since we can take $\Gamma = \mathscr{A}/I$, for any ideal I of $\mathscr{A}$, we conclude that $H^0(M)$ is a flat $\mathscr{A}$-module. If moreover M is a finite $\mathscr{A}$-module, then so is $H^0(M)$ by Proposition 3.2, and therefore in that case we conclude that $H^0(M)$ is a free $\mathscr{A}$-module (since $\mathscr{A}$ is a local ring).

Taking the isomorphisms (4) into account, the long exact sequences coming from (1) and (2) respectively give:

$$0 \to \mathrm{Tor}_1^{\mathscr{A}}(R^i, \Gamma) \xrightarrow{\sigma_i} B^i \otimes \Gamma \xrightarrow{\tau_i} M^{i+1} \otimes \Gamma \to R^i \otimes \Gamma \to 0 \tag{5}$$

$$\begin{aligned} 0 \to \mathrm{Tor}_1^{\mathscr{A}}(H^i(M), \Gamma) \to \mathrm{Tor}_1^{\mathscr{A}}(R^{i-1}, \Gamma) \to \mathrm{Tor}_2^{\mathscr{A}}(H^{i+1}(M), \Gamma) \\ \to H^i(M) \otimes \Gamma \xrightarrow{\phi_i} R^i \otimes \Gamma \xrightarrow{\pi_i} B^i \otimes \Gamma \to 0 \end{aligned} \tag{6}$$

Note that $Q_\bullet \otimes \Gamma$ is the standard resolution of M_Γ so that $H_\Gamma^i(M_\Gamma)$ is the ith cohomology of the complex $H_\Gamma^0(Q_\bullet \otimes \Gamma)$. By Lemma 3.1 we have:

$$Q_i \otimes \Gamma = H^0(U^\flat/U^0, Q_i') \otimes \Gamma \simeq H^0(U_\Gamma^\flat/U_\Gamma^0, Q_i' \otimes \Gamma)$$

Therefore, we find:

$$H_\Gamma^0(Q_i \otimes \Gamma) \simeq H^0(U_\Gamma/U_\Gamma^0, Q_i' \otimes \Gamma) \simeq H^0(U/U^0, Q_i') \otimes \Gamma \simeq M^i \otimes \Gamma$$

It follows that $H^i_\Gamma(M_\Gamma)$ is the kernel of: $R^i_\Gamma \to M^{i+1}_\Gamma$. Thus combining (4) and (5) we obtain a commutative diagram:

$$\begin{array}{ccccccc}
 & & & & 0 & & \\
 & & & & \downarrow & & \\
 & & & & \mathrm{Tor}_1^{\mathscr{A}}(R^i, \Gamma) & & \\
 & & & & \downarrow & & \\
 & H^i(M)_\Gamma & \xrightarrow{\phi_i} & R^i_\Gamma & \xrightarrow{\pi_i} & B^i_\Gamma & \to 0 \\
 & \downarrow \eta_i & & \| & & \downarrow & \\
0 \to & H^i_\Gamma(M_\Gamma) & \xrightarrow{\sigma_i} & R^i_\Gamma & \xrightarrow{\tau_i} & M^{i+1}_\Gamma &
\end{array}$$

By the Snake Lemma, we obtain $\mathrm{Coker}(\eta_i) \simeq \mathrm{Tor}_1^{\mathscr{A}}(R^i, \Gamma)$. Together with (4), this gives a six-term exact sequence relating $H^i(M)_\Gamma$ and $H^i_\Gamma(M_\Gamma)$. Summarizing, we have obtained the:

Theorem. *Assume that $M \in \mathscr{C}^b$ is a free $\mathscr{A}$-module. Then:*

(*i*) *$H^0(M)$ is a flat $\mathscr{A}$-module. If moreover M is finite free then so is $H^0(M)$.*
(*ii*) *For each $i \geqq 0$ there is an exact sequence:*

(7)
$$\begin{aligned} &0 \to \mathrm{Tor}_1^{\mathscr{A}}(H^i(M), \Gamma) \to \mathrm{Tor}_1^{\mathscr{A}}(R^{i-1}, \Gamma) \to \mathrm{Tor}_2^{\mathscr{A}}(H^{i+1}(M), \Gamma) \to \\ &\to H^i(M)_\Gamma \to H^i_\Gamma(M_\Gamma) \to \mathrm{Tor}_1^{\mathscr{A}}(R^i, \Gamma) \to 0 \end{aligned}$$

Remark. Assume that $H^j(M) = 0$ for all $j > i$. Then (7) with i replaced by $i + 1$ gives $\mathrm{Tor}_1^{\mathscr{A}}(R^i, \Gamma) = 0$, hence we obtain $H^i(M)_\Gamma \simeq H^i_\Gamma(M_\Gamma)$. Of course, this also follows from the spectral sequence in 3.4.

3.6. Two special cases are worth recording. Suppose that the $\mathscr{A}$-algebra Γ is flat over $\mathscr{A}$. Then 3.5 (ii) simply reads: $H^i(M)_\Gamma \simeq H^i_\Gamma(M_\Gamma)$ for all $i \geqq 0$.

The other case is when the $\mathscr{A}$-module Γ has projective dimension one. Then $\mathrm{Tor}_j^{\mathscr{A}}(B_i, \Gamma) = 0$ for all $i \geqq 0$, $j \geqq 1$, hence the exact sequence 3.5(2) gives $\mathrm{Tor}_1^{\mathscr{A}}(H^{i+1}(M), \Gamma) \simeq \mathrm{Tor}_1^{\mathscr{A}}(R^i, \Gamma)$ for all $i \geqq 0$. Therefore the six-term sequence 3.5(6) simplifies to the short exact sequence:

(8)
$$0 \to H^i(M)_\Gamma \to H^i_\Gamma(M_\Gamma) \to \mathrm{Tor}_1^{\mathscr{A}}(H^{i+1}(M), \Gamma) \to 0$$

3.7. Now, assume that $\varphi: \mathscr{A} \to \Gamma$ is a specialization of $\mathscr{A}$ into a field Γ such that $\varphi(v) = 1$. Let $M \in \mathscr{C}^b$. We identify the cohomology groups $H^i_\Gamma(M_\Gamma)$ with the sheaf cohomology on the flag variety G_Γ/B_Γ.

Proposition. *$H^i_\Gamma(M_\Gamma) \simeq H^i(G_\Gamma/B_\Gamma, M_\Gamma)$ for all $i \geqq 0$.*

Proof. By [L 6, 8.15] the hyperalgebra of G_Γ identifies with the quotient of U_Γ by the ideal generated by $K_i - 1, i = 1, \ldots, n$. Hence any G_Γ-module is a U_Γ-module, and conversely any locally finite U_Γ-module on which the K_i's act as the identity is a G_Γ-module (in characteristic zero this is well known, and for positive characteristic see e.g. [CPS 2, 9.2]). Similarly, the category of U^b_Γ-modules in $\mathscr{C}^b_\Gamma$ identifies with the category of B_Γ-modules. In positive characteristic, this is proved in [loc. cit., 9.4]. On the other hand, in characteristic zero it is well known that $H^0(U^b_\Gamma/U^0_\Gamma, \Gamma)$ identifies with the coordinate algebra of the unipotent radical of B. It follows that the injective modules in the two categories coincide, hence the result follows as in [loc. cit.]. From this we deduce: $H^0_\Gamma(M) \simeq H^0(G_\Gamma/B_\Gamma, M)$, e.g. because both satisfy Frobenius reciprocity (see 2.12). Moreover, the standard resolution is acyclic for both functors, so that the derived functors also coincide. □

3.8. All that we used about the base change $\mathscr{A} \to \Gamma$ is that the $\mathscr{A}$-module Γ has finite projective dimension, equal to two. Therefore the same argument applies to another base change $\Gamma \to \Gamma'$, if the projective dimension of Γ' as a Γ-module is at most 2. We will use this in the following cases.

3.9. Take $\Gamma = \mathbb{Q} \otimes \mathscr{A}$ and $\Gamma' = \mathbb{Q}$ where $\mathbb{Q}$ is made into a Γ-algebra by taking v to 1. Then, we have:

Corollary. *Let $M \in \mathscr{C}_\Gamma^b$ be Γ-free. Then for each $i \geqq 0$ we have a short exact sequence:*

$$0 \to H_\Gamma^i(M) \otimes_\Gamma \mathbb{Q} \to H^i(G_\mathbb{Q}/B_\mathbb{Q}, M_\mathbb{Q}) \to \operatorname{Tor}_1^\Gamma(H_\Gamma^{i+1}(M), \mathbb{Q}) \to 0$$

Proof. This follows from 3.6(8) together with Proposition 3.7. □

3.10. Let $\wp$ be a prime ideal in $\mathscr{A}$ distinct from (0) and $\gamma\eta$. If Γ is either the residue field of $\mathscr{A}_\wp$, or $\mathscr{A}/\wp$ or the quotient field of the latter, then we have for any $\mathscr{A}$-free $M \in \mathscr{C}^b$ an exact sequence:

$$(9) \qquad 0 \to H^i(M)_\Gamma \to H_\Gamma^i(M_\Gamma) \to \operatorname{Tor}_1^\mathscr{A}(H^{i+1}(M), \Gamma) \to 0$$

As an example, let $l = p^e$ for some $e \geqq 1$ and let ϕ_l be the corresponding cyclotomic polynomial. Then the fraction field of $\mathscr{A}/(\phi_l)$ is $\mathbb{Q}[\zeta]$ where ζ is a primitive l^{th} root of 1.

3.11. Let finally $\Gamma = k$, the residue field of $\mathscr{A}$. Then Γ is an $\mathscr{A}$-module of projective dimension 2, and we can apply Theorem 3.5.

Suppose that $M \in \mathscr{C}^b$ is $\mathscr{A}$-free and has the property that $H^i(G_k/B_k, M_k) = 0$ whenever $i > i_0$. Then it follows from 3.5 and 3.7 that: $H^{i_0}(M) \otimes k \simeq H^{i_0}(G_k/B_k, M_k)$. If M is a finite $\mathscr{A}$-module then so are all $H^j(M)$, as we shall see in the next section. Hence the above isomorphism gives via Nakayama that: if $H^i(G_k/B_k, M_k) = 0$ for $i \geqq i_0$, then $H^i(M) = 0$ for $i \geqq i_0$.

4. Rank one

In this section, we assume $n = 1$ and write F, $K^{\pm 1}$, E for the generators of U'. We compute the structure of $H^i(\lambda)$ for $\lambda \in X$, $i \geqq 0$.

4.1. For $m \in \mathbb{Z}$ let $\lambda_m : U^0 \to \mathscr{A}$ be the character defined by:

$$\lambda_m(K) = v^m, \lambda_m\left(\begin{bmatrix} K; c \\ t \end{bmatrix}\right) = \begin{bmatrix} m + c \\ t \end{bmatrix}, c \in \mathbb{Z}, t \in \mathbb{N}$$

Proposition. *Let $m \in \mathbb{Z}$.*

(*i*) *$H^0(\lambda_m) \neq 0$ if and only if $m \geqq 0$.*

(*ii*) *If $m \geqq 0$ then $H^0(\lambda_m)$ is a free $\mathscr{A}$-module of dimension $(m + 1)$, with a basis: $\{e_0, \ldots, e_m\}$ such that: e_i is of weight λ_{m-2i}, and:*

$$E^{(j)}e_i = \begin{bmatrix} i \\ j \end{bmatrix} e_{i-j}, F^{(j)}e_i = \begin{bmatrix} m - i \\ j \end{bmatrix} e_{i+j}, 0 \leqq i, j \leqq m .$$

Proof. (i) If $H^0(\lambda_m) \neq 0$ there exists $f \in H^0(\lambda_m)$ such that $f(1) = 1$. Then, using 1.11 (1), we have:

$$(F^{(j)}f)(E^{(j)}) = f(E^{(j)}F^{(j)}) = \sum_{t \geqq 0} f\left(F^{(j-t)} \begin{bmatrix} K; 2t - 2j \\ t \end{bmatrix} E^{(j-t)} \right)$$

$$= \sum_{t \geqq 0} F^{(j-t)} \begin{bmatrix} m + 2t - 2j \\ t \end{bmatrix} f(E^{(j-t)}) = \begin{bmatrix} m \\ j \end{bmatrix}$$

If $m < 0$ then $\begin{bmatrix} m \\ j \end{bmatrix} \neq 0$ for all $j \geqq 0$ and since $F^{(j)}f = 0$ for $j \gg 0$ we must therefore have $m \geqq 0$.

(ii) Conversely, if $m \geqq 0$, we prove that $H^0(\lambda_m)$ has dimension $(m + 1)$. Define $e_i : U \to \mathscr{A}$ by:

$$e_i(F^{(r)}uE^{(s)}) = \delta_{0r}\delta_{is}\lambda_m(u),\ u \in U^0,\ r, s \geqq 0$$

Then, using 1.11 (1) again we get:

$$(F^{(r)}e_i)(E^{(s)}) = e_i(E^{(s)}F^{(r)}) = \sum_t e_i\left(F^{(r-t)} \begin{bmatrix} K_i; 2t - r - s \\ t \end{bmatrix} E^{(s-t)} \right)$$

$$= \delta_{i,r+s} \begin{bmatrix} m - i \\ r \end{bmatrix}$$

In particular, if $i > m$ then $F^{(j)} \neq 0$ for all $j \geqq 0$ and, if $i \leqq m$ then we read off the stated action of $F^{(j)}$ on e_i (note that $F^{(j)}e_i = 0$ for $j > m - i$). Similarly, we compute:

$$(E^{(j)}e_i)(E^{(M)}) = e_i(E^{(j)}E^{(M)}) = \begin{bmatrix} j + M \\ j \end{bmatrix} e_i(E^{(j+M)})$$

It follows: $E^{(j)}e_i = \begin{bmatrix} i \\ j \end{bmatrix} e_{i-j}$. □

Proposition 4.2. *Assume $m \geqq -1$. Then $H^i(\lambda_m) = 0$ for $i > 0$.*

Proof. Set $I(m) = H^0(U^\flat/U^0, \lambda_m)$ and observe that the weights of $I(m)$ are $\{\lambda_{m+2i} | i \geqq 0\}$, each occurring with multiplicity one. Moreover, for each $r \geqq 0$ there is an inclusion: $H^0(\lambda_r) \otimes \lambda_{r+m} \subseteq I(m)$. In fact, the U^0-homomorphism $H^0(\lambda_r) \otimes \lambda_{r+m} \to \lambda_m$ (see 4.1. (ii)) gives by Frobenius reciprocity a $U^\flat$-homomorphism $H^0(\lambda_r) \otimes \lambda_{r+m} \to I(m)$.

We set $Q_r = (H^0(\lambda_r) \otimes \lambda_{r+m})/\lambda_m$ and claim that the induced sequence:

$$(E) \qquad 0 \to H^0(\lambda_m) \to H^0(\lambda_r) \otimes H^0(\lambda_{r+m}) \to H^0(Q_r) \to 0$$

is exact. (We have used the tensor identity for the middle term).

In fact, by Theorem 3.5 (i) all terms are finite free $\mathscr{A}$-modules. Hence it is enough to prove that the sequence $(E \otimes k)$ is exact. By Proposition 4.1 we have: $\dim(H^0(\lambda_m)_k) = m + 1$ for all $m \geqq -1$, and so we are done if we check the inequality:

$$\dim(H^0(Q_r)_k) \leqq (r + 1)(r + m + 1) - (m + 1) = r(r + m + 2)\,.$$

But the weights of Q_r are: $\{\lambda_{m+2i} | 1 \leqq i \leqq r\}$ and therefore:

$$\dim(H^0(Q_r)_k) \leqq \sum_{i=1}^{r} \operatorname{rank}_{\mathscr{A}}(H^0(\lambda_{m+2i})) = \sum_{i=1}^{r} (m+2i+1) = r(r+m+2)$$

Since $I(m) = \bigcup_{r \geqq 0} H^0(\lambda_r) \otimes \lambda_{r+m}$ (by weight considerations) we conclude that the map $H^0(I(m)) \to H^0(I(m))/\lambda_m)$ is surjective. It follows that $H^1(\lambda_m) = 0$.

Moreover, $H^i(\lambda_m) \simeq H^{i-1}(I(m)/\lambda_m)$ for $i > 1$, and since all weights $\lambda_t \in I(m)$ satisfy: $t \geqq m \geqq -1$, we conclude by induction on i that $H^i(\lambda_m) = 0$ for all $i \geqq 1$, $m \geqq -1$. □

Proposition 4.3. *The derived functors H^i are zero for $i > 1$.*

Proof. It is enough to prove that $H^i(\lambda_m) = 0$ for all $i > 1, m \in \mathbb{Z}$. We already know this when $m \geqq -1$, so we assume $m < -1$. It follows from Proposition 4.1 that the kernel of $\mathscr{E}v: H^0(\lambda_{-m}) \to \lambda_{-m}$ can be identified with $H^0(\lambda_{-m-1}) \otimes \lambda_{-1}$. Hence, tensoring by λ_m we get the exact sequence:

$$0 \to H^0(\lambda_{-m-1}) \otimes \lambda_{m-1} \to H^0(\lambda_{-m}) \otimes \lambda_m \to \lambda_0 \to 0$$

Then, the tensor identity together with Proposition 4.2 applied to λ_0 give: $H^0(\lambda_{-m-1}) \otimes H^i(\lambda_{m-1}) \simeq H^0(\lambda_{-m}) \otimes H^i(\lambda_m)$ for $i > 1$. Thus, the proposition follows by induction on $|m|$. □

Remark. The proofs of Propositions 4.2–4.3 are copied from Donkin's analogous results for SL_2 ([Do, Section 12.2]).

