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Shafarevich maps and plurigenera of algebraic varieties

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1 Introduction

Let X be a smooth projective variety over $\mathbb{C}$ and let $\tilde{X}$ be its universal cover. The conjecture of [Sh, IX.4.3] asserts that there is a proper morphism $\mathrm{sh}_{\tilde{X}}$: $\tilde{X} \to \mathrm{Sh}(\tilde{X})$ onto a normal Stein space $\mathrm{Sh}(\tilde{X})$. We may assume that $\mathrm{sh}_{\tilde{X}}$ has connected fibers, and then $\mathrm{sh}_{\tilde{X}}$ is unique. The fundamental group $\pi_1(X)$ acts on $\tilde{X}$ properly discontinuously, and this action descends to a proper action of $\pi_1(X)$ on $\mathrm{Sh}(\tilde{X})$. In general .this action has fixed points, but we can still take the quotient $\mathrm{Sh}(X) = \mathrm{Sh}(\tilde{X})/\pi_1(X)$. The morphism $\mathrm{sh}(\tilde{X})$ descends to a morphism

$$\mathrm{sh}_X\colon X \to \mathrm{Sh}(X). \tag{1.1}$$

$\mathrm{Sh}(X)$ will be called the Shafarevich variety of X and sh_X the Shafarevich morphism. At the moment the existence of (1.1) is hypothetical.

It is possible to give an internal characterisation of the Shafarevich morphism without recourse to universal covers. Let $Z \subset X$ be a connected subvariety. It is easy to see that

$$\mathrm{sh}_X(Z) = \text{point} \quad \text{iff} \quad \mathrm{im}\,[\pi_1(Z) \to \pi_1(X)] \quad \text{is finite.} \tag{1.2}$$

The right hand side of (1.2) can be used to define an equivalence relation on the closed points of X by

$$x_1 \sim x_2 \Leftrightarrow \exists \text{ a connected } x_1, x_2 \in Z \subset X \text{ s.t. } \mathrm{im}\,[\pi_1(Z) \to \pi_1(X)] \text{ is finite.} \tag{1.3}$$

If $\mathrm{Sh}(X)$ exists, it is the quotient of X by the equivalence relation $\sim$.

Even if $\mathrm{Sh}(X)$ exist, it is usually singular and sh_X is not flat. Quotients by such equivalence relations are very hard to handle. The problem becomes easier if we want to find the Shafarevich variety only "generically".

1.4. Definition. Let X be a normal and proper variety. A normal variety $\mathrm{Sh}(X)$ and a rational map sh_X: $X \dashrightarrow \mathrm{Sh}(X)$ are called the Shafarevich variety and the Shafarevich map of X if

(1.4.1) sh_X has connected fibers, and

(1.4.2) there are countably many closed subvarieties $D_i \subset X$ ($D_i \neq X$) such that for every irreducible $Z \subset X$ such that $Z \not\subset \cup D_i$,

$$\mathrm{sh}_X(Z) = \text{point iff } \operatorname{im}[\pi_1(\bar{Z}) \to \pi_1(X)] \text{ is finite.}$$

Here $\bar{Z}$ is the normalisation of Z. The change from $\pi_1(Z)$ to $\pi_1(\bar{Z})$ is done for technical reasons and does not effect the results.

It is quite likely that finitely many D_i would be sufficient, but I do not know how to prove this.

It is easy to see that sh_X: $X \to \mathrm{Sh}(X)$ is unique up to birational equivalence if it exists.

I will usually think of $\mathrm{Sh}(X)$ and sh_X as birational equivalence classes of varieties and maps, and the "true" Shafarevich variety $\mathrm{Sh}(X)$ (resp. Shafarevich morphism sh_X) is a distinguished representative (if it exists). The following is the first result:

1.5. Theorem. *For any normal and proper variety X (over $\mathbb{C}$) the Shafarevich map sh_X: $X \dashrightarrow \mathrm{Sh}(X)$ exists.*

In algebraic geometry it is very hard to see the whole fundamental group. Grothendieck defined the algebraic fundamental group of an arbitrary scheme X [SGA1, V] ; we will denote it by $\hat{\pi}_1(X)$. If X is defined over $\mathbb{C}$ then $\hat{\pi}_1(X)$ is the profinite completion of $\pi_1(X)$ [SGA1, XII.5.2]. One can define the algebraic Shafarevich map $\widehat{\mathrm{sh}}_X$: $X \to \widehat{\mathrm{Sh}}(X)$ by replacing the condition "$\operatorname{im}[\pi_1(\bar{Z}) \to \pi_1(X)]$ is finite" in (1.4) by "$\operatorname{im}[\hat{\pi}_1(\bar{Z}) \to \hat{\pi}_1(X)]$ is finite".

The algebraic version of (1.5) is the following:

1.6. Theorem. *For any normal and proper variety X (defined over an algebraically closed field of arbitrary characteristic) the algebraic Shafarevich map $\widehat{\mathrm{sh}}_X$: $X \dashrightarrow \widehat{\mathrm{Sh}}(X)$ exists.*

One aim of the Shafarevich conjecture is to find a way to construct every variety using two simpler building blocs:

Type I: varieties with finite fundamental group, and

Type II: varieties whose universal covers are Stein.

We have to replace type II by one of the the following larger classes:

1.7. Definition. Let X be a normal and proper variety. We say that X has *generically large fundamental group* (resp. *generically large algebraic fundamental group*) if the following equivalent conditions are satisfied:

(1.7.1) sh_X (resp. $\widehat{\mathrm{sh}}_X$) is birational;

(1.7.2) If $x \in X$ is a sufficiently general point and $x \in Z \subset X$ is an irreducible positive dimensional subvariety then

$$\operatorname{im}[\pi_1(\bar{Z}) \to \pi_1(X)] \text{ (resp. } \operatorname{im}[\hat{\pi}_1(\bar{Z}) \to \hat{\pi}_1(X)]) \text{ is infinite.}$$

If X has generically large algebraic fundamental group, then X has generically large fundamental group. The converse is not known.

The fibers of the Shafarevich map either have finite fundamental group or they can be further decomposed by induction on the dimension. $\mathrm{Sh}(X)$ is supposed to be the part which carries the whole fundamental group of X. Choose a smooth model for $\mathrm{Sh}(X)$ and then $\pi_1(\mathrm{Sh}(X))$ is well defined.

Unfortunately it happens frequently that $\pi_1(X)$ is large but $\mathrm{Sh}(X)$ is simply connected. This is caused by possible fixed points of the action of $\pi_1(X)$ on $\mathrm{Sh}(\tilde{X})$. We can however hope that a finite index subgroup of $\pi_1(X)$ acts freely on $\mathrm{Sh}(\tilde{X})$. This happens if $\pi_1(X)$ is residually finite, which is unfortunately not always the case [Tol, Cat, Ko]. Algebraic fundamental groups are by definition residually finite and in the algebraic case the following holds:

1.8. Theorem. *Let X be a smooth and proper variety (over $\mathbb{C}$). Then there is a finite étale cover $X' \to X$ such that*

(1.8.1) $\widehat{\mathrm{sh}}_*\colon \hat{\pi}_1(X') \to \hat{\pi}_1(\widehat{\mathrm{Sh}}(X'))$ *is an isomorphism, and*

(1.8.2) $\widehat{\mathrm{Sh}}(X')$ *has generically large algebraic fundamental group.*

In the topological case there are counterexamples (4.11).

There are two types of classical examples of varieties whose universal covers are Stein:

Flat: quotients of $\mathbb{C}^n$ by a group of translations, called Abelian varieties.

Negatively curved: quotients of bounded symmetric domains by discrete subgroups of the corresponding Lie groups.

The following observation is very useful in distinguishing these two types.

For a smooth variety X let K_X denote the canonical line bundle (i.e. local sections of K_X are the holomorphic $(\dim X)$-forms). For an Abelian variety A the canonical bundle K_A is trivial and so $K_A^{\otimes n}$ has only the constant sections for every n. If X is a quotient of a bounded symmetric domain then $K_X^{\otimes n}$ has lots of sections for $n \gg 1$, in fact sections separate points of X.

1.9. Definition. A smooth proper variety X is *of general type* if sections of $K_X^{\otimes n}$ separate points over an open dense set $U \subset X$ for $n \gg 1$. (We will frequently say that sections generically separate points.)

Products of Abelian varieties and varieties of general type give nearly all varieties with generically large algebraic fundamental group:

1.10. Conjecture. *Let X be a smooth projective variety (over $\mathbb{C}$). Assume that X has generically large fundamental group. Then X has a finite étale cover $p\colon X' \to X$ such that X' is birational to a smooth family of Abelian varieties over a projective variety of general type Z which has generically large fundamental group.*

The conjecture is true if $\dim X \leqq 2$. The general case would need various parts of the Minimal Model Program and a singular generalisation of the Cheeger-Gromoll Splitting Theorem.

Let X be a quotient of a bounded (not necessarily symmetric) domain H (by a fixed point free group). The theory of automorphic forms connects the holomorphic function theory of H and the meromorphic function theory of X. [Si., 6.1] constructs automorphic forms which show that for $n \gg 1$ the sections of $K_X^{\otimes n}$ separate points of X. The choice of n is however not clear from the construction.

Using the Nonvanishing Theorem proved in [Ko3] the presence of a large fundamental group can be exploited to prove the existence of sections of $K_X^{\otimes n}$. (Unfortunately I do not see how to use the theory to produce holomorphic functions on $\tilde{X}$.)

1.11. Definition. Let X be a proper variety and let L be a line bundle on X. We say that L is *big* if sections of $L^{\otimes n}$ generically separate points for $n \gg 1$.

1.12. Theorem. *Let X be a smooth proper variety over $\mathbb{C}$ and let L be a big line bundle on X. Assume that X has generically large algebraic fundamental group. Then* $h^0(X, K_X \otimes L) \geqq 1$.

If L is a power of K_X we can say even more:

1.13. Theorem. *Let X be a smooth projective variety over $\mathbb{C}$. Assume that X has generically large algebraic fundamental group and X is of general type. Then*
(1.13.1) $h^0(X, K_X^{\otimes m}) \geqq 1$ *for* $m \geqq 2$;
(1.13.2) $h^0(X, K_X^{\otimes m}) \geqq 2$ *for* $m \geqq 4$;
(1.13.3) Sections of $K_X^{\otimes m}$ *generically separate points for* $m \geqq 10^{\dim X}$.

1.14. Examples. (1.14.1) There are several examples of surfaces of general type X such that X has generically large algebraic fundamental group and $h^0(X, K_X) = 0$ (see [BPV, VII.11] for a list).

(1.14.2) For every M there are smooth projective varieties of general type X such that $h^0(X, K_X^{\otimes m}) = 0$ for $m \leqq M$ (8.6). In the examples $\dim X$ is roughly $3M$.

For threefolds one can weaken the assumption about the fundamental group further:

1.15. Theorem. *Let X be a smooth projective threefold over $\mathbb{C}$. Assume that $\hat{\pi}_1(X)$ is infinite and X is of general type. Then*
(1.15.1) $h^0(X, K_X^{\otimes m}) \geqq 1$ *for* $m \geqq 2$;
(1.15.2) $h^0(X, K_X^{\otimes m}) \geqq 2$ *for* $m \geqq 4$;
(1.15.3) *The sections of $K_X^{\otimes m}$ generically separate points for* $m \geqq 49$.

In general, the methods of Sects. 8–10 show that in any dimension the worst varieties with respect to existence of sections of $K^{\otimes m}$ are the simply connected ones.

(1.12) is also useful in several other contexts as well. One application is the following characterisation of Abelian varieties (b_1 is the first Betti number):

1.16. Theorem. *Let X be a smooth proper variety over $\mathbb{C}$. If $h^0(X, K_X^{\otimes m}) = 1$ for some $m \geqq 3$ then $b_1(X) \leqq 2\dim X$.*

1.17. Theorem. *Let X be a smooth proper variety over $\mathbb{C}$. The following are equivalent*:
(1.17.1) *X is birational to an Abelian variety*;
(1.17.2) $b_1(X) = 2\dim X$ *and* $h^0(X, K_X^{\otimes 4}) = 1$;
(1.17.3) $b_1(X) = 2\dim X$ *and* $h^0(X, K_X^{\otimes m}) = 1$ *for some* $m \geqq 4$.

For technical reasons the 3 in (1.16) is replaced by 4 in (1.17). It is possible that in both theorems 2 is the optimal value.

2 Definitions and Basic Properties

2.1. Definition. Let X be a normal variety. By a *normal cycle on X* we mean an irreducible and normal variety W together with a finite morphism $w: W \to X$ which is birational to its image.

Let $Z \subset X$ be any closed irreducible subvariety. Let $n: \bar{Z} \to Z \subset X$ be the normalisation. This is a normal cycle on X.

2.2. Definition. Let X be a normal variety. *A family of normal cycles on X* is a diagram

$$\begin{array}{ccc} U & \xrightarrow{u} & X \\ {\scriptstyle p}\downarrow & & \\ S & & \end{array}$$

where

(2.2.1) every connected component of U and of S is of finite type (but there can be infinitely many such components);

(2.2.2) p is flat with irreducible, geometrically reduced and normal fibers;

(2.2.3) for every $s \in S$, $u|U_s$: $U_s \to X$ is a normal cycle. (Here and later U_s stands for the fiber of p over $s \in S$.)

We say that the family $U \to S$ is dominant if u is dominant. We will usually use this notion only if S is irreducible.

2.3. Definition. Notation as above. Assume that everything is defined over a field K.

(2.3.1) We say that $U \to S$ is a *weakly complete family of normal cycles* if for every normal cycle $w: W \to X_L$ defined over a field $L \supset K$ there is a unique morphism s: Spec $L \to S$ such that $W \cong U \times_s S$. (In positive characteristic it is better to restrict to cycles W that are geometrically normal.)

(2.3.2) (over $\mathbb{C}$) We say that $U \to S$ is a *weakly complete family of locally topologically trivial normal cycles* if it is a weakly complete family of normal cycles and p is a locally trivial fibration in the Euclidean topology.

(2.3.3) We will use the first notion when dealing with the algebraic fundamental group and the second one when dealing with the topological fundamental group. We will usually say that $U \to S$ is a *weakly complete family of normal cycles* and we understand that local topological triviality is required in the topological case.

2.4. Proposition. *Let X be a normal variety. There is a weakly complete family of normal cycles*

$$\begin{array}{ccc} U(X) & \xrightarrow{u(X)} & X \\ {\scriptstyle p(X)}\downarrow & & \\ S(X). & & \end{array}$$

Proof. Let $X' \supset X$ be a compactification. If $U(X') \to S(X')$ is a weakly complete family of normal cycles on X' then take $U(X) = u(X')^{-1}(X)$ and $S(X) = p(X')U(X)$. This is a weakly complete family of normal cycles on X. Topological triviality will be discussed at the end of the proof.

Thus assume that X is proper. Start with a family of all cycles or subschemes r: Univ $\to R$ (one can take R to be Chow(X) or Hilb(X)). Replace R by red$R$ and take the largest open subset which parametrises reduced and irreducible cycles or subschemes. We still use R to denote the resulting scheme.

Let $\bar{r}$: $\overline{\text{Univ}} \to R$ be the normalisation. There is a dense open subset $Q_0 \subset R$ such that $\bar{r}$ is flat over Q_0 and every fiber of $\bar{r}$ is the normalisation of the corresponding fiber of r. Restrict r to $R_1 = R - Q_0$ and iterate this procedure. We end up with countably many families $p_i: V_i \to Q_i$ which satisfy the properties (2.2). (In positive characteristic we may have to take a purely inseparable morphism $R' \to R$ and subdivide further.)

In the topological case we need one more step. In general let $f: Y \to Z$ be a proper and flat morphism and let $B \subset Y$ be a closed subvariety. Choose a Whitney stratification of Y such that B is the union of strata. Then Z has an open and dense set $Z^0 \subset Z$ such that every stratum is smooth over Z^0. By [GoMacPh, I.1.11] $f: f^{-1}(Z^0) \setminus B \to Z^0$ is a locally trivial fibration in the Euclidean topology. □

2.4.1. Remark. It is frequently convenient to do a further subdivision of $S(X)$ to achieve that every irreducible component of $S(X)$ is also a connected component. We will assume that this has been done if this makes a proof easier.

The constructed $S(X)$ is not unique but fortunately we do not need uniqueness beyond what is required in (2.3).

2.5. Corollary. *Let X be a normal variety. There are countably many closed subvarieties $D_i \subset X$ ($D_i \neq X$) such that if $w: W \to X$ is a normal cycle and* $\operatorname{im} w \not\subset \cup D_i$ *then there is a unique point $s \in S(X)$ such that the following two statements hold:*

(2.5.1) $[u(X): U_s(X) \to X] \cong [w: W \to X]$.

