Invent. math. 113, 21–39 (1993)

Inventiones mathematicae

Beurling type density theorems in the unit disk

Kristian Seip

Department of Mathematical Sciences, University of Trondheim, N-7034 Trondheim, Norway

Oblatum 9-I-1992 & 10- XII-1992

Summary. We consider two equivalent density concepts for the unit disk that provide a complete description of sampling and interpolation in A^{-n} (the Banach space of functions f analytic in the unit disk with $(1-|z|^2)^n|f(z)|$ bounded). This study reveals a 'Nyquist density': A sequence of points is (roughly speaking) a set of sampling if and only if its density in every part of the disk is strictly larger than n, and it is a set of interpolation if and only if its density in every part of the disk is strictly smaller than n. Similar density theorems are also obtained for weighted Bergman spaces.

1 Introduction and main theorems

In this paper we introduce a notion of density that enables us to describe completely what we call sets of sampling and interpolation for Bergman type spaces on the unit disk. There appears in these results a critical density that resembles the familiar Nyquist density. We reveal thus a basic similarity between spaces of bandlimited functions and Bergman type spaces, not recognized in previous treatments on decomposition and interpolation problems for such spaces [5, 11, 1, 12, 3]. We find also a corresponding resemblance with Bargmann-Fock type spaces, in view of the papers [14, 15].

A main inspiration for this research are two theorems of Beurling for band-limited functions [2], or more precisely, for the Banach space of functions of exponential type at most a, bounded on the real line. We say that a discrete set of real numbers is a set of sampling if the associated restriction operator has a bounded inverse, and that it is a set of interpolation if the interpolation problem associated to the set has a solution for all bounded sequences of complex numbers (we adopt here Landau's terminology from [10]). Roughly speaking, Beurling proved that a discrete set is a set of sampling if and only if its density in every part of the line is larger than a/π, and a set of interpolation if and only if its density in every part of the line is smaller than a/π; here 'density' in an interval means the number of points in the interval divided by its length. These two theorems give, in a remarkably precise manner, meaning to the engineers' notion of the Nyquist density.

The natural counterpart in the unit disk to Beurling's class of bandlimited functions is the L^∞ version of the weighted Bergman space, i.e., the Banach space A^{-n} ($n > 0$), which consists of all functions f analytic in the open unit disk U with

$$\|f\| = \|f\|_n = \sup_{z \in U} (1 - |z|^2)^n |f(z)| < \infty .$$

We say that a sequence of distinct points $\Gamma = \{z_j\}$ is a *set of sampling* for A^{-n} if there exists a positive constant L such that

$$\|f\| \leqq L \sup_j (1 - |z_j|^2)^n |f(z_j)|$$

for all $f \in A^{-n}$. If for every sequence $\{a_j\}$ for which $\{(1 - |z_j|^2)^n a_j\}$ is bounded, there is an $f \in A^{-n}$ with $f(z_j) = a_j$ for all j, we say that Γ is a *set of interpolation* for A^{-n}. We shall focus our attention on A^{-n}, leaving the last section of the paper for some remarks on the corresponding L^2 problems. The L^2 density theorems are not less interesting, but their proofs rely on our work on A^{-n} or on arguments very similar to those used for A^{-n}.

Let now

$$\rho(z, \zeta) = \left| \frac{z - \zeta}{1 - \bar{z}\zeta} \right|,$$

which is the pseudo-hyperbolic distance function on U. We say that a sequence $\Gamma = \{z_j\}$ is *uniformly discrete* (or separated) if

$$\delta = \inf_{j \neq k} \rho(z_j, z_k) > 0.$$

For a uniformly discrete set $\Gamma = \{z_j\}$ and $\frac{1}{2} < r < 1$, let

$$D(\Gamma, r) = \frac{\sum_{1/2 < |z_j| < r} \log \dfrac{1}{|z_j|}}{\log \dfrac{1}{1 - r}}.$$

For every $z \in U$, we form a new sequence

$$\Gamma_z = \left\{ \frac{z_j - z}{1 - \bar{z} z_j} \right\}.$$

The *lower* and *upper uniform densities* of Γ are defined, respectively, as

$$D^-(\Gamma) = \liminf_{r \to 1} \inf_{z \in U} D(\Gamma_z, r)$$

and

$$D^+(\Gamma) = \limsup_{r \to 1} \sup_{z \in U} D(\Gamma_z, r).$$

Our main theorems are given below. Besides the connection to Beurling's work, Theorem 1.2 appears as a natural counterpart to Carleson's interpolation theorem for H^∞ [4].