Proposition 4.4. *Let $m \geqq 0$. Then $H^1(\lambda_{-m-2})$ is a free $\mathscr{A}$-module of dimension $(m+1)$, with a basis $\{f_0, \ldots, f_m\}$ such that f_i is of weight $-m+2i$, and:*

$$E^{(j)} f_i = \begin{bmatrix} i+j \\ i \end{bmatrix} f_{i+j}, \quad F^{(j)} f_i = \begin{bmatrix} m-i+j \\ j \end{bmatrix} f_{i-j}$$

Proof. From the description of $H^0(\lambda_1)$, we obtain, for all $m \geqq 0$, the exact sequence:

$$0 \to \lambda_{-m-2} \to H^0(\lambda_1) \otimes \lambda_{-m-1} \to \lambda_{-m} \to 0$$

When $m = 0$ this gives $H^1(\lambda_{-2}) \simeq H^0(\lambda_0) = \mathscr{A}$, and in this case we take $f_0 = e_0$, the generator of $H^0(\lambda_0)$. Now suppose $m > 0$ and assume the proposition for smaller values of m. Denote by $\{f'_0, \ldots, f'_{m-1}\}$, resp. $\{f''_0, \ldots, f''_{m-2}\}$ the basis for $H^1(\lambda_{-m-1})$, resp. $H^1(\lambda_{-m})$ given by this induction hypothesis. From the above exact sequence we obtain via Propositions 4.1 and 4.3 and the tensor identity 2.16 the exact sequence:

$$0 \to H^1(\lambda_{-m-2}) \to H^0(\lambda_1) \otimes H^1(\lambda_{-m-1}) \xrightarrow{\phi} H^1(\lambda_{-m-1}) \to 0$$

Consider the map ϕ. Since it is a U^0-homomorphism, there exists elements $a_i, b_i \in \mathscr{A}$ such that:

$$\phi(e_0 \otimes f'_i) = a_i f''_i \quad \text{and} \quad \phi(e_1 \otimes f'_i) = b_i f''_{i-1}, \quad 0 \leqq i \leqq m-1 .$$

(Here $\{e_0, e_1\}$ is the basis of $H^0(\lambda_1)$ described in Proposition 4.1.) Since $E.\phi(e_0 \otimes f'_i) = \phi(E.(e_0 \otimes f'_i))$ we get:

$$a_i[i+1] f''_{i+1} = \phi(Ke_0 \otimes Ef'_i) = v[i+1] a_{i+1} f''_{i+1}, \quad \text{hence } a_i = v^{-i} a_0 .$$

Likewise the relation $F.\phi(e_1 \otimes f'_i) = \phi(F.(e_1 \otimes f'_i))$ gives $b_{i+1} = b_1$ for all i. Each of the two previous relations gives then $a_0 = -v^{-1}b_1$. Since ϕ is surjective we see that up to a unit in $\mathscr{A}$ we have:

$$\phi(e_0 \otimes f'_i) = -v^{-i-1}f''_i \quad \text{and} \quad \phi(e_1 \otimes f'_i) = f''_{i-1} \quad \text{for all } i.$$

It follows that $H^1(\lambda_{-m-2}) = \mathrm{Ker}(\phi)$ has a basis consisting of the elements:

$$f_i = e_0 \otimes f'_{i-1} + v^{-i}e_1 \otimes f'_i \quad \text{where } 0 \leqq i \leqq m, (\text{with } f'_{-1} = f'_m = 0)$$

It is now straightforward to check that the action w.r.t. this basis is given by the formulas stated in the propositions. □

Corollary 4.5. *Let $m \geqq -1$. Then the map $T_m: H^1(\lambda_{-m-2}) \to H^0(\lambda_m)$ which takes each f_i to $\begin{bmatrix} m \\ i \end{bmatrix} e_{m-i}$ is a U-homomorphism. Moreover, any U-homomorphism $H^1(\lambda_{-m-2}) \to H^0(\lambda_m)$ is proportional to T_m.*

Proof. This follows from Propositions 4.1 and 4.4. □

4.6. Let $\mathscr{A} \to \Gamma$ be a specialization of $\mathscr{A}$ into a field Γ. Let $\zeta \in \Gamma$ denote the image of v. For $m \geqq 0$ we denote by $L_\Gamma(\lambda_m)$ the simple U_Γ-module with highest weight λ_m. The following corollary was proved by Lusztig in the case $\Gamma = \mathbb{C}$ ([L 3, Proposition 9.2]).

Corollary. (*i*) *Suppose that either ζ is not a root of unity or $\zeta = 1$ and* $\mathrm{char}(\Gamma) = 0$. *Then for all $m \geqq 0$ there is a U_Γ-isomorphism:*

$$H^1_\Gamma(\lambda_{-m-2}) \simeq H^0_\Gamma(\lambda_m)$$

(*ii*) *Suppose that ζ is a primitive l^{th} root of unity, and* $\mathrm{char}(\Gamma) = 0$. *Let $m \geqq -1$.*

(1) *The map $T^\Gamma_m: H^1_\Gamma(\lambda_{-m-2}) \to H^0_\Gamma(\lambda_m)$ is an isomorphism if and only if $m < l$ or $m = al - 1$ for some $a \geqq 0$.*

(2) *If $m = m_1 l + m_0$ with $m_1 > 0$, $0 \leqq m_0 < l - 1$, then:*

$$\mathrm{Im}(T^\Gamma_m) = L_\Gamma(\lambda_m) \text{ and } \mathrm{Coker}(T^\Gamma_m) \simeq L_\Gamma(\lambda_{m_1 l - m_0 - 2})$$

(*iii*) *Suppose that $\zeta = 1$ and* $\mathrm{char}(\Gamma) = p$. *Then the statements in* (*ii*) *remain true with l replaced by p as long as $m < p^2$.*

Proof. Assertion (i) follows from Corollary 4.5 (the conditions in (i) ensure that for any $i \leqq m$, $\begin{bmatrix} m \\ i \end{bmatrix}$ does not specialize to 0 in Γ).

Assertions (ii) and (iii) also follow from Corollary 4.5: write $i = i_1 l + i_0$, $0 \leqq i_0 < l$. Then, by [L 3, Proposition 3.2.(a)] we have:

$$\begin{bmatrix} m \\ i \end{bmatrix}_\zeta = \begin{bmatrix} m_0 \\ i_0 \end{bmatrix}_\zeta \begin{pmatrix} m_1 \\ i_1 \end{pmatrix} \text{ where } \begin{pmatrix} m_1 \\ i_1 \end{pmatrix} \text{ is an ordinary binomial coefficient.} \quad \square$$

4.7. Let the notations and assumptions be as in Corollary 4.6. For uniform notation we take $l = p$. We set: $\chi(m) = \mathrm{ch}(H^0_\Gamma(\lambda_m)) - \mathrm{ch}(H^1_\Gamma(\lambda_m))$, $m \in \mathbb{Z}$. Then an easy computation (compare [A 2]) gives:

Corollary. *Let* $m = m_1 p + m_0$, $m_1 \geqq 0$, $0 \leqq m_0 < p - 1$. *If* $\mathrm{char}(\Gamma) = p$ *we assume that* $m < p^2$. *Then:*

(*i*) $\mathrm{ch}(L_\Gamma(\lambda_m)) = \sum_{j=0}^{m_1} \chi(m - 2jp)$

(*ii*) *There are exact sequences of* $U_\Gamma^\flat$*-modules:*

$$0 \to K_m \to H_\Gamma^0(\lambda_{m+1}) \otimes \lambda_{-1} \to \lambda_m \to 0$$

$$0 \to \lambda_{-m-2} \to K_m \to V_m \to 0$$

$$0 \to C_m \to V_m \to I_m \to 0$$

$$0 \to I_m \to H_\Gamma^0(\lambda_{m-1}) \otimes \lambda_{-1} \to C_m \to 0$$

where C_m *has weights:* $\lambda_{-m-2+2jp}$, $1 \leqq j \leqq m_1$.

5. Vanishing theorems

In this section we study further the functors H^i, $i \geqq 1$. Using both the detailed study from section 4 of the behaviour of these functors for $n = 1$ and the relation obtained in section 3 to the much studied H_k^i we prove that H^i takes a finite object in $\mathscr{C}^\flat$ into a finite object in $\mathscr{C}$, that $H^i = 0$ for $i \geqq |R^+|$, that Kempf's vanishing theorem holds, that there is a Demazure character formula and that in fact a lot of the results in the modular theory carry over to the quantum case.

5.1. Recall that for $i \in \{1, \ldots, n\}$ the (minimal) parabolic subalgebra $U^\flat(i)$ was defined in 2.5. The induction functor $H^0(U^\flat(i)/U^\flat, -)$ and its derived functors will be denoted $H^r(s_i, -)$, and sometimes simply H_i^r.

Let $w \in W$ and let $s = s_{i_1} \ldots s_{i_r}$ be a reduced expression for w. Then we set $H^0(s, -) = H_{i_1}^0 \ldots H_{i_r}^0$ and view it as a functor from $\mathscr{C}^\flat$ to itself. The j'th derived functor is denoted $H^j(s, -)$ and we let $H_k^j(s, -)$ be the analogously defined functors on $\mathscr{C}_k^\flat$. (We shall see later that these functors only depend on w and not on the reduced expression).

Theorem. *Let* $\lambda \in X^+$ *and* $w \in W$. *Then:*

(*i*) *The natural map* $H^0(s, \mathscr{A}_\lambda) \otimes k \to H_k^0(s, k_\lambda)$ *is an isomorphism.*

(*ii*) *The natural map* $H^0(\lambda) \to H^0(s, \lambda)$ *is surjective.*

(*iii*) $H_i^r(H^0(s, \lambda)) = 0$ *for* $r > 0$.

Proof. We proceed by induction on $l(w)$. If $w = 1$ the statements are trivial. So we let $w > 1$ and assume the theorem for all $w' \in W$ of length smaller than $l(w)$.

(i) Set $s' = s_{i_2} \ldots s_{i_r}$. This is a reduced expression for $w' = s_{i_1} w$. By induction hypothesis $H_{i_1}^r(H^0(s', \lambda)) = 0$ for $r > 0$. Then the Remark following Theorem 3.5 together with the induction hypothesis gives the isomorphisms:

$$H^0(s, \lambda) \otimes k = H^0(s_{i_1}, H^0(s', \lambda)) \otimes k \simeq \mathrm{H}_k^0(s_{i_1}, H^0(s', \lambda) \otimes k)$$

$$\simeq H_k^0(s_{i_1}, H_k^0(s', \lambda)) \simeq H_k^0(s, \lambda)$$

(ii) The evaluation map $H^0(\lambda) \to \lambda$ induces a $U^\flat(i_r)$-homomorphism $H^0(\lambda) \to H_{i_r}^0(\lambda)$. This in turn gives a $U^\flat(i_{r-1})$-homomorphism $H^0(\lambda) \to H_{i_{r-1}}^0 H_{i_r}^0(\lambda)$ and the natural map $H^0(\lambda) \to H^0(s, \lambda)$ is obtained by repeating this procedure r times. Let Q denote the cokernel of this map. From Proposition 3.2 we deduce that $H^0(s, \lambda)$ is a finite $\mathscr{A}$-module. Hence so is Q and by Nakayama we are done if we prove that $Q \otimes k = 0$. But we have the commutative diagram

$$\begin{array}{ccccccc} H^0(\lambda) \otimes k & \to & H^0(s, \lambda) \otimes k & \to & Q \otimes k & \to & 0 \\ \downarrow & & \downarrow \wr & & & & \\ H_k^0(\lambda) & \to & H_k^0(s, \lambda) & & & & \end{array}$$

By (i) the second vertical map is an isomorphism, and so is the first one, by Corollary 3.3. Also, the bottom horizontal map is a surjection by [A 4, Theorem 3.2], [RR, Theorem 2]. It follows that $Q \otimes k = 0$.

(iii) Let $\lambda_0 \in X$ denote the trivial character. Then by the tensor identity 2.16 and Proposition 4.2 we have:

$$H_{i_1}^r(H^0(\lambda)) \simeq H_{i_1}^r(\lambda_0) \otimes H^0(\lambda) = 0 \quad \text{for } r > 0$$

Since also $H_{i_1}^r = 0$ for $r \geqq 2$ (Proposition 4.3) we see that (iii) follows from (ii). □

5.2. In order to study further the functors $H^r(s, -)$ from 5.1 we need the following general lemma on composite functors

Lemma. *Let $F_1: \mathscr{D}_1 \to \mathscr{D}_2$ and $F_2: \mathscr{D}_2 \to \mathscr{D}_3$ be left exact additive covariant functors between abelian categories. Suppose $\mathscr{D}_1$ and $\mathscr{D}_2$ have enough injectives.*

(i) If $M \in \mathscr{D}_1$ is acyclic for F_1 and $F_1(M) \in \mathscr{D}_2$ is acyclic for F_2 then M is acyclic for $F_2 \circ F_1$.

(ii) Suppose $M \in \mathscr{D}_1$ has a resolution

$$0 \to M \to I_0 \to I_1 \to \ldots$$

where I_j satisfies the assumptions in (i) for all j. Then $R^j(F_2 \circ F_1)(M)$ is the j^{th} cohomology of the complex $F_2 \circ F_1(I)$.

Proof. (i) Imbed M into an injective object $I \in \mathscr{D}_1$ and denote by Q the quotient I/M. By assumption we get exact sequences:

$$0 \to F_1 M \to F_1 I \to F_1 Q \to R^1 F_1 M = 0$$

$$1 \to F_2 F_1 M \to F_2 F_1 I \to F_2 F_1 Q \to R^1 F_2(F_1 M) = 0$$

It follows that $R^1(F_2 \circ F_1)(M) = 0$. Since $R^j(F_2 \circ F_1)(M) \simeq R^{j-1}(F_2 \circ F_1)(Q)$ for $j > 1$ and since Q also satisfies the assumptions in the lemma we get by induction that $R^j(F_2 \circ F_1)(M) = 0$ for $j > 1$.

(ii) is an obvious consequence of (i). □

Lemma 5.3. (Compare [CPS 2, Proposition 5.5]). *Let $\mu \in X$. Then $H^0(U^\flat/U^0, \mu)$ is the directed union for $m \geqq 0$ of submodules isomorphic to $H^0(m\rho) \otimes (m\rho + \mu)$.*

Proof. Set $H^0(U^\flat/U^0, \mu) = I$ and $V(m) = H^0(m\rho) \otimes (m\rho + \mu)$ for $m \geqq 0$. By Frobenius reciprocity the U^0-homomorphism $V(m) \to \mu$ induces a $U^\flat$-homomorphism $\phi: V(m) \to I$ which is injective on the μ-weight space. By Corollary 3.3 (ii) it follows that ϕ is injective. Therefore $V(m)$ identifies with a $U^\flat$-submodule of I. The same proof shows that $V(m) \otimes k$ identifies with a $U_k^\flat$-submodule of $I \otimes k$. Now, let v be an arbitrary weight of I. Since $\mathrm{ch}(V(m) \otimes k)$ is known by Corollary 3.3 (i), we obtain by Kostant's multiplicity formula (see [CPS 2, Lemma 5.3]) that:

$$\dim_k(I_v \otimes k) = \dim_k(V(m)_v \otimes k) \text{ for } m \text{ large enough .}$$

By Nakayama Lemma we conclude that $I_\nu = V(m)_\nu$ for any such m. It follows that $I = \bigcup_{m \geqq 0} V(m)$. □

Theorem 5.4. *Let $w \in W$ and s a reduced expression of w. Then:*

(*i*) (Demazure vanishing). $H^r(s, \lambda) = 0$ *for any* $\lambda \in X^+$, $r > 0$.

(*ii*) $H^r(s, H^0(U^b/U^0, V)) = 0$ *for any* $V \in \mathscr{C}^0$, $r > 0$.

(*iii*) *If $s_i w < w$ and if s' is a reduced expression of $s_i w$, then for any $M \in \mathscr{C}^b$ there is a spectral sequence:*

$$H_i^r(H^t(s', M)) \Rightarrow H^{r+t}(s, M) .$$

Proof. We use induction on $l(w)$. We can assume $l(w) > 0$ and assertions (i) and (ii) proved for strictly smaller lengths. Let $M \in \mathscr{C}^b$. We have seen in 2.17–2.19 that M has a resolution $0 \to M \to Q_\bullet$ where, for each $j \geqq 0$, $Q_j = H^0(U^b/U^0, Q_j')$ for some $Q_j' \in \mathscr{C}^0$. By assertion (ii) applied to s' we get $H^t(s', Q_j) = 0$ for all $j \geqq 0$, $t > 0$. Therefore, in order to apply Lemma 5.2 it is enough to prove that, for any $Q' \in \mathscr{C}^0$, $Q = H^0(s', H^0(U^b/U^0, Q'))$ is acyclic for H_i^0.

Since $Q' \in \mathscr{C}^0$, we can reduce to the case where U^b acts on Q' by the character χ_μ for some $\mu \in X$. Moreover, taking a finite resolution of Q' by free $\mathscr{A}$-modules on which U^b acts by χ_μ (recall gldim$(\mathscr{A}) < \infty$), we see that we can reduce to the case $Q' = \mathscr{A}_\mu$. Then, by Lemma 5.3 and the tensor identity 2.16, $H^0(s', Q)$ is the directed union of the submodules $H^0(m\rho) \otimes H^0(s', m\rho + \mu)$, $m \geqq 0$. Note that $m\rho + \mu \in X^+$ when $m \gg 0$. Since cohomology commutes with directed unions, we obtain via the tensor identity 2.16 and Theorem 5.1 (iii) that $H_i^r(H^0(s', Q)) = 0$ for $r > 0$.

Hence the conditions of Lemma 5.2 are satisfied and therefore we obtain a spectral sequence:

$$H_i^r(H^t(s', M)) \Rightarrow H^{r+t}(s, M) .$$

Then, using assertion (ii) for s' and the argument above, we obtain, for any $V \in \mathscr{C}^0$ and $r > 0$:

$$H^r(s, H^0(U^b/U^0, V)) \simeq H_i^r(H^0(s', H^0(U^b/U^0, V))) = 0 .$$

Hence assertion (ii) is satisfied for s. Finally, let $\lambda \in X^+$. By assertion (i) applied to s', we have $H^t(s', \lambda) = 0$ for $t > 0$ and therefore for all $r \geqq 0$ the spectral sequence gives $H^r(s, \lambda) \simeq H_i^r(H^0(s', \lambda))$. By Theorem 5.1 (iii) the latter vanishes for $r > 0$. □

Corollary 5.5. *Let $w \in W$, and s a reduced expression.*

(*i*) *If $V \in \mathscr{C}^b$ is a finite $\mathscr{A}$-module then so are all $H^r(s, V)$, $r \geqq 0$.*

(*ii*) $H^r(s, -) = 0$ *for* $r > l(w)$.

Proof. For any $i \in \{1, \ldots, n\}$, Propositions 4.1, 4.3 and 4.4 ensure that, H_i^r takes $\mathscr{A}$-finite modules in $\mathscr{C}^b$ to $\mathscr{A}$-finite modules, for $r = 0, 1$, and vanishes for $r > 1$. Therefore the Corollary follows from Theorem 5.4 (iii). □

5.6. Let s_0 be a reduced expression of the longest element w_0. For $V \in \mathscr{C}^b$ we denote by Φ_V the natural map $H^0(V) \to H^0(s_0, V)$.

Proposition. (*i*) *If $\lambda \in X^+$, then Φ_λ is an isomorphism.*

(*ii*) *If $\mu \in X$ and $V = H^0(U^b/U^0, \mu)$ then Φ_V is an isomorphism.*

(*iii*) *For any $\lambda \in X$, $i \geqq 0$ there is an isomorphism $H^i(\lambda) \simeq H^i(s_0, \lambda)$.*

Proof. (i) Say $s_0 = s_{i_N} \ldots s_{i_1}$. By Frobenius reciprocity, the U^b-homomorphism $H^0(\lambda) \to \lambda$ induces a U^b-homomorphism $H^0(\lambda) \to H_{i_1}^0(\lambda)$, which in turn induces a $U^\flat$-homomorphism $H^0(\lambda) \to H^0_{i_2}H^0_{i_1}(\lambda)$. Repeating this argument N times, we obtain a $U^\flat$-homomorphism $H^0(\lambda) \to H^0(s_0, \lambda)$. The same argument applies to $H^0_k(\lambda)$ and $H^0_k(s_0, \lambda)$, and we obtain a commutative diagram:

$$\begin{array}{ccc} H^0(\lambda) \otimes k & \to & H^0(s_0, \lambda) \otimes k \\ \downarrow & & \downarrow \\ H^0_k(\lambda) & \to & H^0_k(s_0, \lambda) \end{array}$$

The first vertical map is an isomorphism by Corollary 3.3 (iii), and so is the second one by Theorem 5.1 (i). Moreover, by [CPS 1, Theorem 3.1], the bottom map is also an isomorphism. This gives (i).