(2.5.2) *Let $s \in S_j(X) \subset S(X)$ be the irreducible componenet containing s. Then $u_j(X): U_j(X) \to X$ is dominant.*

Proof. Let $S_i(X)$ be the irreducible components of $S(X)$. *Let D_i* be the closure of $u(X)$ $(U_i(X))$ for $U_i(X) \to X$ not dominant. □

2.6. Proposition. *Let $U(X) \to S(X)$ be a weakly complete family of normal cycles. In the algebraic case assume that X is proper. Assume that $s, t \in S(X)$ are in the same irreducible component. Then*

$$\operatorname{im}[\hat{\pi}_1(U_s(X)) \to \hat{\pi}_1(X)] = \operatorname{im}[\hat{\pi}_1(U_t(X)) \to \hat{\pi}_1(X)] \quad \textit{resp.}$$

$$\operatorname{im}[\pi_1(U_s(X)) \to \pi_1(X)] = \operatorname{im}[\pi_1(U_t(X)) \to \pi_1(X)].$$

Proof. The topological case is clear. In the algebraic case let η be the geometric generic point of the corresponding component. By [SGA1, X.2.3] there is a (nonunique) surjective specialisation map $\hat{\pi}_1(U_\eta(X)) \to \hat{\pi}_1(U_s(X))$. Thus

$$\operatorname{im}[\hat{\pi}_1(U_s(X)) \to \hat{\pi}_1(X)] = \operatorname{im}[\hat{\pi}_1(U_\eta(X)) \to \hat{\pi}_1(X)] = \operatorname{im}[\hat{\pi}_1(U_t(X)) \to \hat{\pi}_1(X)].$$

In the nonproper case very little is known about specialisations of $\hat{\pi}_1$, especially in positive characteristic. This is the reason of the properness assumption. □

Let X be a projective variety over $\mathbb{C}$ and let $\tilde{X}$ be its universal cover. If $\tilde{X}$ is Stein then it does not contain positive dimensional proper subspaces. The following notions should be viewed as weaker versions of this property.

2.7. Definition. (2.7.1) Let X be a normal variety. We say that X has *large algebraic fundamental group* (resp. *large fundamental group*) if for every normal cycle $w: W \to X$

$$\operatorname{im}[\hat{\pi}_1(W) \to \hat{\pi}_1(X)] \quad \text{resp.} \quad \operatorname{im}[\pi_1(W) \to \pi_1(X)] \quad \text{is infinite.}$$

(2.7.2) Let X be a normal variety. We say that X has *generically large algebraic fundamental group* (resp. *generically large fundamental group*) if for every normal cycle $w: W \to X$ such that $\operatorname{im} w \not\subset \cup D_i$ (cf. (2.5))

$$\operatorname{im}[\hat{\pi}_1(W) \to \hat{\pi}_1(X)] \quad \text{resp.} \quad \operatorname{im}[\pi_1(W) \to \pi_1(X)] \quad \text{is infinite.}$$

2.8. Remark. Let X be a normal variety over $\mathbb{C}$ and assume that its universal cover $\tilde{X}$ is holomorphically convex. Let $Z \subset X$ be a connected subscheme. Let $\bar{Z}_i$ be the connected components of $\bar{Z}$. [Gu] observed that $\mathrm{im}[\pi_1(Z) \to \pi_1(X)]$ is finite iff $\mathrm{im}[\pi_1(\bar{Z}_i) \to \pi_1(X)]$ is finite for every i. (This is not stated in [Gu] but follows easily from the proof.) Thus if the Shafarevich conjecture is true, it is not important that in (2.7.1) we restricted our attention to normal cycles.

The following proposition shows that in (2.7) we could have considered arbitrary morphisms $W \to X$ or we could have restricted ourselves to $\dim W = 1$.

2.9. Proposition. (2.9.1) [Cam1] *Let X, Y be irreducible normal varieties. Let $f: X \to Y$ be a dominant morphism such that the geometric generic fiber has at most k irreducible components. Then the image of $\pi_1(X) \to \pi_1(Y)$ has index at most k in $\pi_1(Y)$. The same holds for $\hat{\pi}_1$.*

(2.9.2) [D2] *Let $C \subset Y$ be a smooth curve obtained as a complete intersection of general very ample divisors. Then $\hat{\pi}_1(C) \to \hat{\pi}_1(Y)$ and $\pi_1(C) \to \pi_1(Y)$ are surjective.*

Proof. We prove the topological case and give only references to the algebraic case.

Assume first that f is an open immersion and let $h: Y' \to Y$ be an étale cover of Y. Then $X \times_Y Y' \subset Y$ is a Zariski open subset, thus connected if Y' is normal. This shows that $\pi_1(X) \to \pi_1(Y)$ is surjective [SGA1, V.8.2].

We can factor f through a proper morphism, thus we may assume that f itself is proper. Let $Y^0 \subset Y$ be open and let $X^0 = f^{-1}(Y^0)$. By the first step $\pi_1(X^0) \to \pi_1(X)$ and $\pi_1(Y^0) \to \pi_1(Y)$ are surjective, thus it is sufficient to prove (2.9) for $f^0: X^0 \to Y^0$.

By choosing Y^0 suitably we may assume that there is a factorisation $X^0 \to Z^0 \to Y^0$ where $X^0 \to Z^0$ is a topological fiber bundle with connected fibers and $Z^0 \to Y^0$ is finite and étale of degree at most k. Thus $\pi_1(X^0) \to \pi_1(Z^0)$ is surjective [SGA1, IX.4.10, IX.6.11]. Finally $\pi_1(Z^0) \to \pi_1(Y^0)$ is injective and the image has index at most k. □

As an aside, let us note a corollary of (2.9.1) which will be useful later.

2.9.3. Corollary. *Let $g: W \to X$ be a morphism between normal varieties and let $x \in X$ be a very general point. Assume that $x \in \mathrm{im}\, g$ and let $H = \mathrm{im}[\pi_1(W) \to \pi_1(X)]$. Then the normaliser of H has finite index in $\pi_1(X)$. The same holds in the algebraic case if X is proper.*

If W is a general fiber of a morphism $X \to Y$ then $H \lhd \pi_1(X)$.

Proof. Let Z be the closure of $\mathrm{im}\, g$ and let $H' = \mathrm{im}[\pi_1(\bar{Z}) \to \pi_1(X)]$. By (2.9.1) $H < H'$ has finite index.

If $Z = \{x\}$ then $H = \{1\}$. If $\dim Z > 0$ then by (2.5) there is a dominant family of topologically trivial cycles $p: U_i \to S_i$ such that $Z = \mathrm{im}[U_s \to X]$ for some $s \in S$. Then $\mathrm{im}[\pi_1(\bar{Z}) \to \pi_1(U)]$ is a normal subgroup of $\pi_1(U)$. By (2.9.1) $G' = \mathrm{im}[\pi_1(U) \to \pi_1(X)]$ has finite index in $\pi_1(X)$ and is contained in the normaliser of H'. G' acts by conjugation on H' and H has only finitely many conjugates since H' is finitely generated. Thus there is a finite index subgroup $G < G'$ which normalises H.

If W is a general fiber of a morphism $X \to Y$ then take $U_i = X$ and $S_i = Y$. □

The following properties are straightforward:

2.10. Proposition. (2.10.1) *Let $f: X \to Y$ be a generically finite and dominant morphism between irreducible and normal varieties. Assume that Y has generically large algebraic fundamental group (resp. generically large fundamental group). Then X has generically large algebraic fundamental group (resp. generically large fundamental group).*

(2.10.2) *Let $f: X \to Y$ be a proper birational morphism between irreducible and normal varieties. Assume that $f_*: \hat{\pi}_1(X) \to \hat{\pi}_1(Y)$ (resp. $f_*: \pi_1(X) \to \pi_1(Y)$) is an isomorphism. (This holds e.g. if Y is smooth.) Then Y has generically large algebraic fundamental group (resp. generically large fundamental group)* iff *X has.*

(2.10.3) *Let X be a normal variety. Let $Z \subset X$ be a positive dimensional subvariety such that $Z \not\subset \cup D_i$. Assume that X has generically large algebraic fundamental group (resp. generically large fundamental group). Then so does $\bar{Z}$.* □

For the rest of the section we assume that everything is defined over $\mathbb{C}$.

2.11. Definition. Let X be a connected analytic space. By $\tilde{u}: \tilde{X} \to X$ we denote the universal covering space (which is again a connected analytic space).

Let $K \subset \pi_1(X)$ be the kernel of the map $\pi_1(X) \to \hat{\pi}_1(X)$. Let $\hat{u}: \hat{X} \to X$ be the covering space corresponding to K. This will be called the *universal algebraic covering of* X. Observe that usually $\hat{X}$ is not an algebraic variety. Also, usually $\tilde{X} \neq \hat{X}$ [Tol, CatKo].

The above notions can easily be translated to properties of $\tilde{X}$ resp. $\hat{X}$. (2.7) makes sense for arbitrary analytic spaces and we formulate the next result in this form.

2.12. Proposition. *Let X be a connected analytic space.*

(2.12.1) *X has large fundamental group* iff *$\tilde{X}$ does not contain any positive dimensional proper complex subspaces.*

(2.12.2) *X has generically large fundamental group* iff *$\tilde{X}$ does not contain any positive dimensional proper complex subspaces containing a very general point $x \in \tilde{X}$ (i.e. $x \notin \tilde{u}^{-1}(\cup D_i)$).*

(2.12.3) *X has large algebraic fundamental group* iff *$\hat{X}$ does not contain any positive dimensional proper complex subspaces.*

(2.12.4) *X has generically large algebraic fundamental group* iff *$\hat{X}$ does not contain any positive dimensional proper complex subspaces containing a very general point $x \in \hat{X}$ (i.e. $x \notin \hat{u}^{-1}(\cup D_i)$).*

Proof. Let X' stand for $\tilde{X}$ or $\hat{X}$. Let $Y \subset X'$ be an irreducible compact complex subspace with normalisation $\bar{Y}$. The projection morphism $\bar{Y} \to X$ has discrete (hence finite) fibers. Thus Y is an algebraic variety. Let $Z \subset X$ be the image of Y. Then

$$\operatorname{im}[\hat{\pi}_1(\bar{Z}) \to \hat{\pi}_1(X)] \quad \text{resp.} \quad \operatorname{im}[\pi_1(\bar{Z}) \to \pi_1(X)]$$

is the same as the Galois group of $\bar{Y}/\bar{Z}$, in particular finite.

This shows that (up to deck transformations of $X' \to X$) there is a one-to-one correspondence between irreducible compact complex subspaces of X' and irreducible subvarieties $Z \subset X$ such that

$$\operatorname{im}[\hat{\pi}_1(\bar{Z}) \to \hat{\pi}_1(X)] \quad \text{resp.} \quad \operatorname{im}[\pi_1(\bar{Z}) \to \pi_1(X)]$$

is finite. □

2.13. Example. Let X be a small neighbourhood of the minimal resolution of the singularity $(xyz + x^4 + y^4 + z^4 = 0)$. The exceptional set E is a triangle of 3 lines. In the universal cover we obtain an infinite chain of $\mathbb{P}^1$-s. Here the difference between looking at

$$\operatorname{im}[\pi_1(E) \to \pi_1(X)] \quad \text{or} \quad \operatorname{im}[\pi_1(\bar{E}) \to \pi_1(X)]$$

is significant.

3 Construction of the Shafarevich Map

The aim of this section is to construct the Shafarevich map. We will construct a more general version, which we introduce first.

3.1. Definition. Let G be a group and let H_1, H_2 be subgroups. We say that H_1 is *essentially a subgroup of* H_2 if $H_1 \cap H_2$ has finite index in H_1. We denote this relationship by $H_1 \lesssim H_2$.

By definition, $H \lesssim \{1\}$ iff H is finite.

3.2. Definition. Let X be a normal variety. Let $H \triangleleft \pi_1(X)$ be a normal subgroup. A normal variety $\mathrm{Sh}^H(X)$ and a rational map $\mathrm{sh}_X^H: X \dashrightarrow \mathrm{Sh}^H(X)$ are called the *H-Shafarevich variety* and the *H-Shafarevich map of X* if

(3.2.1) sh_X^H has connected fibers, and

(3.2.2) there are countably many closed subvarieties $D_i \subset X$ $(D_i \neq X)$ such that for every closed, irreducible subvariety $Z \subset X$ such that $Z \not\subset \cup D_i$.

$$\mathrm{sh}_X^H(Z) = \text{point} \quad \text{iff} \quad \operatorname{im}[\pi_1(\bar{Z}) \to \pi_1(X)] \lesssim H.$$

It is easy to see that $\mathrm{sh}_X^H: X \to \mathrm{Sh}^H(X)$ is unique up to birational equivalence if it exists.

In the algebraic case we take $H \triangleleft \hat{\pi}_1(X)$ to be a closed subgroup, otherwise the definition is the same.

Clearly $\mathrm{sh}_X = \mathrm{sh}_X^{\{1\}}$.

3.2.3. Example. Let X be an Abelian variety and let $H < \pi_1(X) = H_1(X, \mathbb{Z})$ be a subgroup. Let $A \subset X$ be a maximal dimensional Abelian subvariety such that $\operatorname{im}[\pi_1(A) \to \pi_1(X)] \lesssim H$. Then sh_X^H is the quotient morphism $X \to X/A$. In particular, $\mathrm{Sh}^H(X)$ is also Abelian.

3.3. Proposition. *Let X be a normal variety and let $H \triangleleft \pi_1(X)$. Consider the following diagram:*

$$\begin{array}{ccccccc} T & \xrightarrow{i} & W & \xrightarrow{w} & Z & \xrightarrow{n} & X \\ & & {\scriptstyle p}\downarrow & & & & \\ & & V & & & & \end{array}$$

where all the schemes are irreducible and normal, $p \circ i$ and w are dominant. Let W_{gen} be the geometric generic fiber of p. Assume that

$$\operatorname{im}[\pi_1(T) \to \pi_1(X)] \lesssim H \quad \text{and} \quad \operatorname{im}[\pi_1(W_{gen}) \to \pi_1(X)] \lesssim H.$$

Then

$$\operatorname{im}[\pi_1(Z) \to \pi_1(X)] \lesssim H.$$

The same holds for $\hat{\pi}_1$.

Proof. By (2.9.1) $\operatorname{im}[\pi_1(W) \to \pi_1(Z)]$ has finite index in $\pi_1(Z)$ thus it is sufficient to show that $\operatorname{im}[\pi_1(W) \to \pi_1(X)] \lesssim H$. Replacing W by $W \times_V T$ and V by T we may assume that i is a section $s: V \cong T \to W$. Let V^0 be an open subset (or a purely inseparable cover of an open subset in char. p) and let $W^0 = p^{-1}(V^0)$. By suitable choice of V^0 we may assume that $p^0: W^0 \to V^0$ is flat with only geometrically reduced fibers and that p^0 is locally trivial in the Euclidean topology. We have a right split exact sequence

$$\pi_1(W_{\text{gen}}) \to \pi_1(W^0) \leftrightarrows \pi_1(V^0) \to 1. \tag{3.3.1}$$

Let $H_1 = \operatorname{im}[\pi_1(W_{\text{gen}}) \to \pi_1(X)]$, $H_2 = \operatorname{im}[\pi_1(V^0) \to \pi_1(X)]$, and $H_3 = \operatorname{im}[\pi_1(W^0) \to \pi_1(X)]$. Then $H_1 \lhd H_3$ and $H_3 = H_1 H_2$ by (3.3.1). $H_1 \lesssim H$ and $H_2 \lesssim H$ since $H_2 \subset \operatorname{im}[\pi_1(T) \to \pi_1(X)]$.

Let $H_i' = H_i \cap H$ and let $H_i = \sum_j b_{ij} H_i'$. Since H_1 is normal in H_3,

$$H_3 = H_1 (\cup b_{2j} H_2') = \cup b_{2j} (b_{2j}^{-1} H_1 b_{2j}) H_2' = \bigcup_{j,k} b_{2j} b_{1k} H_1' H_2'.$$

Thus $H_3 \lesssim H$.

In the algebraic case the proof is the same. The existence of (3.3.1) is assured by [SGA1, X.1.4]. □

The following result is the main step in the construction of the H-Shafarevich map:

3.4. Theorem. *Let X be a normal variety. In the algebraic case assume that X is also proper. Fix a normal subgroup $H \lhd \pi_1(X)$. Let*

$$\begin{array}{ccc} U & \xrightarrow{u} & X \\ {\scriptstyle p}\downarrow & & \\ S & & \end{array}$$

be a family of normal cycles. Assume that

(3.4.1.1) *S is irreducible,*

(3.4.1.2) *u is dominant,*

(3.4.1.3) $\operatorname{im}[\pi_1(U_s) \to \pi_1(X)] \lesssim H$,

(3.4.1.4) *$u(U_s) \neq u(U_t)$ for $s \neq t$, and*

(3.4.1.5) $\dim(U/S)$ *is the greatest possible with the above properties.*

(3.4.2) *Assume first that we are in characteristic zero. Then*

(3.4.2.1) *u is birational, and*

(3.4.2.2) *if we have another diagram satisfying* (3.4.1.1–3) (*we can even drop the assumption that $u: U_s' \to u(U_s')$ be birational*)

$$\begin{array}{ccc} U' & \xrightarrow{u'} & X \\ {\scriptstyle p'}\downarrow & & \\ S' & & \end{array} \tag{3.4.2.3}$$

then there is a unique rational map $g: S' \to S$ which completes (3.4.2.3) *to a commutative diagram*

$$\begin{array}{ccc} U' & \xrightarrow{u'} & X \\ {\scriptstyle p'}\downarrow & & \downarrow{\scriptstyle p\circ u^{-1}} \\ S' & \xrightarrow{g} & S. \end{array} \tag{3.4.2.4}$$

The same holds in the algebraic case.