Theorem 1.1 *A sequence Γ of distinct points in U is a set of sampling for A^{-n} if and only if it contains a uniformly discrete subsequence Γ' for which $D^{-}(\Gamma') > n$.*

Theorem 1.2 *A sequence Γ of distinct points in U is a set of interpolation for A^{-n} if and only if Γ is uniformly discrete and $D^{+}(\Gamma) < n$.*

An interesting example of a 'typical' sequence Γ is the following. For $a > 1, b > 0$, let Γ denote the image of $\{a^j(bk + i)\}_{j,k \in \mathbf{Z}}$ under the Cayley transform of the upper half-plane to the unit disk. Then, using the results of [13], it is easily verified that

$$D^{-}(\Gamma) = D^{+}(\Gamma) = \frac{2\pi}{b \log a}.$$

This is an analogue of the standard sampling sets (multiples of the integers) for band-limited functions. [13] was in fact the first paper clearly suggesting the above-mentioned resemblance between spaces of bandlimited functions and Bergman type spaces.

One may define the uniform densities in a slightly different manner, that makes the connection to Beurling's density concept more transparent. For each z, let $n_z(r)$ denote the number of points from Γ_z contained in the disk $|\zeta| < r$, and put

$$N_z(r) = \int_0^r n_z(\tau)\, d\tau.$$

The hyperbolic area of the disk $|\zeta| < r$ is $a(r) = 2r^2(1 - r^2)^{-1}$ (suitably normalized). If we put

$$A(r) = \int_0^r a(\rho)\, d\rho,$$

we easily find that

$$D^{-}(\Gamma) = \liminf_{r \to 1} \inf_{z \in U} \frac{N_z(r)}{A(r)}$$

and

$$D^{+}(\Gamma) = \limsup_{r \to 1} \sup_{z \in U} \frac{N_z(r)}{A(r)}.$$

Beurling's approach suggests that we should consider $n_z(r)/a(r)$. The reason that it is natural to divide *averages* instead, is that the main contribution to $a(r)$ comes from points that lie 'close' to the boundary of the disk; this gives rise to an instability which is removed by taking averages.

It is a relatively simple matter to verify Theorem 1.1 using Beurling's method of proof from [2]. The most difficult part is to prove Theorem 1.2. The crucial ingredients in our proof of this theorem are, in addition to Beurling's techniques, the following. We prove (see Sect. 3) that our density concept is equivalent to one based on Korenblum's description of the zero sets for A^{-n} [9], and we improve (see Sect. 4) Korenblum's main theorem concerning the zero sets.

2 Preliminaries

In this section we describe some notational conventions and introduce some tools to be used in the proofs.

We let

$$\| f | \Gamma \| = \| f | \Gamma \|_n = \sup_{z \in \Gamma} (1 - |z|^2)^n |f(z)|.$$

For any sequence Γ, $L(\Gamma) = L(\Gamma, n)$ will denote the smallest number L such that

$$\| f \| \leqq L \| f | \Gamma \|$$

for all $f \in A^{-n}$. Γ is consequently a set of sampling for A^{-n} if and only if $L(\Gamma) < \infty$.

If $\Gamma = \{z_j\}$ is a set of interpolation for A^{-n}, a standard argument based on the closed graph theorem [8, p. 196], shows that the interpolation can be performed in a stable way. This means that there exists a positive number M such that for every bounded sequence $\{a_j\}$ we can find $f \in A^{-n}$ with $f(z_j) = (1 - |z_j|^2)^{-n} a_j$ for all j, and

$$\| f \| \leqq M \| f | \Gamma \|.$$

The smallest such M is denoted $M(\Gamma) = M(\Gamma, n)$, and we put $M(\Gamma) = \infty$ if Γ is not a set of interpolation for A^{-n}.

We recall the transformation rule of the Bergman kernel:

$$(1) \qquad (1 - \overline{\Phi(\zeta)} \Phi(z))^{-2} \Phi'(z) \overline{\Phi'(\zeta)} = (1 - \bar{\zeta} z)^{-2},$$

Φ a Möbius self map of U; z, ζ arbitrary points in U. Using (1) with $\zeta = z$, we see that the transformations

$$(T_\Phi^n f)(z) = (T_\Phi f)(z) = (\Phi'(z))^n f(\Phi(z)),$$

act isometrically in A^{-n}. This Möbius invariance implies immediately that $L(\Gamma) = L(\Phi\Gamma)$ and $M(\Gamma) = M(\Phi\Gamma)$, and it will permit us to transfer our analysis around an arbitrary point z to 0.