(ii) Since both functors commute with directed unions, then (ii) follows from (i) via Lemma 5.3 and the tensor identity 2.16.

(iii) Recall that in the standard resolution of $\lambda: 0 \to \lambda \to Q_\bullet$, each Q_j is equal to $H^0(U^\flat/U^0, Q'_j)$ for some $Q'_j \in \mathscr{C}^0$ which is a free $\mathscr{A}$-modules (see Lemma 2.18). Hence Q_j is a direct sum of modules of the form considered in (ii), hence (ii) gives an isomorphism between the complexes $H^0(Q_\bullet)$ and $H^0(s_0, Q_\bullet)$ and (iii) follows. □

Corollary 5.7. (Kempf's vanishing.) *Let* $\lambda \in -\rho + X^+$.

(i) $H^i(\lambda) = 0$ *for* $i > 0$.

(ii) Let M *be a finite* $\mathscr{A}$*-module on which* $U^\flat$ *acts by the character* χ_λ. *Then* $H^i(M) = 0$ *for* $i > 0$.

Proof. If $\lambda \in X^+$ then (i) is an immediate consequence of Theorem 5.4(i) and Proposition 5.6 (iii). Now, if $\lambda \notin X^+$ then there exists a simple root α_i such that $\langle \lambda, \alpha_i^\vee \rangle = -1$. Then, by 4.1 and 4.2 we have $H^t(s_i, \lambda) = 0$ for all $t \geqq 0$. Also, by corollary 2.15 we have a spectral sequence

$$H^r(U/U^\flat(i), H^t(s_i, \lambda)) \Rightarrow H^{r+t}(\lambda)$$

and therefore we conclude that $H^m(\lambda) = 0$ for all $m \geqq 0$.

(ii) If M is free then (ii) follows immediately from (i). Now in any case M has a finite resolution by free $\mathscr{A}$-modules, $P^\bullet \to M$. Making the P^j into $U^\flat$-modules via χ_λ we get a resolution of M in $\mathscr{C}^\flat$. By (i) $H^i(P^j) = 0$ for $i > 0$ and this implies the same vanishing for M. □

Theorem 5.8. *(i) For all* $V \in \mathscr{C}^\flat$, $\Phi_V: H^0(V) \to H^0(s_0, V)$ *is an isomorphism.*

(ii) (Serre's theorem) H^j *takes finite* $\mathscr{A}$*-modules in* $\mathscr{C}^\flat$ *to finite* $\mathscr{A}$*-modules in* $\mathscr{C}$.

(iii) (Grothendieck's theorem) $H^j = 0$ *for* $j > N$.

Proof. Note that (ii) and (iii) follow from (i) by Corollary 5.5 and Proposition 5.6 (iii). Also we have already proved (i) for $V = \lambda \in X^+$, see Proposition 5.6 (i). An easy induction on the rank shows that then Φ_V is an isomorphism for all $V \in \mathscr{C}^\flat$ such that V is a finite free $\mathscr{A}$-module and all weights of V are dominant (to carry out the induction step we employ Corollary 5.7 (i) and Theorem 5.4 (i)). This in turn implies the result for any finite $\mathscr{A}$-module V on which $U^\flat$ acts by the character χ_λ, $\lambda \in X^+$ (take a finite free resolution). Now for a general $V \in \mathscr{C}^\flat$ which is a finite $\mathscr{A}$-module we pick $m > 0$ such that for all weights of $V \otimes m\rho$ are dominant. Then the exact sequence in $\mathscr{C}^\flat$

$$0 \to V \to H^0(m\rho) \otimes (m\rho \otimes V) \to Q \to 0$$

gives via the tensor identity the commutative diagram

$$\begin{array}{ccccccc} 0 \to & H^0(V) & \to & H^0(m\rho) \otimes H^0(m\rho \otimes V) & \to & H^0(Q) \\ & \downarrow \Phi_V & & \downarrow 1 \otimes \Phi_{m\rho \otimes V} & & \downarrow \Phi_Q \\ 0 \to & H^0(s_0, V) & \to & H^0(m\rho) \otimes H^0(s_0, m\rho \otimes V) & \to & H^0(s_0, Q) \end{array}$$

By the above $\Phi_{m\rho \otimes V}$ is an isomorphism. Hence Φ_V is injective. As Q is also finite we must as well have that Φ_Q is injective. But then the diagram shows that Φ_V is surjective. □

5.9. We can now derive the:

Proposition. (Braid relations.) *Let $w \in W$, and s, s' two reduced expressions of w. Then, there exists a natural isomorphism of functors: $H^0(s, -) \simeq H^0(s', -)$.*

Proof. We only have to check that the functors $H^0(s_i, -)$, $i = 1, \ldots, n$ satisfy the braid relations, i.e. that if s and s' are the two possible reduced expressions of the longest element of a rank two subgroup, then $H^0(s, -) \simeq H^0(s', -)$. But this follows from Theorem 5.8 (i). □

Let s be a reduced expression of $w \in W$. It follows from Proposition 5.9 that $H^0(s, -)$ can be denoted by $H^0(w, -)$, or simply H_w^0. We shall do this in the sequel. The induction functors $H^0(w, -)$, $w \in W$ compose according to the

Proposition 5.10. *Let $w \in W$, s_i a simple reflection. Then:*

$$H_i^0 H_w^0 = \begin{cases} H_{s_i w}^0 & \text{if } s_i w > w \\ H_w^0 & \text{if } s_i w < w \end{cases}$$

Proof. The first case just follows from the definitions. In the second case we can take a reduced expression of w starting (from the left) with s_i, and therefore for any $M \in \mathscr{C}^b$, $H^0(w, M)$ belongs to $\mathscr{C}^b(i)$. Hence, by the tensor identity 2.16 together with Proposition 4.1 (ii) we obtain:

$$H_i^0(H^0(w, M)) \simeq H_i^0(0) \otimes H^0(w, M) = H^0(w, M) .$$

This proves that $H_i^0 H_w^0 \simeq H_w^0$ in that case. □

5.11. For $\alpha \in R^+$ we let $\Lambda_\alpha^0 : \mathbb{Z}[X] \to \mathbb{Z}[X]$ denote the Demazure operator, see [De 1]. If $\mathscr{A} \to \Gamma$ is a homomorphism into a field Γ then it follows from the results in Section 4 that for all $\lambda \in X$, $i \in \{1, \ldots, n\}$ we have:

$$\Lambda_{\alpha_i}^0 = \operatorname{ch} H_\Gamma^0(s_i, \lambda) - \operatorname{ch} H_\Gamma^1(s_i, \lambda) .$$

It is then standard to derive the following formula from Theorem 5.4

Proposition. *Let $V \in \mathscr{C}_\Gamma^b$, $w \in W$, $s = s_{i_1} \ldots s_{i_r}$ is a reduced expression of w. Then:*

$$\sum_j (-1)^j \operatorname{ch} H_\Gamma^j(w, V) = \Lambda_{i_1}^0 \ldots \Lambda_{i_r}^0(\operatorname{ch} V)$$

Corollary 5.12. (Demazure's character formula.) *Keep the notations of* 5.11, *and let $\lambda \in X^+$. Then:*

$$\operatorname{ch} H_\Gamma^0(w, \lambda) = \Lambda_{i_1}^0 \ldots \Lambda_{i_r}^0(e^\lambda)$$

Proof. This follows from Proposition 5.11 and Theorem 5.4 (i). □

Remark. Taking $w = w_0$ we get via Theorem 5.8 (i) a character formula for $H^0(\lambda)$, $\lambda \in X^+$. As is well known this is equivalent to the Weyl character formula (see [De 1, 5.6]).

5.13. Let $M \in \mathscr{C}$. We say that M has a good filtration if there exists a filtration in $\mathscr{C}$

$$0 = F_0 \subset F_1 \subset \ldots$$

with $\cup F_i = M$ and $F_i/F_{i-1} \simeq H^0(\lambda_i)$ for some $\lambda_i \in X^+$.
Note that if M has a good filtration then M is a free $\mathscr{A}$-module.

Lemma. (Compare [Do, 11.5.3]). *Let $M \in \mathscr{C}$ be a finite free $\mathscr{A}$-module. If $M \otimes k$ has a good filtration (in $\mathscr{C}_k$) then M has a good filtration.*

Proof. We use induction on the rank of M. Choose $\lambda \in X^+$ such that λ is maximal among the weights of M. The U^b-homomorphism $M \to \lambda$ arising from this situation gives by Frobenius reciprocity a U-homomorphism $M \to H^0(\lambda)$. Let M' be the kernel and Q the cokernel of this map. Tensoring by k we see from [W, Lemma 3.1] that $M \otimes k \to H^0(\lambda) \otimes k \simeq H_k^0(\lambda)$ is surjective. Hence $Q \otimes k = 0$, i.e. $Q = 0$. We thus have an exact sequence

$$0 \to M' \to M \to H^0(\lambda) \to 0$$

which remains exact upon tensoring by k. Moreover, M' satisfies the hypothesis of the lemma (use [loc. cit.] again) and we are done by induction. □

Corollary 5.14. *Let $\lambda, \mu \in X^+$. Then $H^0(\lambda) \otimes H^0(\mu)$ has a good filtration.*

Proof. This follows from Lemma 5.13 and [Do], [Ma]. □

5.15. As a preparation for the next result we need the:

Proposition. *Suppose $M, N \in \mathscr{C}$ are finite free $\mathscr{A}$-modules. Then* $\mathrm{Ext}_U^i(M, N)$ *is a finite $\mathscr{A}$-module for all i and vanishes for $i \gg 0$.*

Proof. By Kempf's vanishing theorem 5.7 and the tensor identity 2.16 we get

$$\mathrm{Ext}_U^i(M, N) \simeq \mathrm{Ext}_U^i(M, H^0(N)) \simeq \mathrm{Ext}_{U^b}^i(M, N)$$

This shows that the proposition follows if we prove that for any $\lambda \in X$, $H^i(U^b, \lambda) = \mathrm{Ext}_{U^b}^i(\mathscr{A}, \lambda)$ is $\mathscr{A}$-finite for all i and vanishes for $i \gg 0$.
In the standard U^b-resolution (2.17) of λ

$$0 \to \lambda \to I_0 \to I_1 \to \ldots$$

we have $\mathrm{ht}(\mu - \lambda) \geqq j$ for all weights μ of I_j. Here ht is the usual $\mathbb{Z}$-linear function on the root lattice whose value is 1 on simple roots. Hence $\mathrm{Hom}_{U^b}(\mathscr{A}, I_j) \subseteq (I_j)_0$ is $\mathscr{A}$-finite for all j and vanishes for $j > \mathrm{ht}(-\lambda)$. □

5.16. Let $M \in \mathscr{C}^b$. We say that M has an excellent filtration if there exists a filtration in $\mathscr{C}^b$

$$0 = F_0 \subset F_1 \subset F_2 \subset \ldots$$

with $\cup F_i = M$ and $F_i/F_{i-1} \simeq H^0(w_i, \lambda_i)$ for certain $w_i \in W$, $\lambda_i \in X^+$.
Note that if M has an excellent filtration then M is a free $\mathscr{A}$-module.

Lemma. *Let $M \in \mathscr{C}^b$ be a finite free $\mathscr{A}$-module. If $M \otimes k$ has an excellent filtration (in $\mathscr{C}_k^b$) then M has an excellent filtration.*

Proof. We use induction on the rank of M. Let $\lambda \in X^+$ be a weight of $M \otimes k$ of maximal norm. By [P 1, Proposition 3.1], there exists a surjective $U_k^\flat$-homomorphism $\varphi: M_k \twoheadrightarrow H_k^0(w, \lambda)$, for some $w \in W$. By [loc. cit., Corollaire 2.5], $H_k^0(w, \lambda)$ is injective in the category of B-modules whose weights have norm at most equal to the norm of λ, hence $\mathrm{Ext}^i_{U_k^\flat}(M_k, H_k^0(w, \lambda)) = 0$ for $i > 0$. Via Proposition 5.15 we get using base change arguments (like in Section 3) that

$$\mathrm{Hom}_{U^\flat}(M, H^0(w, \lambda)) \otimes k \simeq \mathrm{Hom}_{U_k^\flat}(M_k, H_k^0(w, \lambda))$$

Therefore, there exists $\psi \in \mathrm{Hom}_U(M, H^0(w, \lambda))$ such that $\psi_k = \varphi$. Now, φ is surjective, hence so is ψ, by Nakayama. Set $K = \mathrm{Ker}(\psi)$. Since $H^0(w, \lambda)$ is free, then K is a direct summand of M, and is therefore free. Also, $K \otimes k$ identifies with $\mathrm{Ker}(\varphi)$, and the latter has an excellent filtration by [loc. cit., Proposition 3.1]. Since K has smaller rank than M we conclude by induction hypothesis that K has an excellent filtration. □

Corollary 5.17. *Let $\lambda, \mu \in X^+$ and $w \in W$. Then $H^0(w, \lambda) \otimes \mu$ has an excellent filtration.*

Proof. This follows from Lemma 5.16 and [Ma], see also [P 2]. □

6. Borel-Weil-Bott theory

In this section we study the modules $H^i(\lambda) = H^i(U/U^\flat, \lambda)$, $\lambda \in X$, $i \geqq 0$ as well as the corresponding modules for U_Γ, Γ an $\mathscr{A}$-algebra.

Proposition 6.1. *Let $\lambda \in X$.*

(i) *$H^0(\lambda) \neq 0$ if and only if $\lambda \in X^+$.*

(ii) *If $\lambda \in X^+$ then $H^0(\lambda)^{U^+}$ is a free $\mathscr{A}$-submodule of rank 1. In fact, $H^0(\lambda)^{U^+} = H^0(\lambda)_\lambda$.*

(iii) *If $\lambda \in X^+$ then λ is the unique maximal weight of $H^0(\lambda)$.*

Proof. Suppose $\mu \in X$ is a weight of $H^0(\lambda)$ and let $f \in H^0(\lambda)_\mu$ be non-zero. Then there exists $r_j \geqq 0$ such that

$$f(E_{i_1}^{(r_1)} \dots E_{i_s}^{(r_s)}) \neq 0$$

for some $i_1, \dots, i_s \in \{1, \dots, n\}$, i.e.

$$Ev(E_{i_1}^{(r_1)} \dots E_{i_s}^{(r_s)} f) \neq 0$$

Since $E_{i_1}^{(r_1)} \dots E_{i_s}^{(r_s)} f$ has weight $\mu + \sum_{j=1}^{s} r_j \alpha_{i_j}$ and since Ev is zero on all but the λ-weight space we conclude that $\mu \leqq \lambda$. This proves (ii) and (iii).

To prove (i) assume first $H^0(\lambda) \neq 0$ and pick $f \in H^0(\lambda)_\lambda \setminus \{0\}$. Using 1.11. (1) we get for $r \geqq 0$:

$$\begin{aligned}(F_i^{(r)} f)(E_i^{(r)}) &= f(E_i^{(r)} F_i^{(r)}) = f\left(\sum_{t=0}^{r} F_i^{(r-t)} \begin{bmatrix} K_i; 2(t-r) \\ t \end{bmatrix} E_i^{(r-t)} \right) \\ &= f\left(\begin{bmatrix} K_i; 0 \\ r \end{bmatrix} \right) = \lambda\left(\begin{bmatrix} K_i; 0 \\ r \end{bmatrix} \right) f(1) \\ &= \begin{bmatrix} \lambda_i \\ r \end{bmatrix}_{d_i} f(1)\end{aligned}$$

Since $\begin{bmatrix} a \\ r \end{bmatrix}_{d_i} \neq 0$ for all r if $a < 0$ we conclude that $F_i^{(r)}f = 0$ for $r \gg 0$ implies $\lambda_i \geqq 0$.

On the other hand suppose $\lambda_i \geqq 0$ for $i = 1, \ldots, n$ and define $f: U \to \mathscr{A}$ by

$$f\left(\prod_{\beta\in R^+} F_\beta^{(M_\beta)} \prod_{i=1}^{n} K_i^{\delta_i} \begin{bmatrix} K_i; 0 \\ t_i \end{bmatrix} \prod_{\beta\in R^+} E_\beta^{(M'_\beta)}\right)$$

$$= \begin{cases} \prod_{i=1}^{N} v^{d_i\lambda_i\delta_i} \begin{bmatrix} \lambda_i \\ t_i \end{bmatrix}_{d_i} & \text{if } M_\beta = M'_\beta = 0 \text{ for all } \beta \\ 0 & \text{otherwise} \end{cases}$$

(we use that the elements of the given form are a basis of U, see [L 6]).

We claim that $f \in H^0(\lambda)$. It is clear that $f \in \mathrm{Hom}_{U^\flat}(U, \lambda)$ so that the only thing we have to verify is that $F_i^{(r)}f = 0$ for $r \gg 0$, $i = 1, \ldots, n$. Noting that f has weight λ we see that

$$F_i^{(r)}f(E_i^{(r)}) = \begin{bmatrix} \lambda_i \\ r \end{bmatrix}_{d_i} f(1) = 0 \quad \text{for } r > \lambda_i$$

The proposition follows. □

Remark. Of course (i) could be also deduced from the corresponding classical result via Theorem 5.1 and 5.6 (iii).

6.2. In the rest of this section Γ will be a field and $\mathscr{A} \to \Gamma$ will be a homomorphism into Γ. The image of v is denoted by ζ. We write $H_\Gamma^i = H^i(U_\Gamma/U_\Gamma^\flat, -)$.

Corollary. *Let $\lambda \in X^+$. Then $H_\Gamma^0(\lambda)$ contains a unique simple U_Γ-module, $L_\Gamma(\lambda)$. It has highest weight λ.*

Proof. Exactly as in the proof of Theorem 6.1 we see that $H_\Gamma^0(\lambda)$ has a 1-dimensional $U_\Gamma^\natural$-socle, namely $H_\Gamma^0(\lambda)_\lambda$. Hence $H_\Gamma^0(\lambda)$ has also a simple U_Γ-socle and this socle contains $H_\Gamma^0(\lambda)_\lambda$. □

Proposition. 6.3. *Assume that $S \in \mathscr{C}_\Gamma$ is a simple U_Γ-module. Then $S \simeq L_\Gamma(\lambda)$ for some $\lambda \in X^+$.*

Proof. It follows from Corollary 1.28 that S is finite dimensional. So, let λ be a maximal weight of S. Then there exists a non-zero $U_\Gamma^\flat$-homomorphism $S \to \lambda$, and by Frobenius reciprocity (2.12), this gives a non-zero U_Γ-homomorphism $S \to H_\Gamma^0(\lambda)$. By Proposition 6.1 (i) and Corollary 6.2 we obtain $\lambda \in X^+$ and $S \simeq L_\Gamma(\lambda)$. □

Theorem 6.4. *Suppose ζ is not a root of 1. Then we have for all $\lambda \in X$ with $\lambda + \rho \in X^+$ and all $w \in W$*

$$H_\Gamma^i(w \cdot \lambda) \simeq \begin{cases} H_\lambda^0(\lambda) & \text{if } i = l(w) \\ 0 & \text{otherwise} \end{cases}$$

Proof. The theorem is proved in the standard way via Theorem 5.8 (iii) from the lemma below. □

Remark. The theorem is also true when $\zeta = \pm 1$ and char $\Gamma = 0$ (the same proof applies). In that case it is equivalent to the classical Borel-Weil-Bott theorem (see e.g. [De 2]).