(3.4.3) *Assume next that we are in positive characteristic. Then u is purely inseparable over an open set. Thus a suitable power of the Frobenius can be factored as*

$$F^m : U \xrightarrow{u} X \xrightarrow{u_m^{-1}} U .$$

In (3.4.2.4) *we need to replace* $p \circ u^{-1}$ *by* $p \circ u_m^{-1}$ *for suitable* $m \gg 1$.

Proof. We prove the assertions (3.4.2.1–3) simultaneously. In case (3.4.2.1) set $U' = U$ etc. Pick a point $x \in X$ such that $x \notin \cup D_i$ and u' is flat over x. Pick $s \in S$ such that $x \in u(U_s)$. Let T be an irreducible component of $u'^{-1}(u(U_s))$ which dominates $u(U_s)$. Let $V = p'(T)$ and $W = p'^{-1}(V)$. Let Z be the normalisation of $u'(W) \subset X$. T, V, W, Z satisfy the assumptions of (3.3). Thus $\mathrm{im}[\pi_1(Z) \to \pi_1(X)] \lesssim H$. x is in the image of $Z \to X$, hence by (2.5) there is a dominant family of normal cycles containing Z and satisfying the assumptions (3.4.1.1-4).

By construction $u(U_s)$ is contained in the closure of Z. By the assumption (3.4.1.5) this implies that $u'(W) = u(U_s)$. At the set theoretic level this means the following:

Let B be the union of the D_i and of the set over which u' is not flat. Then

$$u(U_s) \cap u'(U_t') \not\subset B \Rightarrow u'(U_t') \subset u(U_s).$$

Assume that u is not purely inseparable. Then $u^{-1}(x)$ has at least two connected components, thus there are $s \neq t \in S$ such that $x \in u(U_s) \cap u(U_t)$. Thus $u(U_t) \subset u(U_s)$ which implies $s = t$, a contradiction.

(3.4.2.2) is also clear. Pick general $t \in S'$. Then $u'(U_t')$ is contained in a unique $g(t) \in S$ such that $u(U_{g(t)})$. The correspondence $t \to g(t)$ gives a rational map $g: S' \to S$. In positive characteristic g is unique only up to a purely inseparable map; this accounts for the slightly different formulation. □

3.5. Corollary. *Let X be a normal variety.*

(3.5.1) *Assume that X is defined over $\mathbb{C}$. For any normal subgroup $H \triangleleft \pi_1(X)$ the H-Shafarevich map* $\mathrm{sh}_X^H : X \dashrightarrow \mathrm{Sh}^H(X)$ *exists.*

(3.5.2) *If X is proper (over any algebraically closed field) then for any closed normal subgroup $H \triangleleft \hat{\pi}_1(X)$ the algebraic H-Shafarevich map* $\widehat{\mathrm{sh}}_X^H : X \dashrightarrow \widehat{\mathrm{Sh}}^H(X)$ *exists.*

Proof. Choose $u: U \to X$ as in (3.4). u is birational (resp. generically purely inseparable). Thus $p \circ u^{-1}: X \to S$ is a rational map with connected fibers. (3.4) shows that $\mathrm{Sh}^H(X) = S$ and $\mathrm{sh}_X^H = p \circ u^{-1}$ satisfy the requirements.

In positive characteristic we take sh_X^H to be the Stein factorisation of $p \circ u_m^{-1}$. □

The following result describes the basic functoriality property of the Shafarevich maps.

3.6. Theorem. *Let $f: X \to Y$ be a dominant morphism between normal varieties. Let $H \triangleleft \pi_1(X)$ and $G \triangleleft \pi_1(Y)$ be normal subgroups. Assume that $f_* H \lesssim G$. Then there is* *a rational map* $\mathrm{sh}(f)\colon \mathrm{Sh}^H(X) \dashrightarrow \mathrm{Sh}^G(Y)$ *which makes the following diagram commutative*:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \mathrm{sh}_X^H \downarrow & & \downarrow \mathrm{sh}_Y^G \\ \mathrm{Sh}^H(X) & \xrightarrow{\mathrm{sh}(f)} & \mathrm{Sh}^G(Y). \end{array}$$

The same holds in the algebraic case if X *and* Y *are proper.*

Proof. Let $U_s \subset X$ be a general fiber of sh_X^H and let $V_s = f(U_s)$. By (2.9.1) $\mathrm{im}\,[\pi_1(V_s) \to \pi_1(Y)] \lesssim G$, thus V_s is contained in a fiber of sh_Y^G. This gives $\mathrm{sh}(f)$. □

3.7. Corollary. *Let* $f\colon X \to Y$ *be a finite étale morphism between normal varieties. Then* $\mathrm{Sh}(X)$ *is the normalisation of* $\mathrm{Sh}(Y)$ *in the function field of* X.

3.8. Corollary. *Let* $f\colon X \to Y$ *be a birational map between smooth and proper varieties. Then there is a birational map* $\mathrm{sh}(f)\colon \mathrm{Sh}(X) \dashrightarrow \mathrm{Sh}(Y)$ *which makes the following diagram commutative*:

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \mathrm{sh}_X \downarrow & & \downarrow \mathrm{sh}_Y \\ \mathrm{Sh}(X) & \xrightarrow{\mathrm{sh}(f)} & \mathrm{Sh}(Y). \end{array}$$

The same holds in the algebraic case.

Proof. Let Z be the normalisation of the graph of f. By (3.6) there are maps $\mathrm{sh}(\mathrm{pr}_X)$ and $\mathrm{sh}(\mathrm{pr}_Y)$ where pr_X and pr_Y are the projections. It is sufficient to prove that $\mathrm{sh}(\mathrm{pr}_X)$ and $\mathrm{sh}(\mathrm{pr}_Y)$ are birational.

It is easy to see that $(\mathrm{pr}_X)_*\colon \pi_1(Z) \to \pi_1(X)$ is an isomorphism. Let $U_s \subset X$ be a general fiber of sh_X. There is a proper modification $U_s' \to U_s$ which fits into a diagram

$$\begin{array}{ccc} U_s' & \to & Z \\ \downarrow & & \downarrow \\ U_s & \to & X \end{array}$$

which shows that

$$\mathrm{im}\,[\pi_1(U_s') \to \pi_1(Z)] = \mathrm{im}\,[\pi_1(U_s) \to \pi_1(X)].$$

Therefore U_s' is contained in a fiber of sh_Z. Thus $\mathrm{sh}(\mathrm{pr}_X)$ is birational and similarly for $\mathrm{sh}(\mathrm{pr}_Y)$. □

3.9. Remark. If $f\colon X \to Y$ is a birational map between proper and normal varieties then usually there is no natural map between $\mathrm{Sh}(X)$ and $\mathrm{Sh}(Y)$ since there is not much relationship between $\pi_1(X)$ and $\pi_1(Y)$. The problem is to relate $\pi_1(X)$ to the fundamental group of a resolution. This question will be investigated in Sect. 7.

3.10. Relative Shafarevich maps. Let $f\colon X \to Y$ be a dominant morphism with connected fibers. One can define the relative H-Shafarevich map. In (3.2) we have to add the additional assumption that $f(Z) =$ point. The existence and basic properties can be established the same way as for the absolute version. Thus we have $\mathrm{Sh}_Y^H(X) \to Y$.

Let $X_y \subset X$ be a very general fiber and let $K \triangleleft \pi_1(X_y)$ be the preimage of H under the natural map $\pi_1(X_y) \to \pi_1(X)$. Then the fiber of $\mathrm{Sh}_Y^H(X) \to Y$ over $y \in Y$ is birational to $\mathrm{Sh}^K(X_y)$.

4 Properties of the Shafarevich map

The following theorem asserts that the Shafarevich map is defined on a large open set:

4.1. Theorem. *Let X be a normal and proper variety and let $\mathrm{sh}_X^H : X \dashrightarrow \mathrm{Sh}^H(X)$ be the H-Shafarevich map. There is an open subset $X^0 \subset X$ such that $\mathrm{sh}_X^H | X^0$ is proper.*

The same holds in the algebraic case.

Proof. Let $u : U \to X$ be as in (3.4). Let $E \subset U$ be the union of positive dimensional fibres of u. Assume that $p : E \to S$ is dominant.

Let $s \in S$ be a general point. By assumption there is an $x \in X$ and an irreducible curve $T \subset u^{-1}(x)$ such that $s \in p(T) \subset E$ is one dimensional. Apply (3.3) to $T = T$, $V = p(T)$, $W = p^{-1}(p(T))$ and $Z =$ the normalisation of $u(W)$. By construction $\dim Z = \dim(U/S) + 1$, $\mathrm{im}[\pi_1(Z) \to \pi_1(X)] \lesssim H$ and the image of Z is not contained in $\cup D_i$. This is a contradiction, thus there is an open subset $S^0 \subset S$ such that $u : p^{-1}(S^0) \to X$ is quasifinite. Let $X^0 = u(p^{-1}(S^0))$. Then $\mathrm{sh}_X^H | X^0 : X^0 \to S^0$ is a proper morphism.

The algebraic case can be proved the same way. □

4.1.1. Remark. If X is not proper the above proof shows that there is an open $X^0 \subset X$ such that the fibers of $\mathrm{sh}_X^H | X^0$ are closed in X.

4.2. Corollary. *Let X be a smooth and proper variety. Assume that* rank $\mathrm{Pic}(X) = 1$. *If $\pi_1(X)$ is infinite then X has generically large fundamental group. If $\hat{\pi}_1(X)$ is infinite then X has generically large algebraic fundamental group.*

Proof. Assume the contrary and let $\mathrm{sh}_X : X \dashrightarrow \mathrm{Sh}(X)$ be the Shafarevich map. Let $D' \subset \mathrm{Sh}(X)$ be an effective divisor and let $D \subset X$ be the closure of its pull back. D is disjoint from the general fiber of sh_X, thus D is not ample. This is impossible since rank $\mathrm{Pic}(X) = 1$. □

4.2.1. Remark. It is possible that under the above assumptions X has large fundamental group. This is indeed the case if $\mathrm{Sh}(X)$ exists and is projective.

The Albanese morphism is a special case of H-Shafarevich maps:

4.3. Proposition. *Let X be a smooth and proper variety. Let* $\mathrm{alb} : X \to A(X) \to \mathrm{Alb}(X)$ *be the Stein factorisation of the Albanese morphism and let $H = [\pi_1(X), \pi_1(X)]$ be the commutator subgroup. Then* $\mathrm{Sh}^H(X) = A(X)$.

Proof. $\pi_1(\mathrm{Alb}(X)) = (\pi_1(X)/H)/(\text{torsion})$ and an Abelian variety has large fundamental group. Thus for an irreducible $Z \subset X$, $\mathrm{im}[\pi_1(\bar{Z}) \to \pi_1(X)] \lesssim H$ iff $\mathrm{alb}_X(Z) = \text{point}$. □

4.3.1. Remark. Assume that $\pi_1(X)$ is Abelian. Then $A(X) = \mathrm{Sh}(X)$.

One would like to compare the fundamental group of X with the fundamental group of $\mathrm{Sh}(X)$. This question does not make sense at the moment since $\mathrm{Sh}(X)$ is only a birational equivalence class and the fundamental group is not a birational invariant for normal varieties. We can however restrict our attention to smooth models of $\mathrm{Sh}(X)$ and define $\pi_1(\mathrm{Sh}(X))$ as the fundamental group of a smooth model. If X is not smooth then in general we do not have any map from $\pi_1(X)$ to $\pi_1(\mathrm{Sh}(X))$, thus let us assume that X itself is smooth. By choosing a suitable model we may assume that $\mathrm{sh}_X: X \to \mathrm{Sh}(X)$ is a morphism. In general the fundamental group of $\mathrm{Sh}(X)$ can be very small:

4.4. Example. Let C be a hyperelliptic curve with the involution τ and let S be a K3 surface with a fixed point free involution σ. Let $X = C \times S/(\tau, \sigma)$. The Shafarevich map is the natural morphism $X \to C/\tau \cong \mathbb{P}^1$ and the general fiber is S. Thus $\mathrm{im}[\pi_1(S) \to \pi_1(X)] = 1$ and $\pi_1(\mathrm{Sh}(X)) = 1$ but $\pi_1(X)$ is infinite.

In many cases the above pathology can be eliminated by taking a finite étale cover:

4.5. Theorem. *Let X be a smooth and proper variety over $\mathbb{C}$. Assume that $\pi_1(X)$ is residually finite (i.e., the intersection of all finite index subgroups is $\{1\}$). Then there is a finite étale cover $X' \to X$ such that*

(4.5.1) $\mathrm{sh}: \pi_1(X') \to \pi_1(\mathrm{Sh}(X'))$ *is an isomorphism, and*

(4.5.2) $\mathrm{Sh}(X')$ *has generically algebraic large fundamental group.*

We formulate separately the algebraic case:

4.5′. Theorem. (char = 0) *Let X be a smooth and proper variety. There is a finite étale cover $X' \to X$ such that*

(4.5′.1) $\widehat{\mathrm{sh}}_*: \hat{\pi}_1(X') \to \hat{\pi}_1(\widehat{\mathrm{Sh}}(X'))$ *is an isomorphism, and*

(4.5′.2) $\widehat{\mathrm{Sh}}(X')$ *has generically large algebraic fundamental group.*

More generally, for every $H \lhd \hat{\pi}_1(X)$ there is a finite étale cover $X' \to X$ such that

(4.5′.3) $\widehat{\mathrm{Sh}}^{H'}(X')$ *has generically large algebraic fundamental group, where $H' = H \cap \hat{\pi}_1(X')$.*

We will prove a more general result about arbitrary morphisms. First some notation.

4.6. Definition. Let $H < G$. By $\bar{H} < G$ we denote the *closure of H* in the profinite topology of G (i.e., $\bar{H}$ is the intersection of all finite index subgroups of G which contain H).

4.7. Notation. Let $f: X \to Y$ be a dominant morphism between proper varieties. Let $G \subset \pi_1(X)$ be a subgroup of finite index and let $X(G) \to X$ be the corresponding étale cover. Let $Y(G)$ be the normalisation of Y in $X(G)$. Let $f(G): X(G) \to Y(G)$ and $q(G): Y(G) \to Y$ be the natural morphisms.

4.8. Theorem. (char = 0) *Let $f: X \to Y$ be a dominant morphism between smooth and proper varieties with connected general fiber F. Let $H = \mathrm{im}[\pi_1(F) \to \pi_1(X)]$. Then*

(4.8.1) *There is a finite index normal subgroup $H < \Gamma \lhd \pi_1(X)$ such that*

$$\ker[\pi_1(X(\Gamma)) \to \pi_1(Y(\Gamma))] \subset \bar{H}.$$

(4.8.2) *If $H = \bar{H}$ then*

$$\pi_1(F) \to \pi_1(X(\Gamma)) \to \pi_1(Y(\Gamma)) \to 1 \text{ is exact.}$$

4.8.3. Remark. The condition $H = \bar{H}$ is satisfied in two important cases:

(4.8.3.1) In the algebraic case H is closed since any algebraic fundamental group is compact.

(4.8.3.2) If $\pi_1(X)$ is residually finite and H is finite.

Proof. By (2.9.3) $H \triangleleft \pi_1(X)$, hence $\bar{H}$ is the intersection of all finite index normal subgroups of $\pi_1(X)$ which contain H.

Let $W \subset Y$ be the open subset over which f is smooth. By blowing up we may assume that $Y - W = \cup B_j$ is a divisor with normal crossings only.

Let $G \triangleleft \pi_1(X)$ be a normal subgroup of finite index. If $H < G$ then $\deg(Y(G)/Y) = \deg(X(G)/X)$. $q(G)$ is étale over W thus it ramifies only along $\cup B_j$. In particular $Y(G)$ has only quotient singularities for every G.

The main technical lemma is the following:

4.8.4. Lemma. *Notation as above. Let $f^{-1}(B_j) = \sum_i b_{ij} B_{ij}$. Let $q_j(G)$ be the ramification index of $q(G)$ over the generic point of B_j.*

Then $q_j(G) | b_{ij}$ for every i, j.