An important feature of A^{-n} is the following compactness property: If $\{f_k\}$ is a sequence in the ball

$$\{ f \in A^{-n} : \| f \| \leqq R \},$$

then there is a subsequence $\{f_{k_l}\}$ converging pointwise and uniformly on compact sets to some function in the ball. This is immediate from the definition of A^{-n} and a normal family argument.

A sequence Q_j of closed sets converges strongly to Q, denoted $Q_j \to Q$, if $[Q, Q_j] \to 0$; here $[\cdot,\cdot]$ denotes the Hausdorff distance (with respect to pseudo-hyperbolic distance) between two closed sets. Q_j converges weakly[1] to Q, denoted $Q_j \rightharpoonup Q$, if for every compact set D, $(Q_j \cap D) \cup \partial D \to (Q \cap D) \cup \partial D$.

Following Beurling, for a closed set Γ, we let $W(\Gamma)$ denote the collection of weak limits of the sequences $\Phi\Gamma$, Φ ranging over the Möbius self-maps of U. The compactness property and the Möbius invariance of A^{-n} make $W(\Gamma)$ an important tool in our analysis; as in [14], we find that most of Beurling's arguments concerning $W(\Gamma)$ can be carried over to our situation.

The following lemma will play the role that Bernstein's theorem does for bandlimited functions.

---

[1] Note that in this definition we have eliminated an obvious error in Beurling's notes. Unfortunately, the same error appears in [14], where a corresponding correction is required.

Lemma 2.1 *For $f \in A^{-n}$ we have for $S(w) = (1 - |w|^2)^n f(w)$,*

$$||S(z)| - |S(\zeta)|| \leqq u(\rho(z, \zeta)) \|f\|,$$

where

$$u(r) = \min_{0 < \rho < 1 - r} \{(1 - r^2)^{-n} - 1 + (r/\rho)(1 - (r + \rho)^2)^{-n}\}.$$

Proof. We consider

$$S_\zeta(w) = (1 - |w|^2)^n (T_{\Phi_\zeta} f)(w) = (1 - |w|^2)^n f_\zeta(w),$$

where Φ_ζ is the Möbius transformation that interchanges 0 and ζ, i.e.,

$$\Phi_\zeta(z) = \frac{\zeta - z}{1 - \bar{\zeta} z}.$$

Observe first that $|S(z)| = |S_\zeta(\Phi_\zeta(z))|$, in view of (1) and the fact that Φ_ζ is involutory. Thus,

$$\begin{aligned} ||S(z)| - |S(\zeta)|| &= ||S_\zeta(\Phi_\zeta(z))| - |S_\zeta(0)|| \leqq |S_\zeta(\Phi_\zeta(z)) - S_\zeta(0)| \\ &\leqq (1 - (1 - \rho(z, \zeta))^n)|f_\zeta(\Phi_\zeta(z))| + |f_\zeta(\Phi_\zeta(z)) - f_\zeta(0)|. \end{aligned}$$

From this the result follows by an application of Cauchy's formula. □

Note that $u(r) = O(r)$ as $r \to 0$.

Let A_0^{-n} denote the class of functions f for which

$$(1 - |z|^2)^n |f(z)| \to 0$$

as $|z| \to 1$; it is easily seen that A_0^{-n} is a closed subspace of A^{-n}. We may as well consider sets of sampling and interpolation for A_0^{-n}; a sequence of distinct points $\{z_j\}$ in U is said to be a set of interpolation for A_0^{-n} if for every sequence $\{a_j\}$ for which $(1 - |z_j|^2)^n |a_j| \to 0$ as $|z_j| \to 1$, there is an $f \in A_0^{-n}$ with $f(z_j) = a_j$ for all j. We may replace the numbers $L(\Gamma)$ and $M(\Gamma)$ by the corresponding numbers for A_0^{-n}, which we denote by $L_0(\Gamma)$ and $M_0(\Gamma)$. It is then easy to check that all that was said above is also true for A_0^{-n}.

Bruna and Pascuas proved that the sets of interpolation for A_0^{-n} and for A^{-n} are the same [3]. This follows from an argument based on the fact that A^{-n} is the second dual of A_0^{-n} [16]. We shall at a certain stage find it convenient to make use of this result. (It may likewise be proved that the sets of sampling for A_0^{-n} coincide with the sets of sampling for A^{-n}, but this will not be needed.)

