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Finiteness properties of local cohomology modules (an application of D-modules to Commutative Algebra)

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Summary. The main goal of this paper is to establish finiteness properties of local cohomology modules in characteristic 0 that would be analogous to those proven by C. Huneke and R. Sharp in characteristic $p > 0$. Our method, based on the theory of algebraic D-modules, seems to be the first application of D-modules to Commutative Algebra.

0 Introduction

Throughout this paper R is a commutative Noetherian ring. If M is an R-module and Y a locally closed subscheme of Spec R, we denote by $H_Y^i(M)$ the i-th local cohomology module of M with support in Y. If Y is closed in Spec R with defining ideal $I \subset R$, $H_Y^i(M)$ is denoted by $H_I^i(M)$. These local cohomology modules have been studied by a number of authors. See, for example, Faltings [F1, F2], Grothendieck [G1, G2], Hartshorne [Ha1, Ha2, Ha3], Hartshorne and Speiser [Ha-Sp], Hochster and Roberts [Ho-R], Huneke and Koh [Hu-K], Huneke and Lyubeznik [Hu-Ly], Huneke and Sharp [Hu-Sh], Ogus [O], Peskine and Szpiro [P-Sz] and Sharp [Sh]. Yet despite this extensive effort, the structure of these modules is still full of mystery. In most cases one cannot even tell if a given local cohomology module is zero. When it is non-zero, it is rarely finitely generated, even if M is. The more finiteness properties one could prove about them, the better understanding of their structure one would achieve.

In the case that $\dim R/I = 0$, the structure of $H_I^i(M)$, for a finitely generated M, has been extensively studied by Grothendieck [G2] and is well-understood. Although not necessarily finitely generated, it is Artinian. In particular, $\mathrm{Hom}_R(R/I, H_I^i(M))$ is finitely generated. This lead Grothendieck to conjecture that $\mathrm{Hom}_R(R/I, H_I^i(M))$ is always finitely generated, if M is [G2, Exp. XIII, 1.1]. Grothendieck's conjecture was shown to be false by Hartshorne [Ha2, Sect. 3],

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who gave an example of a (non-regular) three-dimensional local domain R and an ideal $I \subset R$ such that $H_I^2(R)$ is supported only at the maximal ideal, but is not Artinian, so even $\mathrm{Hom}_R(K, H_I^2(R))$, where K is the residue field of R, is not finitely generated.

If R is regular, $\mathrm{Hom}_R(R/I, H_I^i(R))$ need not be finitely generated either. In fact, Huneke and Koh [Hu-K, 2.3i] proved, that if R is any regular ring and r is bigger than the height of all minimal primes of I, $\mathrm{Hom}_R(R/I, H_I^i(R))$ is finitely generated for all $i \geqq r$ if and only if $H_I^i(R) = 0$ for all $i \geqq r$. Huneke and Koh also proved [Hu-K,2.3ii], that if R is a regular ring containing a field of characteristic $p > 0$ and i is any integer bigger than the height of all minimal primes of I, then $\mathrm{Hom}_R(R/I, H_I^i(R))$ is finitely generated if and only if $H_I^i(R) = 0$. In this paper we extend this latter result to the case of a regular ring containing a field of any characteristic (see (3.5) and (3.6e)).

On the bright side, the finiteness of $\mathrm{Hom}_R(K, H_I^i(R))$, where R is local and K is the residue field of R, has been known in some important cases. Ogus [O, 2.7] proved that if R is regular local, contains a field of characteristic 0, and $H_I^i(R)$, for all $i > r$, where r is some integer, are supported only at the maximal ideal, then $\mathrm{Hom}_R(K, H_I^i(R))$ are finitely generated for all $i > r$, while Hartshorne and Speiser [Ha-Sp, 2.4] proved, that if R is regular local, contains a field of characteristic $p > 0$, and $H_I^i(R)$ is supported only at the maximal ideal, then $\mathrm{Hom}_R(K, H_I^i(R))$ is finitely generated and, moreover, $H_I^i(R)$ is injective.

Huneke and Sharp [Hu-Sh] made a remarkable breakthrough. They generalized the above-mentioned results of Hartshorne and Speiser by proving that if R is any regular ring containing a field of characteristic $p > 0$, the local cohomology modules of R have the following properties:

(i) $H_m^j(H_I^i(R))$ is injective, where m is any maximal ideal of R.
(ii) $\mathrm{inj.dim}_R H_I^i(R) \leqq \dim_R H_I^i(R)$.
(iii) The set of the associated primes of $H_I^i(R)$ is finite.
(iv) All the Bass numbers of $H_I^i(R)$ are finite.

Here $\mathrm{inj.dim}_R H_I^i(R)$ stands for the injective dimension of $H_I^i(R)$, i.e. the length of its minimal injective resolution, $\dim_R H_I^i(R)$ stands for the dimension of the support of $H_I^i(R)$ in $\mathrm{Spec}\, R$ and the j-th Bass number of $H_I^i(R)$ with respect to a prime ideal P of R is defined as $\mu_j(P, H_I^i(R)) = \mathrm{length}_{K(R/P)}(\mathrm{Ext}^j_{R_P}(K(R/P), (H_I^i(R))_P))$, where $K(R/P)$ is the fraction field of R/P (see [Ba]). In particular, (iv) implies that if R is local with residue field K and maximal ideal m, then $\mathrm{length}_K(\mathrm{Hom}_R(K, H_I^i(R))) = \mu_0(m, H_I^i(R))$ is finite.

The main purpose of this paper is to obtain characteristic 0 analogs of these results of Huneke-Sharp. We prove, in particular, that if R is any regular ring containing a field of characteristic 0 and $Y \subset \mathrm{Spec}\, R$ is a locally closed subscheme, the local cohomology modules $T(R) = H_Y^i(R)$ have the following properties:

(0.1) $H_m^j(T(R))$ is injective, where m is any maximal ideal of R (see (3.4a)).
(0.2) $\mathrm{inj.dim}_R T(R) \leqq \dim_R T(R)$ (see (3.4b)).
(0.3) For every maximal ideal m of R, the number of associated primes of $T(R)$ contained in m is finite (see (3.4c)).
(0.4) All the Bass numbers of $T(R)$ are finite (see (3.4d)).

Our results are more general than all the previous ones not only in that we assume Y to be just locally closed, rather than closed ((0.1) and (0.2) hold even more generally, for $T(R) = H^i_{\psi/\varphi}(R)$ where $\varphi \subset \psi$ are two arbitrary families of supports on $\mathrm{Spec}\, R$). In fact, we prove that (0.1)–(0.4) hold for a considerably larger class of functors. Namely, if $Y_1 \subset Y_2$ are closed subsets of $\mathrm{Spec}\, R$ and T is either the kernel, or the image, or the cokernel of any of the three natural transformation $H^i_{Y_1}(-) \to H^i_{Y_2}(-)$, $H^i_{Y_2}(-) \to H^i_{Y_2 - Y_1}(-)$, or $H^i_{Y_2 - Y_1}(-) \to H^{i+1}_{Y_1}(-)$, then $T(R)$ satisfies (0.1)–(0.4). So does a composition of a finite number of such functors. In particular, if $Y_1, Y_2, \ldots, Y_t$ are locally closed subsets of $\operatorname{Spec} R$, the module $T(R) = H^{i_1}_{Y_1}(H^{i_2}_{Y_2}(\ldots H^{i_t}_{Y_t}(R)\ldots))$ satisfies (0.1)–(0.4).

Our method is completely different from that of Huneke and Sharp [Hu-Sh] and is, in fact, quite new in Commutative Algebra. While Huneke and Sharp use the Frobenius morphism, we use D-modules. Our results (0.1)–(0.4) proven in Theorem 3.4 follow from our results on D-modules proven in Theorem 2.4. To the best of our knowledge, this is the first application of D-modules to Commutative Algebra.

Our results together with those of [Hu-Sh] imply that if R is any regular ring containing a field, then all the Bass numbers of $H^i_I(R)$ are finite. In the last section of this paper we show that this fact enables one to define a new set of numerical invariants of any local ring A containing a field (of any characteristic). Namely, we prove, that if $\pi: R \to A$ is a surjection with R a regular local ring of dimension n containing a field and $I = \operatorname{Ker} \pi$, then $\mu_p(m, H^{n-i}_I(B))$, the p-th Bass number of $H^{n-i}_I(R)$ with respect to the maximal ideal, depends neither on R, nor on π, but is an invariant of A, p and i.

1 Preliminaries

Throughout the whole paper R is a commutative Noetherian ring. In this section we introduce functors $\mathscr{T}$ and T that are the main objects of study in this paper.