Proof. Fix i, j. Let $\Delta \subset Y$ be a small disc transversal to B_j at a general point. Let $D \subset f^{-1}(\Delta)$ be a neighborhood of a general point of $B_{ij} \cap f^{-1}(\Delta)$. Let Δ' be a component of $q(G)^{-1}(\Delta)$. Choose coordinates $x_1, \ldots, x_k$ on D, z on Δ and z' on Δ' such that $f|D$ and $q(G)|\Delta'$ are given by

$$z = x_1^{b_{ij}} \quad \text{and} \quad z = (z')^{q_j(G)}.$$

By assumption $X(G) \to X$ is étale which means that

$$\overline{D \times_\Delta \Delta'} \to D$$

is étale where $\overline{}$ denotes normalisation. Explicit computation yields that $q_j(G)/b_{ij}$. □

4.8.5. Lemma. *Notation as above. Let $G_1, G_2 \triangleleft \pi_1(X)$ be two normal subgroups. If $G_1 < G_2$ then $q_j(G_1) \geqq q_j(G_2)$.*

Proof. This is clear since $q(G_1)$ factors as $q(G_1)\colon Y(G_1) \to Y(G_2) \to Y$. □

4.8.6. Corollary. *Notation as above. There is a finite index normal subgroup $H < G^1 \triangleleft \pi_1(X)$ such that for every finite index normal subgroup $H < G \triangleleft G^1$ the induced morphism $Y(G) \to Y(G^1)$ is étale in codimension one.*

Proof. Choose $H < G^1$ such that $q_j(G^1)$ is the maximal possible for every j. □

Back to the proof of (4.8). Let G^1 be as in (4.8.6) and let $\cup V_j = \operatorname{Sing} Y(G^1)$ be a Whitney stratification. For each stratum pick a transversal slice $0 \in T_j$ and let L_j be the fundamental group of $T_j - \{0\}$. As we remarked earlier, $Y(G^1)$ has quotient singularities which implies that every L_j is finite.

Given a normal subgroup $H < G \triangleleft G^1$ the resulting cover $Y(G) \to Y(G^1)$ induces covers of $T_j - \{0\}$ corresponding to normal subgroups $L_j(G) \triangleleft L_j$. If $G_2 < G_1$ then $L_j(G_2) < L_j(G_1)$. Choose $H < G^2 < G^1$, $G^2 \triangleleft \pi_1(X)$ such that every $L_j(G^2)$ is the smallest possible. Thus if $H < G < G^2$ then $Y(G) \to Y(G^2)$ is étale everywhere.

Look at the diagram

$$\begin{array}{ccc} X(G) & \xrightarrow{p} & X(G^2) \times_{Y(G^2)} Y(G) \\ \downarrow & & \downarrow \\ X(G^2) & = & X(G^2). \end{array}$$

As we noted at the beginning, $\deg(Y(G)/Y(G^2)) = \deg(X(G)/X(G^2))$, thus p has degree one and it is étale, hence an isomorphism. Thus every étale Galois cover of $X(G^2)$ which is trivial on F is obtained from a Galois étale cover of $Y(G^2)$ via base change. Therefore every finite index normal subgroup $H < G \triangleleft G^2$ is the preimage of a finite index normal subgroup of $\pi_1(Y(G^2))$. Set $\Gamma = G^2$. □

4.9. Remark. The proof has three problems in positive characteristic. First of all we need resolution of singularities. In (4.8.4) one would have to conisider the possibility of wild ramification. Also, the singularities of $Y(G)$ may not be quotient.

4.10. Proof of (4.5) Let us prove the algebraic version.

Let U_s be a general fiber of sh_X. By assumption $\mathrm{im}[\hat{\pi}_1(U_s) \to \hat{\pi}_1(X)]$ is finite, thus there is a finite index normal subgroup $G \subset \hat{\pi}_1(X)$ such that

$$G \cap \mathrm{im}[\hat{\pi}_1(U_s) \to \hat{\pi}_1(X)] = 1.$$

Replacing X with its cover corresponding to G we are reduced to the situation when $\mathrm{im}[\hat{\pi}_1(U_s) \to \hat{\pi}_1(X)] = 1$.

Let $\Gamma \triangleleft \pi_1(X)$ be as in (4.8). By (3.7) $\widehat{\mathrm{Sh}}(X(\Gamma))$ is birational to $Y(\Gamma)$ and we will see in (7.8) that $\hat{\pi}_1(Y(\Gamma)) \cong \hat{\pi}_1(\widehat{\mathrm{Sh}}(X(\Gamma)))$ since $Y(\Gamma)$ has only quotient singularities.

Let $F \subset X$ be a general fiber of the H-Shafarevich map. By (4.8) we can choose $X' \to X$ such that

$$\hat{\pi}_1(F) \to \hat{\pi}_1(X') \to \hat{\pi}_1(\widehat{\mathrm{Sh}}^{H'}(X')) \to 1$$

is exact where $H' = H \cap \hat{\pi}_1(X')$. I claim that $\widehat{\mathrm{Sh}}^{H'}(X')$ has generically large algebraic fundamental group. Let $Z \subset \widehat{\mathrm{Sh}}^{H'}(X')$ be a subvariety containing a very general point. Let $Z' = (\widehat{\mathrm{sh}}^{H'}_{X'})^{-1}(Z)$. Then

$$\frac{\mathrm{im}[\hat{\pi}_1(\bar{Z}') \to \hat{\pi}_1(X')]}{\mathrm{im}[\hat{\pi}_1(F) \to \hat{\pi}_1(X')]} = \mathrm{im}[\hat{\pi}_1(\bar{Z}) \to \hat{\pi}_1(\widehat{\mathrm{Sh}}^{H'}(X'))].$$

The left hand side is infinite by the definition of the H'-Shafarevich map. Thus $\widehat{\mathrm{Sh}}^{H'}(X')$ has generically large algebraic fundamental group. □

4.11. Example. This example shows that (4.5) fails if $\pi_1(X)$ is not residually finite.

Let X be a smooth projective variety of dimension n with fundamental group Γ. Let L be a very ample line bundle on X and let $c(L) \in H^2(X, \mathbb{Z}_r)$ be the mod r reduction of $c_1(L)$. $c(L)$ corresponds to a central extension

$$0 \to \mathbb{Z}_r \to \Delta \to \Gamma \to 1.$$

Choose a finite morphism $g: X \to \mathbb{P}^n$ such that $g^*\mathcal{O}(1) = L$. Let $r: \mathbb{P}^n \to \mathbb{P}^n$ be the rth power map $(x_0: \dots : x_n) \mapsto (x_0^r: \dots : x_n^r)$. Let $X_r = X \times_{(g,r)} \mathbb{P}^n$. As in [Cat Tov] one can show that $\pi_1(X_r) \cong \Delta$.

Let S be a surface and L_S a line bundle on S. Let $X = S \times \mathbb{P}^{n-2}$ and pick L such that

$$c(L) = (c(L_S), 0) \in H^2(S, \mathbb{Z}_r) \times H^2(\mathbb{P}^{n-2}, \mathbb{Z}_r) \cong H^2(X, \mathbb{Z}_r).$$

Thus we obtain $f: X_r \to S$. If $n \geq 3$ then f has connected fibers. Let $F \subset X_r$ be the general fiber. By construction $\mathrm{im}[\pi_1(F) \to \pi_1(X_r)] \subset \mathbb{Z}_r$. Therefore, if S has generically large fundamental group, then $f = \mathrm{sh}_{X_r}$.

For suitable choices of S and L_S (see [CatKo]) every finite index subgroup of Δ contains $\mathbb{Z}_r$. Thus for any finite étale cover $X' \to X_r$ the kernel of the induced map $\pi_1(X') \to \pi_1(\mathrm{Sh}(X'))$ is exactly $\mathbb{Z}_r$.

5 Fundamental group and the Kodaira dimension

The aim of this section is to describe some relationships between the Kodaira dimension and the fundamental group of a variety. All varieties considered in this section are defined over $\mathbb{C}$. The results remain true over any field of characteristic zero.

Sometimes it will be necessary to assume various results from Mori's program which are known at the moment only in dimension three. See e.g. [Ko2] or [Ko et al.] for introductions.

Let X be a smooth projective variety. Its basic algebro-geometric invariant is the Kodaira dimension denoted by $\kappa(X)$. If X is covered by rational curves (i.e. uniruled) then $\kappa(X) = -\infty$ and conjecturally the converse also holds.

Since $\pi_1(\mathbb{P}^1) = 1$ we can expect that varieties with lots of rational curves have small fundamental groups. Our first aim is to see that this is indeed the case.

5.1. Definition. Let X be a smooth proper variety. We say that X is *rationally connected* if any two points can be connected by an irreducible rational curve. See [KoMiMo, 2.1] for further equivalent conditions.

In characteristic zero it is known that if X is rationally connected then $h^i(X, \mathcal{O}_X) = 0$ for $i > 0$ and X is simply connected [Cam1, 3.4–5; KoMiMo, 2.5].

5.2. Theorem. *Let $f: X \to Y$ be a dominant morphism between smooth and proper varieties with connected fibers. Assume that general fibers are rationally connected. Then $f_*: \pi_1(X) \to \pi_1(Y)$ is an isomorphism.*

Proof. By choosing suitable birational models we may asume that the following additional assumptions are satisfied:

Let $Y^s \subset Y$ be the open set over which f is smooth. Then $D = Y \setminus Y^s$ is a normal crossing divisor and $E = f^{-1}(D)$ is also a normal crossing divisor.

Let $D^s \subset D$ be the dense open set over which $f|E$ is semi-smooth (i.e. it is smooth on the irreducible components of E, on the irreducible components of the double locus, triple locus etc.). Let $Y^0 = Y^s \cup D^s$ and let $X^0 = f^{-1}(Y^0)$.

5.2.1. Lemma. *Every fiber of $f^0: X^0 \to Y^0$ is simply connected.*

Using (5.2.1) let us finish the proof of (5.2). The lemma implies that $\pi_1(X^0) \cong \pi_1(Y^0)$. Since $Y \setminus Y^0$ has codimension two, $\pi_1(Y) \cong \pi_1(Y^0)$ and by (2.9.1) we have a surjection $\pi_1(X^0) \twoheadrightarrow \pi_1(X)$. Putting these together we obtain

$$\pi_1(X^0) \twoheadrightarrow \pi_1(X) \xrightarrow{f_*} \pi_1(Y) \cong \pi_1(Y^0),$$

and the two ends are isomorphic. This shows (5.2).

In order to prove (5.2.1) we need two further results which are very useful in many other situations as well.

5.2.2. Lemma. *Let $f: X \to \Delta$ be a proper morphism from a normal analytic space to a small disc. By shrinking Δ we may assume that f is a locally topologically* *trivial fibration over $\Delta - 0$. Let X_g be the fiber over a point $\eta \in \Delta - 0$. Let $T: \pi_1(X_g) \to \pi_1(X_g)$ be the monodromy representation obtained from going around in $\Delta - 0$. Let $X_0 = \sum m_i X_0^i$ be the central fiber and let r be the gcd of the m_i. Then there is an exact sequence (unique up to monodromy)*

$$\pi_1(X_g) \xrightarrow{s} \pi_1(X) \to \mathbb{Z}_r.$$

Proof. Let $D^i \subset X$ be a disc transversal to X_0^i at a general point. m_i is the degree of $D^i \to \Delta$. Choose a triangulation of X which induces a triangulation of X_0 and D^i. Use this triangulation to obtain X_0 as a retract of X. This gives a continuous map $t: X \setminus X_0 \to X_0$. The general fiber of t over X_0^i is connected (the punctured disc D^i) and all the fibers are also connected since $X \setminus X_0$ is connected in the neighborhood of any point of X_0 (X is normal). Thus we have a surjection $\pi_1(X \setminus X_0) \to \pi_1(X_0)$. $X \setminus X_0$ is a fiber bundle over a punctured disc which gives an exact sequence

$$\pi_1(X_g) \to \pi_1(X \setminus X_0) \to \mathbb{Z} \to 1. \tag{5.2.2.1}$$

Let $t \in \pi_1(X \setminus X_0)$ be a lifting of $1 \in \mathbb{Z}$. Let $\gamma_i \in \pi_1(X \setminus X_0)$ be the element corresponding to a loop in D^i. Then there are $p_i \in \pi_1(X_g)$ such that $t^{m_i} = \gamma_i p_i$. By construction $s(\gamma_i) = 1$ thus $s(t)^{m_i} \in s(\pi_1(X_g))$ for every i. This implies that $s(t)^r \in s(\pi_1(X_g))$. □

5.2.3. Lemma [St, 2.14] *Let X be a complex space and let $D \subset X$ be a divisor such that D as a complex space is proper and has normal crossings only. Assume that D is a retract of X (topologically). Then the restriction maps $h^i(X, \mathcal{O}_X) \to h^i(D, \mathcal{O}_D)$ are surjective.*

Proof. This is what the proof of [St, 2.14] gives. Observe that it is sufficient to assume that D has DuBois singularities. □

Proof of (5.2.1) Pick $y \in Y^0$. If $y \in Y^s$ then X_y is rationally connected [KoMiMo, 2.4] thus it is simply connected. If $y \in D^s$ then let $\Delta \subset Y^0$ be a small disc transversally intersecting D^s at y. After base change we have a morphism $f': X' \to \Delta$ such that X' is smooth, $X'_y = X_y$ is a normal crossing divisor and the general fiber of f' is rationally connected. $R^i f'_* \mathcal{O}_{X'}$ is torsion free (see e.g. [Ko1, Step 6 on p. 20]) thus it is zero for $i > 0$. By (5.2.3) this implies that $h^i(\operatorname{red} X_y, \mathcal{O}) = 0$ for $i > 0$ and $\chi(\operatorname{red} X_y, \mathcal{O}) = 1$.

From (5.2.2) we conclude that $\pi_1(X') \cong \pi_1(X_0)$ is finite cyclic, say of order r. Let $X'' \to X'$ be the universal cover and let $f'': X'' \to \Delta''$ be the induced morphism with connected fibers. The general fiber is unchanged, so the same argument as before shows that $\chi(\operatorname{red} X''_y, \mathcal{O}) = 1$. $\operatorname{red} X''_y \to \operatorname{red} X'_y$ is étale of degree r which shows that $r = 1$. □

5.3. Corollary. *Let X' be a smooth and proper variety. There is a variety X birational to X' and a dominant morphism with connected fibers $g: X \to Z$ onto a smooth proper variety Z such that*

(5.3.1) *Z is not uniruled, and*
(5.3.2) *$g_*: \pi_1(X) \to \pi_1(Z)$ is an isomorphism.*

Proof. If X' is not uniruled then take $X = Y = X'$. Otherwise by [Cam2; KoMiMo, 2.7] there is a morphism $g: X \to Y$ as in (5.2). Continue with Y replacing X'. The dimension of the target drops at each step so eventually we stop with some $g: X \to Z$. □

The next step is to consider varieties with $\kappa = 0$. At least conjecturally the situation is again very simple:

5.4. Conjecture. *Let X be a smooth and proper variety with $\kappa(X) = 0$. Then:*

(5.4.1) *X has a finite étale cover X' which is birational to the product of a simply connected and of an Abelian variety.*

(5.4.2) *$\pi_1(X)$ has a finite index Abelian subgroup.*

5.5 Corollary – Conjecture. *Let X be a smooth proper variety with $\kappa(X) = 0$. The following are equivalent:*

(5.5.1) *X has a generically large algebraic fundamental group,*

(5.5.2) *X has a generically large fundamental group,*

(5.5.3) *X has a finite étale cover which is birational to an Abelian variety.*

(5.6). Implications between the conjectures.

(5.6.1) Clearly (5.4.1) $\Rightarrow$ (5.4.2). The converse seems more delicate:

5.6.2. Claim. *Let X be a projective variety (smooth or with terminal singularities) such that $\kappa(X) = 0$. Assume that $\pi_1(X)$ is Abelian. Then*

(5.6.2.1) *The Albanese morphism* $\mathrm{alb}: X \to \mathrm{Alb}(X)$ *is surjective and coincides with the Shafarevich map.*

(5.6.2.2) *If X has a generically large fundamental group, then X is birational to an Abelian variety.*

(5.6.2.3) *If X has terminal singularities and K_X is numerically trivial then X has a finite étale cover X' which is isomorphic to the product* $\mathrm{Alb}(X') \times Z$ *where Z has terminal singularities, $K_Z = 0$ and Z is simply connected.*

Proof. Let $\mathrm{alb}_X: X \to \mathrm{Alb}(X)$ be the Albanese morphism. Since $\kappa(X) = 0$ [Ka1] (or see (10.1)) implies that alb_X is surjective with connected fibers. This shows the first two claims.

The third one is a consequence of [Ka2]. □

(5.6.3) Clearly (5.5.3) $\Rightarrow$ (5.5.1) $\Rightarrow$ (5.5.2) is true in all dimensions. (5.6.2.2) shows that if (5.4.2) holds then (5.5.2) $\Rightarrow$ (5.5.3).

Finally let us look at varieties with positive Kodaira dimension. As usual, varieties of general type are too general to deal with. We will be interested in the case $0 < \kappa(X) < \dim X$. By the Iitaka fibration theorem X is birational to a smooth variety X' such that there is a morphism with connected fibers $\phi: X' \to I(X)$ such that for very general $z \in I(X)$ the fiber $X_z = \phi^{-1}(z)$ satisfies

$$\dim X_z = \dim X - \kappa(X) \quad \text{and} \quad \kappa(X_z) = 0.$$

Therefore by (2.10.3) if $\dim X_z \leqq 2$ (and conjecturally always) X_z has a finite étale cover which is birational to an Abelian variety.

5.7. Definition. An Abelian scheme over a variety Y is a proper and smooth morphism $f: X \to Y$ with a section $s: Y \to X$ such that every fiber is an Abelian variety.