The following simple fact will be used repeatedly. Let ω denote Lebesgue measure on **C**, and for a domain $S \subset U$, let $S^+ = \{z: \rho(z, S) < \delta\}$. If f is analytic in S^+ and

$$\delta = \inf_{j \neq k} \rho(z_j, z_k) > 0,$$

we have

$$\sum_{z_j \in S} (1 - |z_j|^2)^s |f(z_j)|^2 \leqq C(\delta) \int_{S^+} (1 - |z|^2)^{s-2} |f(z)|^2 \, d\omega(z) \tag{2}$$

whenever $s > 0$ (both sides may be infinite). This is easy to see, e.g., as a consequence of the Cauchy-Schwarz inequality and the following reproducing formula,

$$f(z) = C(\delta, s) \int_{\rho(z,\zeta)<\delta} (1-\bar{\zeta}z)^{-s} f(\zeta)(1-|\zeta|^2)^{s-2}\, d\omega(\zeta);$$

for $z = 0$ this formula is a consequence of the mean value theorem, and for general z it follows from this special case by a change of variables by a Möbius self-map of the unit disk. Note in particular that (2) yields the estimate

$$\sum_{z_j \in S} \log\frac{1}{|z_j|} \leqq C(\delta) \int_{S+} (1-|\zeta|^2)^{-1}\, d\omega(\zeta). \tag{3}$$

3 An equivalent density concept

In this section we introduce another way of measuring density, based on Korenblum's description of zero sets of functions in A^{-n}. We shall prove that this density concept is equivalent to that introduced above.

For an arbitrary finite subset F of the unit circle ∂U, let $\{I_k\}$ denote the set of complementary arcs of F. We put

$$\hat{\kappa}(F) = \sum_k \frac{|I_k|}{2\pi}\left(\log\frac{2\pi}{|I_k|} + 1\right),$$

which is called the *Carleson characteristic* of F. The normalized angular distance on ∂U is defined by

$$d(e^{it}, e^{is}) = \min_{k\in\mathbf{Z}} \frac{|t-s+2\pi k|}{\pi}.$$

For a finite set $F \subset \partial U$ and parameters $0 < a < 1/2$, $1 \leqq q$, define

$$G_{F;q,a} = \left\{z \in \bar{U}\colon\ 1-|z| \geqq a d^q\left(\frac{z}{|z|}, F\right),\ |z| > \frac{1}{2}\right\}.$$

For a sequence of points Γ from U we put

$$\sigma_z(F, q, a) = \sum_{\zeta \in \Gamma_z \cap G_{F;q,a}} \log\frac{1}{|\zeta|},$$

and define

$$m_z^-(\alpha; q, a) = \inf_F (\sigma_z(F, q, a) - \alpha\hat{\kappa}(F))$$

and

$$m_z^+(\alpha; q, a) = \sup_F (\sigma_z(F, q, a) - \alpha\hat{\kappa}(F)).$$

The *lower* and *upper uniform Korenblum densities* are then

$$D_K^-(\Gamma) = \sup\{\alpha\colon \inf_z m_z^-(\alpha; 1, a) > -\infty\}$$

and

$$D_K^+(\Gamma) = \inf\{\alpha: \sup_z m_z^+(\alpha; 1, a) < +\infty\}.$$

It is easy to see that these definitions are independent of a.

Korenblum's notion of κ-area suggests that there is a close connection between our two density concepts. The κ-area of a measurable set $S \subset U$ is

$$\kappa A(S) = \frac{1}{2\pi}\int_S \frac{d\omega(z)}{1-|z|}.$$

We observe that

$$\kappa A(\{|z| < r\}) = \log\frac{1}{1-r} - r,$$

while on the other hand, we easily verify that

(4) $$|\kappa A(G_{F;1,a}) - \hat{\kappa}(F)| \leqq C,$$

C independent of F.