A family of supports ψ on $X = \operatorname{Spec} R$ is a set of closed subsets of X such that every closed subset of every $Z \in \psi$ belongs to ψ and $Z', Z'' \in \psi$ implies $Z' \cup Z'' \in \psi$. For an abelian sheaf S on X the group of the global sections of S whose support is contained in ψ is denoted by $\Gamma_\psi(X, S)$. If $\varphi \subset \psi$ is another family of supports, $\Gamma_\psi(X, S)/\Gamma_\varphi(X, S)$ is denoted by $\Gamma_{\psi/\varphi}(X, S)$. The i-th right derived functors of $\Gamma_\psi(X, -)$ and $\Gamma_{\psi/\varphi}(X, -)$ are denoted respectively, by $H^i_\psi(X, -)$ and $H^i_{\psi/\varphi}(X, -)$. Clearly, $H^i_\psi(X, -)$ is a special case of $H^i_{\psi/\varphi}(X, -)$ with $\varphi = \emptyset$. These functors are related by the following long exact sequence [Ha1, IV.1]:

$$
\begin{aligned}
&0 \to H^0_\varphi(X, -) \to H^0_\psi(X, -) \to H^0_{\psi/\varphi}(X, -) \\
&\to H^1_\varphi(X, -) \to H^1_\psi(X, -) \to H^1_{\psi/\varphi}(X, -) \ldots
\end{aligned} \tag{1.1}
$$

One of our main objects of study in this paper will be composite functors $\mathscr{T}$ of the form $\mathscr{T} = \mathscr{T}_1 \circ \mathscr{T}_2 \circ \ldots \circ \mathscr{T}_t$, where each $\mathscr{T}_j$ is either $H^{i_j}_{\psi_j/\varphi_j}(X, -)$, or the kernel of any arrow appearing in (1.1) with $\varphi = \varphi_j$ and $\psi = \psi_j$ where $\varphi_j \subset \psi_j$ are two families of supports on X. Since (1.1) is exact, allowing $\mathscr{T}_j$ to be the image, or the cokernel of any arrow appearing in (1.1) produces the same class of functors.

If Y is a closed subset of X and φ_Y is the set of all closed subsets of Y, the functors $\Gamma_{\varphi_Y}(X, -)$ and $H^i_{\varphi_Y}(X, -)$ are written as $\Gamma_Y(X, -)$ and $\Gamma^i_Y(X, -)$. More generally, if Y is a locally closed subset of X, i.e. $Y = Y'' - Y'$, where $Y' \subset Y''$ are two closed subsets of X, the functors $\Gamma^i_{\varphi_{Y''}/\varphi_{Y'}}(X, -)$ and $H^i_{\varphi_{Y''}/\varphi_{Y'}}(X, -)$ are written as $\Gamma_Y(X, -)$ and $H^i_Y(X, -)$; they depend only on Y, but not on Y', Y'' [G1, p. 1-2]. Clearly, $H^i_Y(X, -)$ is a special case of $H^i_{\psi/\varphi}(X, -)$. If $Y' \subset Y''$ are two closed subsets of X, (1.1) takes the following form:

$$\begin{aligned}(1.2)\qquad & 0\to H^0_{Y'}(X,-)\to H^0_{Y''}(X,-)\to H^0_{Y''-Y'}(X,-)\\ & \to H^1_{Y'}(X,-)\to H^1_{Y''}(X,-)\to H^1_{Y''-Y'}(X,-)\dots\end{aligned}$$

A special case of functors $\mathcal{T}$ are composite functors T of the form $T=T_1\circ T_2\circ\dots\circ T_t$, where each T_j is either $H^{i_j}_{Y_j}(X,-)$ with Y_j a locally closed subscheme of X, or the kernel of any arrow appearing in (1.2) with $Y'=Y'_j$ and $Y''=Y''_j$, where $Y'_j\subset Y''_j$ are two closed subschemes of X. Since (1.2) is exact, allowing T_j to be the image, or the cokernel of any arrow appearing in (1.2) produces the same class of functors.

If M is an R-module, we denote by $M^\sim$ the associated quasicoherent sheaf on X. If G is a functor from the category of abelian sheaves on X to the category of abelian groups, we denote $G(M^\sim)$ by $G(M)$. For every $r\in R$, the multiplication by $r\in R$ on M induces a map $M^\sim\to M^\sim$, which, if G is covariant, in turn induces a map $G(M)\to G(M)$ which we also call the multiplication by r. If G is both covariant and additive, the multiplication by elements of R thus defined gives $G(M)$ a structure of R-module. In particular, $\mathcal{T}(M)$ and $T(M)$ are R-modules, since $\mathcal{T}$ and T are covariant and additive. Every natural transformation $\eta: G\to G'$ with both G and G' covariant and additive, induces a homomorphism of R-modules $\eta': G(M)\to G'(M)$. In particular, since all arrows appearing in (1.1) and (1.2) are natural transformations, (1.1) and (1.2), applied to $M^\sim$, become exact sequences of R-modules and homomorphisms of R-modules. We denote $\Gamma_{\psi/\varphi}(X,M^\sim)$, $H^i_{\psi/\varphi}(X,M^\sim)$, $\Gamma_Y(X,M^\sim)$ and $H^i_Y(X,M^\sim)$ by, respectively, $\Gamma_{\psi/\varphi}(M,)$, $H^i_{\psi/\varphi}(M)$, $\Gamma_Y(M)$ and $H^i_Y(M)$. If Y is closed in X and $I\subset R$ is the defining ideal of Y, we denote $\Gamma_Y(M)$ and $H^i_Y(M)$ by, respectively, $\Gamma_I(M)$ and $H^i_I(M)$.

(1.2) with $Y'=Y$ and $Y''=X$ and the vanishing of $H^i(X,M^\sim)$ for $i>0$, imply an exact sequence $0\to H^0_Y(M)\to M\to H^1(X-Y,M^\sim)\to H^1_Y(M)\to 0$ and isomorphisms $H^i_Y(M)=H^{i-1}(X-Y,M^\sim)$ for $i\geqq 2$. Let $f_1,\dots,f_s\in R$ generate the defining ideal of Y. Since $H^*(X-Y,M^\sim)$ are the cohomology modules of the Čech complex of $M^\sim|(X-Y)$ for the covering of $X-Y$ by Spec R_{f_i}, splicing the above exact sequence with this Čech complex and considering the above isomorphisms we get a complex

$$(1.3)\qquad 0\to M\to\oplus M_{f_i}\to\oplus M_{f_if_j}\to\oplus M_{f_if_jf_k}\to\dots$$

whose i-th cohomology module is $H^i_Y(M)$ for all $i\geqq 0$. Here the map $M_{f_{i_1}\dots f_{i_j}}\to M_{f_{k_1}\dots f_{k_{j+1}}}$ induced by the corresponding differential is the natural localization (up to sign) if $\{i_1,\dots,i_j\}$ is a subset of $\{k_1,\dots,k_{j+1}\}$ and is 0 otherwise.

Let P be a prime ideal of R. By definition, the i-th Bass number of M with respect to P is $\mu_i(P,M)=\mathrm{length}_{K(R/P)}(\mathrm{Ext}^i_{R_P}(K(R/P),M_P))$, where $K(R/P)$ is the fraction field of R/P. We need the following lemma.

Lemma 1.4 *Let P be a prime of R and let M be an R-module such that $(H^i_P(M))_P$ are injective for all i. Let J^* be a minimal injective resolution of M. Then all the differentials in the complex $(\Gamma_P(J^*))_P$ are zero, $(H^i_P(M))_P=(\Gamma_P(J^i))_P$ and $\mathrm{Ext}^i_{R_P}(K(R/P),M_P)=\mathrm{Hom}_{R_P}(K(R/P),(H^i_P(M))_P)=\mathrm{Hom}_{R_P}(K(R/P),(\Gamma_P(J^i))_P)$. Hence $\mu_i(P,M)=\mu_0(P,H^i_P(M))$.*

Proof. If J is an injective R-module, $J^\sim$ is flasque, so $J^{*\sim}$ is a flasque resolution of $M^\sim$. Since flasque sheaves are acyclic for the functor $\Gamma_Y(X,-)$ [Ha1, IV, 1], the i-th cohomology module of $\Gamma_P(J^*)$ is $H^i_P(M)$, so the i-th cohomology module of

$(\Gamma_P(J^*))_P$ is $(H^i_P(M))_P$. Assume s is the smallest integer with non-zero $d_s : (\Gamma_P(J^s))_P \to (\Gamma_P(J^{s+1}))_P$. Then $\operatorname{Ker} d_s = H^s((\Gamma_P(J^*))_P) = (H^s_P(M))_P$ is injective, so $(\Gamma_P(J^s))_P = \operatorname{Ker} d_s \oplus J'$. Since J^* is minimal, the map $\operatorname{Hom}_{R_P}(K(R/P), (\Gamma_P(J^s))_P) \to \operatorname{Hom}_{R_P}(K(R/P), (\Gamma_P(J^{s+1}))_P)$ induced by d_s, is zero, so the map $\operatorname{Hom}_{R_P}(K(R/P), J') \to \operatorname{Hom}_{R_P}(K(R/P), (\Gamma_P(J^{s+1}))_P)$ is zero as well. Since J' is supported only at the maximal ideal of R_P and the map $J' \to (\Gamma_P(J^{s+1}))_P$ is injective, $J' = 0$. So $(\Gamma_P(J^s))_P = \operatorname{Ker} d_s$ and $d_s = 0$. This contradiction proves that all the differentials in the complex $(\Gamma_P(J^*))_P$ are zero. Hence $(H^i_P(M))_P = H^i((\Gamma_P(J^*))_P) = (\Gamma_P(J^i))_P$. Since $\operatorname{Ext}^i_{R_P}(K(R/P), (M_P) = H^i(\operatorname{Hom}_{R_P}(K(R/P), (J^*)_P) = H^i(\operatorname{Hom}_{R_P}(K(R/P), (\Gamma_P(J^*))_P) = \operatorname{Hom}_{R_P}(K(R/P), (\Gamma_P(J^i))_P)$, the lemma is proven.