Every such $X \to Y$ is a C^∞-fiber bundle with fiber $\mathbb{R}^{2g}/\mathbb{Z}^{2g}$, but the complex structure of the fibers may vary.

5.8. Theorem. *Let X be a smooth projective variety. Assume that X has generically large fundamental group and $\kappa(X) \geqq \dim X - 2$. Then X has a finite étale cover $X' \to X$ such that*

(5.8.1) *$X' \to I(X')$ is birational to an Abelian scheme over a proper variety,*

(5.8.2) *$I(X')$ is of general type,*

(5.8.3) *(a smooth model of) $I(X')$ has generically large fundamental group.*

5.8.4. Remark. If $\kappa(X) = \dim X - 1$ then X' is birational to the product of $I(X')$ and of an elliptic curve. For $\kappa(X) \leqq \dim X - 2$ there are other examples too (6.2.1).

Proof. The hardest part is (5.8.1) and we postpone its proof to the next section. Assuming it, the other statements follow from the next result about Abelian schemes:

5.9. Proposition. *Let $f\colon X \to Y$ be an Abelian scheme over a smooth and proper variety Y. Then*

(5.9.1) *$\kappa(X) = \kappa(Y)$, and*

(5.9.2) *X is birational to $\mathrm{Sh}(X) \times_{\mathrm{Sh}(Y)} Y$.*

Proof. We may replace Y with a suitable finite étale cover, hence we may assume that $X \to Y$ admits a level three structure. The main point is that there is a universal family $U_3 \to \mathscr{A}_3$ over the moduli space of Abelian varieties with level three structure. (We omitted the dimension and the polarisation from the subscripts.)

Let $u\colon Y \to \mathscr{A}_3$ be the induced morphism, let Z be a desingularisation of the image and let $U_Z \to Z$ be the pull back of U_3 to Z.

Let us prove first (5.9.2). $\mathscr{A}_3$ is the quotient of the Siegel upper half space, thus Z has generically large fundamental group. Thus there is a map $\mathrm{Sh}(Y) \to Z$. By choosing suitable birational models we may assume that $Y \to \mathrm{Sh}(Y) \to Z$ are all morphisms. Since $X = Y \times_Z U_Z$ we obtain a dominant morphism $X \to \mathrm{Sh}(Y) \times_Z U_Z$. I claim that this is the Shafarevich map of X. Let $F \subset Y$ be a general fiber of $Y \to \mathrm{Sh}(Y)$. Then $f^{-1}(F) \subset X$ is the direct product of F and of an Abelian variety A_F. Let $F_1 \subset f^{-1}(F)$ be a horizontal section (isomorphic to F). We need to show that $\mathrm{im}\,[\pi_1(F_1) \to \pi_1(X)]$ is finite. Look at the commutative diagram with exact rows:

$$\begin{array}{ccccccc} 0 \to & \pi_1(A_F) & \to & \pi_1(X) & \to & \pi_1(Y) \to 1 \\ & \cong\downarrow & & \downarrow & & \downarrow \\ 0 \to & \pi_1(A_F) & \to & \pi_1(\mathrm{Sh}(Y) \times_Z U_Z) & \to & \pi_1(\mathrm{Sh}(Y)) \to 1. \end{array}$$

F_1 is mapped to a point by $X \to \mathrm{Sh}(Y) \times_Z U_Z$, thus

$$\mathrm{im}\,[\pi_1(F_1) \to \pi_1(X)] \cap \mathrm{im}\,[\pi_1(A) \to \pi_1(X)] = \{1\}.$$

By assumption $\mathrm{im}\,[\pi_1(F_1) \to \pi_1(Y)]$ is finite, hence so is $\mathrm{im}\,[\pi_1(F_1) \to \pi_1(X)]$. Thus $\mathrm{Sh}(X) \sim \mathrm{Sh}(Y) \times_Z U_Z$ and

$$\mathrm{Sh}(X) \times_{\mathrm{Sh}(Y)} Y \sim U_Z \times_Z \mathrm{Sh}(Y) \times_{\mathrm{Sh}(Y)} Y \sim U_Z \times_Z Y \sim X.$$

We use the same notation for (5.9.1). By (5.9.3) Z has general type. Let $z \in Z$ be a very general point. By [Ka1, V] we see that

$$\kappa(X) = \kappa(X_z) + \dim Z \quad \text{and} \quad \kappa(Y) = \kappa(Y_z) + \dim Z,$$

where X_z and Y_z denote a very general fiber. As was noted above, $X_z = Y_z \times A_z$ for some Abelian variety A_z, thus $\kappa(X_z) = \kappa(Y_z)$. $\square$

5.9.3. Lemma. *Let Z be a desingularisation of a proper subvariety of $\mathscr{A}_3$. Then Z is of general type.*

Proof. I thank D. Toledo for explaining this simple proof to me.

More generally, let X be a quotient of a bounded symmetric domain H and let $g:Z \to M \subset X$ be a desingularisation of a compact subvariety. Let h be the invariant Hermitian metric on the holomorphic tangent bundle of H and let

$$\Theta : T_H \to \Omega^{1,1} \otimes T_H$$

be the curvature tensor. For a holomorphic tangent vector u set $\Theta(u) = (\Theta u, \bar{u})$ which is a (1, 1) covector. Θ is Griffiths seminegative and H has negative holomorphic sectional curvatures. I.e., for any two nonzero holomorphic tangent vectors u, v

$$\sqrt{-1}\,\Theta(u)(v, \bar{v}) \leqq 0 \quad \text{and} \quad \sqrt{-1}\,\Theta(u)(u, \bar{u}) < 0.$$

Both of these properties are inherited by the tangent bundle of any submanifold $M \subset X$. We are interested in the canonical bundle K_M, which is the determinant of the cotangent bundle. Its curvature Θ_K is minus the trace of the curvature tensor Θ. Given u, fix an orthonormal basis $v_1 = u, v_2, \ldots$ of the holomorphic tangent vectors. Then

$$\sqrt{-1}\,\Theta_K(u, \bar{u}) = -\sum_i \sqrt{-1}\,\Theta(v_i)(u, \bar{u}) > 0.$$

Thus if $M \subset X$ is a compact and smooth subvariety then K_M is even ample. In general by choosing Z suitably we can achieve that

$$T'_Z \overset{\text{def}}{=} \text{saturation of } \operatorname{im}[T_Z \to g^* T_X]$$

is a subbundle and the above computations yield that $\det^{-1} T'_Z$ is nef and big on Z. It is a subsheaf of $\mathcal{O}(K_Z)$, hence $\mathcal{O}(K_Z)$ itself is big. □

6 Fiber spaces of abelian varieties

The aim of this section is to prove (5.8.1) in a more general form.

6.1. Definition. Let $Y \subset X$ be a closed and irreducible subvariety. We say that X has generically large fundamental group on Y (resp. X has generically large algebraic fundamental group on Y) if the following condition is satisfied:

If $x \in Y$ is a sufficiently general point and $x \in Z \subset Y$ is an irreducible positive dimensional subvariety then

$$\operatorname{im}[\pi_1(\bar{Z}) \to \pi_1(X)] \quad (\text{resp. } \operatorname{im}[\hat{\pi}_1(\bar{Z}) \to \hat{\pi}_1(X)]) \text{ is infinite.}$$

We will be especially interested in the case when Y is a general fiber of a morphism $X \to X'$.

6.2. Examples. (6.2.1) Let $h: U \to V$ be a smooth morphism between smooth and proper varieties. Assume that h has a section $s: V \to U$ and that every fiber of h is an Abelian variety of dimension g (i.e. U/V is an Abelian group scheme over a proper base).

There is a split exact sequence

$$0 \to \mathbb{Z}^{2g} \to \pi_1(U) \leftrightarrows \pi_1(V) \to 1.$$

$\pi_1(V)$ acts on $\mathbb{Z}^{2g}$ by conjugation, which is the same as the monodromy action on $H_1(A_v, \mathbb{Z})$ for a smooth fiber A_v. If $Z \subset A_v$ then

$$\operatorname{im}[\pi_1(\bar{Z}) \to \pi_1(A_v)] = \operatorname{im}[\pi_1(\bar{Z}) \to \pi_1(U)].$$

By (4.3) U has generically large fundamental group on A_v.

If the monodromy representation in $\operatorname{Aut}(H_1(A_v, \mathbb{Z}))$ has a finite image, then U becomes a product after a finite étale cover. This is always the case if $g = 1$. However for $g \geqq 2$ there are many examples where the fibers of h have variable moduli:

Let Univ $\to \mathscr{A}_{g,3}$ be the universal family over the moduli space of Abelian varieties of dimension g with a level three structure (and with some polarisation). (See [Ch] for a good introduction to these notions.) Let $\bar{\mathscr{A}}_{g,3}$ be the Satake compactification. $\bar{\mathscr{A}}_{g,3} - \mathscr{A}_{g,3}$ has codimension g in $\bar{\mathscr{A}}_{g,3}$. Thus by taking generic hyperplane sections we obtain a proper and smooth subvariety $Z \subset \mathscr{A}_{g,3}$ ($\dim Z = g - 1$). $\text{Univ}_Z \to Z$ gives an Abelian scheme where the fibers have variable moduli.

(6.2.2) Let A be an Abelian variety and let τ_A be an automorphism of finite order m. Let E be an elliptic curve and let τ_E be a translation of order m. $\tau = (\tau_A, \tau_E)$ is a fixed point free automorphism of order m of $A \times E$. Let C be a curve with an order m automorphism σ. Let $X = A \times E \times C/(\tau, \sigma)$, $Y = C/\sigma$ and let $f\colon X \to Y$ be the induced morphism. For general $y \in Y$ the fiber X_y is isomorphic to $A \times E$. Since $A \times E \times C \to X$ is étale, X has generically large fundamental group on X_y. The monodromy around a fixed point of σ is a power of τ, thus in general it is nontrivial on $H_1(A \times E, \mathbb{Z})$.

6.3. Theorem. *Let $f\colon X \to Y$ be a dominant morphism with connected fibers between smooth and proper varieties. Let $y \in Y$ be a very general point. Assume that*

(6.3.1) *X_y has a finite étale cover which is birational to an Abelian variety, and*

(6.3.2) *X has generically large fundamental group on X_y.*

Then X has a finite étale cover $X' \to X$ such that $f'\colon X' \to Y'$ is birational an Abelian group scheme over a proper base (6.2.1), *where $X' \to Y' \to Y$ is the Stein factorisation of $X' \to X \to Y$.*

Proof. The proof will be done in several steps.

6.4. Reduction to the case of Abelian general fiber. The main point is a group theoretic result whose proof we postpone to the end.

Let $G = \pi_1(X)$ and let $H = \operatorname{im}[\pi_1(X_y) \to \pi_1(X)]$. By (2.9.3) $H \lhd G$, H is finitely generated, H has a finite index Abelian subgroup and H is infinite. Choose $G_1 < G$ as in (6.4.3.2) and take the corresponding cover $X_1 \to X$. Let $X_1 \to Y_1 \to Y$ be the Stein factorisation of $X_1 \to X \to Y$. Let X_{1y} be a general fiber of $X_1 \to Y_1$. Then

$$\operatorname{im}[\pi_1(X_{1y}) \to \pi_1(X_1)] \subset H \cap G_1,$$

thus it is Abelian. I claim that X_{1y} is birational to an Abelian variety. Let $F \subset X_{1y}$ be a general fiber of the Albanese morphism of X_{1y}. Then $\operatorname{im}[\pi_1(F) \to \pi_1(X_{1y})]$ has finite image in any commutative quotient of $\pi_1(X_{1y})$, thus it has finite image in $\pi_1(X_1)$. Therefore F is zero dimensional and X_{1y} is Abelian by (10.1). $\square$

The group theoretic results are the following:

6.4.1. Proposition. *Let G be a group and let $H \triangleleft G$ be a normal subgroup. Assume that H is finitely generated and residually finite. Let $H_1 < H$ be a finite index subgroup. Then G has a finite index subgroup $G_1 < G$ such that $H \cap G_1 < H_1 \cdot Z(H)$ where $Z(\ )$ denotes the center. Equivalently,*

$$\ker [\hat{H} \to \hat{G}] < Z(\hat{H}).$$

6.4.2. Remark. In general one can not choose G_1 such that $H \cap G_1 < H_1$. Also, it is necessary to assume that H is finitely generated.

Proof. Since H is finitely generated, finite index characteristic subgroups give a basis for the profinite topology. Thus we may assume that $H_1 < H$ is normal.

If $K < H$ is characteristic of finite index, let $q_K\colon H \to H/K$ be the quotient map. G acts on H/K by conjugation. Let $G_K \triangleleft G$ be the subgroup acting trivially. Thus

$$G_K \cap H = q_K^{-1} Z(H/K).$$

Let $h_i \in H$ be coset representatives for H_1. If $h_i \notin Z(H)$ then there is an h_i' such that $[h_i, h_i'] \neq 1$. Choose $K < H_1$ (characteristic of finite index in H) such that if $h_i \notin Z(H)$ then $[h_i, h_i'] \notin K$. Thus

$$\forall i : h_i \notin Z(H) \Rightarrow q_K(h_i) \notin Z(H/K).$$

This implies that

$$G_K \cap H = q_K^{-1} Z(H/K) < H_1 \cdot Z(H). \quad \square$$

6.4.3. Corollary. *Let G be a group and let $H \triangleleft G$ be a normal subgroup. Assume that H is finitely generated and let $H_1 < H$ be a finite index subgroup. Then*

(6.4.3.1) If *H_1 has an infinite Abelian quotient, then G has a finite index subgroup $G_1 < G$ such that $H \cap G_1$ has an infinite Abelian quotient.*

(6.4.3.2) If *H_1 is Abelian then G has a finite index subgroup $G_1 < G$ such that $H \cap G_1$ is also Abelian.*

Proof. We may assume that H_1 is a finite index characteristic subgroup. We can replace G by $G/[H_1, H_1]$, thus we may assume that H_1 is Abelian. Hence (6.4.3.2) implies (6.4.3.1). $H_1 \cdot Z(H)$ is Abelian, thus (6.4.1) implies (6.4.3.2). $\square$

From now on we assume that the general fiber of f is an Abelian variety.

6.5. Notation. (6.5.1) Let $Y^0 \subset Y$ be the open set over which f is smooth, $X^0 = f^{-1}(Y^0)$ and $f^0 : X^0 \to Y^0$ the restriction. By blowing up we may assume that $\cup D_i = Y - Y^0$ is a divisor with normal crossings.

(6.5.2) Pick a very general point $y \in Y$ and let $V = H_1(X_y, \mathbb{Q})$. Let $H = \operatorname{im}[\pi_1(X_y) \to \pi_1(X)]$ and let $H' = H/(\text{torsion})$. H' is a free Abelian group. Let $W \subset V$ be the kernel of

$$V \to H' \otimes \mathbb{Q}.$$

By (6.3.1) if $Z \subset X_y$ is a subvariety then

$$\operatorname{im}[H_1(\bar{Z}, \mathbb{Q}) \to V] \not\subset W. \tag{6.5.2.1}$$

(6.5.3) Let $T\colon \pi_1(Y^0, y) \to \operatorname{Aut}(H_1(X_y, \mathbb{Z}))$ be the monodromy representation. Let $M_{\mathbb{Z}}$ be the image of T and let $M < \operatorname{GL}(V)$ be the Zariski closure of $M_{\mathbb{Z}}$. Let $M^0 \triangleleft M$ be the connected component. By [D1, 4.2.6] M^0 is semisimple.

(6.5.4) For every D_i choose a disc Δ_i intersecting D_i transversally at a general point. Let γ_i' be a small loop in Δ_i around $D_i \cap \Delta_i$. Let δ_i be a path from y to the origin of γ_i', and set $\gamma_i = \delta_i^{-1}\gamma_i'\delta_i \in \pi_1(Y^0, y)$. Let $T_i = T(\gamma_i)$.

6.6. Finiteness of local monodromies.

6.6.1. Proposition. *Notation and assumptions as above. Then T_i has finite order for every i.*

Proof. Let us analyse the situation locally around D_i. By base change we obtain the family $X_i \to \Delta_i$ with central fiber X_{i0} and general fiber X_{ig}. As in (5.2.2) there is a semidirect product

$$0 \to \pi_1(X_{ig}) \to \pi_1(X_i \backslash X_{i0}) \to \mathbb{Z} \to 0.$$

Let t_i be a lifting of γ_i' to $\pi_1(X_i \backslash X_{i0})$. Then

$$t_i^{-1} h t_i = T_i(h),$$

where we identified $\pi_1(X_{ig})$ and $\pi_1(X_y)$ using the path δ_i. (Since $\pi_1(X_{ig})$ is commutative, it does not matter which lifting we choose.)

By (5.2.2) there is an exact sequence

$$\pi_1(X_g) \xrightarrow{s} \pi_1(X_i) \to \mathbb{Z}_{r_i},$$

thus the image of t^{r_i} in $\pi_1(X_i)$ is contained in $s(\pi_1(X_g))$. Therefore $s(t^{r_i})$ commutes with every element of $s(\pi_1(X_g))$ and hence $s(h) = s(T^{r_i}(h))$ for every $h \in \pi_1(X_g)$.