We now prove:

Proposition 3.1 *If Γ is uniformly discrete, we have $D_K^-(\Gamma) = D^-(\Gamma)$ and $D_K^+(\Gamma) = D^+(\Gamma)$.*

Proof. To see that $D^-(\Gamma) \geqq D_K^-(\Gamma)$ and $D^+(\Gamma) \leqq D_K^+(\Gamma)$, let F consist of N equidistant points on ∂U. Then $\hat{\kappa}(F) = \log N + 1$, and we see that $G_{F;1,a}$ contains the disk $U_N = \{|z| < 1 - (C/N)\}$, with C a constant depending on a. We also find that

$$\sup_N \kappa A(G_{F;1,a} \backslash U_N) < \infty,$$

and therefore, by (3), that

$$\sum_{z_j \in G_{F,1,a}\backslash U_N} \log\frac{1}{|z_j|} \leqq C,$$

With C depending only on δ and a. Thus the estimate holds for all Γ_z, C independent of z. Hence,

$$D(\Gamma_z, 1-(C/N)) = \frac{\sigma_z(F, 1, a)}{\hat{\kappa}(F)} + O((\log N)^{-1}),$$

from which the stated inequalities follow.

We next prove that $D^-(\Gamma) \leqq D_K^-(\Gamma)$. Let $\alpha = D^-(\Gamma)$ and for any $\varepsilon > 0$, let r be so large that

(5) $$D(\Gamma_z, s) \geqq \alpha - \varepsilon$$

for all z and $s \geqq r$. Choose also a sufficiently small $a > 0$ so that for each F and each $\zeta \in \partial G_{F;1,a}$, $|\zeta| < 1$, we have

(6) $$\rho(\zeta, G_{F;1,1/4}) \geqq r;$$

it is easy to check that this is possible.

Consider an arbitrary F, fix z, and put $\{z_j\} = \Gamma_z \cap G_{F;1,a}$. Let

$$r_k = \tanh\left(\frac{k}{2}\log\frac{1+r}{1-r}\right) = \frac{1-\left(\frac{1-r}{1+r}\right)^k}{1+\left(\frac{1-r}{1+r}\right)^k},$$

k a nonnegative integer, so that

$$\rho(r_k, r_{k+1}) = r. \tag{7}$$

We put $A_k = \{r_k \leqq |\zeta| < r_{k+2}\}$, $k = 0, 1, 2, \ldots$, and correspondingly we define

$$f_k(\zeta) = \prod_{z_j \in A_k} \frac{z_j - \zeta}{1 - \bar{z}_j \zeta}.$$

We apply Jensen's formula to f_k in the disk $|\zeta| \leqq r_{k+1}$:

$$\sum_{z_j \in A_k} \log|z_j| = \sum_{z \in A_k \cap A_{k-1}} \log\frac{|z_j|}{r_{k+1}} + \frac{1}{2\pi}\int_0^{2\pi} \log|f_k(r_{k+1}e^{i\theta})|\, d\theta. \tag{8}$$

Let

$$J_k(x) = \{\theta \colon r_{k+1}e^{i\theta} \in G_{F;1,x}\}.$$

From (8) we deduce that

$$\sum_{z_j \in A_{k+1} \cap A_k} \log\frac{1}{|z_j|} \geqq |J_k(1/4)|(\alpha - \varepsilon)|\log(1-r)| - C|J_{k-1}(a)|, \tag{9}$$

with C depending on δ, but not on r. Here we have applied (5) to those points $r_{k+1}e^{i\theta}$ in the integral in (8) with $\theta \in J_k(1/4)$; this is possible by (6) and (7). One arrives then at (9) after checking that

$$\log\frac{1}{r_{k+1}} \sum_{z_j \in A_k \cap A_{k-1}} 1 \leqq C|J_{k-1}(a)|,$$

which is a consequence of (2).

Note next that

$$\log(1-r_k) - \log(1-r_{k+1}) = \log\frac{1+r}{1-r} - \log\frac{1+r_{k+1}}{1+r_k}. \tag{10}$$

Hence, upon summing (9) over k, we obtain

$$\sigma_z(F;1,a) \geqq (\alpha-\varepsilon)\kappa A(G_{F;1,1/4}) - C|\log(1-r)|^{-1}\kappa A(G_{F;1,a}) - C',$$

C' depending on a, and thus on r. In view of (4), this implies

$$\sigma_z(F;1,a) \geqq (\alpha - \varepsilon - C(\delta)|\log(1-r)|^{-1})\hat{\kappa}(F) - c(\delta, r).$$

Since the estimate is independent of z, we conclude that $D^-(\Gamma) \leqq D_K^-(\Gamma)$.