2 *D*-modules

The main result of this section is Theorem 2.4 that gives information about the injective dimension, the associated primes and the Bass numbers of D-modules. It is of independent interest plus it will be used in the next section to obtain similar results about modules of the form $\mathscr{T}(R)$ and $T(R)$, which is the main goal of this paper.

Let K be a subring of R. We denote by $D(R, K)$ the subring of $\operatorname{Hom}_K(R, R)$ generated by the K-linear derivations $R \to R$ and the multiplications by elements of R. By a $D(R, K)$-module we always mean a left $D(R, K)$-module. The injective ring homomorphism $R \to D(R, K)$ that sends r to the map $R \to R$ which is the multiplication by r, gives $D(R, K)$ a structure of R-algebra. Every $D(R, K)$-module M is automatically an R-module via this map. We denote by $M^\sim$ the associated quasicoherent sheaf on $\operatorname{Spec} R$.

Examples 2.1(i) The natural action of $D(R, K)$ on R makes R a $D(R, K)$-module.

(ii) If M is a $D(R, K)$-module and $S \subset R$ is a multiplicative system of elements, M_S carries a natural structure of $D(R, K)$-module. Namely, for $r \in R$ we set $r(m/s) = (rm)/s$, for a derivation d we define $d(m/s)$ using the quotient rule, i.e. $d(m/s) = (sd(m) - d(s)m)/s^2$ and this uniquely extends to an action of $D(R, K)$ on M_S. In particular, $D(R, K)_S$ has a natural structure of $D(R, K)$-module. This implies that $D(R, K)_S$ has a natural ring structure and M_S has a natural structure of $D(R, K)_S$-module.

(iii) Let M be a $D(R, K)$-module and let G be any covariant additive functor from the category of sheaves of K-modules on $\operatorname{Spec} R$ to the category of abelian groups. We claim that $G(M)$ has a natural structure of $D(R, K)$-module and every natural transformation $\eta : G \to G'$ induces a homomorphism of $D(R, K)$-modules $\eta' : G(M) \to G'(M)$. Indeed, for all $f \in R$, M_f carries a natural structure of $D(R, K)$-module, so there exists a natural ring homomorphism $h_f : D(R, K) \to \operatorname{Hom}_K(M_f, M_f)$ that sends each $\delta \in D(R, K)$ to the action of δ on M_f. Since $\operatorname{Spec} R_f$ form a base for the topology of $\operatorname{Spec} R$, the h_f's patch up to give a ring homomorphism $h : D(R, K) \to \operatorname{End}(M^\sim)$, where the endomorphisms are taken in the category of sheaves of K-modules. Since $G(M)$ has a natural structure of $\operatorname{End}(M^\sim)$-module, h gives it a natural structure of $D(R, K)$-module. Since η' is a homomorphism of $\operatorname{End}(M^\sim)$-modules, it is a homomorphism of $D(R, K)$-modules via h.

(iv) Let M be a $D(R, K)$-module. Since $\mathscr{T}$ and T are additive and covariant, (iii) implies, that $\mathscr{T}(M)$ and $T(M)$ have a natural structure of $D(R, K)$-modules.

All functors involved in (1.1) and (1.2) are additive and covariant and all arrows are natural transformations. So, (1.1) and (1.2) applied to $M^{\sim}$, become sequences of $D(R, K)$-modules and homomorphisms of $D(R, K)$-modules.

(v) If $R = K[[X_1, \ldots, X_n]]$ is the ring of formal power series in n variables over K, the K-linear derivations form a free R-module on the n generators $d_1, \ldots, d_n$, where $d_j: R \to R$ is the partial differentiation with respect to X_j. $D(R, K)$ is generated by $d_1, \ldots, d_n$ as an R-algebra and is a free left (as well as right) R-module on the monomials $d_i^{i_1} \ldots d_n^{i_n}$, with $i_j \geqq 0$ (we set $d_1^0, \ldots, d_n^0 = 1$). $D(R, K)$ may be thought of as the associative non-commutative R-algebra with generators $d_1, \ldots, d_n$ and relations $d_i d_j = d_j d_i$ and $d_i r - r d_i = \partial r/\partial X_i$ for all i, j and all $r \in R$.

In general, the ring $D(R, K)$ does not have too many good properties. But if K is a field of characteristic 0 and R is a ring of formal power series in a finite number of variables over K, then $D(R, K)$ is left and right Noetherian [Bj, 3.1.6]. This implies that every finitely generated $D(R, K)$-module is Noetherian. In addition there exists a remarkable class of finitely generated $D(R, K)$-modules, called holonomic $D(R, K)$-modules [Bj, p. 100]. Some of the properties of holonomic modules are as follows:
(2.2a) R with its natural structure of $D(R, K)$-module is holonomic [Bj, 3.3.2].
(2.2b) If M is holonomic and $f \in R$, then M_f is holonomic [Bj, 3.4.1].
(2.2c) The holonomic modules form an abelian subcategory of the category of $D(R, K)$-modules, which is closed under formation of submodules, quotient modules and extensions. (A proof of this is completely analogous to the proof of [Bj, 1.5.2].)
(2.2d) If M is holonomic, then $T(M)$ is holonomic. Indeed, as $T = T_1 \circ T_1 \circ \ldots \circ T_t$, by induction on t it is enough to prove that $T_t(M)$ is holonomic. It follows from the definition of T_t that $T_t(M)$ is a $D(R, K)$-submodule of $H_Y^i(M)$, where Y is locally closed in Spec R. So by (2.2c) it is enough to prove that $H_Y^i(M)$ is holonomic. By (1.2), there exists an exact sequence $H_{Y''}^i(M) \to H_Y^i(M) \to H_{Y'}^{i+1}(M)$, where Y' and Y'' are closed. So, by (2.2c) it is enough to prove that $H_Y^i(M)$ is holonomic, where Y is closed in Spec R. By (2.2b) all modules appearing in (1.3) are holonomic. All the maps $M_{f_{i_1} \ldots f_{i_j}} \to M_{f_{k_1} \ldots f_{k_j} f_{k_{j+1}}}$ induced by the differentials of (1.3) are either 0 or the natural localizations (up to sign), so every arrow in (1.3) is a homomorphism of $D(R, K)$-modules and by (2.2c) $H_Y^i(M)$ is holonomic.
(2.2e) A holonomic module is semisimple, i.e. has a finite filtration with simple quotients [Bj, 2.7.13]. (Of course, a simple $D(R, K)$-module is one with no non-trivial $D(R, K)$-submodules.)
(2.2f) A simple holonomic module M has just one associated prime [Bj, 3.3.16–17]. Let Q be the associated prime of M and let $M_0 = \{m \in M \mid Qm = 0\}$. Then there exists a non-zero element $h \in R/Q$ such that $(M_0)_h$ is a finitely generated $(R/Q)_h$-module. This is because, by Noether normalization, there exists a linear change of variables such that R/Q is finite over $R' = K[[X_1, \ldots, X_k]]$ where $k = \dim R/Q$, and by [Bj, p. 109, lines 3–6] there exists a non-zero $h \in R'$ such that $(M_0)_h$ is a finitely generated R'_h-module.