Since $\pi_1(X_g) \to \pi_1(X)$ factors through $\pi_1(X_g) \to \pi_1(X_i)$, this implies that

$$\operatorname{im}(1 - T_i^{r_i}) \subset W. \tag{6.6.2}$$

The rest is a formal argument using the already established facts and it will be used again in (6.7).

Let $M_1 \lhd M$ be the normal subgroup generated by the elements $T_i^{r_i}$. M^0 is semisimple, hence so is M_1^0. In particular V is completely reducible as an M_1-module. We can decompose it as $V = I + N$ where I is the trivial representation part and N is the nontrivial part.

6.6.3. Claim. *N is the unique smallest M-invariant subspace containing* $\operatorname{im}(1 - T_i^{r_i})$ *for every i. Thus $N \subset W$.*

Proof. $T_i^{r_i} \in M_1$, thus $\operatorname{im}(1 - T_i^{r_i}) \subset N$. Let N' be the smallest M-invariant subspace containing $\operatorname{im}(1 - T_i^{r_i})$ for every i. $T_i^{r_i}$ acts trivially on V/N', and so do all conjugates and the group they generate. By complete reducibililty this implies that $N = N'$. □

By [D2, 4.2.8] $N \subset V = H_1(X_y, \mathbb{Q})$ is a sub Hodge structure. Thus there is an Abelian subvariety $B \subset X_y$ such that

$$\operatorname{im}[H_1(B, \mathbb{Q}) \to V] \subset N \subset W.$$

By (6.5.2.1) this implies that $N = 0$, thus $T_i^{r_i} = 1$ for every i. □

6.7. After an étale cover, local monodromies become trivial. Fix $m \geqq 3$. $mH < H$ is a characteristic subgroup, thus $\pi_1(X)$ acts by conjugation on H/mH. Let $G_1 \lhd \pi_1(X)$ be the subgroup that acts trivially. $H < G_1$ since H is Abelian. By taking the corresponding étale cover of X we are reduced to the situation when $\pi_1(Y^0)$

and so $M_{\mathbb{Z}}$ acts trivially on H/mH thus also on H'/mH'. By (6.7.1) every torsion element of $M_{\mathbb{Z}}$ acts trivially on H'.

By (6.6), $T_i \in M_{\mathbb{Z}}$ has finite order, thus it acts trivially on H'. This means that $\operatorname{im}(1 - T_i) \subset W$ for every i.

We can proceed exactly as in (6.5) to conclude that $T_i = 1$ for every i. □

6.7.1. Proposition (Minkowski) *If* $t \in \operatorname{Aut}(\mathbb{Z}^k)$ *has finite order and* t *acts trivially on* $\mathbb{Z}^k/m\mathbb{Z}^k$ *for some* $m \geqq 3$ *then* $t = \mathrm{id}$. □

6.8. The case when f has a section. Assume that $f: X \to Y$ is a dominant morphism between smooth proper varieties such that the general fiber is an Abelian variety. Assume that all local monodromies are trivial on $H_1(X_y, \mathbb{Z})$. Assume furthermore that f has a rational section $s: Y \dashrightarrow X$.

6.8.1. Claim. *Notation and assumptions as above. There is a finite étale cover* $Y' \to Y$ *such that* $X \times_Y Y' \to Y'$ *is birational to an Abelian scheme over* Y'.

Proof. Let $U \subset Y$ be an open set over which f is smooth and s is a morphism. $f: X_U \to U$ is an Abelian scheme. Let $X_U(3) \subset X_U$ be the subscheme of 3-torsion points. $X_U(3) \to U$ is a finite étale cover. The monodromy of $X_U(3)$ around a boundary point $y' \in Y - U$ is the same as the monodromy on $H_1(X_y, \mathbb{Z})/3H_1(X_y, \mathbb{Z})$ which is assumed to be trivial. Thus the normalisation of Y in $X_U(3)$ is étale over Y. Therefore we can choose a finite étale cover $Y' \to Y$ such that $X_U(3) \times_U U'$ is a union of disjoint copies of U' where $U' \subset Y'$ is the preimage of U. Thus $X_U \times_U U'$ is an Abelian scheme with a level three structure.

We obtain a morphism $U' \to \mathscr{A}_3$ which extends to $Y' \to \mathscr{A}_3$ since there are no local monodromies. Thus

$$X \times_Y Y' \quad \text{is birational to} \quad \operatorname{Univ} \times_{\mathscr{A}_3} Y'. \qquad \square$$

6.9. Completion of the proof. Fix $f: X \to Y$ such that X, Y are smooth and projective, the general fiber of f is an Abelian variety and there are no local monodromies on $H_1(X_y, \mathbb{Z})$.

We would like to apply (6.8.1) thus we need to create a section of f. We assumed that X was algebraic, thus there is an irreducible subvariety $S \subset X$ such that $f: S \to Y$ is dominant and generically finite. S is a multivalued rational section.

Let $A^0(X/Y) = \operatorname{Pic}^0(\operatorname{Pic}^0(X/Y)/Y)$ be the "naive" relative Albanese variety. (See [Grot, 236-16] for the "true" Albanese.) (The definition makes sense over $Y^0 \subset Y$ and we compactify it in some way.) The fiber of $A^0(X/Y) \to Y$ over a point $y \in Y^0$ is isomorphic to X_y but the isomorphism is not canonical. Using the multisection S we can define a rational map as follows. Let $s_1(y), \ldots, s_k(y) \in X_y$ be the points of $S \cap X_y$ ($k = \deg(S/Y)$). Given $x \in X_y$ let $\operatorname{alb}_S(x) = kx - \sum s_i(y)$. (It is easy to see that this does not depend on the choice of the origin on X_y). After suitable birational modifications we obtain a morphism

$$\operatorname{alb}_S : X \to A^0(X/Y).$$

On the fibers over $y \in Y^0$ this is multiplication by k followed by a suitable translation.

$A^0(X/Y) \to Y$ is a family of Abelian varieties with a section. $H_1(A^0(X/Y)_y, \mathbb{Z})$ is canonically isomorphic to $kH_1(X_y, \mathbb{Z})$, thus local monodromies of $A^0(X/Y) \to Y$ are trivial. Therefore by (6.8.1) after a suitable étale cover we may assume that $A^0(X/Y) \to Y$ is an Abelian scheme.

Let $k: A^0(X/Y) \to A^0(X/Y)$ be multilpication by k in the fibers. k is étale. Thus

$$X' = A^0(X/Y) \times_{k, \mathrm{alb}_S} X \to X \tag{6.9.1}$$

is étale. Furthermore for $y \in Y^0$ the fiber X'_y is the union of disjoint copies of $A^0(X/Y)_y$. Thus the pull back of the zero section of $A^0(X/Y)$ via the first projection in (6.9.1) produces a section of $X' \to Y'$ where $X' \to Y' \to Y$ is the Stein factorisation of $X' \to Y$.

We can apply (6.8.1) to $X' \to Y'$ to obtain (6.3). □

7 Fundamental groups of resolutions

Let X be a normal analytic space and let $f: Y \to X$ be a resolution of singularities. $\pi_1(Y) \to \pi_1(X)$ is surjective but in general it has a large kernel. The aim of this section is to find local conditions on X which ensure that the above map is an isomorphism.

7.1. Definition. (7.1.1) Let X be a normal analytic space and left $f: Y \to X$ be a resolution of singularities. $\pi_1(Y)$ is independent of the choice of Y. It will be denoted by $\pi_1(\mathrm{Res}\, X)$.

(7.1.2) We say that $\mathrm{Res}\, X$ is *locally simply connected* (resp. *locally algebraically simply connected*) if every point $x \in X$ has a contractible neighborhood $x \in U \subset X$ such that $\pi_1(f^{-1}(U)) = 1$ (resp. $\hat{\pi}_1(f^{-1}(U)) = 1$).

If X is an algebraic variety (over any field) instead of a contractible neighborhood U one can take the Henselisation of the local ring of $x \in X$. At least in characteristic zero, all the results of this chapter go through without changes.

7.2. Lemma. *Let X be a normal analytic space.*

(7.2.1) *If* $\mathrm{Res}\, X$ *is locally simply connected (resp. locally algebraically simply connected) then* $\pi_1(\mathrm{Res}\, X) \to \pi_1(X)$ *(resp.* $\hat{\pi}_1(\mathrm{Res}\, X) \to \hat{\pi}_1(X)$*) is an isomorphism.*

(7.2.2) *Let $f: Y \to X$ be a proper bimeromorphic morphism, Y normal. If Y is (algebraically) simply connected and* $\mathrm{Res}\, Y$ *is locally (algebraically) simply connected then* $\mathrm{Res}\, X$ *is (algebraically) simply connected.*

7.3. Lemma. *Let $p: X_1 \to X_2$ be a proper and dominant morphism between irreducible normal analytic spaces. Assume that* $\pi_1(\mathrm{Res}\, X_1)$ *(resp.* $\hat{\pi}_1(\mathrm{Res}\, X_1)$ *is trivial. Then* $\pi_1(\mathrm{Res}\, X_2)$ *(resp.* $\hat{\pi}_1(\mathrm{Res}\, X_2)$*) is finite. (In fact their order is bounded by* $\deg p$.)

Proof. Consider the following commutative diagram

$$\begin{array}{ccccc} X_1 & \xleftarrow{f_1} & Y_1 & \xleftarrow{e_1} & Y'_1 = Y_1 \times_{Y_2} Y'_2 \\ p\downarrow & & p_Y \downarrow & & p' \downarrow \\ X_2 & \xleftarrow{f_2} & Y_2 & \xleftarrow{e_2} & Y'_2 \end{array} \tag{7.3.1}$$

where f_2 is a resolution, f_1 is a resolution which dominates Y_2 and e_2 is an étale cover. By construction e_1 is étale, thus Y'_1 is a disjoint union of several copies of Y_1. Thus $p_Y: Y_1 \to Y_2$ factors through Y'_2. Therefore $\deg e_2 \leqq \deg p$. □

7.4. Lemma. *Let $p: X_1 \to X_2$ be a finite and dominant morphism between irreducible normal analytic spaces. Assume that*

(7.4.1) *X_1 has rational singularities;*

(7.4.2) *$\pi_1(\operatorname{Res} X_1)$ (resp. $\hat{\pi}_1(\operatorname{Res} X_1)$) is trivial, and*

(7.4.3) *$\pi_1(X_2)$ (resp. $\hat{\pi}_1(\operatorname{Res} X_2)$) is trivial.*

Then $\pi_1(\operatorname{Res} X_2)$ (resp. $\hat{\pi}_1(\operatorname{Res} X_2)$) is trivial.

Proof. The question is clearly local, thus we may assume that X_2 is a contractible neighborhood of a point $x \in X_2$. Consider the diagram (7.3.1). We may assume that $D_2 = \operatorname{red} f_2^{-1}(x)$ is a divisor with normal crossings only. By (7.2) $\pi_1(\operatorname{Res} X_2)$ (resp. $\hat{\pi}_1(\operatorname{Res} X_2)$) is finite, thus $Y'_2 \to Y_2$ is finite. Let X' be the normalisation of X_2 in Y'_2 and $f': Y'_2 \to X'$ the induced morphism. The natural morphisms $r: X_1 \to X'$ and $q: X' \to X_2$ are finite and dominant. In particular, X' also has rational singularities. Let $x' \in X'$ be the preimage of $x \in X_2$. Let $D' = \operatorname{red} f'^{-1}(x')$. $D' \to D_2$ is a finite étale morphism thus D' is a divisor with normal crossings only and $\chi(D', \mathcal{O}) = \deg q \cdot \chi(D_2, \mathcal{O})$.

Since X' has rational singularities, $\chi(D', \mathcal{O}) = 1$ by (5.2.3). Thus $\deg q = 1$. □

Following [Ko et al., 2.13] we use klt as an abbreviation for "Kawamata log terminal". (Note that the same notion is called "log terminal" in [KaMM, 0-2-10].) Results about klt pairs (X, Δ) are mentioned for completeness sake. They are not used in this article.

7.5. Theorem. *Let X be a normal analytic space.*

(7.5.1) *If (X, Δ) is* klt *for some Δ then* $\operatorname{Res} X$ *is locally algebraically simply connected.*

(7.5.2) *If X has quotient singularities then* $\operatorname{Res} X$ *is locally simply connected.*

Proof. The second claim follows from (7.4). The first one is local, so consider $x \in X_2 \subset X$ and let $Y'_2 \to Y_2 \to X_2$ be as in (7.3.1). Let X' be the normalisation of X_2 in Y'_2. $p: X' \to X_2$ is étale over the smooth points of X, thus $(X', p^*\Delta)$ is klt by [Ko et al., 20.3]. Hence X' has rational singularities by [KaMM, 1-3-6] and (7.4) applies. □

7.5.3. Remark. It is quite likely that if (X, Δ) is klt for some Δ then $\operatorname{Res} X$ is locally simply connected. This is true in dimension 3:

7.6. Theorem. *Let X be a three dimensional normal analytic space. If (X, Δ) is* klt *for some Δ then* $\operatorname{Res} X$ *is locally simply connected.*

Proof. The problem is local thus we may assume that X is simply connected.

By [Ko et al., 6.11.1] there is a small projective morphism $f_1: (X_1, \Delta_1) \to (X, \Delta)$ such that X_1 is $\mathbb{Q}$-factorial and $K + \Delta_1 \equiv f_1^*(K + \Delta)$. Therefore $R^1(f_1)_*\mathcal{O}_{X_1} = 0$ [KaMM, 1-2-6] hence the exceptional set of f_1 is a tree of rational curves. In particular X_1 is simply connected and by (7.2.2) it is sufficient to prove that $\operatorname{Res} X_1$ is locally simply connected.

Since X_1 is $\mathbb{Q}$-factorial, X_1 is klt. Let $X_2 \to X_1$ be its index one cover. Then X_2 is canonical and by (7.4) it is sufficient to prove that $\operatorname{Res} X_2$ is locally simply connected. This will be done by following the resolution procedure of [Re2, §2].

Let $x \in X$ be an index 1 canonical point with general hyperplane section $x \in H$. Let $f: X' \to X$ be the weighted blow-up specified by [Re2, 2.11] (almost always this is the ordinary blow up of x). Let $S \subset X'$ be the reduced exceptional divisor. Let

$H' \to H$ be the corresponding blow up of H with exceptional curve E. Applying induction and (7.2.2) it is sufficient to prove two statements:

(7.6.1) S is simply connected; and

(7.6.2) if Y is terminal of index one then Res Y is locally simply connected.

E is a general hyperplane section of S, thus by (2.9.2) $\pi_1(E) \to \pi_1(S)$ is surjective. H is a minimally elliptic surface singularity [Re2, 2.6] and these are well understood. From [L, 3.4] it follows that E is either simply connected, or a cycle of rational curves or an elliptic curve. In all these cases $\pi_1(E)$ is Abelian and therefore $\pi_1(S) \to \hat{\pi}_1(S)$ is injective. By (7.5.1) $\hat{\pi}_1(S) = 1$, thus $\pi_1(S) = 1$.

If $y \in Y$ is terminal of index one, it is a hypersurface double point and $Y - \{y\}$ is simply connected [Mi]. Thus Res Y is locally simply connected. □

7.7. *Example.* Let S be a surface with $q = p_g = 0$ and let X be the cone over the embedding $S \subset \mathbb{P}^n$ given by a sufficiently ample complete linear system. X has rational singularities and $\pi_1(X) = 1$. However $\pi_1(\operatorname{Res} X) = \pi_1(S)$ which can be quite large (cf. [BPV, VII.11]).

For ease of reference we summarise our results:

7.8. Theorem. *Let X be a normal analytic space and let $f: Y \to X$ be a resolution of singularities. Then*

(7.8.1) *$\pi_1(Y) \to \pi_1(X)$ is an isomorphism if either X has quotient singularities or $\dim X \leq 3$ and X has log terminal singularities.*

(7.8.2) *$\hat{\pi}_1(Y) \to \hat{\pi}_1(X)$ is an isomorphism if (X, Δ) is* klt *for some Δ. In particular this holds if X has log terminal singularities.*

8 Nonvanishing theorems

In this section we prove a rather strong nonvanishing result for varieties with generically large algebraic fundamental group. Let us first recall the following:

8.1 Theorem [Ko3, 3.2] *Let $g: X \to S$ be a surjective morphism, X smooth and projective. Let $U \subset S$ be a dense open set. Let L be a nef and big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on S, N a Cartier divisor on X, M and Δ $\mathbb{Q}$-divisors on X. Assume that:*

(8.1.1) Supp Δ *is a normal crossing divisor and* $\lfloor \Delta \rfloor = \emptyset$;

(8.1.2) *$N|g^{-1}(U)$ is linearly equivalent to an effective divisor;*

(8.1.3) *M is nef and either big on the general fiber of g or numerically trivial on X;*

(8.1.4) $N \equiv K_X + \Delta + M + g^*L$;

(8.1.5) If $Z \xleftarrow{p} U \xrightarrow{u} X$ *is a dominant family of normal cycles (Z irreducible) then*

$$L^{\dim(U/Z)} \cdot U_{\text{gen}} \geqq \left(\frac{(\dim S)^2 + \dim S + 1}{2} \right)^{\dim(U/Z)}.$$

Then $h^0(X, N) \neq 0$. More generally, if X_g is the generic fiber of g then $H^0(X, N) \to H^0(X_g, N|X_g)$ is surjective.