We follow the same pattern in order to prove that $D_K^+(\Gamma) \geqq D^+(\Gamma)$. Choose r so large that

$$D(\Gamma_z, s) \leqq \alpha + \varepsilon \tag{11}$$

for all z and $s \geqq r$. Let a be as above, and make the same definitions with one exception: Now $\{z_j\} = \Gamma_z \cap G_{F;1,1/4}$. Jensen's formula applied to f_k as above is again our starting point:

$$\sum_{z_j \in A_k} \log|z_j| = \sum_{z_j \in A_k \cap A_{k-1}} \log \frac{|z_j|}{r_{k+1}} + \frac{1}{2\pi} \int_0^{2\pi} \log|f_k(r_{k+1} e^{i\theta})|\, d\theta. \tag{12}$$

Apply first (11) to those points $r_{k+1} e^{i\theta}$ in the integral in (12) for which $\theta \in J_k(a)$. We see that we also need a bound on the contribution to $|f_k(r_{k+1} e^{i\theta})|$ from those points z_j for which $\rho(z_j, r_{k+1} e^{i\theta}) \geqq r$ and $\rho(z_j, r_{k+1} e^{i\theta}) \leqq 1/2$. To treat the points for which $\rho(z_j, r_{k+1} e^{i\theta}) \geqq r$, put

$$S_{w,r} = \left\{ \zeta : \rho(\zeta, w) > r, |\zeta| \leqq \frac{|w| + r}{1 + |w| r} \right\},$$

and check by direct computation that

$$\int_{S_{w,r}} (1 - \rho(\zeta, w))(1 - |\zeta|^2)^{-2} d\omega(\zeta) \leqq C_0, \tag{13}$$

C_0 an absolute constant. This can be done by transporting w to 0, so that we get

$$\int_{S_{w,r}} (1 - \rho(\zeta, w))(1 - |\zeta|^2)^{-2} d\omega(\zeta) \leqq \int_{\Sigma_r} (1 - |\zeta|)^{-1} d\omega(\zeta),$$

where $\Sigma_r = \{\zeta : |\zeta| > r, |\zeta - \frac{1-r}{2}| < \frac{1+r}{2}|\}$. It is easy to show that the latter integral is bounded by a constant which does not depend on r. Applying (2) in an appropriate way, we find by (13) that

$$\frac{1}{2\pi} \int_{J_k(a)} \log|f_k(r_{k+1} e^{i\theta})| d\theta \geqq -J_k(a)((\alpha + \varepsilon)|\log(1 - r)| + C_0' + C),$$

where C corresponds to the contribution from the points for which $\rho(z_j, r_{k+1} e^{i\theta}) \leqq 1/2$; this bound depends on δ.

As to the integral along the remaining part, $J_k^* = \partial U \backslash J_k(a)$, an elementary computation, using (6), gives

$$\frac{1}{2\pi} \int_{J_k^*} \log|f_k(r_{k+1} e^{i\theta})| d\theta \geqq -C|J_k(1/4)|(1 - r)(\log(1 - r_k) - \log(1 - r_{k+2})).$$

We sum (12) over k, use (10), and obtain

$$\sigma_z(F; 1, a) \leqq (\alpha + \varepsilon + C|\log(1 - r)|^{-1}) \kappa A(G_{F;1,a}) + C',$$

C' depending only on a, and hence,

$$\sigma_z(F; 1, a) \leqq (\alpha + \varepsilon + C(\delta)|\log(1 - r)|^{-1}) \hat{\kappa}(F) + c(\delta, r).$$

The estimate is independent of z, and so $D^+(\Gamma) \leqq D_K^+(\Gamma)$. □

4 A density theorem for zero sets

In this section we add an argument to Korenblum's analysis in [9] in order to obtain a sharp density theorem for the zero sets for A^{-n}.

We first recall Korenblum's theorem [9, p. 192]: *For* Γ (a sequence of not necessarily distinct points with no accumulation point in U) *to be the zero set of a function in* A^{-n}, *it is necessary that* $m_0^+(\alpha; 1, a) < \infty$ *for all* $\alpha > 2n$, *and sufficient that* $m_0^+(\alpha; 1, a) < \infty$ *for some* $\alpha < n/2$. We prove now that the latter condition can be sharpened to read $\alpha < n$. More precisely, we prove the following (to make our result fit into the rest of our exposition, we disregard the points close to 0, which are of no interest in any case).

Lemma 4.1 *Let* Γ *be a sequence of not necessarily distinct points from the annulus* $\frac{1}{2} < |z| < 1$. *Then if* $m_0^+(\alpha; 1; a) < \infty$, *we can, for every* $q > 1$, *find an analytic function* g *vanishing on* Γ *with* $g(0) = 1$, and

$$|g(z)| \leqq C(q, m_0^+(\alpha; 1; a))(1 - |z|)^{-q\alpha}.$$

Proof. The lemma is proved by modifying Korenblum's proof of the sufficiency part of Theorem 1 in [9, p. 192].