Proposition 2.3 *Let K be a field of characteristic 0, let $R = K[[X_1, \ldots, X_n]]$ be a ring of formal power series in n variables over K and let m be the maximal ideal of R. Then as an R-module, $D(R, K)/D(R, K)m$ is isomorphic to $E_R(K)$, the injective hull of the residue field of R in the category of R-modules.*

Proof. Since R is regular, $E_R(K) = H^n_m(R)$. Since m is generated by $X_1, \ldots, X_n$, it follows from (1.3) that $E_R(K)$ is the vector space over K spanned by the monomials $X_1^{i_1} \ldots X_n^{i_n}$ with all $i_j \leqq -1$ and with a natural R-module structure (that is, if $X^\alpha = X_1^{\alpha_1} \ldots X_n^{\alpha_n} \in R$ and $X^\beta = X_1^{\beta_1} \ldots X_n^{\beta_n} \in E_R(K)$, then $X^\alpha X^\beta = X_1^{\alpha_1+\beta_1} \ldots X_n^{\alpha_n+\beta_n}$, if $\alpha_i + \beta_i < 0$ for all i and $X^\alpha X^\beta = 0$ otherwise). It follows from (2.1v) that $D(R, K)/D(R, K)m$ is a free K-module on the monomials $d_1^{i_1} \ldots d_n^{i_n}$ with $i_j \geqq 0$. The K-linear map $D(R, K)/D(R, K)m \to E_R(K)$ that sends $d_1^{i_1} \ldots d_n^{i_n}$ to $(-1)^{i_1+\cdots+i_n}(i_1!) \ldots (i_n!)X_1^{-i_1-1} \ldots X_n^{-i_n-1}$ is an isomorphism of R-modules. q.e.d.

Our main result in this section is the following.

Theorem 2.4 *Let K be a field of characteristic 0, let $R = K[[X_1, \ldots, X_n]]$ be a ring of formal power series in n variables over K, let m be the maximal ideal of R and let M be a $D(R, K)$-module.*

(a) *If* $\dim_R M = 0$, *then M is a direct sum of copies of $D(R, K)/D(R, K)m$.*

(b) $\operatorname{inj.dim}_R M \leqq \dim_R M$.

(c) *If M is finitely generated, the set of the associated primes of M is finite. (Of course, by an associated prime of M we mean a prime of R associated to the R-module M).*

(d) *If M is holonomic, all the Bass numbers of M are finite.*

Proof. (a) The socle of $D(R, K)/D(R, K)m$ is the one-dimensional vector space over K spanned by 1. (The socle of an R-module is the submodule annihilated by m). Let $\{e_i\}_{i\in I}$ be a K-basis of the socle of M. There is a homomorphism of $D(R, K)$-modules $(D(R, K)/D(R, K)m)^I \to M$, that sends the element 1 of the i-th copy of $D(R, K)/D(R, K)m$ to e_i. This map is injective, because it induces an isomorphism on the socles and $(D(R, K)/D(R, K)m)^I$ is supported only at m. By (2.3) $(D(R, K)/D(R, K)m)^I$ is an injective R-module, so $M = (D(R, K)/ D(R, K)m)^I \oplus N$, where N is an R-module supported only at m. Since the map on the socles is an isomorphism, $N = 0$, so $M = (D(R, K)/D(R, K)m)^I$. This proves (a).

(b) Let P be a minimal prime of M and let $(R_P)^\wedge$ be the completion of the local ring R_P with respect to its maximal ideal. We claim that M_P has a natural structure of $D((R_P)^\wedge, K')$-module, where $K' \subset (R_P)^\wedge$ is a suitable coefficient field of $(R_P)^\wedge$. Indeed, M_P has a natural structure of R_P-module and since every element of M_P is annihilated by a power of P, M_P has a natural structure of $(R_P)^\wedge$-module. Let the height of P be h. By Noether normalization, we can assume, after a possible change of variables, that R/P is finite over $S = K[[X_{h+1}, \ldots, X_n]]$. Let Ω be the module of continuous S-linear differentials of R and let $d : R \to \Omega$ be the canonical S-linear derivation. So, Ω is the free R-module of rank h on $dX_1, \ldots, dX_h$. We denote the R-module of S-linear derivations $R \to R$ by $\operatorname{Der}_S R$. The universal property of Ω implies that the map $\operatorname{Hom}_R(\Omega, R) \to \operatorname{Der}_S R$ that sends $f \in \operatorname{Hom}_R(\Omega, R)$ to the derivation $fd : R \to R$ is an isomorphism of R-modules. Since Ω and R are finitely generated, $(\operatorname{Der}_S R)_P = (\operatorname{Hom}_R(\Omega, R))_P = \operatorname{Hom}_{R_P}(\Omega_P, R_P)$. Let $K(S)$ be the fraction field of S. Then $d : R \to \Omega$ extends via the quotient rule to the $K(S)$-linear derivation $d_P : R_P \to \Omega_P$. Let $Z_1, \ldots, Z_h \in R$ generate $P_P \subset R_P$. Since Ω_P is a free R_P-module of rank h on $dZ_1, \ldots, dZ_h$, we get $K(S)$-linear derivations $\delta_i = f_i d_P : R_P \to R_P$ $(i = 1, \ldots, h)$, where $f_i \in \operatorname{Hom}_{R_P}(\Omega_P, R_P)$ is defined by $f_i(dZ_j) = 0$ if $i \neq j$ and $f_i(dZ_i) = 1$. Since $\delta_i((P_P)^t) \subset (P_P)^{t-1}$, the δ_i's uniquely extend to derivations $\delta_i^\wedge : (R_P)^\wedge \to (R_P)^\wedge$. Let K' be the algebraic closure of $K(S)$ in $(R_P)^\wedge$. Every $K(S)$-linear derivation $(R_P)^\wedge \to (R_P)^\wedge$ is automatically K'-linear. Hence the $\delta_i^\wedge$'s are K'-linear. By Cohen's structure theorem $(R_P)^\wedge = K'[[Z_1, \ldots, Z_h]]$. The only K'-linear derivation $(R_P)^\wedge \to (R_P)^\wedge$ that sends Z_i to 1 and Z_j to 0 for all $j \neq i$ is the partial differentiation with respect to Z_i. Hence $\delta_i^\wedge$ is nothing but the partial differentiation with respect to Z_i. So $\delta_i^\wedge \delta_j^\wedge = \delta_j^\wedge \delta_i^\wedge$ and $\delta_i^\wedge r^\wedge - r^\wedge \delta_i^\wedge = \delta_i^\wedge(r)^\wedge$ for all i, j and all $r \in (R_P)^\wedge$, where we denote by $r^\wedge : (R_P)^\wedge \to (R_P)^\wedge$ (resp. $\delta_i^\wedge(r)^\wedge$: $(R_P)^\wedge \to (R_P)^\wedge$) the multiplication by r (resp. $\delta_i^\wedge(r)$). By (2.1ii) $D(R, S)_P$ forms a ring. By restricting the above relations to $R_P \subset (R_P)^\wedge$ we see that $\delta_i \delta_j = \delta_j \delta_i$ and $\delta_i r - r\delta_i = \delta_i(r)$ in $D(R, S)_P$ for all i, j and all $r \in R_P$. Since $D(R, S)$ is a subring of $D(R, K)$ and M_P is a $D(R, K)$-module, M_P is a $D(R, S)$-module, hence also a $D(R, S)_P$-module. So, $\delta_i^* \delta_j^* = \delta_j^* \delta_i^*$ and $\delta_i^* r^* - r^* \delta_i^* = \delta_i(r)^*$ for all i, j and all $r \in R_P$, where $\delta_i^* : M_P \to M_P$, $r^* : M_P \to M_P$ and $\delta_i(r)^* : M_p \to M_P$ are induced by the $D(R, S)_P$-module structure. Note that $r^* : M_P \to M_P$ (resp. $\delta_i(r)^* : M_P \to M_P$) is nothing but the multiplication by r (resp. $\delta_i(r)$) induced by the R_P-module structure. For each $r \in (R_P)^\wedge$ and $v \in M_P$ there exists $r' \in R_P$ such that $\delta_i^* r'^*(v) = \delta_i^* r^*(v)$, $r'^* \delta_i^*(v) = r^* \delta_i^*(v)$ and $\delta_i(r')^*(v) = \delta_i^\wedge(r)^*(v)$, so $\delta_i^* r^* - r^* \delta_i^* = \delta_i^\wedge(r)^*$ for all $r \in (R_P)^\wedge$. Since $\delta_i^\wedge(r) = \partial r/\partial Z_i$, (2.1v) implies that by letting d_i act on M_P via δ_i^* we get a well-defined ring homomorphism $D((R_P)^\wedge, K') \to \mathrm{End}_{K'}(M_P)$. This makes M_P a $D((R_P)^\wedge, K')$-module and proves the claim.

So, by (a) and (2.3), M_P is a direct sum of copies of $E_{(R_P)^\wedge}((R_P)^\wedge/P(R_P)^\wedge)$. But as an R-module $E_{(R_P)^\wedge}((R_P)^\wedge/P(R_P)^\wedge)$ is isomorphic to $E_R(R/P)$, so M_P is an injective R-module.