In order to apply (8.1) to varieties with generically large algebraic fundamental group, we need the following:

8.2. Lemma. *Let Y be a normal and proper variety with generically large algebraic fundamental group. Let L be a nef and big divisor on Y and let $M > 0$. Then there is a finite étale cover $m: Y' \to Y$ such that if $Z \xleftarrow{p} U \xrightarrow{u} Y'$ is a dominant family of normal cycles on Y' (Z irreducible) then $(m^*L)^{\dim(U/Z)} \cdot U_{\text{gen}} \geqq M$.*

Proof. Let $p(Y): U(Y) \to S(Y)$ be a weakly complete family of normal cycles. Let $m: Y' \to Y$ be a finite étale cover. Let $U(Y') = U(Y) \times_Y Y'$ and let $p(Y'): U(Y') \to S(Y')$ be the Stein factorisation of $U(Y') \to S(Y)$. It is clear that $p(Y'): U(Y') \to S(Y')$ is a weakly complete family of normal cycles on Y'.

Let $S(Y) = \cup S_i$ be the irreducible components. If $s \in S_i$ then the top selfintersection of $u(Y)^*L|U_s$ depends only on i. This value will be denoted by $\deg_L U_i/S_i$.

Assume that $m: Y' \to Y$ is étale and Galois corresponding to a normal subgroup $G \triangleleft \hat{\pi}_1(Y)$. Let $s' \in S_i'$ be a preimage of s under the morphism $S_i' \to S_i$. Let $u^*G < \hat{\pi}_1(U_{i,s})$ be the preimage of G under the natural homomorphism $\hat{\pi}_1(U_{i,s}) \to \hat{\pi}_1(Y)$.

By construction

$$\deg(U'_{i,s'}/U_{i,s}) = |\hat{\pi}_1(U_{i,s}) : u^*G|, \quad \text{thus}$$

$$\deg_{m^*L} U_i'/S_i' = |\hat{\pi}_1(U_{i,s}) : u^*G|^{\dim U_{i,s}} \cdot \deg_L U_i/S_i.$$

There are only finitely many indices i such that $\deg_L U_i/S_i \leqq M$. By suitable indexing we may assume that these are $i = 1, \ldots, k$. Choose $r \in \mathbb{N}$ such that

$$\min_i \deg_L U_i/S_i \geqq M/r.$$

Since Y has generically large algebraic fundamental group, we can choose a normal subgroup $G \triangleleft \hat{\pi}_1(Y)$ such that if $m: Y' \to Y$ is the corresponding cover then

$$\deg(U'_{i,s'}/U_{i,s}) = |\hat{\pi}_1(U_{i,s}) : u^*G| \geqq r \quad \text{for } i = 1, \ldots, k.$$

Thus $\deg_{m^*L} U_i'/S_i' \geqq M$ for every i. □

8.3. Theorem. *Let $g: X \to S$ be a surjective morphism between normal and proper varieties. Let L be a big $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor on S, N a Cartier divisor on X and Δ an effective $\mathbb{Q}$-divisor on X. Let X_g be the general fiber of g. Assume that:*

(8.3.1) *X_g is smooth, $\operatorname{Supp} \Delta|X_g$ is a normal crossing divisor and $\llcorner \Delta|X_g \lrcorner = \emptyset$;*

(8.3.2) *$N|X_g$ is linearly equivalent to an effective divisor;*

(8.3.3) *M is nef and either big on the general fiber of g or numerically trivial on X;*

(8.3.4) *$N \equiv K_X + \Delta + M + g^*L$;*

(8.3.5) *S has generically large algebraic fundamental group.*

Then $h^0(X, N) \neq 0$.

Proof. We can write $L \equiv L' + E$ where L' is ample and E is effective (both $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisors). We can incorporate $g^*(E)$ into Δ thus we may assume that L is nef and big (even ample).

Let $f: X' \to X$ be a log resolution and write $f^*(K_X + \Delta) = K_{X'} + \Theta$. By our assumptions $\Theta = \Delta' + H' - H''$, where Δ' is an effective divisor with normal crossings only such that $\llcorner \Delta' \lrcorner = \emptyset$, H' is an effective integral divisor disjoint from $f^{-1}(X_g)$ and H'' is an f-exceptional integral divisor. Hence

$$f^*N + H'' - H' \equiv K_{X'} + \Delta' + f^*M + (gf)^*L, \quad \text{and}$$

$$H^0(X', f^*N + H'' - H') \subset H^0(X', f^*N + H'') = H^0(X, N).$$

Therefore it is sufficient to show that (8.3) holds under the additional assumptions that X is smooth and Δ is an effective divisor with normal crossings only such that $\llcorner \Delta \lrcorner = \emptyset$.

Let $m: S' \to S$ be étale. By base change we obtain a commutative diagram

$$\begin{array}{ccc} X' & \xrightarrow{g'} & S' \\ {\scriptstyle m_X}\downarrow & & \downarrow{\scriptstyle m} \\ X & \xrightarrow{g} & S. \end{array}$$

8.3.6. Lemma. $h^0(X', m_X^* N) = \deg(m) h^0(X, N)$.

Proof. By [Ko1, 2.1; EV1, 3.1]

$$h^0(X, N) = h^0(S, g_* \mathcal{O}_X(N)) = \chi(S, g_* \mathcal{O}_X(N)), \quad \text{and}$$
$$h^0(X', m_X^* N) = h^0(S', g'_* \mathcal{O}_{X'}(m_X^* N)) = \chi(S', m^* g_* \mathcal{O}_X(N)).$$

m is étale, thus the Euler characteristic of a sheaf is multipled by the degree under pull back. □

By (8.2) we can choose $m: S' \to S$ in such a way that $X' \to S'$ satisfies all the assumptions of (8.1). Thus $h^0(X', m_X^* N) \neq 0$ which implies that $h^0(X, N) \neq 0$. □

8.3.7. Remark. In (8.3) it is not true that $h^0(X, N) \to h^0(X_g, N|X_g)$ is surjective. For example let X be an Abelian variety with a symmetric ample line bundle N and let $g: X \to S \cong X$ be multiplication by m. Let $L = N/m^2$, so $N \equiv g^* L$. $h^0(X_g, N|X_g) = m^{2 \dim X}$, thus surjectivity fails for $m \gg 1$. However I do not know any examples where g has connected fibers.

8.3.8. Example. Let $F_n = \mathrm{Proj}_{\mathbb{P}^1}(\mathcal{O} + \mathcal{O}(n))$ $(n > 0)$. Let f be a fiber of the projection to $\mathbb{P}^1$ and let e be the unique section with selfintersection $-n$. Let $L = \mathcal{O}(e + f)$. Then L is big and $h^0(X, K_X \otimes L^{\otimes m}) = 0$ for $m \leqq n + 1$. Of course F_n is simply connected.

8.4. Corollary. *Let Y be a normal proper variety and let D be a big $\mathbb{Q}$-Cartier Weil-divisor on Y. Assume that resolutions of Y have generically large algebraic fundamental group.*

Then $h^0(Y, \omega_Y(D)) > 0$.

Proof. Let $g: X \to Y$ be a log resolution of (Y, D), $D' = \ulcorner g^* D \urcorner$. There is a natural injective map $g_* \omega_X(D') \to \omega_Y(D)$, thus it is sufficient to prove (8.4) for Y smooth. This becomes a special case of (8.3) by setting $X = S = Y$. □

8.5. Corollary. *Let X be a smooth projective variety. Assume that X is of general type and it has generically large algebraic fundamental group. Then*

$$P_m(X) > 0 \quad \text{for } m \geqq 2.$$

8.6. Examples. In these examples $X_d \subset \mathbb{P}(a_0, \ldots, a_k)$ stands for a general hypersurface of degree d in the indicated weighted projective space. We refer to [F] for the definitions and further notation. We use the standard but slightly misleading notation $\mathbb{P}(a_0^{s_0}, \ldots, a_n^{s_n})$ to denote a weighted projective space of dimension $-1 + \sum s_i$ where we have s_j coordinates with weight a_i.

8.6.1. Proposition. *Given $a_0, \ldots, a_k$ and M there is a smooth projective variety of general type X such that*

$$P_m(X) = h^0(\mathbb{P}(a_0, \ldots, a_k), \mathcal{O}(m)) \quad \textit{for } m \leqq M.$$

Proof. Let $r = M \prod a_i$ and let s, t be natural numbers to be chosen later such that $s, t \geqq \max\{a_i, r+1\}$. Choose $d \gg 1$ such that $r(r+1)|d$ and then choose suitable s, t such that $d = 1 + sr + t(r+1) + \sum a_i$. We claim that a desingularisation of

$$X_d \subset \mathbb{P}(a_0, \ldots, a_k, r^s, (r+1)^t)$$

is a required example. By the choices we made $\mathcal{O}(K_{X_d}) \cong \mathcal{O}(1)$ thus

$$\begin{aligned} h^0(X_d, \mathcal{O}(mK_{X_d})) &= h^0(\mathbb{P}(a_0, \ldots, a_k, r^s, (r+1)^t), \mathcal{O}(m)) \\ &= h^0(\mathbb{P}(a_0, \ldots, a_k), \mathcal{O}(m)) \quad \text{for } m \leqq M. \end{aligned}$$

We still need to check that X_d has canonical singularities. Since $r(r+1)|d$, $|\mathcal{O}_{\mathbb{P}}(d)|$ is base point free, hence by [Re2, 1.13] it is sufficient to prove that $\mathbb{P}(a_0, \ldots, a_k, r^s, (r+1)^t)$ has canonical singularities. This is implied by the following special case of [Re2, 3.1]:

Let $\mathbb{Z}_m$ act on $\mathbb{A}^n$ by $(x_1, \ldots, x_n) \mapsto (\varepsilon^{b_1}x_1, \ldots, \varepsilon^{b_n}x_n)$ where ε is a primitive mth root of unity. Assume that $\#\{i|(m, b_i) = 1\} \geqq m$. Then $\mathbb{A}^n/\mathbb{Z}_m$ has canonical singularities. □

We mention three concrete examples:

(8.6.2) $X_d = X_{3r(r+1)} \subset \mathbb{P}(r^{r+2}, (r+1)^{2r-1})$ has terminal singularities and satisfies

$$P_1(X_d) = \ldots = P_{r-1}(X_d) = 0.$$

(8.6.3) $X_d = X_{3r(r+1)} \subset \mathbb{P}(1, r^{r+3}, (r+1)^{2r-2})$ has terminal singularities and satisfies

$$P_1(X_d) = \ldots = P_{r-1}(X_d) = 1.$$

(8.6.4) By [F, II.5.1] $X_{46} \subset \mathbb{P}(4, 5, 6, 7, 23)$ has terminal singularities,

$$P_1(X) = P_2(X) = P_3(X) = 0 \quad \text{and} \quad P_4(X) = \ldots = P_9(X) = 1.$$

The general analog of (8.5) is slightly technical. Let X be a normal variety and let Δ be an effecive $\mathbb{Q}$-divisor. Assume that $K + \Delta$ is $\mathbb{Q}$-Cartier and big. Let $f: X' \to X$ be a resolution and let $K_{X'} + \Delta' \equiv f^*(K_X + \Delta)$. Let E be a sufficiently large multiple of the reduced exceptional divisor. Then

$$f_*\mathcal{O}(mK_{X'} + \llcorner m\Delta' \lrcorner + E) = \mathcal{O}(mK_X + \llcorner m\Delta \lrcorner).$$

We would like to apply (8.4) to X'. This can be done with the set-up $D = (m-1)K_{X'} + \ulcorner (m-1)\Delta' \urcorner + E$. Thus we obtain:

8.7. Theorem. *Notation as above. Assume that resolutions of X have generically large algebraic fundamental group. Assume that $\llcorner m\Delta \lrcorner \geqq (m-1)\Delta$ for some m. Then*

$$h^0(X, \mathcal{O}(mK_X + \llcorner m\Delta \lrcorner)) > 0.$$

The extra condition is satisfied if $\Delta = 0$ or if every coefficient in Δ is of the form $1 - 1/r$ for some $r \in \mathbb{N}$.

9 Plurigenera in étale covers

Let X be a smooth projective variety of general type. Assume that K_X is ample. By Kodaira's vanishing theorem $h^0(X, \mathcal{O}(mK_X)) = \chi(X, \mathcal{O}(mK_X))$ for $m \geqq 2$. Thus, for instance, plurigenera are multiplicative in étale covers. Mori's program asserts that even if K_X is not ample, one can find a suitable birational model of X where K becomes ample. Thus plurigenera should be multiplicative in étale covers for every smooth projective variety of general type. The aim of this section is to prove this claim and then to use it to get further lower bounds on the plurigenera of varieties with generically large algebraic fundamental group.

It will be useful to handle a more general situation:

9.1. Notation. Let X be a smooth proper variety. Let M be a Cartier divisor on X, Δ an effective $\mathbb{Q}$-divisor such that $\operatorname{Supp}\Delta$ has normal crossings, $\llcorner \Delta \lrcorner = 0$ and L a nef $\mathbb{Q}$-divisor. Assume that:

(9.1.1) $M \equiv a(K_X + \Delta) + L$ for some $a > 1$, and

(9.1.2) L or M is big.

Let $f: Y \to X$ be a birational morphism whose exceptional divisor has normal crossings. Let $f^*(K_X + \Delta) = K_Y + \Delta'$. By our assumptions $-\llcorner \Delta' \lrcorner$ is effective. Let E be the effective part of $-\llcorner a\Delta' \lrcorner$ and let $\Delta_Y = (a\Delta' + E)/a$. Then Δ_Y is an effective $\mathbb{Q}$-divisor such that $\operatorname{Supp}\Delta_Y$ has normal crossings and $\llcorner \Delta_Y \lrcorner = 0$. Furthermore

$$f^*M + E \equiv a(K_Y + \Delta_Y) + f^*L \quad \text{and} \quad h^0(Y, f^*M + E) = h^0(X, M).$$

In particular, as long as we are interested only in $h^0(X, M)$, we can perform a series of blow-ups and by replacing M by $f^*M + E$ we may continue to assume that our divisor is of the form specified above.

9.2. Proposition. *Notation as above. Let T be a numerically trivial Cartier divisor on X. Then*

$$H^0(X, \mathcal{O}(M + T)) = H^0(X, \mathcal{O}(M)).$$

Proof. Let T', T'' be numerically trivial Cartier divisors on X. Assume first that $|r(M + T')| \neq \emptyset$ for some $r > 0$ and let $|r(M + T')| = |F| + B$ where B is the fixed part. Choosing $r \gg 1$ we may assume that F is big if M is. By blowing up we may assume that $\operatorname{Supp}(\Delta + B)$ has normal crossings only and $|F|$ is free. By (9.2.3)

$$\ulcorner B/r \urcorner \geqq \llcorner (a-1)B/ar + \Delta \lrcorner \stackrel{\text{def}}{=} D(r),$$

thus $\dim|M + T' - D(r)| = \dim|M + T'|$ by (9.2.2). On the other hand,

$$M + T' - D(r) \equiv K_X + \frac{a-1}{ar}F + \frac{1}{a}L + \left\{\frac{a-1}{ar}B + \Delta\right\}.$$

By assumption either F or L is big, thus

$$h^0(X, \mathcal{O}(M + T')) = h^0(X, \mathcal{O}(M + T' - D(r))) = \chi(X, \mathcal{O}(M + T' - D(r)).$$

Furthermore,

$$\begin{aligned} h^0(X, \mathcal{O}(M + T' + T'')) &\geqq h^0(X, \mathcal{O}(M + T' + T'' - D(r))) \\ &= \chi(X, \mathcal{O}(M + T' + T'' - D(r))) \\ &= \chi(X, \mathcal{O}(M + T' - D(r))) \\ &= h^0(X, \mathcal{O}(M + T')). \end{aligned} \tag{9.2.1}$$

Apply the above inequality for $T' = 0, T'' = T$ and then for $T' = T, T'' = -T$.

We still have to deal with our assumption that $|r(M + T')| \neq \emptyset$. If $|M + T'| = \emptyset$ for every T' then (9.2) is clearly satisfied. If $|M + T'| \neq \emptyset$ for some T' then by (9.2.1) $|M + T| \neq \emptyset$ for every T and the argument applies. □

9.2.2. Lemma. *Let X be a normal and proper variety and let L be a Weil divisor on X. Let $|rL| = |F| + B$ where B is the fixed part. Then $|L| = |L - \ulcorner B/r \urcorner| + \ulcorner B/r \urcorner$.*

9.2.3. Lemma. *Let B and Δ be effective Weil divisors on a normal variety such that $\llcorner \Delta \lrcorner = 0$. Let $0 < b < 1$. Then $\ulcorner B \urcorner \geqq \llcorner bB + \Delta \lrcorner$.*

The following application of (9.2) will be used in Sect. 10.