Korenblum's starting point is the assumption that $m_0^+(\alpha; 1; a) < \infty$ (which is the condition (T_α) in his terminology) and that $q > 2$. He then constructs (see Sect. 3.5 in [9]) a function g as required by our lemma. We show now that Korenblum's proof can be modified to work when assuming $q \gtrsim 1$ instead of $q > 2$.

Consider Korenblum's proof that his function $\tilde{f}$ defined by (3.5.2) in [9, p. 198] satisfies condition (ii), i.e., the growth estimate

$$|\tilde{f}(z)| \leqq C(1 - |z|)^{-q\alpha}.$$

Let $\tilde{\Gamma}$ denote a finite part of Γ, and for every $z \in U$, put $\zeta = z/|z|$, $G_\zeta = G_{\{\zeta\};q,a}$. Let $a < 1/4$ and define

$$S(z) = \exp\left\{ \sum_{z_j \in \tilde{\Gamma}, z_j \notin G_\zeta} \log|z_j| \frac{\zeta_j + z}{\zeta_j - z} \right\},$$

where $\zeta_j = z_j/|z_j|$. Korenblum shows that

$$(14) \qquad \left| [S(z)]^{-1} \prod_{z_j \in \tilde{\Gamma}} \frac{z_j - z}{1 - \bar{z}_j z} \right| \leqq \exp\left\{ C_1 \sum_{z_j \notin G_\zeta} \frac{(1 - |z_j|)^2}{|\zeta_j - z|^2} \right\},$$

and he proves that the right-hand side is bounded by a constant whenever $q > 2$.

The idea is now to multiply the left-hand side of (14) (and thus $\tilde{f}$) by a suitable auxiliary function in order to deal with the expression on the right-hand side. To this end, put

$$w_j = (1 + (1 - |z_j|)^{\frac{1}{q}})\zeta_j,$$

and introduce the function

$$v(z) = \sum_j \frac{(1 - |z_j|)^2}{|w_j - z|^2};$$

$v(z)$ is well-defined in U, in fact subharmonic there, since $m_0^+(\alpha; 1, a) < \infty$ implies that

$$(15) \qquad \sum_j (1 - |z_j|)^{1+\varepsilon} = C(\varepsilon, m_0^+(\alpha; 1, a)) < \infty$$

for every $\varepsilon > 0$ (see [9, p. 193]). A straightforward computation shows that

$$\int_0^{2\pi} \frac{d\theta}{|w_j - re^{i\theta}|^2} \leqq C(1 - |z_j|)^{2-\frac{1}{q}},$$

C an absolute positive constant, and thus

$$\lim_{r \to 1} \int_0^{2\pi} v(re^{i\theta}) d\theta \leqq C \sum_j (1 - |z_j|)^{2-\frac{1}{q}}.$$

By (15) and the assumption that $q > 1$, the function

$$u(z) = \lim_{r \to 1} \frac{1}{2\pi} \int_0^{2\pi} P(z, re^{i\theta}) v(re^{i\theta}) d\theta$$

($P(z, \zeta)$ denotes the Poisson kernel) is therefore a harmonic majorant of $v(z)$ in U.

It is easily verified that

(16) $$\sum_{\alpha_j \notin G_\zeta} \frac{(1 - |z_j|)^2}{|\zeta_j - z|^2} \leqq C_2 v(z),$$

C_2 a positive constant depending on a. We replace the function $\tilde{f}(z)$, defined by (3.5.2) in [9], by the function

$$\tilde{g}(z) = \tilde{f}(z) \exp\left\{ -\lim_{r \to 1} \frac{C_1 C_2}{2\pi} \int_0^{2\pi} \left(\frac{re^{i\theta} + z}{re^{i\theta} - z} v(re^{i\theta}) - v(re^{i\theta}) \right) d\theta \right\},$$

with C_1 as in (14) and C_2 as in (16), and see that our job is to estimate

$$\left| [S(z)]^{-1} \prod_{z_j \in \tilde{\Gamma}} \frac{z_j - z}{1 - \bar{z}_j z} \exp\left\{ -\lim_{r \to 1} \frac{C_1 C_2}{2\pi} \int_0^{2\pi} \frac{re^{i\theta} + z}{re^{i\theta} - z} v(re^{i\theta}) d\theta \right\} \right|$$

instead of the left-hand side of (14). In view of (14) and (16), this expression is bounded by 1.