Now we use induction on $d = \dim_R M$. The case $d = 0$ follows from (a) and (2.3). Assume $d > 0$ and let $M' = \Gamma_\varphi(M)$, where φ is the family of all closed subsets of Spec R of dimension $< d$. This is a $D(R, K)$-module by (2.1iv), so, by induction inj. $\dim_R M' < d$. The resulting short exact sequence $0 \to M' \to M \to M'' \to 0$ shows that it is enough to prove that inj. $\dim_R M'' \leqq d$. In other words, we can assume that $M = M''$, i.e. that all the associated primes of M have dimension d. Let $\{P_j\}$ be the associated primes of M. Since M has no embedded associated primes, the natural map $M \to \oplus M_{P_i}$ is an injective homomorphism of $D(R, K)$-modules and its cokernel has dimension $< d$. Since this cokernel is a $D(R, K)$-module, its injective dimension is $< d$ by induction. Since $\oplus M_{P_i}$ is injective, inj. $\dim_R M \leqq d$. This proves (b).

(c) We claim there is a finite filtration of M by $D(R, K)$-submodules $0 = M_0 \subset M_1 \subset \ldots \subset M_i = M$ such that M_j/M_{j-1} has only one associated prime. For let P_1 be a maximal element in the set of the associated primes of M. Then $\Gamma_{P_1}(M) \subset M$ is non-zero and has only one associated prime, namely, P_1. Set $M_1 = \Gamma_{P_1}(M)$. By (2.1iv) this is a $D(R, K)$-submodule of M, so M/M_1 is a $D(R, K)$-module. Let P_2 be a maximal element in the set of the associated primes of M/M_1. Then $\Gamma_{P_2}(M/M_1)$ is a non-zero $D(R, K)$-submodule of M/M_1 and has only one associated prime, namely, P_2. Set M_2 to be the preimage of $\Gamma_{P_2}(M/M_1)$ in M. Since M is Noetherian [Bj, 3.1.6], this process eventually stops. This proves the claim. The set of the associated primes of M is contained in the union of the sets of the associated primes of all M_j/M_{j-1}. This proves (c).

(d) Let P be a prime ideal of R. By (2.1iv) $H_P^i(M)$ is a $D(R, K)$-module. Since P is a minimal prime of $H_P^i(M)$, it follows like in the proof of (b) that $(H_P^i(M))_P$ is an injective R-module. So, (1.4) implies that $\mu_i(P, M) = \mu_0(P, H_P^i(M))$. By (2.2d) $H_P^i(M)$ is holonomic, so it is enough to prove, that if N is holonomic and Supp $N \subset V(P)$, then $\mu_0(P, N)$ is finite. By (2.2e) there is a finite filtration of N with simple quotients. By (2.2c) these quotients are holonomic. A short exact sequence $0 \to N' \to N \to N'' \to 0$ gives an exact sequence $0 \to \mathrm{Hom}_{R_P}(K(R/P), N'_P)$

$\to \mathrm{Hom}_{R_P}(K(R(P), N_P) \to \mathrm{Hom}_{R_P}(K(R/P), N''_P)$, hence, if $\mu_0(P, N')$ and $\mu_0(P, N'')$ are finite, so is $\mu_0(P, N)$. So by induction on the length of the filtration, it is enough to consider the case that N is simple. By (2.2f) N has just one associated prime Q. If $Q \neq P$, then $N_P = 0$ and $\mu_0(P, N) = 0$. So, assume $Q = P$. Set $N_0 = \{v \in N \mid Pv = 0\}$. By (2.2f) there is a non-zero element $h \in R/P$ such that $(N_0)_h$ is a finitely generated $(R/P)_h$-module. Since $\mathrm{Hom}_{R_P}(K(R/P), N_P) = (N_0)_P = ((N_0)_h)_P$, we are done.

Corollary 2.5 *Let R be a ring of formal power series in a finite number of variables over a field of characteristic 0 and let G be any additive covariant functor from the category of abelian sheaves on* Spec R *to the category of abelian groups. Then inj.* $\dim_R(G(R)) \leq \dim_R(G(R))$. *In particular, if* $\dim_R(G(R)) = 0$, *then $G(R)$ is injective. For example, inj.* $\dim_R \mathscr{T}(R) \leq \dim_R \mathscr{T}(R)$ *and, in particular, if* $\dim_R(\mathscr{T}(R)) = 0$, *then $\mathscr{T}(R)$ is injective.*

Proof. This follows from (2.1iii) and (2.4b).

Remark 2.6 In [Ly] we prove that a result completely analogous to (2.5) holds if R is a ring of formal power series in a finite number of variables over a field of characteristic $p > 0$.

Corollary 2.7 *Let R be a ring of formal power series in a finite number of variables over a field of characteristic 0. Then the set of the associated primes of $T(R)$ is finite and all the Bass numbers of $T(R)$ are finite.*

Proof. This follows from (2.2d) and (2.4.c, d).

Question 2.8 If R is a ring of formal power series in a finite number of variables over a field of characteristic 0 and $K, K' \subset R$ are two different coefficient fields of R (of course, K is isomorphic to K'), then $D(R, K)$ is isomorphic to $D(R, K')$. Call this ring D. A priori $D(R, K)$ and $D(R, K')$ give $T(R)$ two different structures of D-module. In other words, the structure of D-module on $T(R)$ depends on the choice of a coefficient field. To what extent is this structure independent of the choice of a coefficient field? For example, is the length of $T(R)$ as a holonomic D-module independent of the choice of a coefficient field?

Remark 2.9 Properties (2.2a)–(2.2f) are valid also in the case that K is an algebraically closed field of characteristic 0 and R is a regular domain finitely generated as K-algebra [Bj, 3.2]. In this case results analogous to (2.4) and (2.5) also hold and their proofs are practically the same as above.

3 The main result

The main result of this section (and the whole paper) is Theorem 3.4. It establishes properties of the injective dimension, the associated primes and the Bass numbers of modules of the form $\mathscr{T}(R)$ and $T(R)$, that are analogous to those proven in the above Theorem 2.4 for D-modules. Our method is to reduce to results of the preceding section by localization and completion. We begin with a few preliminary remarks concerning the behavior of the functors $\mathscr{T}$ and T under ring homomorphisms.

Let $R^\wedge$ be another commutative Noetherian ring and let $g: R \to R^\wedge$ be a ring homomorphism. For a locally closed subscheme Y of $X = \mathrm{Spec}\, R$ we denote by

$Y^\wedge$ its preimage in $X^\wedge = \operatorname{Spec} R^\wedge$. If φ is a family of supports on $\operatorname{Spec} R$, we denote by $\varphi^\wedge$ the family of supports on $\operatorname{Spec} R^\wedge$ consisting of all the closed subsets of all the $Y^\wedge$, where $Y \in \varphi$. If $\mathcal{T} = H^i_{\psi/\varphi}(X, -)$ (resp. $T = H^i_Y(X, -)$), we denote by $\mathcal{T}^\wedge$ (resp. $T^\wedge$) the functor $H^i_{\psi^\wedge/\varphi^\wedge}(X^\wedge, -)$ (resp. $H^i_{Y^\wedge}(X^\wedge, -)$). If $\mathcal{T}$ (resp. T) is the kernel of an arrow appearing in (1.1) (resp. (1.2)), we denote by $\mathcal{T}^\wedge$ (resp. $T^\wedge$) the kernel of the same arrow with X replaced by $X^\wedge$ and ψ and φ (resp. Y'' and Y') replaced by $\psi^\wedge$ and $\varphi^\wedge$ (resp. $Y''^\wedge$ and $Y'^\wedge$). If $\mathcal{T} = \mathcal{T}_1 \circ \mathcal{T}_2 \circ \ldots \circ \mathcal{T}_t$ (resp. $T = T_1 \circ T_2 \circ \ldots \circ T_t$)), we set $\mathcal{T}^\wedge = \mathcal{T}_1^\wedge \circ \mathcal{T}_2^\wedge \circ \ldots \mathcal{T}_t^\wedge$ (resp. $T^\wedge = T_1^\wedge \circ T_2^\wedge \circ \ldots \circ T_t^\wedge$).