Lemma 6.5. *Let $\zeta \in \Gamma$ be as in 6.4 and suppose $\mu \in X$, $i \in \{1, \dots, n\}$ such that $\mu_i \geqq 0$. Then*

$$H_\Gamma^{j+1}(s_{\alpha_i} \cdot \mu) \simeq H_\Gamma^j(\mu), \quad j \geqq 0$$

Proof. For convenience we drop Γ from the notation.

As in 5.1 we set $H_i^j = H^j(U^b(i)/U^b, -)$. We can apply the rank 1 results from section 4 to get

$$H_i^j(s_{\alpha_i} \cdot \mu) \simeq \begin{cases} H_i^0(\mu) & \text{if } j = 1 \\ 0 & \text{if } j \neq 1 \end{cases}$$

and

$$H_i^j(\mu) = 0 \quad \text{for } j > 0$$

Since $H^0 = H^0(U/U^b(i), -) \circ H_i^0$ we conclude from Corollary 2.9 (ii) that

$$H^{j+1}(s_{\alpha_i} \cdot \mu) \simeq H^j(\mu) \qquad \square$$

Lemma 6.6. *Let ζ be the image of v in Γ. If $\zeta \neq 1$ is a primitive l^{th} root of 1, then $l = p^e$ for some positive integer e.*

Proof. Denote the homomorphism $\mathscr{A} \to \Gamma$ by φ. Then $\zeta = \varphi(v)$. Suppose $\zeta^l = 1$ and $\zeta^m \neq 1$ for any integer $0 < m < l$. Then the polynomial $v^l - 1$ belongs to $\wp = \ker \varphi$. We can assume that $l = p^e q$ where q is prime to p. Then $v^l - 1 = (v^{p^e} - 1)R(v)$ for $R(v) = \sum_{i=0}^{q-1} v^{ip^e}$. Since $R(1) = q$, then $R(v) \notin \mathfrak{m}$ hence $R(v) \notin \wp$. Since $\wp$ is a prime ideal, $v^{p^e} - 1 \in \wp$ and $\zeta^{p^e} = 1$ and hence $l = p^e$. $\square$

Theorem 6.7. *Let* $\operatorname{char} \Gamma = 0$ *and suppose $\zeta \neq 1$ is an l'th root of 1 where $l = p^e$ for some $e > 0$. Then for all $\lambda \in X$ with $\lambda + \rho \in X^+$ and all $w \in W$ we have*

(i) *If $\langle \lambda + \rho, \alpha^\vee \rangle \leqq l$ for all $\alpha \in R^+$ then*

$$H_\Gamma^j(w \cdot \lambda) \simeq \begin{cases} H_\Gamma^0(\lambda) & \text{if } j = l(w) \\ 0 & \text{otherwise} \end{cases}$$

(ii)

$$H_\Gamma^j(lw \cdot \lambda + (l-1)\rho) \simeq \begin{cases} H_\Gamma^0(l\lambda + l(\rho - 1)) & \text{if } j = l(w) \\ 0 & \text{otherwise} \end{cases}$$

Proof. The same proof as in 6.4 applies appealing this time to Corollary 4.6 (ii). $\square$

7. Serre duality and complete reducibility

Preserve notation from section 6 and let Γ still denote a field. Here we prove that the cohomology modules H_Γ^i, $i = 0, 1, \dots, N$ satisfy Serre duality. The Serre duality combined with the results from Section 6 easily gives the irreducibility of $H_\Gamma^0(\lambda)$, $\lambda \in X^+$ when ζ is not a root of unity. It is also needed for the proof of the linkage principle in the next section.

Lemma 7.1. *$H^N(-2\rho) \simeq \mathscr{A}$ and $H_\Gamma^N(-2\rho) \simeq \Gamma$*

Proof. By Theorem 5.8 (iii) and Theorem 3.5 we have $H^N(-2\rho) \otimes \Gamma \simeq H_\Gamma^N(-2\rho)$ for any $\mathscr{A}$-algebra Γ. Taking $\Gamma = k$ and recalling 3.11 and Serre duality over G_k/B_k we obtain: $H^N(-2\rho) \otimes k \simeq H^0(G_k/B_k, 0)^* = k$. Taking now $\Gamma = \mathscr{A}'$ and applying Theorem 6.4 we get: $H^N(-2\rho) \otimes \Gamma \simeq H_\Gamma^0(0) = \Gamma$. Then, by 1.21 we conclude that $H^N(-2\rho)$ is a rank one free $\mathscr{A}$-module. The lemma follows. □

7.2. Let $V_1, V_2 \in \mathscr{C}^b$. By Frobenius reciprocity the evaluations $H^0(V_1) \to V_1$ and $H^0(V_2) \to V_2$ give a homomorphism $H^0(V_1) \otimes H^0(V_2) \to H^0(V_1 \otimes V_2)$ which is functorial in both V_1 and V_2. If V_1 (say) is flat as an $\mathscr{A}$-module we get therefore corresponding natural homomorphisms

$$H^i(V_1) \otimes H^j(V_2) \to H^{i+j}(V_1 \otimes V_2), \quad i, j \geqq 0$$

In particular, if we denote by V^* and V^t the two U-module structures on $\mathrm{Hom}_{\mathscr{A}}(V, \mathscr{A})$ (see 1.18), we obtain for any flat $V \in \mathscr{C}^b$ a pairing

$$a) \qquad H^i(V) \times H^{N-i}(V^t \otimes -2\rho) \to \mathscr{A}$$

by composing the above homomorphism

$$H^i(V) \otimes H^{N-i}(V^t \otimes -2\rho) \to H^N(V \otimes V^t \otimes -2\rho)$$

with the map $H^N(V \otimes V^t \otimes -2\rho) \to H^N(-2\rho) \simeq \mathscr{A}$ induced by the natural homomorphism $V \otimes V^t \to \mathscr{A}$. Likewise for $V \in \mathscr{C}_\Gamma^b$ we have a pairing

$$b) \qquad H_\Gamma^i(V) \times H_\Gamma^{N-i}(V^t \otimes -2\rho) \to \Gamma$$

Theorem 7.3. *Let $V \in \mathscr{C}_\Gamma^b$ be finite dimensional. Then the pairing 7.2 b) is non-singular, i.e. it induces for each $i \geqq 0$ an isomorphism in $\mathscr{C}_\Gamma$*

$$H_\Gamma^i(V) \simeq H_\Gamma^{N-i}(V^t \otimes -2\rho)^*$$

Proof. We shall first observe that if $\lambda \in X^+$ then the homomorphism $H^0(\lambda) \to H^N(-\lambda - 2\rho)^*$ coming from 7.2 a) is an isomorphism. This follows from Serre duality in the classical case via the commutative diagram (compare [A 3, Proposition 2.10]))

$$\begin{array}{ccc} H^0(\lambda) \otimes k & \to & H^N(-\lambda - 2\rho)^* \otimes k \\ \downarrow \wr & & \downarrow \wr \\ H_k^0(\lambda) & \simeq & H_k^N(-\lambda - 2\rho)^* \end{array}$$

Hence the theorem holds for $i = 0$ and $V = \lambda \in X^+$. An easy induction gives then that it also holds when the weights of V are all dominant.

For a general V we then choose $m \geqq 0$ such that $V \otimes m\rho$ has only dominant weights. The short exact sequence

$$0 \to V \to H^0(m\rho) \otimes m\rho \otimes V \to Q \to 0$$

gives rise to the commutative diagram (using the tensor identity)

$$\begin{array}{ccccccc} 0 \to & H_\Gamma^0(V) & \to & H_\Gamma^0(m\rho) \otimes H_\Gamma^0(m\rho \otimes V) & \to & H_\Gamma^0(Q) \\ & \downarrow & & \downarrow & & \downarrow \\ 0 \to & H^N(V^t \otimes -2\rho)^* & \to & H_\Gamma^0(m\rho) \otimes H_\Gamma^N(-(m+2)\rho \otimes V^t)^* & \to & H_\Gamma^N(Q^t \otimes -2\rho)^* \end{array}$$

By the above the middle vertical map is an isomorphism. Hence the left vertical map is injective. The analogous diagram for Q then gives that the right vertical map is injective and the diagram then implies the surjectivity of the left vertical map.

Fix $i > 0$. We get via Corollary 5.7 (i) the commutative diagram

$$\begin{array}{ccc}
H^0_\Gamma(m\rho) \otimes H^{i-1}_\Gamma(m\rho \otimes V) & \to & H^0_\Gamma(m\rho) \otimes H^{N-i+1}_\Gamma(-(m+2)\rho \otimes V^t)^* \\
\downarrow & & \downarrow \\
H^{i-1}_\Gamma(Q) & \to & H^{N-i+1}_\Gamma(Q^t \otimes -2\rho)^* \\
\downarrow & & \downarrow \\
H^i_\Gamma(V) & \to & H^{N-1}_\Gamma(V^t \otimes -2\rho) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}$$

from which the theorem follows by induction on i. □

Corollary 7.4. *Let $\lambda \in X^+$. Then up to a scalar there is a unique non-zero U_Γ-homomorphism $H^N_\Gamma(w_0 \cdot \lambda) \to H^0_\Gamma(\lambda)$ and its image is $L_\Gamma(\lambda)$.*

Proof. By Theorem 7.3 there is an isomorphism $H^N_\Gamma(w_0 \cdot \lambda) \simeq H^0_\Gamma(-w_0\lambda)^*$. It follows that $H^N_\Gamma(w_0 \cdot \lambda)$ has a unique maximal submodule M, and $H^N_\Gamma(w_0 \cdot \lambda)/M \simeq L_\Gamma(\lambda)$. Since any composition factor of M has highest weight less than λ and since $H^0_\Gamma(\lambda)$ has socle $L_\Gamma(\lambda)$ it follows that M is killed by any U_Γ-homomorphism $H^N_\Gamma(w_0 \cdot \lambda) \to H^0_\Gamma(\lambda)$. □

Corollary 7.5. *Let $\lambda \in X^+$. If $H^N_\Gamma(w_0 \cdot \lambda) \simeq H^0_\Gamma(\lambda)$ then $H^0_\Gamma(\lambda)$ is irreducible.*

Corollary 7.6. *Assume that ζ is a primitive l^{th} root of unity. Let $\lambda = (l-1)\rho + l\mu$, for some $\mu \in X^+$. Then $H^0_\Gamma(\lambda)$ is simple.*

Proof. Note that $w_0 \cdot \lambda = (l-1)\rho + lw_0 \cdot \mu$. Hence, by Theorem 6.7 (ii) the hypothesis of Corollary 7.5 is satisfied. □

Remark. Let $H^0_\Gamma((l-1)\rho)$ be denoted by St. Then St* is also a simple U_Γ-module with highest weight $(l-1)\rho$. Hence St* $\simeq$ St by Corollary 6.2.

Corollary 7.7. (Lusztig [L 6, 7.2], Rosso [R 2, Partie C], Xi [X, Theorem 2.4]). *Suppose ζ is not a root of* 1. *Then*

(*i*) *For any $\lambda \in X^+$, $H^0_\Gamma(\lambda)$ is irreducible and isomorphic to $H^0_\Gamma(-w_0\lambda)^*$.*

(*ii*) *Any finite dimensional U_Γ-module in $\mathscr{C}_\Gamma$ is completely reducible.*

Proof. (i) follows from Theorems 6.4 and 7.3 and Corollary 7.5.

Let us prove (ii). By (i) it is enough to prove that for $\lambda, \mu \in X^+$ any extension

$$(1) \qquad 0 \to H^0_\Gamma(\lambda) \to M \to H^0_\Gamma(\mu) \to 0$$

in $\mathscr{C}_\Gamma$ is split. Assume firstly that $\mu \not> \lambda$. Then λ is a maximal weight of M, and it follows that there exists a non-zero U^b-homomorphism $M \to H^0_\Gamma(\lambda)_\lambda$ (when $\mu = \lambda$ we have to use the hypothesis that $M \in \mathscr{C}_\Gamma$, namely that M is the sum of its weight spaces). By Frobenius reciprocity 2.12 this gives a U_Γ-homomorphism $M \to H^0_\Gamma(\lambda)$ which splits the exact sequence.

Assume now that $\mu > \lambda$. Dualizing the exact sequence (1) and taking into account the isomorphisms in assertion (i), we obtain an exact sequence

(2) $$0 \to H^0_\Gamma(-w_0\mu) \to M^t \to H^0_\Gamma(-w_0\lambda) \to 0$$

which is split since $-w_0\lambda \not> -w_0\mu$. Taking duals again we obtain a splitting of (1). □

8. The linkage and translation principles

In this section Γ denotes a field and $\mathscr{A} \to \Gamma$ a homomorphism which takes v into a (primitive) l'th root of 1 where via Lemma 6.6 l has to be some p^e for e a positive integer. We prove that the linkage and translation principles for semi-simple algebraic groups over a field of prime characteristic have direct analogues for U_Γ. The arguments, however, follow the very same paths as in the modular case. We give only brief indications of proofs.

8.1. Let $\lambda, \mu \in X$. We say that μ is strongly linked to λ if there exist $\lambda_1, \ldots, \lambda_r \in X$, $\beta_1, \ldots, \beta_{r-1} \in R^+$, $m_1, \ldots, m_{r-1} \in \mathbb{N}$ such that

$$\mu = \lambda_1 \leqq s_{\beta_1} \cdot \lambda_1 + m_1 l \beta_1 = \lambda_2 \leqq \ldots \leqq s_{\beta_{r-1}} \cdot \lambda_{r-1} + m_{r-1} l \beta_{r-1} = \lambda_r = \lambda$$

Theorem. *Let μ, $\lambda + \rho \in X^+$ and $w \in W$, $i \geqq 0$. If $L_\Gamma(\mu)$ is a composition factor of $H^i_\Gamma(w \cdot \lambda)$ then μ is strongly linked to λ.*

Proof. Apply the rank 1 case in section 4 together with induction on λ and Corollary 7.4 (compare [A 2]). □

Corollary 8.2. *Let $V \in \mathscr{C}_\Gamma$ and suppose V is indecomposable. If $\lambda, \mu \in X^+$ such that $L_\Gamma(\lambda)$ and $L_\Gamma(\mu)$ both are composition factors of V, then $\mu \in W_l \cdot \lambda$.*

Proof. It is enough to verify that any extension

$$0 \to L_\Gamma(\lambda) \to V \to L_\Gamma(\mu) \to 0$$

with $V \in \mathscr{C}_\Gamma$ and $\lambda, \mu \in X^+$, $\mu \notin W_l \cdot \lambda$ splits. To see this we may assume $\mu \not> \lambda$ (dualize if necessary). Then λ is a maximal weight of V, i.e. we have a U^b-homomorphism $V \to \lambda$ which by Frobenius reciprocity gives a U_Γ-homomorphism $V \to H^0_\Gamma(\lambda)$. Since $\mu \notin W_l \cdot \lambda$ we see by Theorem 8.1 that $L_\Gamma(\mu)$ is not a composition factor of $H^0_\Gamma(\lambda)$. Hence the map $V \to H^0_\Gamma(\lambda)$ has image $L_\Gamma(\lambda)$ and the sequence is split. □

8.3. Let C denote the bottom alcove in X^+, i.e.

$$C = \{\lambda \in X \mid 0 < \langle \lambda + \rho, \alpha^\vee \rangle < l, \alpha \in R^+\}$$

and set

$$\bar{C} = \{\lambda \in X \mid 0 \leqq \langle \lambda + \rho, \alpha^\vee \rangle \leqq l, \alpha \in R^+\}$$

Note that $C \neq \emptyset$ if and only if $l \geqq h$ (the Coxeter number).

For $\lambda, \mu \in \bar{C}$ we define the translation functor $T^\mu_\lambda : \mathscr{C}_\Gamma \to \mathscr{C}_\Gamma$ as follows

$$T^\mu_\lambda V = pr_\mu(V \otimes H^0_\Gamma(\tau(\mu - \lambda)))$$

Here $pr_\mu : \mathscr{C}_\Gamma \to \mathscr{C}_\Gamma$ is the projection onto the biggest submodule (summand according to Corollary 8.2) whose composition factors have highest weights in $W_l \cdot \mu$ and $\tau \in W$ is chosen such that $\tau(\mu - \lambda) \in X^+$.

We have (compare [Ja 3], Chapter II.7)

Theorem. *Suppose* $\lambda \in C$, $\mu \in \bar{C}$.

(i) $T_\lambda^\mu H_\Gamma^i(w \cdot \lambda) \simeq H_\Gamma^i(w \cdot \mu)$, $i \geqq 0$, $w \in W_l$.

(ii) *If* $w \in W_l$ *such that* $w \cdot \lambda \in X^+$ *then*

$$T_\lambda^\mu L_\Gamma(w \cdot \lambda) = \begin{cases} L_\Gamma(w \cdot \mu) & \text{if } w \cdot \mu \text{ is in the upper closure of } w \cdot C \\ 0 & \text{otherwise} \end{cases}$$

(iii) *Suppose* $\{y \in W_l | y \cdot \mu = \mu\} = \{1, s\}$ *and let* $w \in W_l$ *such that* $w \cdot \lambda < ws \cdot \lambda$. *Then there is an exact sequence*

$$0 \to H_\Gamma^0(w \cdot \lambda) \to T_\mu^\lambda H_\Gamma^0(w \cdot \mu) \to H_\Gamma^0(ws \cdot \lambda) \to$$

$$\dots$$

$$\to H_\Gamma^i(w \cdot \lambda) \to T_\mu^\lambda H_\Gamma^i(w \cdot \mu) \to H_\Gamma^i(ws \cdot \lambda) \to$$

$$\dots$$

Proof. (i) and (iii) follow from the linkage principle 8.1 and the tensor identity 2.16 via a close analysis of the weights of $w \cdot \lambda \otimes H_\Gamma^0(\tau(\mu - \lambda))$, resp. $w \cdot \mu \otimes H_\Gamma^0(\tau(\lambda - \mu))$. (ii) follows from (i) by recalling that $L_\Gamma(w \cdot \lambda)$ is the image of $H_\Gamma^N(w_0 w \cdot \lambda) \to H_\Gamma^0(w \cdot \lambda)$, see Corollary 7.4. □

8.4. As in the modular case we get the following corollary, sometimes called the translation principle.

Corollary. (*i*) *Let* $\lambda, \lambda' \in C$, $w, y \in W_l$. *Then*

$$[H_\Gamma^i(w \cdot \lambda) : L_\Gamma(y \cdot \lambda)] = [H_\Gamma^i(w \cdot \lambda') : L_\Gamma(y \cdot \lambda')] \quad \text{for all } i$$

(*ii*) *Let* $\lambda \in C$, $\mu \in \bar{C}$, $y \in W_l$. *If* $y \cdot \mu$ *is in the upper closure of* $y \cdot C \subset X^+$ *then for all* i

$$[H_\Gamma^i(w \cdot \lambda) : L_\Gamma(y \cdot \lambda)] = [H_\Gamma^i(w \cdot \mu) : L_\Gamma(y \cdot \mu)] = [H_\Gamma^i(ws \cdot \lambda) : L_\Gamma(y \cdot \lambda)]$$

for all $s \in W_l$ *with* $s \cdot \mu = \mu$.