9.3. Proposition. *Let X be a smooth projective variety of general type and let $f: X \to E$ be a morphism onto an elliptic curve with general fiber X_g. Then for every $m \geqq 2$*

$$P_m(X) \geqq 1 \Leftrightarrow P_m(X_g) \geqq 1.$$

Proof. The implication $\Rightarrow$ is clear.

Assume that $P_m(X_g) \geqq 1$. $f_*\mathcal{O}(mK_X)$ is a vector bundle on E which can be written as the sum of indecomposable vector bundles F_i.

$$P_m(X) = h^0(E, f_*\mathcal{O}(mK_X)) \geqq \sum \max\{0, \deg F_i\}.$$

Thus we are done unless $\deg F_i \leqq 0$ for every i. By [Ka2], $\deg F_i \geqq 0$ for every i, hence in fact $\deg F_i = 0$.

By the classification of [A], for every i there is a line bundle L_i of degree zero on E such that $h^0(F_i \otimes L_i) \geqq 1$ (in fact $= 1$). Thus $h^0(X, \mathcal{O}(mK_X) \otimes f^*L_1) \geqq 1$. By (9.2) this shows that $P_m(X) \geqq 1$. □

9.3.1. Remark. The above argument proves that $f_*\mathcal{O}(mK_X)$ is an ample vector bundle (or zero) for $m \geqq 2$. More general results were proved by [EV2] under a different set of assumptions.

9.4. Proposition. *Let $p: X' \to X$ be a finite ètale morphism between smooth and proper varieties. We keep the notation and assumptions of (9.1). Let $M' = p^*M$. Then*

$$h^0(X', \mathcal{O}(M')) = \deg p \cdot h^0(X, \mathcal{O}(M)).$$

Proof. Let $X'' \to X' \to X$ be the Galois closure of X'/X. By going from X to X'' and then from X'' to X' we are reduced to the case when p is Galois. Let $d = \deg p$.

If $|dM| = \emptyset$ then the norm map $n: |M'| \to |dM|$ shows that $|M'| = \emptyset$ and we are done. Otherwise let $|drM| = |F| + B$ where B is the fixed part. If M is big, choose $r \gg 1$ such that F is also big. By blowing up we may assume that Supp $(B + \Delta)$ has normal crossings only and $|F|$ is free. In general p^*B may not be the fixed part of $|drM'|$. However by looking at the norm map $n: |rM'| \to |drM|$ we see that p^*B/d is contained in the base locus of $|rM'|$. Thus by (9.2.3) $\llcorner (a-1)B/adr + \Delta \lrcorner \overset{\text{def}}{=} D(r)$ is contained in the base locus of $|M'|$. Furthermore

$$M - D(r) \equiv K_X + \frac{a-1}{ar} F + \frac{1}{a} L + \left\{ \frac{a-1}{ar} B + \Delta \right\}.$$

Therefore vanishing holds for $M - D(r)$ and for $M' - p^*D(r)$. Thus

$$\begin{aligned} h^0(X', \mathcal{O}(M')) &= h^0(X', \mathcal{O}(M' - p^*D(r))) \\ &= \chi(X', \mathcal{O}(M' - p^*D(r))) \\ &= d\cdot\chi(X, \mathcal{O}(M - D(r))) \\ &= d\cdot h^0(X, \mathcal{O}(M - D(r))) \\ &= d\cdot h^0(X, \mathcal{O}(M)). \quad \square \end{aligned}$$

9.4.1. Remark. (9.4) should be viewed as a nonabelian version of (9.2). Assume that T in (9.2) is torsion of order k in $\operatorname{Pic}(X)$. Then T determines an Abelian Galois cover $f: X' \to X$ and

$$f_*\mathcal{O}_{X'} = \sum_{i=0}^{k-1} \mathcal{O}_X(iT).$$

Thus (9.2) implies (9.4) in this case.

The results of Sect. 8 allow us to find a pluricanonical divisor. The following result will enable us to obtain a pluricanonical pencil. The examples given in (8.6) show that (9.5) fails for simply connected varieties.

9.5. Theorem. *Let X be a smooth proper variety. Assume that $\hat{\pi}_1(X) \neq 1$. Let $M_1 \equiv a_1(K_X + \Delta_1) + L_1$ and $M_2 \equiv a_2(K_X + \Delta_2) + L_2$ be as in* (9.1). *If $h^0(X, M_1) \geqq 1$ and $h^0(X, M_2) \geqq 1$ then*

$$h^0(X, M_1 + M_2) \geqq h^0(X, M_1) + h^0(X, M_2).$$

Proof. Let $p: X' \to X$ be the étale cover whose existence is assumed; $d = \deg p \geqq 2$. By (9.4) and (9.5.1)

$$\begin{aligned} h^0(X, M_1 + M_2) &= \frac{1}{d} h^0(X', M_1' + M_2') \\ &\geqq \frac{1}{d}(dh^0(X, M_1) + dh^0(X, M_2) - 1) \\ &= h^0(X, M_1) + h^0(X, M_2) - 1/d. \end{aligned}$$

$h^0(X, M_1 + M_2)$ is an integer, thus we are done. $\square$

9.5.1. Lemma. *Let X be a normal and proper variety and let L, M be effective Weil divisors on X. Then $h^0(X, L + M) \geqq h^0(X, L) + h^0(X, M) - 1$.*

9.5.2. Corollary. *Let X be a smooth proper variety of general type. Assume that $\hat{\pi}_1(X) \neq 1$. If $P_k(X) \geqq 1$ and $P_m(X) \geqq 1$ for some $k, m \geqq 2$ then $P_{k+m}(X) \geqq P_k(X) + P_m(X)$.*

Proof. Set $kK_X = M_1$ and $mK_X = M_2$ in (9.5). $\square$

9.6. Remark. Let (Y, D) be a klt pair. Let M_Y be a $\mathbb{Q}$-Cartier Weil divisor on X such that $M_Y \equiv a(K_Y + D) + L_Y$ where L_Y is a $\mathbb{Q}$-Cartier $\mathbb{Q}$-divisor. Let $f: X \to Y$ be a log resolution and let $K_X + D_X = f^*(K_Y + D)$ where $-\llcorner D_X \lrcorner$ is effective by the definition of klt. $f_*(\mathcal{O}_X(\llcorner f^*M \lrcorner)) = \mathcal{O}_Y(M)$ thus if E is an effective f-exceptional divisor then

$$h^0(X, \mathcal{O}_X(\llcorner f^*M \lrcorner + E) = h^0(Y, \mathcal{O}_Y(M)).$$

Let E be the effective part of $-\llcorner aD_X - \{f^*M\}\lrcorner$ and let $\Delta = (aD_X + E)/a$. Then

$$\llcorner f^*M \lrcorner + E \equiv a(K_X + \Delta) + f^*L.$$

This shows that (9.2, 9.4, 9.5) are valid for (Y, D) klt.

10 Fiber spaces and plurigenera

Let $f: X \to S$ be a dominant morphism between smooth projective varieties with general fiber X_g. Iitaka's problem [I] asks about relating the plurigenera of X, S and X_g. Here we will study this problem assuming that S has generically large algebraic fundamental group.

The original Iitaka problem compares the asymptotic behavior of the plurigenera. The first nonasymptotic results appeared in [Ko1] and the method was further developed in [EV1]. Here we improve the results in case S has generically large algebraic fundamental group. The main application is the following:

10.1. Theorem. *Let X be a proper and smooth threefold of general type. Assume that $\hat{\pi}_1(X)$ is infinite. Then*

$$P_m(X) > 0 \quad \text{for} \quad m \geqq 2.$$

The main idea is of course to apply nonvanishing (7.3). Unfortunately mK_X is not of the form required. One of the most important results about this question is the weak positivity of $K_{X/S}$, due to [V]. Here we need a corollary of it which is modeled after similar applications in [EV1; Mo].

10.2. Proposition. *Let $f: X \to S$ be a surjective morphism between smooth proper varieties. Let E be a Cartier divisor on X, L a big $\mathbb{Q}$-Cartier divisor on S and B an effective $\mathbb{Q}$-divisor on X. Let X_g be the generic fiber of f. Assume that $E \equiv 2f^*L + B$ and $E|X_g \sim 0$. (In particular, B is disjoint from X_g.) Then for every $m \geqq 1$ there are*

(10.2.1) *a smooth projective birational model $f': X' \xrightarrow{p} X \to S$; (Let X'_g be the generic fiber of f'.)*

(10.2.2) *an effective $\mathbb{Q}$-divisor Δ' on X' such that $\llcorner \Delta'|X'_g \lrcorner = \emptyset$ and* Supp $\Delta'|X'_g$ *has normal crossings only;*

(10.2.3) *a Cartier divisor N' on X' such that $N' \equiv K_{X'} + \Delta' + f'^*L$;*

(10.2.4) *a map*

$$f'_*\mathcal{O}_{X'}(N') \to f_*\mathcal{O}_X(K_X + (m-1)K_{X/S} + E)$$

which is an isomorphism at the generic point of S.

Proof. By [V] there is an $r > 0$ such that the restriction map

$$H^0(X, \mathcal{O}(rm(m-1)K_{X/S} + rmf^*L)) \otimes \mathbb{C}(X) \to H^0(X_g, \mathcal{O}(rm(m-1)K_{X_g}))$$

is surjective. Take a general divisor

$$D \in |rm(m-1)K_{X/S} + rmf^*L|.$$

Choose $p: X' \to X$ such that

(i) Supp D' has normal crossings only, where $D' \in |rm(m-1)K_{X'/S} + rmf'^*L|$ is the corresponding divisor; and

(ii) $|mK_{X'_g}| = |H| + F$ where F is the fixed part and $|H|$ is base point free. Set

$$N' = K_{X'} + (m-1)K_{X'/S} - \llcorner D'/rm \lrcorner + p^*E, \quad \text{and} \quad \Delta' = \{D'/rm\} + p^*B.$$

The only condition that needs checking is (10.2.4). $D'|X'_g = \bar{H} + r(m-1)F$ where $\bar{H} \in |r(m-1)H|$ is a general smooth member. Thus

$$N'|X'_g \sim mK_{X'_g} - \llcorner (1-1/m)F \lrcorner \quad \text{and} \quad K_{X'} + (m-1)K_{X'/S} + p^*E|X'_g \sim mK_{X'_g}.$$

Since $\llcorner (1-1/m)F \lrcorner \leqq F$, we obtain that the natural morphism

$$H^0(X'_g, N'|X'_g) \to H^0(X'_g, mK_{X'_g} + p^*E|X'_g) \cong H^0(X_g, mK_{X_g} + E|X_g)$$

is an isomorphism. Thus

$$f'_*\mathcal{O}_{X'}(N') \to f'_*\mathcal{O}_{X'}(K_{X'} + (m-1)K_{X'/S} + p^*E) \cong f_*\mathcal{O}_X(K_X + (m-1)K_{X/S} + E)$$

is an isomorphism at the generic point of S. □

10.3. Corollary. *Let $f: X \to S$ be a surjective morphism between smooth proper varieties with general fiber X_g. Assume that S has generically large algebraic fundamental group. Let D be a big Cartier divisor on S. Then*

$$h^0(X, \mathcal{O}(K_X + (m-1)K_{X/S} + f^*D)) > 0 \Leftrightarrow h^0(X_g, \mathcal{O}(mK_{X_g})) > 0.$$

Proof. If $h^0(X_g, \mathcal{O}(mK_{X_g})) = 0$ then $f_*\mathcal{O}(K_X + (m-1)K_{X/S} + f^*D)) = 0$.

If $h^0(X_g, \mathcal{O}(mK_{X_g})) > 0$ then choose $L = D/2$ and $B = 0$ in (10.2) and construct X' as there. By (7.3) $h^0(X', \mathcal{O}(N')) > 0$ hence by (10.2.4)

$$h^0(X, \mathcal{O}(K_X + (m-1)K_{X/S} + f^*D)) > 0. \quad \square$$

The following is the first application to Iitaka-type problems:

10.4. Theorem. *Let $f: X \to S$ be a surjective morphism between smooth proper varieties with general fiber X_g. Assume that S is of general type and it has generically large algebraic fundamental group. Assume furthermore that $P_m(X_g) \neq 0$ for some $m \geqq 2$. Then*

$$P_m(X) \geqq P_{m-2}(S).$$

Proof. Choose $D = K_S$ in (10.3). We obtain that

$$h^0(S, \mathcal{O}(2K_S) \otimes f_*\mathcal{O}(mK_{X/S})) = h^0(X, \mathcal{O}(K_X + (m-1)K_{X/S} + f^*K_S)) > 0.$$

This gives an injection

$$\mathcal{O}((m-2)K_S) \to f_*\mathcal{O}(mK_X). \qquad \square$$

10.5. Remarks. (10.5.1) By choosing $D = jK_S, (j = 1, \ldots, m-1)$ we can get the slightly stronger result

$$P_m(X) \geqq \max\{P_0(S), \ldots, P_{m-2}(S)\}.$$

In particular $P_m(X) \geqq 1$.

(10.5.2) It is possible that $P_m(X) \geqq P_m(X_g)P_{m-2}(S)$ if f has connected fibers.

10.6. Proof of (10.1). By (8.4) it is sufficient to prove this for a suitable finite étale cover $p: X' \to X$. By (4.5') and (5.8) there is a finite étale cover $p: X' \to X$ and

a morphism $f': X' \to S$ where S has generically large algebraic fundamental group and either S is of general type or S is Abelian. We distinguish four cases:
(i) $\dim S = 3$. We are done by (8.5).
(ii) $\dim S = 1, 2$ *and S is of general type.* The general fiber of f' is a curve or a surface of general type, thus in both cases $P_m(X'_g) > 0$ for $m \geqq 2$. (10.4) gives that $P_m(X') > 0$ for $m \geqq 2$.
(iii) $\dim S = 2$ *and S is Abelian.* The general fiber of f' is a curve of positive genus, thus (8.10) applies.
(iv) $\dim S = 1$ *and S is Abelian.* The general fiber of f' is a surface of general type, thus $P_m(X'_g) > 0$ for $m \geqq 2$. (9.3) gives that $P_m(X') > 0$ for $m \geqq 2$. □

10.7. Proof of (1.13) and (1.15) (8.5)⇒(1.13.1); (1.13.1) and (9.5.2)⇒(1.13.2); (1.13.2) and [Kol, 4.6]⇒(1.13.3);
(9.1)⇒(1.15.1); (1.15.1) and (9.5.2)⇒(1.15.2); (1.15.2) and [Kol, 4.8]⇒(1.15.3). □

11 Albanese morphism

Let X be a smooth proper variety. Let $\mathrm{alb}: X \to \mathrm{Alb}(X)$ be the Albanese morphism. It was understood early on that the methods of the Iitaka conjecture work especially well to study the structue of this morphism. The main result of this chapter is to put the previous characterisations of Abelian varieties [U; KaV; Ka1; Kol; Mo] into nearly final form:

11.1. Theorem. *Let X be a smooth proper variety. The following are equivalent:*
(11.1.1) *X is birational to an Abelian variety;*
(11.1.2) *$q(X) = \dim X$ and $P_4(X) = 1$;*
(11.1.3) *$q(X) = \dim X$ and $P_m(X) = 1$ for some $m \geqq 4$.*

11.1.4. Correction to [Kol]. [Kol, 4.4] gives an example of a nonabelian compact complex S surface such that $q(S) = 2$ and $P_3(S) = 1$. Unfortunately S is not Kähler and its Albanese is one dimensional.

It is possible that $q(X) = \dim X$ and $P_m(X) = 1$ for some $m \geqq 2$ already implies that X is birational to an Abelian variety (even for X Kähler). For algebraic surfaces this was checked in [Kol, 4.5]. The higher dimensional situation is unknown.

The proof rests on the following result which improves [Kol, 5.1; Mo, 8.1].

11.2. Theorem. *Let X be a smooth proper variety.*

If $P_3(X) = 1$ or $0 < P_m(X) \leqq 2m - 6$ for some m then $\mathrm{alb}: X \to \mathrm{Alb}(X)$ *is surjective.*

Proof. Assume that $\mathrm{alb}: X \to \mathrm{Alb}(X)$ is not surjective. By [U, 10.9] there is a morphism $f: X \to S$ where S is a variety of general type which is birational to a subvariety of an Abelian variety. By [GriH, 4.14] $|K_S|$ gives a generically finite map, thus for $\dim S \geqq 2$

$$P_r(S) \geqq h^0(\mathbb{P}^{\dim S}, \mathcal{O}(r)) = \binom{r + \dim S}{\dim S} \geqq 2r.$$

For $\dim S = 1$ we get $P_r(S) \geqq 2r - 1$ and $P_1(S) \geqq 2$. Thus (10.4) implies (11.2).

11.3. Proof of (11.1) It is clear that (11.1.1)⇒(11.1.2)⇒(11.1.3). (11.1.3) and (11.2) imply that $\text{alb}(X)$ is onto, hence generically finite by dimension comparison. By [Ko1, 4.3] $P_m(X) \geqq 2$ for $m \geqq 4$ unless $\text{alb}(X)$ is birational. □

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