Lemma 3.1 *If M is an $R^\wedge$-module which is flat over R and N is any R-module, then there are isomorphisms $\mathcal{T}^\wedge(M \otimes_R N) = M \otimes_R \mathcal{T}(N)$ and $T^\wedge(M \otimes_R N) = M \otimes_R T(N)$ which are functorial in N.*

Proof. Since T is a special case of $\mathcal{T}$, it is enough to prove that there is an isomorphism $\mathcal{T}^\wedge(M \otimes_R N) = M \otimes_R \mathcal{T}(N)$ which is functorial in N. If P is a prime ideal of R, then $E_R(R/P)$ is divisible by every $r \in R \backslash P$. Since M is a flat R-module, $M \otimes_R E_R(R/P)$ also is divisible by every $r \in R \backslash P$. Since $\operatorname{Supp}(M \otimes_R E_R(R/P)) \subset V(P)$, the sheaf $(M \otimes_R E_R(R/P))^\sim$ is constant on its support, therefore it is flasque. So, $(M \otimes_R J)^\sim$ is flasque for every injective R-module J, because J is a direct sum of modules of the form $E_R(R/P)$. Hence if J^* is an injective resolution of an R-module N, then $(M \otimes_R J^*)^\sim$ is a flasque resolution of $(M \otimes_R N)^\sim$. Since flasque sheaves are acyclic for $\Gamma_{\psi/\varphi}(X, -)$ [Ha1, IV, 1], $H^i_{\psi/\varphi}(X, (M \otimes_R N)^\sim) = H^i(\Gamma_{\psi/\varphi}(M \otimes_R J^*))$. Since M is flat, the functors $\Gamma_{\psi/\varphi}(M \otimes_R -)$ and $M \otimes_R \Gamma_{\psi/\varphi}(-)$ are isomorphic, so

$$(3.2) \qquad H^i_{\psi/\varphi}(M \otimes_R N) = M \otimes_R H^i_{\psi/\varphi}(N).$$

Every map $N \to N'$ lifts to a chain map of injective resolutions which is unique up to homotopy. This implies that (3.2) is functorial in N. If L is an $R^\wedge$-module, we denote by ${}_RL$ the same L regarded as an R-module via g. If J is an injective $R^\wedge$-module, $J^\sim$ is flasque, hence so is $({}_RJ)^\sim$. So, if J^* is a resolution of L by injective $R^\wedge$-modules, $H^i_{\psi/\varphi}({}_RL)$ is the i-th cohomology module of the complex $\Gamma_{\psi/\varphi}({}_RJ^*)$. Since $H^i_{\psi^\wedge/\varphi^\wedge}(L)$ is the i-th cohomology module of the complex $\Gamma_{\psi^\wedge/\varphi^\wedge}(J^*)$, and since $\Gamma_{\psi/\varphi}({}_R-) = \Gamma_{\psi^\wedge/\varphi^\wedge}(-)$,

$$(3.3) \qquad H^i_{\psi/\varphi}({}_RL) = H^i_{\psi^\wedge/\varphi^\wedge}(L).$$

(3.2) and (3.3) with $L = M \otimes_R N$ imply (3.1) in the case that $\mathcal{T}$ is $H^i_{\psi/\varphi}(X, -)$. Since M is flat over R, (3.1) also follows in the case that $\mathcal{T}$ is the kernel of any arrow appearing in (1.1). The general case follows from these special cases by induction on t. q.e.d.

Now we are ready for our main result.

Theorem 3.4 *Let K be a field of characteristic* 0 *and let R be any regular K-algebra.*

(a) *Let m be a maximal ideal of R. Then $H^i_m(\mathcal{T}(R))$ is an injective R-module.*

(b) *inj.* $\dim_R(\mathcal{T}(R)) \leq \dim_R(\mathcal{T}(R))$. *In particular, if* $\dim_R(\mathcal{T}(R)) = 0$, *then $\mathcal{T}(R)$ is injective.*

(c) *For every maximal ideal m of R the set of the associated primes of $T(R)$ contained in m is finite.*

(d) *All the Bass numbers of $T(R)$ are finite.*

Proof. (a) Let $R^\wedge$ be the completion of R with respect to m. Then (3.1), applied to the functor $H^j_m(\mathscr{T}(-))$ shows that $H^j_{mR^\wedge}(\mathscr{T}^\wedge(R^\wedge)) = R^\wedge \otimes_R H^j_m(\mathscr{T}(R))$. Since $H^j_m(\mathscr{T}(R))$ is supported only at m, $R^\wedge \otimes_R H^j_m(\mathscr{T}(R)) = H^j_m(\mathscr{T}(R))$, so $H^j_{mR^\wedge}(\mathscr{T}^\wedge(R^\wedge)) = H^j_m(\mathscr{T}(R))$. By one of Cohen's structure theorems $R^\wedge$ is a ring of power series in several variables over a coefficient field K' of $R^\wedge$. By (2.1iv) $H^j_{mR^\wedge}(\mathscr{T}^\wedge(R^\wedge))$ is a $D(R^\wedge, K')$-module, and since its dimension is 0, (2.4a) and (2.3) show that $H^j_{mR^\wedge}(\mathscr{T}^\wedge(R^\wedge))$ is a direct sum of copies of $E_{R^\wedge}(R^\wedge/mR^\wedge) = E_R(R/m)$, hence injective. This proves (a).

(b) inj. $\dim_R(\mathscr{T}(R)) \leqq \dim_R(\mathscr{T}(R))$ if and only if inj. $\dim_{R_P}(\mathscr{T}(R)_P) \leqq \dim_{R_P}(\mathscr{T}(R)_P)$ for all primes P of R. Since (3.1) with $N = R$ and $M = R^\wedge = R_P$ implies that $\mathscr{T}(R)_P = \mathscr{T}^\wedge(R_P)$, we can assume that R is local. Let m be the maximal ideal of R and let J^* be a minimal injective resolution of $\mathscr{T}(R)$. By induction on $\dim_R \mathscr{T}(R))$ we can assume, that for all non-maximal primes P of R, inj. $\dim_{R_P}(\mathscr{T}(R)_P) \leqq \dim_{R_P}(\mathscr{T}(R)_P) \quad < \dim_R(\mathscr{T}(R))$. Hence all J^i with $i \geqq \dim_R(\mathscr{T}(R))$ are supported at m, i.e. $\Gamma_m(J^i) = J^i$ for $i \geqq \dim_R(\mathscr{T}(R))$. Now (a) and (1.4) imply that all the differentials $J^i \to J^{i+1}$ are zero for $i \geqq \dim_R(\mathscr{T}(R))$. Since J^* is minimal, $J^i = 0$ for $i > \dim_R(\mathscr{T}(R))$. This proves (b).

(c) If P is an associated prime of $T(R)$ contained in m, it is necessarily the restriction to R of an associated prime of $R^\wedge \otimes_R T(R)$, where $R^\wedge$ is the completion of R with respect to m. By one of Cohen's structure theorems $R^\wedge = K'[[X_1, \ldots, X_n]]$ where $K' = R/m$. By (3.1) with $N = R$ and $M = R^\wedge$, $R^\wedge \otimes_R T(R) = T^\wedge(R^\wedge)$. By (2.2d) this is a holonomic $D(R^\wedge, K')$-module, so by (2.4c) it has finitely many associated primes. This proves (c).

(d) Let P be a prime ideal of R. By definition $\mu_i(P, T(R)) = \text{length}_{K'} \operatorname{Ext}^i_{R_P}(K', T(R)_P)$, where K' is the field of fractions of R/P. By (3.1) with $N = R$ and $M = R^\wedge = R_P$, we get $T(R)_P = T^\wedge(R_P)$, so $\mu_i(P, T(R)) = \mu_i(P_P, T^\wedge(R_P))$. Therefore, we can assume that R is local and P is the maximal ideal of R. By (a) and (1.4), $\mu_i(P, T(R)) = \text{length}_{K'} \operatorname{Hom}_R(K', H^i_P(T(R)))$. Let $R^\wedge$ be the completion of R with respect to its maximal ideal. Then $R^\wedge = K'[[X_1, \ldots, X_h]]$ and $PR^\wedge = (X_1, \ldots, X_h)$. Lemma 3.1 applied to $H^i_P(T(-))$ shows that $H^i_{PR^\wedge}(T^\wedge(R^\wedge)) = R^\wedge \otimes_R H^i_P(T(R))$. But $R^\wedge \otimes_R H^i_P(T(R)) = H^i_P(T(R))$ as $\dim_R H^i_P(T(R)) = 0$. So, $\text{length}_{K'} \operatorname{Hom}_R(K', H^i_P(T(R))) = \text{length}_{K'} \operatorname{Hom}_{R^\wedge}(K', H^i_{PR^\wedge}(T^\wedge(R^\wedge)))$, that is, $\mu_i(P, T(R)) = \mu_0(PR^\wedge, H^i_{PR^\wedge}(T^\wedge(R^\wedge)))$. By (2.2d) $H^i_{PR^\wedge}(T^\wedge(R^\wedge))$ is a holonomic $D(R^\wedge, K')$-module. So (2.4d) implies that $\mu_0(PR^\wedge, H^i_{PR^\wedge}(T^\wedge(R^\wedge)))$ is finite. This proves (d) and the theorem.