9. Finite dimensional U_Γ-modules

Let $\mathscr{A} \xrightarrow{f} \Gamma$ be a specialization of $\mathscr{A}$ into a field Γ, and let $\wp = \mathrm{Ker}(f)$. Let ζ be the image of v in Γ. If ζ is a root of unity, then it has order $l = p^e$ for some $e \geqq 0$, by Lemma 6.6. In particular, if $\mathrm{char}(\Gamma) \neq 0$ then $\zeta = 1$.

9.1. By abuse of notation, we still denote by $\chi_{\sigma,\lambda}$ the character $\chi_{\sigma,\lambda} \otimes 1$ of U_Γ^0. Then, we have the:

Lemma. *The characters* $\chi_{\sigma,\lambda}$ *of* U_Γ^0 *are pairwise distinct.*

Proof. Assume that $\chi_{\sigma,\lambda} = \chi_{\tau,\mu}$. If ζ is not a root of unity, then clearly $\mu = \lambda$ and $\tau = \sigma$. Assume now that $\zeta = 1$. Then $\tau = \sigma$, and for each i, and $t \geqq 0$, the integers

$\binom{\lambda_i}{t}$ and $\binom{\mu_i}{t}$ are equal, modulo $\wp \cap \mathbb{Z}$. It is well known that this implies $\lambda_i = \mu_i$.

Assume finally that $\zeta \neq 1$ is a root of unity, of odd order $l = p^e$ $(e > 0)$. Necessarily, $\mathrm{char}(\Gamma) = 0$. From $\sigma_i \zeta^{d_i \lambda_i} = \tau_i \zeta^{d_i \mu_i}$, we obtain $\zeta^{2d_i(\lambda_i - \mu_i)} = 1$. Since $2d_i$ has no common factor with l, we conclude that l divides $\lambda_i - \mu_i$. It follows that $\sigma_i = \tau_i$. Then the equality: $\chi_{\sigma,\lambda}\left(\begin{bmatrix} K_i \\ l \end{bmatrix}\right) = \chi_{\sigma,\mu}\left(\begin{bmatrix} K_i \\ l \end{bmatrix}\right)$ implies, by [L 3, 3.3. (b)] that $\lambda = \mu$. □

9.2. If M is a U_Γ-module, we set: $\mathcal{O}_\sigma(M) = \oplus_{\lambda \in X} M_{\sigma,\lambda}$. This is a U_Γ-submodule of M (see 1.4). Our aim in this section is to prove the:

Theorem. *Let M be a finite dimensional U_Γ-module. Then $M = \oplus_\sigma \mathcal{O}_\sigma(M)$.*

The proof splits into three different cases.

9.3. ζ is not a root of unity. Then all E_i and F_i are nilpotent on M. We reproduce the argument given in [R 1]. From the equality $K_i^{-1} F_i K_i = \zeta^{2d_i} F_i$, it follows that, if z is an eigenvalue of F_i on M then so is $\zeta^{2d_i} z$. Since M is finite dimensional and ζ not a root of unity, this implies that 0 is the only possible eigenvalue. Hence F_i is nilpotent on M.

Say $F_i^{(r)} M = 0$. For $t \geqq 1$, set

$$\gamma_t = \prod_{s=1}^{2t-1} (K_i \zeta^{r-s} - K_i^{-1} \zeta^{s-r}) .$$

Using the commutation formula 1.11 (1) we prove by induction on t that $\gamma_t F_i^{(r-t)} M = 0$. Therefore, $\prod_{s=1}^{2r-1} (K_i^2 - \zeta^{2(s-r)})$ annihilates M. Since the polynomial $\prod_{s=1}^{2r-1} (X - \zeta^{2(s-r)})$ has distinct roots, we obtain that K_i^2 is diagonalizable. Hence, so is K_i, with eigenvalues $\pm \zeta^t$, $|t| \leqq r-1$. But U_Γ^0 is generated by the K_i's, since ζ is not a root of unity, and therefore we conclude that M is the (direct) sum of weight spaces $M_{\sigma,\lambda}$. □

9.4. In view of 9.2, Corollary 7.7 can be restated in the form:

Theorem. (Lusztig [L 6], Rosso [R 1–2], Xi [X]). *Assume that ζ is not a root of unity. Then any finite dimensional U_Γ-module is completely reducible.*

9.5. $\zeta = 1$. In that case, each K_i is in the center of U_Γ and satisfies $K_i^2 = 1$. It follows that $M = \oplus_{\sigma \in \Sigma} M_\sigma$, where M_σ is the U_Γ-submodule of M on which each K_i acts by $\sigma_i = \pm 1$.

Following ([L 3]), we say that M is of type σ if $M = M_\sigma$. Recall that we denote by Γ_σ the U_Γ-module Γ on which U_Γ acts by the character ε_σ (see 1.6). If M is of type σ then $M \otimes \Gamma_\sigma$ is of type 1, and conversely. Therefore, we can assume that M is of type 1. In that case M is a module for the algebra $U_\Gamma/(K_i - 1)$. By [L 6, 8.15] this algebra identifies with $\bar{U}_\Gamma$, the hyperalgebra of the algebraic group G_Γ. Moreover, $U_\Gamma^0/(K_i - 1)$ identifies with the hyperalgebra $\bar{U}_\Gamma^0$ of a maximal torus, and χ_λ corresponds to the usual character χ_λ of $\bar{U}_\Gamma^0$. Therefore, the weight spaces of M, considered as a U_Γ or $\bar{U}_\Gamma$-module, are the same. But, as a $\bar{U}_\Gamma^0$-module, M is the (direct) sum of weight spaces M_λ, with $\lambda \in X$. If $\mathrm{char}(\Gamma) = 0$, this is well known, and for $\mathrm{char}(\Gamma) > 0$ this was proved in [S], [CPS 2]. □

9.6. ζ is a primitive l^{th} root of unity, where $l = p^e$, $e > 0$. The rest of this Section will be devoted to that case.

Necessarily $\mathrm{char}(\Gamma) = 0$, and we can follow the arguments in [L 3]. Each K_i^l is in the center of U_Γ and satisfies $K_i^{2l} = 1$. Then M is the direct sum of the U_Γ-submodules M_σ, $\sigma \in \Sigma$, where M_σ is the subspace on which each K_i^l acts by σ_i. Following [loc. cit.], we say that M has type σ if $M = M_\sigma$. Again, by tensoring with Γ_σ, we can reduce to the case where M has type 1.

9.7. So, let $\mathscr{F}_\Gamma$ be the category of all type 1 finite dimensional U_Γ-modules, and let $\mathscr{C}_\Gamma^f$ be the subcategory consisting of those $M \in \mathscr{F}_\Gamma$ such that $M = \oplus_{\nu \in X} M_\nu$. Observe that $\mathscr{C}_\Gamma^f$ is closed under formation of submodules, quotient modules and tensor products.

Our aim is to prove that in fact any $M \in \mathscr{F}_\Gamma$ belongs to $\mathscr{C}_\Gamma^f$. By [loc. cit., Proposition 6.4], it is so if M is simple, because in that case M is isomorphic to some $L(\lambda)$, $\lambda \in X^+$.

9.8. We follow the ideas of the proof of [CPS 2, 9.4]. Let St denote the U_Γ-module $H_\Gamma^0((l-1)\rho) \in \mathscr{C}_\Gamma^f$. Note that St is self-dual and simple, see Remark 7.6. But St has an even more striking property:

Theorem. St *is a projective object in* $\mathscr{F}_\Gamma$.

Proof. It is enough to prove that $\mathrm{Ext}^1_{U_\Gamma}(\mathrm{St}, L(\lambda)) = 0$ for all $\lambda \in X^+$. From the exact sequence: $0 \to L(\lambda) \to H^0(\lambda) \to Q(\lambda) \to 0$ we get an exact sequence:

$$\mathrm{Hom}_{U_\Gamma}(\mathrm{St}, Q(\lambda)) \to \mathrm{Ext}^1_{U_\Gamma}(\mathrm{St}, L(\lambda)) \to \mathrm{Ext}^1_{U_\Gamma}(\mathrm{St}, H^0(\lambda)) .$$

If $(l-1)\rho$ is not linked to λ, then St is not a composition factor of $H^0(\lambda)$, by 8.1, and therefore $\mathrm{Hom}_{U_\Gamma}(\mathrm{St}, Q(\lambda)) = 0$. On the other hand, if $(l-1)\rho$ is linked to λ then $\lambda = (l-1)\rho + l\mu$ for some $\mu \in X^+$, and then $Q(\lambda) = 0$ by Corollary 7.6. Hence it is enough to prove that $\mathrm{Ext}^1_{U_\Gamma}(\mathrm{St}, H^0(\lambda)) = 0$. Since St is self-dual, this will follow from the:

Lemma 9.9. *Let* $\lambda, \mu \in X^+$. *Then* $\mathrm{Ext}^1_{U_\Gamma}(H^0(\mu)^*, H^0(\lambda)) = 0$.

Proof. By proposition 3.3, what we have to prove is that $\mathrm{Ext}^1_{U_\Gamma}(D(\lambda), D(\mu)^t) = 0$ for all $\lambda, \mu \in X^+$. Note that the highest weight of $D(\mu)^t$ is $-w_0\mu = \mu^*$. Assume firstly that $\lambda \not< \mu^*$, and consider an exact sequence:

$$(1) \qquad 0 \to D(\mu)^t \to M \to D(\lambda) \to 0 .$$

Since the extreme terms belong to $\mathscr{C}_\Gamma^f$, we obtain that $M = \oplus_{\nu \in X} M_{(\nu)}$, where $M_{(\nu)}$ denotes the generalized eigenspace:

$$M_{(\nu)} = \{x \in M \mid (u - \chi_\nu(u))^2 x = 0 \text{ for all } u \in U_\Gamma^0\} .$$

From the commutation relations in [L 6, 6.5. (a3–5)] we deduce that for all $j \in \{1, \ldots, n\}$, $r \in \mathbb{N}$ there exists a bijection $\phi_{jr}: U^0 \to U^0$ such that $uE_j^{(r)} = E_j^{(r)}\phi_{jr}(u)$ and $\chi_\nu(\phi_{jr}(u)) = \chi_{\nu + r\alpha_j}(u)$ for all $u \in U^0$, $\nu \in X$. Let $x \in M_{(\nu)}$. Then, for all $u \in U_\Gamma^0$

$$0 = E_j^{(r)}(\phi_{jr}(u) - \chi_\nu(\phi_{jr}(u)))^2 x = (u - \chi_{\nu + r\alpha_j}(u))^2 E_j^{(r)} x .$$

This proves that $E_j^{(r)} M_{(\nu)} \subseteq M_{(\nu + r\alpha_j)}$. By maximality of λ among the generalized eigenspaces of M, we conclude that $E_j^{(r)} M_{(\lambda)} = 0$ for all j, and $r > 0$. Also, we can take $s \gg 0$ such that $F_j^{(s)} M_{(\lambda)} = 0$. From the commutation relation 1.11(1), we deduce that $\begin{bmatrix} K_j \\ s \end{bmatrix} M_{(\lambda)} = 0$. Take s to be a multiple lr of l. Since M is of type 1, then any K_j acts semi-simply on M, with eigenvalues ζ^m, $0 \leq m < l$. Therefore, by [L 3, 4.2–3], we conclude that $\prod_{t=0}^{r-1} \left(\begin{bmatrix} K_j \\ l \end{bmatrix} - t \right)$ annihilates $M_{(\lambda)}$. Since the polynomial $\prod_{t=0}^{r-1}(X - t)$ has distinct roots, we obtain that the action of $\begin{bmatrix} K_j \\ l \end{bmatrix}$ on $M_{(\lambda)}$ is diagonalizable. Since the K_j's are also diagonalizable, we conclude that $M_{(\lambda)}$ consists of vectors of weight λ. Therefore, a highest weight vector $v_\lambda \in D(\lambda)$ can be lifted to a highest weight vector $x_\lambda \in M_\lambda$. Let N be the U_Γ-submodule of M generated by x_λ. Then N maps onto $D(\lambda)$. On the other hand, by 1.20 (ii) we have $\dim N \leq \dim D(\lambda)$. It follows that N maps isomorphically onto $D(\lambda)$, and this gives a splitting of the exact sequence (1). Hence the lemma is proved in the case $\lambda \not< \mu^*$.

Now, assume that $\lambda < \mu^*$ and consider an exact sequence: $0 \to D(\mu)^t \to M \to D(\lambda) \to 0$. Then we obtain an exact sequence: $0 \to D(\lambda)^* \to M^* \to D(\mu) \to 0$, which is split by the previous argument, since $\mu \not< \lambda^*$. Then taking t-duals we obtain a splitting of the original sequence. Hence Lemma 9.9 is proved, as well as Theorem 9.8. □

9.10. Since St is self-dual, then it is also an injective object in $\mathscr{F}_\Gamma$. Then, the lemma below produces more projective and injective objects.

Lemma. *Let E be a finite dimensional U_Γ-module. Then* St $\otimes$ *E and E* $\otimes$ St *are both projective and injective objects in $\mathscr{F}_\Gamma$.*

Proof. By 1.18, for any U_Γ-modules M, N there are isomorphisms:

$$\mathrm{Hom}_{U_\Gamma}(M \otimes E, N) \simeq \mathrm{Hom}_{U_\Gamma}(M, N \otimes E^*)$$

$$\text{and } \mathrm{Hom}_{U_\Gamma}(E \otimes M, N) \simeq \mathrm{Hom}_{U_\Gamma}(M, E^t \otimes N)$$

The lemma follows. □

Lemma 9.11. *Let $\lambda \in X^+$. Then there exists an imbedding of $L(\lambda)$ into* St $\otimes$ *E for some $E \in \mathscr{C}_\Gamma^f$.*

Proof. By Lusztig's tensor product theorem [L 3, Theorem 7.4] and Lemma 9.10, we can reduce to the case where λ is restricted. In that case, $\mu = (l - 1)\rho - \lambda$ belongs to X^+, and then the $U_\Gamma^\flat$-homomorphism $L(\lambda) \otimes L(\mu) \to (l - 1)\rho$ induces by Frobenius reciprocity a non-zero U_Γ-homomorphism $L(\lambda) \otimes L(\mu) \to \mathrm{St}$, which in turn corresponds to an injective U_Γ-homomorphism $L(\lambda) \hookrightarrow \mathrm{St} \otimes L(\mu)^*$. □

Theorem 9.12. (*i*) *Any $M \in \mathscr{F}_\Gamma$ belongs to $\mathscr{C}_\Gamma^f$. In other words, $\mathscr{F}_\Gamma = \mathscr{C}_\Gamma^f$.*

(*ii*) *The category $\mathscr{F}_\Gamma$ has enough injectives. Moreover any indecomposable injective is a direct summand of some* St $\otimes$ *E, $E \in \mathscr{C}_\Gamma^f$.*

(*iii*) *Injective modules in $\mathscr{F}_\Gamma$ are projective, and conversely.*

Proof. Let $M \in \mathscr{F}_\Gamma$. Since the socle S of M is a direct sum of $L(\lambda)$'s, $\lambda \in X^+$, then by 9.11 we can imbed S into some $\mathrm{St} \otimes E$, $E \in \mathscr{C}_\Gamma^f$. By 9.10, the latter is an injective object in $\mathscr{F}_\Gamma$, hence we obtain an imbedding of M into $\mathrm{St} \otimes E$. Since the latter belongs to $\mathscr{C}_\Gamma^f$, we conclude that M also belongs to $\mathscr{C}_\Gamma^f$.

This proves (i) as well as (ii). As for (iii), it follows from (ii) and 9.10 that any injective is also projective. Now, if $M \in \mathscr{F}_\Gamma$ is projective, then M^* is injective, hence also projective, and therefore M is also injective. □

9.13. Assertion 9.12 (i) concludes the proof of Theorem 9.2. □

10. Sum formulas

In this section we define some filtrations of the cohomology modules for quantum algebras following the method used by the first author in the modular case and we prove sum formulas analogous to the Jantzen sum formula by using exactly the same arguments as in Jantzen's book [Ja 3]. Moreover, since the usual case is also a specialization of the quantum case, we get something new for modular representations. However, we expect that in the lowest p^2-alcove the new filtrations coincide with Jantzen's filtration. For this, we formulate some conjectures.

Lemma 10.1. *Any prime ideal in $\mathscr{A}$ other than $\mathfrak{m}$ is principal.*

Proof. Since $\mathscr{A}$ has dimension 2 then any prime ideal $\wp \neq \mathfrak{m}$ has height at most one and is therefore principal, since $\mathscr{A}$ is a unique factorization domain. □

Lemma 10.2. *Let $\wp$ be a prime ideal in $\mathscr{A}$ other than $\mathfrak{m}$ and 0. Then*

(i) $\mathscr{A}_\wp$ is a discrete valuation ring.

(ii) $\mathscr{A}/\wp$ is a discrete valuation ring if and only if the generator of $\wp$ can be written as a linear combination of p and $(v-1)$ with at least one of the coefficients invertible in $\mathscr{A}$, i.e. $\wp = (a_1 p + a_2(v-1))$ with $a_1, a_2 \in \mathscr{A}$ such that either a_1 or a_2 (or both) is a unit of $\mathscr{A}$.

Proof. (i) This is clear from Lemma 10.1.

(ii) The "if" part is easy because in that case $\mathfrak{m}/\wp$ is generated either by $p + \wp$ or by $(v-1) + \wp$. Now suppose $\mathfrak{m}/\wp$ is generated by $g + \wp$ with $g \in \mathscr{A}$. Then g is in $\mathfrak{m}$ and we can write $g = a_1 p + a_2(v-1)$. If a_1 and a_2 were both in $\mathfrak{m}$ then $g \in \mathfrak{m}^2$ and $\mathfrak{m}/\wp = (\mathfrak{m}/\wp)^2$, in contradiction with Nakayama's Lemma. So at least one of them is not in $\mathfrak{m}$, hence a unit in $\mathscr{A}$. □

Remark. In fact the prime ideal in (ii) is generated either by $\varepsilon p + (v-1)^t$ or by $\varepsilon p^t + (v-1)$ where ε is a unit in $\mathscr{A}$ and t is a positive integer.

10.3. For a positive integer l,

$$\frac{v^l - 1}{v - 1} = v^{l-1} + v^{l-2} + \ldots + v + 1 \in \mathfrak{m}$$

if and only if $p \mid l$. Moreover we have that for any positive integer e,

$$\frac{v^{el} - 1}{v^e - 1} = (v^e)^{l-1} + (v^e)^{l-2} + \ldots + v^e + 1 \in \mathfrak{m}$$

if and only if $p \mid l$.

Lemma. *(i) Let* $\phi_p(v) = \dfrac{v^p - 1}{v - 1}$. *Then* $\phi_p(v^{p^e}) \in \mathscr{m}$ *and it generates a prime ideal of* $\mathscr{A}$ *satisfying the condition stated in Lemma* 10.2 *(ii).*

(ii) Let $\wp$ *be a prime ideal of* $\mathscr{A}$ *different from* $\mathscr{m}$ *and those in (i). Then the Borel-Weil-Bott theorem holds over* $\mathscr{K} = \mathscr{A}_\wp/\wp$ *(see Theorem* 6.4*).*

Proof. (i) $\phi_p(v^{p^e}) \equiv (v^{p^e})^{p-1} \equiv (v-1)^{p^e(p-1)} \bmod(p)$. We can write $\phi_p(v^{p^e}) = (v-1)^{p^e(p-1)} + a(v)p$ for some $a(v) \in \mathscr{A}$. Specializing v to 1 we get $a(1)p = p$, i.e. $a(1) = 1$. Hence $a(v) \notin \mathscr{m}$ is a unit in $\mathscr{A}$.

(ii) The homomorphism $\mathscr{A} \to \mathscr{K}$ does not take v to a root of 1. □

Remark. In fact $\dfrac{v^{p^e} - 1}{v - 1} = \prod_{i=0}^{e-1} \phi_p(v^{p^i})$.

10.4. Let Γ denote either $\mathscr{A}_\wp$ for $\wp$ a prime ideal of $\mathscr{A}$ given in Lemma 10.3 (i) or $\mathscr{A}/\wp$ for a prime ideal of $\mathscr{A}$ satisfying the condition in Lemma 10.2 (ii) but not those appearing in Lemma 10.3 (i). Then Γ is a discrete valuation ring with unique maximal ideal here denoted by $\mathbf{q}$. Denote by $v_\mathbf{q}$ the valuation on the fraction field Γ'.

If $a \in \Gamma$ then we set $v(\Gamma/(a)) = v_\mathbf{q}(a)$. Extending v by linearity, to each finitely generated torsion Γ-module V we get associated an element $v(V) \in \mathbb{Z}$.

Let $\varphi: M \to M'$ be a homomorphism between two finitely generated Γ-modules. Suppose $\varphi \otimes 1: M \otimes_\Gamma \Gamma' \to M' \otimes_\Gamma \Gamma'$ is an isomorphism. Then the cokernel of the induced map $\varphi_f: M_f \to M'_f$ on the free parts of M and M' is a torsion module. We set

$$v(\varphi) = v(\operatorname{coker}(\varphi_f))$$

If in the above setting $V \in \mathscr{C}_\Gamma^0$ (resp. $M, M' \in \mathscr{C}_\Gamma^0$), then we define

$$v^c(V) = \sum_{\mu \in X} v(V_\mu) e^\mu \in \mathbb{Z}[X]$$

respectively

$$v^c(\varphi) = v^c(\operatorname{coker} \varphi)$$

10.5. Fix $\lambda \in X^+$ and $w \in W$. Let $w_0 = s_{j_1} \dots s_{j_N}$ be a reduced expression for w_0. By the vanishing theorems 5.7 and 5.8 we have $H_\Gamma^N(w_0 \cdot \lambda) = H^N(w_0 \cdot \lambda) \otimes \Gamma$ and $H_\Gamma^0(\lambda) = H^0(\lambda) \otimes \Gamma$. Using this also in the rank 1 case we get by Corollary 4.5 a natural homomorphism (compare Lemma 6.5)

$$H_\Gamma^{j+1}(s_{j_{r+1}} \dots s_{j_1} \cdot \lambda) \to H_\Gamma^j(s_{j_r} \dots s_{j_1} \cdot \lambda)$$

for $j, r \geqq 0$. Denote by T_{w_0} the composite of

$$H^N(w_0 \cdot \lambda) \to H^{N-1}(s_{j_{N-1}} \dots s_{j_1} \cdot \lambda) \to \dots \to H^1(s_{j_1} \cdot \lambda) \to H^0(\lambda)$$

Let M_t denote the torsion submodule of a Γ-module M. We see from Lemma 10.3 that $H_\Gamma^i(w \cdot \lambda)_t = H_\Gamma^i(w \cdot \lambda)$ for $i \neq l(w)$ and $T_{w_0} \otimes 1: H_\Gamma^N(w_0 \cdot \lambda) \otimes_\Gamma \Gamma' \to H_\Gamma^0(\lambda) \otimes_\Gamma \Gamma'$ is an isomorphism. Hence $v^c(T_{w_0})$ is defined, and we find

Proposition.

$$v^c(T_{w_0}) = -\sum_{\alpha \in R^+} \sum_{m=1}^{\langle \lambda + \rho, \alpha^\vee \rangle - 1} v_\mathbf{q}([m]) \chi(\lambda - m\alpha)$$

$\left(\textit{Recall that } [m] = \dfrac{v^m - v^{-m}}{v - v^{-1}} \right)$

Proof. Let $T_m : H^1_\Gamma(\lambda_{-m-2}) \to H^0_\Gamma(\lambda_m)$ be the homomorphism considered in Corollary 4.5. Since

$$v_{\mathbf{q}}\left(\begin{bmatrix} m \\ i \end{bmatrix}\right) = \sum_{j=1}^{i} v_{\mathbf{q}}([m-j+1]) - v_{\mathbf{q}}([j])$$

we easily get $v^c(T_m) = -\sum_{j=1}^m v_{\mathbf{q}}([j])\chi(\lambda_{m-2j})$. This proves the proposition in case $n = 1$. The general case then follows just as in the modular case, see [A 3], by noting that $H^j_\Gamma(\lambda)_t = 0 = H^j_\Gamma(w_0 \cdot \lambda)_t$ for all j when $\lambda \in X^+$ (Kempf vanishing theorem 5.7 and Serre duality 7.3). □

10.6. Remark. For each $\mu \in X$, let D_μ denote the determinant of the restriction of T_{w_0} to the μ-weight space of $H^N_\Gamma(w_0 \cdot \lambda)$. Also, for $v \in X$ let $(v:\mu)$ denote the coefficient of e^μ in $\chi(v)$. Set

$$\Delta_\mu = \prod_{\alpha \in R^+} \prod_{m=1}^{\langle \lambda + \rho, \alpha^\vee \rangle - 1} [m]^{(s_\alpha \cdot \lambda + m\alpha : \mu)}$$

By proposition 10.5, D_μ and Δ_μ have the same $\wp$-valuation, for any height one prime ideal $\wp$ contained in $\mathfrak{m} = (v-1, p)$. Since p is an arbitary odd prime (distinct from 3 if (a_{ij}) has a component of type G_2), it follows that D_μ and Δ_μ only differ by a unit in $S^{-1}\mathbb{Z}[v, v^{-1}]$, where S denotes the complement of $\bigcup_{p \neq 2,3}(v-1, p)$. We are indebted to G. Lusztig for this observation.

10.7. By the vanishing theorem we know that both $H^N_\Gamma(w_0 \cdot \lambda)$ and $H^0_\Gamma(\lambda)$ are free Γ-modules. Define a filtration of $H^N_\Gamma(w_0 \cdot \lambda)$ as follows:

$$H^N_\Gamma(w_0 \cdot \lambda)^j = \{x \in H^N_\Gamma(w_0 \cdot \lambda) \mid T_{w_0} x \in \mathbf{q}^j H^0_\Gamma(\lambda)\}$$

This is clearly a U_Γ-filtration of $H^N_\Gamma(w_0 \cdot \lambda)$ and if we let $\hat{\Gamma}$ denote the residue field of Γ and $H^N_{\hat{\Gamma}}(w_0 \cdot \lambda)^j$ the image in $H^N_{\hat{\Gamma}}(w_0 \cdot \lambda) \simeq H^N_\Gamma(w_0 \cdot \lambda) \otimes_\Gamma \hat{\Gamma}$, then this gives a $U_{\hat{\Gamma}}$-filtration of $H^N_{\hat{\Gamma}}(w_0 \cdot \lambda)$.

Note that $H^N_{\hat{\Gamma}}(w_0 \cdot \lambda)^1$ is the kernel of the homomorphism

$$T_{w_0} \otimes 1 : H^N_{\hat{\Gamma}}(w_0 \cdot \lambda) \to H^0_{\hat{\Gamma}}(\lambda)$$

Hence by Corollary 7.4 we see that $H^N_{\hat{\Gamma}}(w_0 \cdot \lambda)^1$ is the maximal proper submodule of $H^N_{\hat{\Gamma}}(w_0 \cdot \lambda)$.

From Proposition 10.5 by using standard arguments (compare [A 3]) we have the following sum formula

Theorem.

$$\sum_{j \geq 1} \operatorname{ch} H^N_{\hat{\Gamma}}(w_0 \cdot \lambda)^j = \sum_{\alpha \in R^+} \sum_{\substack{m \\ 0 < m < \langle \lambda + \rho, \alpha^\vee \rangle}} v_{\mathbf{q}}([m])\chi(s_\alpha \cdot \lambda + m\alpha)$$

Remark. If $w \in W$, then there are similar filtrations of $H^{l(w)}_\Gamma(w \cdot \lambda)_f \otimes_\Gamma \hat{\Gamma}$, compare [A 3]. The sum formulas for $w \neq 1, w_0$ will in general involve non-zero contributions from the torsion in $H^i(w \cdot \lambda), i > 0$.

10.8. Notations are as above. By 10.3 and an easy calculation we have

Lemma.

$$v_{\mathfrak{q}}([m]) = v_{\mathfrak{q}}\left(\frac{v^m - 1}{v - 1}\right) = v_{\mathfrak{q}}\left(\frac{v^{p^e} - 1}{v - 1}\right) = \sum_{i=0}^{e-1} v_{\mathfrak{q}}(\phi_p(v^{p^i}))$$

for any positive integer m, where $e = v_p(m)$.

10.9. Let $\varphi: \mathscr{A} \to \mathscr{K}$ be a homomorphism into a feld $\mathscr{K}$ which takes v into a primitive p^e'th root of 1 where e is a positive integer (see Lemma 6.6). Then $\ker\varphi = (\phi_p(v^{p^{e-1}}))$. Denote $\mathscr{A}_{\ker\phi}$ by Γ and the residue field of Γ by $\hat{\Gamma}$. Then φ factors into $\mathscr{A} \to \Gamma \to \hat{\Gamma} \to \mathscr{K}$. By Lemma 10.8, we have:

$$\begin{aligned} v_{(\phi_p(v^{p^{e-1}}))}([l]) &= \sum_{i=0}^{v_p(l)-1} v_{(\phi_p(v^{p^{e-1}}))}(\phi_p(v^{p^i})) \\ &= \begin{cases} 1 & \text{if } p^e \mid l \text{(i.e. } v_p(l) \geqq e) \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

Therefore, we get the following sum formula for the filtration of $H^N_{\mathscr{K}}(w_0 \cdot \lambda)$ (where we define $H^N_{\mathscr{K}}(w_0 \cdot \lambda)^j = H^N_{\Gamma}(w_0 \cdot \lambda)^j \otimes \mathscr{K}$)

$$\sum_{j \geqq 1} \operatorname{ch} H^N_{\mathscr{K}}(w_0 \cdot \lambda)^j = \sum_{\alpha \in R^+} \sum_{0 < mp^e < \langle \lambda + \rho, \alpha^\vee \rangle} \chi(s_\alpha \cdot \lambda + mp^e \alpha)$$

10.10. Suppose Γ is $\mathscr{A}/\wp$ where $\wp$ is a prime ideal of $\mathscr{A}$ generated by an element in $\mathscr{m}$ which can be written as the combination of p and $(v - 1)$ with at least one of the coefficients a unit in $\mathscr{A}$ but is not generated by any $\phi_p(v^{p^e})$ for e a positive integer. Then the (unique) maximal ideal of Γ is $\mathfrak{q} = \mathscr{m}/\wp$ and the residue field $\hat{\Gamma}$ is k. Moreover $H^i_{\hat{\Gamma}}(w \cdot \lambda) = H^i_k(w \cdot \lambda)$ is just the usual cohomology module for the algebraic group over k. Since there are (infinitely) many such prime ideals we (in 10.7) get many filtrations and sum formulas for $H^N_k(w_0 \cdot \lambda)$. However, $H^N_k(w_0 \cdot \lambda)^1$ defined by any $\wp$ will always be the same (in fact, it is the maximal proper submodule).

10.11. If we take $\wp = (v - 1)$, then $\mathscr{A}/\wp = \mathbb{Z}_p$ and we get exactly the usual Jantzen filtration and sum formula for $H^N_k(w_0 \cdot \lambda)$. Moreover we have

Proposition. *Assume* $\wp$ *is generated by* $(v - 1) + \varepsilon p^t$ *where* ε *is a unit in* $\mathscr{A}$ *and* t *is a positive integer. Then* ($\mathfrak{q} = \mathscr{m}/\wp$)

(*i*) $v_{\mathfrak{q}}\left(\frac{v^l - 1}{v - 1}\right) = v_p(l)$ *and the sum formula of* $\{H^N_k(w_0 \cdot \lambda)^j\}$ *associated to* $\wp$ *is the Jantzen formula.*

(*ii*) *If* t *is large enough (with respect to* $\langle \lambda + \rho, \alpha_0^\vee \rangle$*) then the filtration of* $H^N_k(w_0 \cdot \lambda)$ *defined by* $\wp$ *is the Jantzen filtration*

Proof. Denote $v_p(v)$ by e and write $l = p^e l'$. Then

$$\frac{v^l - 1}{v - 1} = \frac{v^{p^e} - 1}{v - 1} \cdot \frac{(v^{p^e})^{l'} - 1}{v^{p^e} - 1}$$

Hence $v_{\mathfrak{q}}\left(\frac{v^l - 1}{v - 1}\right) = v_{\mathfrak{q}}\left(\frac{v^{p^e} - 1}{v - 1}\right)$ since $\frac{(v^{p^e})^{l'} - 1}{v^{p^e} - 1} \notin \mathscr{m}$. Note that

$(v-1) \equiv \varepsilon p^t \bmod(\wp)$. So

$$\frac{v^{p^e}-1}{v-1} = (v-1)^{p^e-1} + \sum_{i=1}^{p^e-1} \frac{(p^e-1)!}{i!(p^e-i)!} p^e (v-1)^{i-1}$$

$$\equiv (-\varepsilon)^{p^e-1} p^{t(p^e-1)} + \sum_{i=1}^{p^e-1} \frac{(p^e-1)!}{i!(p^e-i)!} (-\varepsilon)^{i-1} p^{t(i-1)+e} \bmod(\wp)$$

Since $p > 2$ then $p^e - 1 > e$. We get $v_{\mathfrak{q}}\left(\frac{v^{p^e}-1}{v-1}\right) = e$ which proves (i).

(ii) Denote the natural map

$$H^N_{\mathscr{A}}(w_0 \cdot \lambda) \to H^N_\Gamma(w_0 \cdot \lambda)$$

by π_1 where $\Gamma = \mathscr{A}/\wp$ and

$$H^N_{\mathscr{A}}(w_0 \cdot \lambda) \to H^N_{\mathbf{Z}_p}(w_0 \cdot \lambda)$$

by π_2. By definition

$$H^N_\Gamma(w_0 \cdot \lambda)^j = \{x \in H^N_\Gamma(w_0 \cdot \lambda) \mid T_{w_0}(x) \in \mathbf{q}^j H^0_\Gamma(\lambda)\}$$

$$H^N_{\mathbf{Z}_p}(w_0 \cdot \lambda)^j = \{x \in H^N_{\mathbf{Z}_p}(w_0 \cdot \lambda) \mid T_{w_0}(x) \in p^j H^0_{\mathbf{Z}_p}(\lambda)\}$$

and then

$$\pi_1^{-1}(H^N_\Gamma(w_0 \cdot \lambda)^j) = \{x \in H^N_{\mathscr{A}}(w_0 \cdot \lambda) \mid T_{w_0}(x) \in p^j H^0(\lambda) + ((v-1) + \varepsilon p^t) H^0(\lambda)\}$$

$$\pi_2^{-1}(H^N_{\mathbf{Z}_p}(w_0 \cdot \lambda)^j) = \{x \in H^N_{\mathscr{A}}(w_0 \cdot \lambda) \mid T_{w_0}(x) \in p^j H^0(\lambda) + (v-1) H^0(\lambda)\}$$

So for $j \leqq t$,

$$\pi_1^{-1}(H^N_\Gamma(w_0 \cdot \lambda)^j) = \pi_2^{-1}(H^N_{\mathbf{Z}_p}(w_0 \cdot \lambda))$$

Looking at the image of them in $H^N_k(w_0 \cdot \lambda)$, we get that the two filtrations of $H^N_k(w_0 \cdot \lambda)$ have the same top t submodules. If t is large enough, e.g. t is larger than the length of the usual filtration, (i) forces that the two filtrations are the same. □

Corollary. (of the proof). *In the filtration of $H^N_k(w_0 \cdot \lambda)$ defined by $\wp$ the first t terms coincide with the corresponding terms in the Jantzen filtration.*

Remark. We think that it is reasonable to conjecture that all the filtrations above are identical with the Jantzen filtration.

10.12. Let us take the opposite case by assuming $\wp = (p)$. Then $\Gamma = \mathscr{A}/\wp = \mathbf{F}_p/[v]_{(v-1)}$, $\mathbf{q} = (v-1)$ and we get a filtration of $H^N_k(w_0 \cdot \lambda)$ denoted here by $H^N_k(w_0 \cdot \lambda)^j_{(p)}$ in order to distinguish it from the usual one. Also there is a sum formula which looks a little different

$$\sum_{j \geqq 0} \operatorname{ch} H^N_k(w_0 \cdot \lambda)^j_{(p)} = \sum_{\alpha \in R^+} \sum_{0 < mp < \langle \lambda + \rho, \alpha^\vee \rangle} (p^{v_p(mp)} - 1) \chi(s_\alpha \cdot \lambda + mp\alpha)$$

because here $v_{(v-1)}\left(\frac{v^l - 1}{v-1}\right) = p^{v_p(l)} - 1$ over $\mathbf{F}_p[v]_{(v-1)}$.