It is worth pointing out that for any maximal ideal m of R (3.4) implies that $H^i_m(T(R))$ is a direct sum of a finite number of copies of $E_R(R/m)$. Indeed, by (3.4a) $H^i_m(T(R))$ is injective, hence a direct sum of copies of $E_R(R/m)$. The number of those equals $\mu_0(m, H^i_m(T(R))$, hence is finite by (3.4d).

Corollary 3.5 *Let K be a field of characteristic* 0, *let R be any regular K-algebra, let I be an ideal of R and let i be an integer bigger than the height of all the minimal primes of I. Then* $\operatorname{Hom}_R(R/I, H^i_I(R))$ *is finitely generated if and only if* $H^i_I(R) = 0$.

Proof. Assume $H^i_I(R) \neq 0$. Let P be a minimal prime in the support of $H^i_I(R)$. Then $\dim_{R_P}((H^i_I(R))_P = H^i_{IR_P}(R_P)) = 0$, so by (3.4b) $H^i_{IR_P}(R_P)$ is injective, i.e. a direct sum of copies of $E_{R_P}(R_P/P_P)$. Since $H^i_{IR_P}(R_P) \neq 0$, $i \leqq \dim R_P$, so $\dim R_P$ is bigger than the height of the minimal primes of I. So $\dim_{R_P} R_P/IR_P > 0$, which implies that $\operatorname{Hom}_{R_P}(R_P/IR_P, E_{R_P}(R_P/P_P))$ is not finitely generated. Hence, neither is $\operatorname{Hom}_{R_P}(R_P/IR_P, H^i_{IR_P}(R_P))$. Since $\operatorname{Hom}_{R_P}(R_P/IR_P, H^i_{IR_P}(R_P)) = \operatorname{Hom}_{R_P}(R/I, H^i_I(R)))_P$, it follows that $\operatorname{Hom}_R(R/I, H^i_I(R))$ is not finitely generated. q.e.d.

Corollary 3.6 *Let R be any regular ring containing a field (of any characteristic) and let I be an ideal of R. Then*

(a) $H^i_m(H^i_I(R))$ is injective for every maximal ideal m of R.

(b) *inj.*$\dim_R(H^i_I(R)) \leqq \dim_R(H^i_I(R))$. *In particular, if* $\dim_R(H^i_I(R)) = 0$, *then* $H^i_I(R)$ *is injective.*

(c) *For every maximal ideal m of R the set of the associated primes of $H^i_I(R)$ contained in m is finite.*

(d) *All the Bass numbers of $H^i_I(R)$ are finite.*

(e) *If i is bigger than the height of all the minimal primes of I, then* $\mathrm{Hom}_R(R/I, H^i_I(R))$ *is finitely generated if and only if* $H^i_I(R) = 0$.

Proof. (a)–(d) follow from (3.4) in characteristic 0 and from [Hu-Sh] in characteristic $p > 0$, while (e) follows from (3.5) in characteristic 0 and from [Hu-K, 2.3ii] in characteristic $p > 0$.

Remarks 3.7 (i) It follows from (2.9) that if R is a regular ring which is finitely generated as an algebra over a field of characteristic 0, the set of the associated primes of $T(R)$ is finite. In [Ly] we prove that if R is a regular ring containing a field of characteristic $p \geqq 0$, the set of the associated primes of $T(R)$ also is finite. Of course, one expects that for every regular ring R the set of the associated primes of $T(R)$ is finite, but this remains to be proven.

(ii) The set of the associated primes of $\mathscr{T}(R)$ need not be finite. For example, if φ is the set of all 0-dimensional subsets of Spec R, then the associated primes of $H^h_\varphi(R)$ are precisely the maximal ideals of R height h.

(iii) Of course, one expects that the statement of (3.4) is valid for every regular ring R. We can prove this provided R contains a field of characteristic $p > 0$ [Ly].

Questions 3.8 Let R be any regular ring.

(i) Are all the Bass numbers of $\mathscr{T}(R)$ finite?

(ii) Let G be any additive covariant functor from the category of abelian sheaves on Spec R to the category of abelian groups. Is it true that inj. $\dim_R(G(R)) \leqq \dim_R(G(R))$? (Cf. (2.5), (2.6) and (2.9))

We do not know the answers to these questions even in the case that R contains a field.

4 New numerical invariants of local rings

The goal of this section is to prove the following.

Theorem-Definition 4.1 *Let A be a local ring which admits a surjective ring homomorphism $\pi : R \rightarrow A$, where R is a regular ring of dimension n containing a field. Set $I = \mathrm{Ker}\,\pi$ and let m be the maximal ideal of R. Then $\mu_p(m, H^{n-i}_I(R))$ is finite and depends only on A, i and p, but neither on R nor on π. We denote this invariant by $\lambda_{p,i}(A)$.*

Proof. It follows from (3.6d) that $\mu_p(m, H^{n-i}_I(R))$ is finite. It remains to prove that it depends neither on R nor on π. We need a couple of lemmas.

Lemma 4.2 $\mu_p(m, H^{n-i}_I(R)) = \mu_p(mR^\wedge, H^{n-i}_{IR^\wedge}(R^\wedge))$, *where $R^\wedge$ is the completion of R with respect to m.*

Proof. By definition $\mu_p(m, H_I^{n-i}(R)) = \text{length}_K(\text{Ext}_R^p(K, H_I^{n-i}(R)))$ and $\mu_p(mR^\wedge, H_{IR^\wedge}^{n-i}(R^\wedge)) = \text{length}_K(\text{Ext}_{R^\wedge}^p(K, H_{IR^\wedge}^{n-i}(R^\wedge)))$, where K is the residue field of both R and $R^\wedge$. By (3.4a) and (1.4), $\text{Ext}_R^p(K, H_I^{n-i}(R)) = \text{Hom}_R(K, H_m^p(H_I^{n-i}(R)))$ and $\text{Ext}_{R^\wedge}^p(K, H_{IR^\wedge}^{n-i}(R^\wedge)) = \text{Hom}_{R^\wedge}(K, H_{mR^\wedge}^p(H_{IR^\wedge}^{n-i}(R^\wedge)))$. By (3.1) $H_{mR^\wedge}^p(H_{mR^\wedge}^{n-i}(R^\wedge)) = R^\wedge \otimes_R H_m^p(H_I^{n-i}(R))$. But $H_m^p(H_I^{n-i}(R))$ is supported at m, so $R^\wedge \otimes_R H_m^p(H_I^{n-i}(R)) = H_m^p(H_I^{n-i}(R))$. Hence $\text{length}_K(\text{Hom}_R(K, H_m^p(H_I^{n-i}(R)))) = \text{length}_K(\text{Hom}_{R^\wedge}(K, H_{mR^\wedge}^p(H_{IR^\wedge}^{n-i}(R^\wedge))))$. This proves (4.2).

Lemma 4.3 *Assume R is complete and let $g: R' \to R$ be a surjective ring homomorphism, where R' is complete local of dimension n'. Set $I' =$ Ker(πg) and let m' be the maximal ideal of R'. Then $\mu_p(m, H_I^{n-i}(R)) = \mu_p(m', H_{I'}^{n'-i}(R'))$.*