But if λ is in the lowest p^2-alcove, i.e. $\langle \lambda + \rho, \alpha_0^\vee \rangle < p^2$, then the formula is

$$\sum_{j \geqq 0} \operatorname{ch} H^N_k(w_0 \cdot \lambda)^j_{(p)} = \sum_{\alpha \in R^+} \sum_{0 < mp < \langle \lambda + \rho, \alpha^\vee \rangle} (p-1) \chi(s_\alpha \cdot \lambda + mp\alpha)$$

So we would like to conjecture that this filtration is in fact the Jantzen filtration "magnified" by $p-1$. That means, this filtration consists of the same submodules as in the Jantzen filtration, each repeated $p-1$ times.

10.13. Let $\wp$ be a prime ideal of $\mathscr{A}$ generated by $\varepsilon(v-1)^t + p$ for ε a unit in $\mathscr{A}$ and t an integer. Let $\Gamma = \mathscr{A}/\wp$ and $\mathbf{q} = \mathscr{m}/\wp$.

Lemma. *(i) If* $t < p-1$, $v_{\mathbf{q}}\left(\dfrac{v^p-1}{v-1}\right) = v_{\mathbf{q}}(\phi_p(v)) = t$.

(ii) If $t > p-1$, $v_{\mathbf{q}}(\phi_p(v)) = p-1$.

(iii) If $t = p-1$, $v_{\mathbf{q}}(\phi_p(v)) \geqq p-1$. *In this case* $v_{\mathbf{q}}(\phi_p(v))$ *depends on* ε.

Proof. Note that

$$\phi_p(v) = (v-1)^{p-1} + \sum_{i=0}^{p-2} \frac{(p-1)!}{(i+1)!(p-i-1)!} p(v-1)^i$$

$$\equiv (v-1)^{p-1} - \sum_{i=0}^{p-2} \frac{(p-1)!}{(i+1)!(p-i-1)!} \varepsilon(v-1)^{t+i} \bmod(\wp)$$

and $\mathbf{q}$ is generated by $(v-1)+\wp$. So (i) and (ii) are clear. For (iii) we must look at $(1+\varepsilon)(v-1)^{p-1}$ which depends on $v_{\mathbf{q}}(1+\varepsilon)$. □

Proposition. *Suppose* λ *is in the lowest* p^2*-alcove.*

(i) The filtration of $H_k^H(w_0\cdot\lambda)$ *defined by* $(\varepsilon(v-1)^t+p)$ *with* ε *a unit in* $\mathscr{A}$ *and* t *an integer different from* $p-1$ *has sum formula*

$$\sum_{j\geq 0} \operatorname{ch} H_k^N(w_0\cdot\lambda)^j = \sum_{\alpha\in R^+} \sum_{0<mp<\langle\lambda+\rho,\alpha^\vee\rangle} \min(p-1,t)\chi(s_\alpha\cdot\lambda + mp\alpha)$$

(ii) If $t > p-1$ *then the top* t *terms of the filtration in (i) coincide with the corresponding terms of the filtration defined by* (p) *(through* $\mathbf{F}_p[v]_{(v-1)}$*).*

10.14. If Γ is a Dedekind domain (or even a p.i.d.) we can always find a basis for $H_\Gamma^N(w_0\cdot\lambda)$ and another for $H_\Gamma^0(\lambda)$ such that the matrix of $T_{w_0}: H_\Gamma^N(w_0\cdot\lambda) \to H_\Gamma^0(\lambda)$ with respect to these bases is diagonal. Since $\mathscr{A}$ is not Dedekind we don't know whether T_{w_0} can be diagonalized or not. Assuming that this can be done, we have the

Proposition. *Let* ε *be a unit in* $\mathscr{A}$ *and* t *an integer. Assume* T_{w_0} *can be diagonalized over* $\mathscr{A}$*. Then*

(i) The filtration of $H_k^N(w_0\cdot\lambda)$ *defined by* $\wp = ((v-1)+\varepsilon p^t)$ *coincides with the Jantzen filtration.*

(ii) Suppose λ *is in the lowest* p^2*-alcove. The filtration defined by* $\wp = (\varepsilon(v-1)^t + p)$ is the Jantzen filtration "magnified" by $v_{\mathscr{m}/\wp}(\phi_p)$.

Proof. Let $\{x_i\}_{i=1}^n$ and $\{y_i\}_{i=1}^n$ be basis of $H^N(w_0\cdot\lambda)$ and $H^0(\lambda)$ respectively such that $T_{w_0}(x_i) = a_i y_i$ for $a_i = a_i(v) \in \mathscr{A}$. Then $\det(T_{w_0}) = \prod_{i=1}^n a_i$ which is the product of polynomials of the form $\dfrac{v^m - v^{-m}}{v - v^{-1}}$ together with a unit in $\mathscr{A}$ by Corollary 4.5, and hence each a_i is a product of polynomials of the form $\phi_p(v^{p^e})$ and some unit in $\mathscr{A}$. So by Proposition 10.11

$$v_{(\mathscr{m}/((v-1)+\varepsilon p^t))}(a_i) = v_{(\mathscr{m}/(v-1))}(a_i) = v_p(a_i(1))$$

and if λ is in the lowest p^2-alcove, the only factor of a_i belonging to $\mathscr{m}$ is $\phi_p(v)$ which occurs $v_p(a_i(1))$ times, and we have

$$\begin{aligned} v_{(\mathscr{m}/(\varepsilon(v-1)^t+p))}(a_i) &= v_{(\mathscr{m}/(\varepsilon(v-1)^t+p))}(\phi_p(v)^{v_p(a_i(1))}) \\ &= v_p(a_i(1))\, v_{(\mathscr{m}/(\varepsilon(v-1)^t+p))}(\phi_p(v)) \end{aligned}$$

Let π be the natural map $H^N_{\mathscr{A}}(w_0\cdot\lambda)\to H^N_k(w_0\cdot\lambda)$ and denote the filtration of $H^N_k(w_0\cdot\lambda)$ defined by $\wp$ by $H^N_k(w_0\cdot\lambda)^j_{\wp}$.

For (i) it is easy to see that both $H^N_k(w_0\cdot\lambda)^j_{\wp}$ and $H^N_k(w_0\cdot\lambda)^j$ have a basis consisting of those $\pi(x_i)$ with $v_p(a_i(1))\geqq j$. So they are equal.

(ii) Let $l=v_{\mathscr{m}/\wp}(\phi_p)$ where $\wp=((\varepsilon(v-1)^t+p))$. For $i=1,2,\dots,l$, $H^N_k(w_0\cdot\lambda)^{(j-1)l+i}_{\wp}$ have the common basis consisting of those $\pi(x_i)$ with $v_p(a_i(1))\geqq j$ which is a basis of $H^N_k(w_0\cdot\lambda)^j$. □

10.15. Let us formulate the

Conjecture. $T_{w_0}: H^N(w_0\cdot\lambda)\to H^H(\lambda)$ *can be diagonalized over* $\mathscr{A}$

Let $\varphi:\mathscr{A}\to\mathscr{K}$ be a homomorphism into a field $\mathscr{K}$ which takes v to a primitive p^{th} root of unity. Let $\{H^N_{\mathscr{K}}(w_0\cdot\lambda)^j\}$ resp. $\{H^N_k(w_0\cdot\lambda)^j\}$ denote the filtration of $H^N_{\mathscr{K}}(w_0\cdot\lambda)$ resp. the Jantzen filtration of $H^N_k(w_0\cdot\lambda)$. If λ is in the lowest p^2-alcove then the two sum formulas of the above two fltrations coincide. Moreover we have

Remark. (i) If we assume the conjecture, then

$$\operatorname{ch} H^N_{\mathscr{K}}(w_0\cdot\lambda)^j = \operatorname{ch} H^N_k(w_0\cdot\lambda)^j$$

for each $j=0,1,\dots,$.

Indeed take bases $\{x_i\}$ for $H^N(w_0\cdot\lambda)$ and $\{y_i\}$ for $H^0(\lambda)$ such that $T_{w_0}(x_i)=a_iy_i$ and each x_i lies in a weight space. Let $\wp=\ker\varphi$ which is generated by $\phi_p(v)$. The same argument as in Lemma 10.3 shows that $v_{\wp}(a_i)=v_p(a_i(1))$ over $\mathscr{A}_{\wp}$. If we denote the natural maps $H^N(w_0\cdot\lambda)\to H^N_{\mathscr{K}}(w_0\cdot\lambda)$ and $H^N(w_0\cdot\lambda)\to H^N_k(w_0\cdot\lambda)$ by π_1 and π_2, respectively, then $H^N_{\mathscr{K}}(w_0\cdot\lambda)^j$ has a basis consisting of those $\pi_1(x_i)$ with $v_p(a_i(1))\geqq j$ while $H^N_k(w_0\cdot\lambda)^j$ has one consisting of $\pi_2(x_i)$ with the same i's. So (i) follows.

(ii) Since $H^N_{\mathscr{K}}(w_0\cdot\lambda)^1$, resp. $H^N_k(w_0\cdot\lambda)^1$, is the maximal proper submodule of $H^N_{\mathscr{K}}(w_0\cdot\lambda)$, resp. $H^N_k(w_0\cdot\lambda)$, the theorem implies in particular

$$\operatorname{ch} L_{\mathscr{K}}(\lambda)=\operatorname{ch} L_k(\lambda)$$

where $L_{\mathscr{K}}(\lambda)$, resp. $L_k(\lambda)$, is the irreducible module for $U_{\mathscr{K}}$, resp. U_k, with highest weight λ. That is, our conjecture implies Lusztig's conjecture [L 3].

11. Examples

Once the linkage and translation principles are established and the sum formula is proved, we can easily obtain the results analogous to those in the modular case which are consequences of the corresponding principles and formula. In this section we illustrate this by showing that it gives the characters of all simple $U_{\mathscr{K}}$-modules when U corresponds to a Cartan matrix of rank 2 or of type A_3. The result verifies Lusztig's conjecture [L 3] for these types.

11.1. Let U be the quantum group corresponding to the Cartan matrix of type A_2. Let $l = p^e$ for $p > 2$ a prime and e a positive integer. Let $\mathscr{A} \to \mathscr{K}$ be a homomorphism into a field $\mathscr{K}$ taking v into a primitive l'th root of 1. Let $\lambda \in X^+$. From Theorem 10.7 it follows that

(1) If $\langle \lambda + \rho, \alpha_1^\vee + \alpha_2^\vee \rangle \leqq l$, then $L_{\mathscr{K}}(\lambda) = H^0_{\mathscr{K}}(\lambda)$.

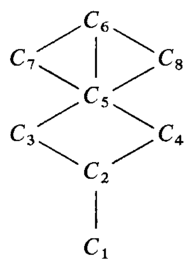
(2) If $\langle \lambda + \rho, \alpha_i^\vee \rangle < l, i = 1, 2$, and $\langle \lambda + \rho, \alpha_1^\vee + \alpha_2^\vee \rangle > l$, then we have the exact sequence

$$0 \to L_{\mathscr{K}}(s_{\alpha_1+\alpha_2} \cdot \lambda + l(\alpha_1 + \alpha_2)) \to H^3_{\mathscr{K}}(w_0 \cdot \lambda) \to L_{\mathscr{K}}(\lambda) \to 0$$

Either from the translation principle 8.3–8.4 or directly from Theorem 10.7, this gives us $\operatorname{ch} L_{\mathscr{K}}(\lambda)$ for all l-restricted λ's (i.e. for $\{\lambda \in X^+ | \langle \lambda, \alpha_i^\vee \rangle < l, i = 1, 2\}$). Now by Lusztig's tensor product theorem [L 3] we can find $\operatorname{ch} L_{\mathscr{K}}(\lambda)$ for general $\lambda \in X^+$. It is immediate to check that the results agree with Lusztig's conjecture.

11.2. Consider now type A_3.

The set of l-restricted weights divides into 6 alcoves, $C_1, \ldots, C_6$, which are ordered in the usual way as follows:



In this diagram we have also included the 2 non-l-restricted alcoves C_7 and C_8 which are less than C_6. If $\lambda_1 \in C_1$, we let λ_i be the W_l-conjugated element in C_i, $i = 2, \ldots, 8$. Then we have

$$\operatorname{ch} L_{\mathscr{K}}(\lambda_1) = \chi(\lambda_1)$$

$$\operatorname{ch} L_{\mathscr{K}}(\lambda_2) = \chi(\lambda_2) - \chi(\lambda_1)$$

$$\operatorname{ch} L_{\mathscr{K}}(\lambda_3) = \chi(\lambda_3) - \chi(\lambda_2) + \chi(\lambda_1)$$

$$\operatorname{ch} L_{\mathscr{K}}(\lambda_4) = \chi(\lambda_4) - \chi(\lambda_2) + \chi(\lambda_1)$$

$$\operatorname{ch} L_{\mathscr{K}}(\lambda_5) = \chi(\lambda_5) - \chi(\lambda_4) - \chi(\lambda_3) + \chi(\lambda_2) - 2\chi(\lambda_1)$$

$$\operatorname{ch} L_{\mathscr{K}}(\lambda_6) = \chi(\lambda_6) - \chi(\lambda_7) - \chi(\lambda_8) - \chi(\lambda_5) + 2\chi(\lambda_4) + 2\chi(\lambda_3) - 4\chi(\lambda_2) + 5\chi(\lambda_1)$$

This is obtained by combining Theorem 10.7 and Corollary 8.4 (compare [Ja 2]). As in 11.1, we then get all $\operatorname{ch} L_{\mathscr{K}}(\lambda)$ for λ l-restricted by applying Theorem 8.3 and finally all $\operatorname{ch} L_{\mathscr{K}}(\lambda)$, $\lambda \in X^+$ from the tensor product theorem.

Again it is easy to check that the results agree with Lusztig's conjecture [L 3]. It is enough to verify this for λ l-restricted because by Kato's result [K] the conjecture "respects" the tensor product theorem.

11.3. The same argument can be given in the case of a Cartan matrix of type B_2 or G_2. In summary, we have

Theorem. *Assume U corresponds to a Cartan matrix of type A_2, B_2, G_2 or A_3, $\mathscr{K}$ is as above and $k = \mathbf{F}_p$. Then for all j*

$$\operatorname{ch} H^N_{\mathscr{K}}(w_0 \cdot \lambda)^j = \operatorname{ch} H^N_k(w_0 \cdot \lambda)^j$$

for λ l-restricted. And for $\lambda = \lambda^0 + p\lambda^1$ such that $\lambda^0 \in X_1$ and λ_1 is in the lowest alcove we have

$$\operatorname{ch} L_{\mathscr{K}}(\lambda) = \operatorname{ch} L_k(\lambda)$$

Proof. If λ is restricted then all the irreducible factors of $H^N_{\mathscr{K}}(w_0 \cdot \lambda)^j$ have multiplicity 1. So the sum formula tells us exactly how $\operatorname{ch} H^N_{\mathscr{K}}(w_0 \cdot \lambda)^j$ looks when expressed as a linear combination of $\operatorname{ch} L_{\mathscr{K}}(\lambda)$'s, which is in fact the same as $\operatorname{ch} H^N_k(w_0 \cdot \lambda)^j$ expressed in terms of $\operatorname{ch} L_k(\lambda)$'s. Now by induction one gets easily that $\operatorname{ch} L_{\mathscr{K}}(\mu) = \operatorname{ch} L_k(\mu)$ and $\operatorname{ch} H^N_{\mathscr{K}}(w_0 \cdot \lambda)^j = \operatorname{ch} H^N_k(w_0 \cdot \lambda)^j$ for λ restricted and μ strongly linked to λ (when μ is not restricted we use the tensor product theorem).

For $\lambda = \lambda^0 + p\lambda^1$ with $\lambda^0 \in X_1$ and λ^1 in the lowest alcove, we use the tensor product theorem and get the result easily since it is true for λ restricted.

12. Appendix: quantum SL_n (by P. Polo)

In this section we prove that for a Cartan matrix of type A_{n-1} the quantum coordinate algebra defined in Section 1 coincides with the one studied in [PW 1–2].

12.1. Coefficient spaces over $\mathscr{A}$. Let V be a U-module. As usual, $\operatorname{Hom}(V, \mathscr{A})$ is denoted by V^*. This is made into a U-module as follows: if $\varphi \in V^*$, $v \in U$, $x \in V$ then $(u \cdot \varphi)(x) = \varphi(S(u)x)$. Then, there is a $U \otimes U$-homomorphism $\mathbf{c}: V^* \otimes V \to U^*$ defined by:

$$\mathbf{c}(\varphi \otimes x)(u) = \varphi(ux), \quad \text{for } \varphi \in V^*, x \in V, u \in U.$$

If several modules are involved, we will write $\mathbf{c}_V$, etc. in order to avoid confusion. The image of $\mathbf{c}_V$ is denoted by $\mathbf{c}(V)$ and called the coefficient space of V. If $V \in \mathscr{C}$, then $\mathbf{c}(V)$ is a $U \otimes U$-submodule of $\mathscr{A}[U]$.

Let E be a U-submodule of V. Set $Q = V/E$, and let π be the projection $V \to Q$. We assume that Q is a free $\mathscr{A}$-module, so that the transposed map $V^* \xrightarrow{\sigma} E^*$ is surjective. Let $x \in E$, $\varphi \in E^*$, and $\psi \in V^*$ such that $\sigma(\psi) = \varphi$. We claim that the element $\mathbf{c}_V(\psi \otimes x) \in \mathbf{c}(V)$ only depends on x and φ and not on the choice of ψ. To see this, let $u \in U$. Then $\mathbf{c}(\psi \otimes x)(u) = \psi(ux)$. But E is a U-submodule of V, hence $ux \in E$ and therefore $\psi(ux) = \varphi(ux)$. This proves our claim. Hence, there exists a well-defined $\mathscr{A}$-linear map $\beta: E^* \otimes E \to \mathbf{c}(V)$ such that:

$$\beta(\varphi \otimes x)(u) = \varphi(ux) \text{ for all } \varphi \in E^*, x \in E, u \in U\ .$$

Observe that β is a $U \otimes U$-homomorphism: if $u_1, u_2, u \in U$, $\varphi \otimes x \in E^* \otimes E$ then:

$$\beta(x_1 \varphi \otimes u_2 x)(u) = (u_1 \varphi)(u u_2 x) = \varphi(S(u_1) u u_2 x) = ((u_1 \otimes u_2)\beta(\varphi \otimes x))(u)\ .$$

Finally, it is clear from the definition of β that $\operatorname{Ker}(\beta) = \operatorname{Ker}(\mathbf{c}_E)$. Hence β factors through an injective $U \otimes U$-homomorphism $\mathbf{c}(E) \subsetneq \mathbf{c}(V)$.

Now, let $y \in Q$, $\theta \in Q^*$, and $z \in V$ such that $\pi(z) = y$. We identify Q^* with the subspace: $E^\perp = \{\eta \in V^* \mid \eta(E) = 0\}$. Again, $\mathbf{c}_V(\theta \otimes z)$ only depends on θ and y, and not on the choice of z. Indeed, let $u \in U$. Then $\mathbf{c}_V(\theta \otimes z)(u) = \theta(uz)$. But this only depends on the image of uz in $V/E = Q$, namely uy. Hence, there exists a well-defined $\mathscr{A}$-linear map $\gamma: Q^* \otimes Q \to \mathbf{c}(V)$ such that:

$$\gamma(\theta \otimes y)(u) = \theta(uy) \text{ for all } \theta \in Q^*, y \in Q, u \in U.$$

Then it is immediate that γ is a $U \otimes U$-homomorphism, and that $\operatorname{Ker}(\gamma) = \operatorname{Ker}(\mathbf{c}_Q)$. Therefore, γ factors through an injective $U \otimes U$-homomorphism $\mathbf{c}(Q) \subsetneq \mathbf{c}(V)$.

We record the results in the following:

Lemma. *Let* $0 \to E \to V \to Q \to 0$ *be an exact sequence of U-modules, such that Q is a free* $\mathscr{A}$*-module. Then* $\mathbf{c}(E)$ *and* $\mathbf{c}(Q)$ *are* $U \otimes U$*-submodules of* $\mathbf{c}(V)$.

12.2. For any $\lambda \in X^+$, we denote $D(\lambda)$ by $E(\lambda)$, and denote by $c(\lambda)$ its coefficient space. If $\varphi \in \mathscr{A}[U]$ is an element of weight v, then by Corollary 1.30 there exists $\lambda, \mu \in X^+$, with $\lambda + w_0\mu = v$, such that φ belongs to the coefficient space of $E(\lambda) \otimes E(\mu)$. But the latter is nothing but $\mathbf{c}(\lambda)\mathbf{c}(\mu)$ (multiplication in $\mathscr{A}[U]$), and therefore we obtain that $\mathscr{A}[U]$ is generated as an algebra by the coefficient spaces $\mathbf{c}(\lambda)$, $\lambda \in X^+$.

12.3. From now on, we assume that the Cartan matrix A is of type A_{n-1}. Let $\omega_1 \in X^+$ be the fundamental weight such that $E(\omega_1) := V$ is the natural representation of U. We will prove that $\mathscr{A}[U]$ is generated as an algebra by the subspace $\mathbf{c}(V)$.

The dual module V^t is isomorphic to $H^0(\omega_{n-1})$. Let $\lambda \in X^+$. Since ω_{n-1} is minuscule, then all weights of the U^b-module $H^0(\omega_{n-1}) \otimes \mathscr{A}_\lambda$ belong to $-\rho + X^+$. By the tensor identity 2.16 and Kempf's vanishing 5.7 we conclude that $H^0(\omega_{n-1}) \otimes H^0(\lambda) \simeq H^0(H^0(\omega_{n-1}) \otimes \mathscr{A}_\lambda)$ has a good filtration. From this we easily obtain the:

Lemma. *Let M be an* $\mathscr{A}$*-finite U-module with a good filtration. Then* $H^0(\omega_{n-1}) \otimes M$ *has a good fltration.*

From the lemma it follows that $(V^t)^{\otimes m}$ has a good filtration, for all $m \geqq 1$. Taking $*$-duals, we obtain that $V^{\otimes m}$ has a *Weyl filtration*, i.e. a sequence of U-submodules: $0 = M_0 \subseteq M_1 \subseteq \ldots \subseteq M_s = V^{\otimes m}$ such that each M_i/M_{i-1} is isomorphic to some Weyl module $E(\lambda_i)$. Observe that each $V^{\otimes m}/M_i$ is a free $\mathscr{A}$-module, and therefore by Lemma 12.1 each $\mathbf{c}(\lambda_i)$ is a $U \otimes U$-submodule of $\mathbf{c}(V^{\otimes m}) = (\mathbf{c}(V))^m$.

12.4. Hence, in order to prove that $\mathbf{c}(V)$ generates the algebra $\mathscr{A}[U]$, we only have to prove that any $E(\lambda)$, $\lambda \in X^+$ appears as a subquotient in some $V^{\otimes m}$. This reduces to a statement about characters, and therefore can be checked in the classical case. Namely, let $G = SL_n(\mathbb{C})$ and let V be the natural representation of G. Then the coordinate algebra $\mathbb{C}[G]$ is generated by the functions x_{ij}, $1 \leqq i, j \leqq n$, which are the coefficients of the representation of G on V. Therefore, any finite dimensional subspace of $\mathbb{C}[G]$ is contained in $\sum_{s \leqq t} \mathbf{c}(V^{\otimes s})$, for some $t \geqq 0$. But each $V^{\otimes s}$ is completely reducible, and if $m_s(\mu)$ denotes $[V^{\otimes s}: E(\mu)]$ for $\mu \in X^+$, it is easily checked that:

$$\mathbf{c}(V^{\otimes s}) = \bigoplus_{\mu \in \Omega(s)} E(\mu)^* \otimes E(\mu), \text{ where } \Omega(s) = \{\mu \mid m_s(\mu) > 0\}.$$

On the other hand, $\mathbb{C}[G] = \bigoplus_{\lambda \in X^+} E(\lambda)^* \otimes E(\lambda)$, by the Peter-Weyl theorem. Since the $E(v)^* \otimes E(v)$, where $v \in X^+$, are pairwise non-isomorphic simple $G \times G$-modules, it follows from the above decomposition of $\mathbf{c}(V^{\otimes s})$ that any $E(\lambda)$ is a subquotient of some $\mathbf{c}(V^{\otimes s})$. Hence, we have proved the:

Proposition. *Assume that U is associated to a Cartan matrix of type A_{n-1}. Then $\mathscr{A}[U]$ is generated as an algebra by the coefficient space $\mathbf{c}(V) = \mathbf{c}(\omega_1)$.*

Remark. In fact, the Proposition is true for any Cartan matrix A, if we take V to correspond to a faithful representation of the simply-connected semi-simple algebraic group G associated to A. If A is of classical type, or E_6 or E_7, we can take V to be a direct sum of minuscule representations, and then Lemma 12.3 still holds. For types E_8, F_4, G_2, we take $V = E(\omega)$, where $\omega \in X^+$ is the highest short root. Then Lemma 12.3 can still be proved by elementary *ad hoc* methods ([P 1, Propositions 3.6–8]). In fact, the lemma is a particular case of a general result on good filtrations, (see [Do], [Ma], and 5.14).

12.5. The quantum symmetric and exterior algebras

Following [PW 1] we define the quantum symmetric and exterior algebras of V as follows. As usual, let $T(V)$ denote the tensor algebra of the free $\mathscr{A}$-module V. Consider the following $\mathscr{A}$-submodules of $V \otimes V$:

$$M = \mathscr{A}\text{-span}\{x_i \otimes x_j - v x_j \otimes x_i \mid 1 \leqq i < j \leqq n\}$$

$$N = \mathscr{A}\text{-span}\{x_i \otimes x_i, x_i \otimes x_j + v^{-1} x_j \otimes x_i \mid 1 \leqq i < j \leqq n\} .$$

Let $\langle M \rangle$, $\langle N \rangle$ be the two sided ideals of $T(V)$ generated by M, N, respectively. Then set: $S_q(V) = T(V)/\langle M \rangle$ and $\Lambda_q(V) = T(V)/\langle N \rangle$.

Since $v + v^{-1}$ is a unit in $\mathscr{A}$, we easily obtain that $V \otimes V = M \oplus N$ (direct sum of $\mathscr{A}$-modules). From this it follows that the union of the given generators of M, N form an $\mathscr{A}$-basis of $V \otimes V$. Hence, they respectively form an $\mathscr{A}$-basis of M, N, which are therefore free.

12.6. U-module structures. Clearly, $T(V)$ is a graded U-module. We leave it to the reader to check that both M and N are U-submodules of $V \otimes V$. Therefore, both $S_q(V)$ and $\Lambda_q(V)$ are graded U-modules.

12.7. Some relations. Now, we describe some relations among the elements of $\mathbf{c}(V)$, which generate the algebra $\mathscr{A}[U]$. Firstly, we observe that since $V^* \otimes V$ is a simple $U \otimes U$-module, then the non-zero $U \otimes U$-homomorphism $\mathbf{c}: V^* \otimes V \to \mathbf{c}(V)$ is an isomorphism. Let x_1 be a generator of the $\mathscr{A}$-module V_{ω_1}, and set $x_{i+1} = F_i x_i$ for all $1 \leqq i \leqq n-1$. Then, x_i has weight $\omega_i - \omega_{i-1}$ (with the convention $\omega_0 = \omega_n = 0$), $E_i x_{i+1} = x_i$ for all $1 \leqq i \leqq n-1$, and $\{x_1, \ldots, x_n\}$ is an $\mathscr{A}$-basis of the $\mathscr{A}$-module V (free of rank n). Let $\{\delta_1, \ldots, \delta_n\}$ be the $\mathscr{A}$-basis of V^*, dual to the basis $\{x_1, \ldots, x_n\}$ of V. For all i, j we denote by X_{ij} the image of $\delta_i \otimes x_j$ in $\mathbf{c}(V) \subseteq \mathscr{A}[U]$.

Observe that, by definition of multiplication, we have:

$$(X_{ij} X_{lm})(u) = (\delta_i \otimes \delta_l)(\Delta(u)(x_i \otimes x_m)) \text{ for all } u \in U .$$

From the direct sum decomposition: $V \otimes V = M \oplus N$ (as U-modules), we obtain relations among the $X'_{ij}s$. Firstly, we observe that the elements

$\{\delta_i \otimes \delta_j - v\delta_j \otimes \delta_i | 1 \leqq i < j \leqq n\}$ form an $\mathscr{A}$-basis of the orthogonal $M^\perp$ of M. Similarly, the elements $\{\delta_i \otimes \delta_i, \delta_i \otimes \delta_j + v^{-1}\delta_j \otimes \delta_i | 1 \leqq i < j \leqq n\}$ form an $\mathscr{A}$-basis of $N^\perp$. Again, both are U-submodules of $V^* \otimes V^*$, and $V^* \otimes V^* = M^\perp \oplus N^\perp$.

Clearly, if $\varphi \in M^\perp$ (resp. $N^\perp$) and $x \in M$ (resp. N), then $\mathbf{c}(\varphi \otimes x) = 0$. Applying this to: $\varphi = \delta_i \otimes \delta_j - v\delta_j \otimes \delta_i$, $x = x_l \otimes x_l$, we obtain:

(1) $$X_{il}X_{jl} - vX_{jl}X_{il} = 0 \text{ for all } l, i < j$$

Similarly, we obtain the relations:

(2) $$X_{li}X_{lj} - vX_{lj}X_{li} = 0 \text{ for all } l, i < j$$

(3) $$X_{li}X_{mj} - X_{mj}X_{li} = 0 \text{ if } l < m \text{ and } i > j$$

(4) $$X_{li}X_{mj} - X_{mj}X_{li} - (v - v^{-1})X_{lj}X_{mi} = 0 \text{ if } l < m \text{ and } i < j$$

12.8. The determinant. Now, consider the U-module $L = \Lambda_q^n(V)$. From the definition of $\Lambda_q(V)$, we obtain that L is generated as an $\mathscr{A}$-module by the image of the element $x_1 \otimes \ldots \otimes x_n$, which we denote by $x_1 \wedge \ldots \wedge x_n$. Moreover, we claim that L is a free (rank one) $\mathscr{A}$-module. For this we observe that by [PW 1, Theorem 3.3.1] both $L \otimes \mathscr{A}'$ and $L \otimes k$ are 1-dimensional. By Nakayama's lemma, this shows that L is a free, rank one, $\mathscr{A}$-module (see 1.21). Now, we claim that U acts on L via the character ε. In fact, since $V \in \mathscr{C}$, then $V^{\otimes n}$, $L \in \mathscr{C}$. But L has rank one, hence the only weight $v \in X$ that can occur in L is ε.

Also, by definition of the "coordinate" functions X_{ij} we have: $ux_j = \sum_{i=1}^n X_{ij}(u)x_i$ for all $u \in U$, $1 \leqq j \leqq n$. Combined with the fact that $x_{\sigma(1)} \wedge \ldots \wedge x_{\sigma(n)} = (-v)^{l(\sigma)}x_1 \wedge \ldots \wedge x_n$ for all $\sigma \in S_n$, this gives, for all $u \in U$:

$$u \cdot (x_1 \wedge \ldots \wedge x_n) = \left(\sum_{\sigma \in S_n} (-v)^{l(\sigma)} X_{\sigma(1)1}(u) \ldots X_{\sigma(n)n}(u) \right) x_1 \wedge \ldots \wedge x_n .$$

Since L is a free $\mathscr{A}$-module, we conclude that: $\sum_{\sigma \in S_n} (-v)^{l(\sigma)} X_{\sigma(1)1} \ldots X_{\sigma(n)n} = \varepsilon$. Let us denote the L.H.S. by D. Since ε is the identity element of the algebra $\mathscr{A}[U]$, this can be rewritten as:

(5) $$D = \sum_{\sigma \in S_n} (-v)^{l(\sigma)} X_{\sigma(1)1} \ldots X_{\sigma(n)n} = 1$$

12.9. The isomorphism. So far, we have obtained that $\mathscr{A}[U]$ is a quotient algebra of the algebra $\mathscr{M}$, defined by the generators X_{ij}, $1 \leqq i, j \leqq n$ and the relations (1)–(5) above. This latter algebra is the one introduced in [PW 1] (up to the change $v \mapsto v^{-1}$).

Now, we prove that the surjection $\mathscr{M} \overset{\varphi}{\twoheadrightarrow} \mathscr{A}[U]$ is actually an isomorphism. We know already that $\varphi_k : \mathscr{M} \otimes k \to k[U]$ is an isomorphism, since $k[U] \simeq k[SL_n]$ is generated by the coordinate functions X_{ij} subject to the sole relation $\det(X_{ij}) = 1$. Let $\mathscr{K} = \mathrm{Ker}(\varphi)$. Since $\mathscr{A}[U]$ is free $\mathscr{A}$-module by Theorem 1.33, then: $\mathscr{M} \simeq \mathscr{A}[U] \oplus \mathscr{K}$.

Our immediate goal is to prove that $\mathscr{M}$ is also a free $\mathscr{A}$-module. For this, we introduce the algebra $\tilde{\mathscr{M}}$, only subject to the relations (1)–(4). By the arguments of [PW 1, Theorem 3.5.1] we obtain that $\tilde{\mathscr{M}}$ is a free $\mathscr{A}$-module with a basis consisting of the monomials $\prod_{ij} X_{ij}^{r_{ij}}$, $r_{ij} \geqq 0$, where the product is taken in some fixed total order on the set $\{1, \ldots, n\}^2$. Also, $\tilde{\mathscr{M}}$ is a graded integral domain.

We fix some total order on $\{1, \ldots, n\}^2$ and define Ξ to be the set of all monomials $\prod_{ij} X_{ij}^{r_{ij}}$ such that at least one of $d_{11}, \ldots, d_{nn}$ is zero. For $r \geqq 0$, let Ξ_r be the set of such monomials of degree $\leqq r$, let $\tilde{\mathscr{M}}(r)$ be the $\mathscr{A}$-span of all monomials of degree $\leqq r$, and let $\tilde{\mathscr{N}}(r)$ be the $\mathscr{A}$-span of Ξ_r. Then, we have the:

Lemma 12.10. *$\tilde{\mathscr{M}}(r) = (D-1)\tilde{\mathscr{M}}(r-n) \oplus \tilde{\mathscr{N}}(r)$, and $\tilde{\mathscr{N}}(r)$ is a free $\mathscr{A}$-module with basis Ξ_r.*

Proof. Set $\Upsilon_r = \{(D-1)x_s \mid 1 \leqq s \leqq t\} \cup \Xi_r$, where $\{x_s\}_{s=1}^t$ is an $\mathscr{A}$-basis of $\tilde{\mathscr{M}}(r-n)$. We claim that Υ_r is an $\mathscr{A}$-basis of $\tilde{\mathscr{M}}(r)$. Indeed, $\tilde{\mathscr{M}}(r) \otimes k$ is generated by the image of Υ_r. Hence by Nakayama $\tilde{\mathscr{M}}(r)$ is generated by Υ_r. Moreover:

$$\operatorname{rank}_{\mathscr{A}} \tilde{\mathscr{M}}(r) = \dim_k(\tilde{\mathscr{M}}(r) \otimes k) = |\Upsilon_r| .$$

It follows that Υ_r is an $\mathscr{A}$-basis of $\tilde{\mathscr{M}}(r)$, and therefore $\tilde{\mathscr{M}}(r) = (D-1)\tilde{\mathscr{M}}(r-n) \oplus \tilde{\mathscr{N}}(r)$, and Ξ_r is an $\mathscr{A}$-basis of $\tilde{\mathscr{N}}(r)$. □

12.11. Let $\mathscr{M}(r)$ denote the image of $\tilde{\mathscr{M}}(r)$ in $\mathscr{M}$. Since $\tilde{\mathscr{M}}$ is a graded integral domain, we have: $\mathscr{M}(r) \simeq \tilde{\mathscr{M}}(r)/((D-1)\tilde{\mathscr{M}}(r-n))$. From this we deduce the:

Corollary. (*i*) *$\mathscr{M}(r)$ is freely generated by the image of Ξ_r.*
(*ii*) *$\mathscr{M}$ is a free $\mathscr{A}$-module, with basis Ξ.*

12.12. Finally, we obtain the:

Proposition. *φ is an isomorphism. In other words, $\mathscr{A}[U]$ identifies with the quantum SL_n introduced in* [PW 1].

Proof. As a direct summand of $\mathscr{M}$, the $\mathscr{A}$-module $\mathscr{K}$ is projective, and is therefore free, since $\mathscr{A}$ is a local ring (see 1.32). On the other hand, $\mathscr{K} \otimes k = 0$ since φ_k is injective. It follows $\mathscr{K} = 0$, hence φ is an isomorphism. □

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Note added in proof

The categories $\mathscr{C}^{I,J}$ of 2.2 are abelian categories. One has to check that if $M \in \mathscr{C}^{I,J}$ and N is a submodule of M, then $N = \oplus_\lambda N_\lambda$. This obtains by the usual argument, as follows. If $x \in N$ then $x = x_1 + \ldots + x_t$, where $x_i \in M_{\lambda_i}$ and $\lambda_i \neq \lambda_j$ if $i \neq j$. One proves that all $x_i \in N$ by induction on t. For each $u \in U^0$ one has $ux - \lambda_t(u)x = \sum_{i=1}^{t-1} (\lambda_i - \lambda_t)(u)x_i$. By Lemma 9.1 the characters λ_i remain pairwise distinct after reduction modulo $\mathscr{m}$. Hence there exists $u_0 \in U^0$ such that, for all $i \in \{1, \ldots, t-1\}$, $(\lambda_i - \lambda_t)(u_0) \notin \mathscr{m}$. Then each $(\lambda_i - \lambda_t)(u_0)$ is invertible in $\mathscr{A}$, and by induction hypothesis one obtains $x_i \in N$ for all $i = 1, \ldots, t-1$, and then also $x_t \in N$.