Proof. Since R is regular, Ker g is generated by $n' - n$ elements that form part of a minimal system of generators of the maximal ideal of R'. By induction on $n' - n$ we are reduced to the case that $n' - n = 1$, so Ker g is an ideal generated by one element $f \in m' \backslash m'^2$. By Cohen's structure theorem $R' = K[[X_1, \ldots, X_{n+1}]]$ and by a change of variables we can assume $f = X_{n+1}$. We identify R with the subring $K[[X_1, \ldots, X_n]]$ of R'. If M is an R-module, we set $g^\#(M) = \oplus_{j=1}^{\infty} MX_{n+1}^{-j}$ and we make it into a R'-module as follows: $rX_{n+1}^{j_1} vX_{n+1}^{-j_2} = rvX_{n+1}^{j_1-j_2}$, if $j_1 < j_2$ and $rX_{n+1}^{j_1} vX_{n+1}^{-j_2} = 0$ otherwise (here $r \in R$, $rX_{n+1}^{j_1} \in R'$, $v \in M$ and $vX_{n+1}^{-j_2} \in MX_{n+1}^{-j_2}$). If $h: M \to N$ is a homomorphism of R-modules, we define $g^\#(h): g^\#(M) \to g^\#(N)$ by $g^\#(h)(vX_{n+1}^{-j}) = h(v)X_{n+1}^{-j}$. This gives us a covariant exact functor $g^\#: R$-mod $\to R'$-mod. Note that socle $(g^\#(M)) = (\text{socle}(M))X_{n+1}^{-1}$. In particular, the socles of M and $g^\#(M)$ have same lengths. The composition of functors $\Gamma_{I'}(-) = \Gamma_I(\Gamma_{(X_{n+1})}(-))$ leads to the spectral sequence $E_2^{p,q} = H_I^p(H_{(X_{n+1})}^q(R')) \Rightarrow H_{I'}^{p+q}(R')$. It follows from (1.3) that $H_{(X_{n+1})}^q(R')$ is the q-th cohomology module of the complex $0 \to R' \to R'_{X_{n+1}} \to 0$. Hence $H_{(X_{n+1})}^1(R') = g^\#(R)$ and $H_{(X_{n+1})}^q(R') = 0$ for $q \neq 1$. So, the above spectral sequence implies that $H_{I'}^{n+1-i}(R') = H_I^{n-i}(g^\#(R))$. Let $f_1, \ldots, f_s \in R$ generate I. It follows from (1.3) that $H_I^{n-i}(g^\#(R)) = g^\#(H_I^{n-i}(R))$. If J is an injective R-module, then $g^\#(J)$ is an injective R'-module, because $\text{Hom}_{R'}(-, g^\#(J)) = g^\#(\text{Hom}_R(-, J))$, so, if $\text{Hom}_R(-, J)$ is exact, $\text{Hom}_{R'}(-, g^\#(J))$ also is exact. So, if J^* is a minimal injective resolution of $H_I^{n-i}(R)$ in the category of R-modules, then $g^\#(J^*)$ is an injective resolution of $g^\#(H_I^{n-i}(R)) = H_{I'}^{n+1-i}(R')$ in the category of R'-modules. Since J^* is minimal, the differentials induce zero maps on the socles of J^*. Hence the differentials induce zero maps on the socles of $g^\#(J^*)$. Since $\mu_p(m, H_I^{n-i}(R))$ equals the length of the socle of J^p, and $\mu_p(m', H_I^{n+1-i}(R'))$ equals the length of the socle of $g^\#(J^p)$ and since the two lengths coincide, (4.3) is proven.

Let $\pi': R' \to A$ and $\pi'': R'' \to A$ be surjections with $R' = K[[X_1, \ldots, X_{n'}]]$ and $R'' = K[[Y_1, \ldots, Y_{n''}]]$. Let $I' = \text{Ker}\,\pi'$ and let $I'' = \text{Ker}\,\pi''$. Let $R''' = R' \hat{\otimes}_K R''$ be the complete tensor product, $\pi''' = \pi' \hat{\otimes}_K \pi'': R' \hat{\otimes}_K R'' \to A$ and $I''' = \text{Ker}\,\pi'''$. Let m', m'' and m''' be the maximal ideals of R', R'' and R'''. Since π''' factors through π', (4.3) shows that $\mu_p(m''', H_{I'''}^{n'+n''-i}(R''')) = \mu_p(m', H_{I'}^{n'-i}(R'))$. Since π''' factors through π'', (4.3) shows that $\mu_p(m''', H_{I'''}^{n'+n''-i}(R''')) = \mu_p(m'', H_{I''}^{n''-i}(R''))$. So, $\mu_p(m', H_{I'}^{n'-i}(R')) = \mu_p(m'', H_{I''}^{n''-i}(R''))$. This proves (4.1).

A complete local ring containing a field is always a surjective image of a regular local ring containing a field. So, if A is a local ring containing a field, but not necessarily a surjective image of a regular local ring containing a field, one can set

$\lambda_{p,i}(A) = \lambda_{p,i}(A^\wedge)$, where $A^\wedge$ is the completion of A with respect to the maximal ideal. Lemma 3.2 shows that this coincides with our original definition in the case that A is a surjective image of a regular local ring containing a field.

Set $d = \dim A$. Here are some elementary properties of $\lambda_{p,i}(A)$:
(4.4i) $\lambda_{p,i}(A) = 0$ if $i > d$ (because $H_I^{n-i}(R) = 0$ for $i > \dim(R/I)$).
(4.4ii) $\lambda_{p,i}(A) = 0$ if $p > i$ (because inj. $\dim_R H_I^{n-i}(R) \leqq \dim_R H_I^{n-i}(R) \leqq i$).
(4.4iii) $\lambda_{d,d}(A) \neq 0$. Indeed, in view of (1.4) and (3.4a) all we have to prove is that $H_m^d(H_I^{n-d}(R)) \neq 0$. Let $I = I' \cap I''$, where all the minimal primes of I' have dimension d and all the minimal primes of I'' have dimension $< d$, so $H_{I''}^{n-i}(R) = 0$ for all $i \geqq d$. All the minimal primes of $I' + I''$ have dimension $< (d-1)$, so $H_{I'+I''}^{n-i}(R) = 0$ for $i \geqq d-1$. Hence Mayer–Vietoris implies that $H_I^{n-d}(R) = H_{I'}^{n-d}(R)$, so we can replace I by I', i.e. we can assume that I is equidimensional. There exists a spectral sequence $E_2^{p,q} = H_m^p(H_I^q(R)) \Rightarrow H_m^{p+q}(R)$. Since $H_I^{n-i}(R) = 0$ for all $i > d$, $E_2^{i,n-i} = 0$ for all $i > d$. If $P \supset I$ is a prime ideal, $(H_I^{n-i}(R))_P = H_{I_P}^{n-i}(R_P) = 0$ for $n-i > \dim R_P$, i.e. for $i < \dim R/P$. If $i = \dim R/P$, Hartshorne's local vanishing theorem [Ha3, 3.1] implies that $(H_I^{n-i}(R))_P = H_{I_P}^{n-i}(R_P) = 0$ if $\dim R_P/I_P > 0$. So, a prime of dimension $\geqq i$ belongs to the support of $H_I^{n-i}(R)$ if and only if it is a minimal prime of I of dimension i. But all minimal primes of I have dimension $\geqq d$, so if $i < d$, then $\dim_R H_I^{n-i}(R) < i$ and so $H_m^i(H_I^{n-i}(R)) = 0$. So, $E_2^{i,n-i} = 0$ for $i \neq d$. Since $H_m^n(R) \neq 0$, we conclude that $H_m^d(H_I^{n-d}(R)) \neq 0$.
(4.4iv) If A is analytically normal, $\lambda_{d,d}(A) = 1$. To prove this it is enough to show in view of (1.4) and (3.4a), that all the differentials that come into and go out of $E_r^{d,n-d}$ are 0, so $E_2^{d,n-d} = H_m^d(H_I^{n-d}(R)) = H_m^n(R)$. The outgoing differentials land in $E_r^{d+r,n-d-(r-1)}$ which vanishes because $H_I^{n-i}(R) = 0$ for all $i > d$. The incoming differentials come from $E_r^{d-r,n-d+(r-1)}$, so it is enough to prove that $E_2^{d-r,n-d+(r-1)} = H_m^{d-r}(H_I^{n-d+(r-1)}(R)) = 0$ for all $r \geqq 2$. We can assume R and A complete and, in particular, excellent. If P is a minimal prime of $H_I^{n-d+(r-1)}(R)$, then R_P/I_P is normal and excellent, so its completion is normal and by [Hu-Ly, 4.4] I_P is formally geometrically irreducible [Hu-Ly, 3.6], so [Hu-Ly, 2.9] implies that $H_{I_P}^{n-d+(r-1)}(R_P) = 0$ for $n-d+(r-1) \geqq \dim R_P - 1$. This inequality reduces to $\dim R/P \geqq d-r$. Hence $\dim_R H_I^{n-d+(r-1)}(R) < d-r$ and we are done.
(4.4v) If A is a complete intersection, $\lambda_{d,d}(A) = 1$ and $\lambda_{i,d}(A) = 0$ for all $i < d$. Indeed, $H_I^i(R) = 0$ if $i \neq n-d$, so the spectral sequence $E_2^{p,q} = H_m^p(H_I^q(R)) \Rightarrow H_m^{p+q}(R)$ shows that $H_m^i(H_I^{n-d}(R)) = H_m^{n-d+i}(R)$. Hence $H_m^i(H_I^{n-d}(R)) = 0$ if $i \neq d$ and $H_m^d(H_I^{n-d}(R)) = H_m^n(R)$.

Question 4.5 Is it true that $\lambda_{d,d}(A) = 1$ for all A?

Finally, it is worth pointing out that if V is a scheme of finite type over the complex numbers C and A is the local ring of V at a closed point $q \in V$, then $\lambda_{p,i}(A)$ are related to the singular topology of V in a neighborhood of q. For example, it follows from [O, 2.3] and [Ha4, IV.3.1] that if q is an isolated singular point of V, then $\lambda_{0,i}(A)$, for all $i < \dim A$, equals the dimension of $H_q^i(V, C)$ as a complex vector space, where $H_q^i(V, C)$ is the i-th singular local cohomology group of V with support in q and with coefficients in C.

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