The rest of the work needed to obtain g as a limit of the functions $\tilde{g}$ (modulo a constant depending on $m_0^+(\alpha; 1, a)$) is verbatimly as that done by Korenblum in Sect. 3.5 in [9], to which we refer for details. □

5 Proof of Theorem 1.1

We first make a simple observation.

Lemma 5.1 *Let Γ and Γ' be two sequences of distinct points from U. Then*

$$|L(\Gamma)^{-1} - L(\Gamma')^{-1}| \leqq u([\Gamma, \Gamma']).$$

Proof. This follows from Lemma 2.1 (see Theorem 2 in [2, p. 344]). □

An immediate consequence of this lemma is the following.

Lemma 5.2 *If Γ is a set of sampling for A^{-n}, then Γ contains a uniformly discrete subsequence that is also a set of sampling for A^{-n}.*

In the sequel, we therefore assume that Γ is uniformly discrete.

We turn to the proof of Theorem 1.1.

Proof of the necessity. Put $\alpha = D^-(\Gamma)$, and assume $L(\Gamma) < \infty$. Let $\varepsilon_j \to 0$, and pick a sequence of points z_j such that

$$D(\Gamma_{z_j}, r_j) \leqq \alpha + \varepsilon_j, \tag{17}$$

with, say, $r_j > 1 - \varepsilon_j$. For each z_j, put $\Gamma_{z_j} = \Gamma_j = \{z_k^{(j)}\}$, and construct a new sequence of points $\Gamma'_j = \{\zeta_k^{(j)}\}$ by letting

$$\zeta_k^{(j)} = \frac{|z_k^{(j)}| + \delta_0}{1 + \delta_0 |z_k^{(j)}|} \frac{z_k^{(j)}}{|z_k^{(j)}|},$$

where $\delta_0 > 0$ is chosen such that $u(\delta_0) L(\Gamma) < 1$. By Lemma 5.1, this implies that

$$L(\Gamma'_j) \leqq \frac{L(\Gamma)}{1 - L(\Gamma) u(\delta_0)} < \infty. \tag{18}$$

An elementary computation shows that

$$D(\Gamma'_j, r_j) \leqq (1 - \delta_0) D(\Gamma_j, r_j) + C |\log(1 - r_j)|^{-1}. \tag{19}$$

On the other hand, consider the function

$$f_j(z) = \prod_{|\zeta_k^{(j)}| < r_j} \frac{1}{|\zeta_k^{(j)}|} \frac{\zeta_k^{(j)} - z}{1 - \overline{\zeta_k^{(j)}} z}.$$

We have $\| f_j \| \geqq 1$, and

$$\| f_j | \Gamma'_j \| \leqq \exp\left(\sum_{|\zeta_k^{(j)}| < r_j} \log \frac{1}{|\zeta_k^{(j)}|} - n \log \frac{1}{1 - r_j^2} \right).$$

Thus, by (19), (18), and (17), we have

$$(1 - \delta_0)(\alpha + \varepsilon_j) + C |\log \varepsilon_j|^{-1} \geqq n,$$

C depending only on $L(\Gamma)$ and δ_0. Since $\varepsilon_j \to 0$, we have proved that $\alpha > n$. □

Proof of the sufficiency. We first note that if every $\Gamma_0 \in W(\Gamma)$ is a set of uniqueness for A^{-n}, i.e., $f(z) = 0$ for all $z \in \Gamma_0$ and $f \in A^{-n}$ imply $f \equiv 0$, then Γ is a set of sampling for A^{-n}. This follows by an argument based on the compactness property (see the proof of Theorem 3 in [2, p. 345]).

We assume then that $\alpha = D^-(\Gamma) > n$. Pick an arbitrary $\Gamma_0 = \{\zeta_k\} \in W(\Gamma)$, and suppose there exists an $f \in A^{-n}$ with $f(\zeta_k) = 0$ for every $\zeta_k \in \Gamma_0$. We may assume that $0 \notin \Gamma_0$ and $f(0) = 1$. Since

$$D^-(\Gamma_0) \geqq \alpha,$$

it follows that

$$D(\Gamma_0, r) \geqq \alpha - \varepsilon,$$

$\varepsilon = \frac{1}{2}(\alpha - n)$, for all sufficiently large r. But then, by Jensen's formula, we have

$$\sup_\theta \log |f(re^{i\theta}| \geqq (n + \varepsilon) \log \frac{1}{1 - r}$$

for all sufficiently large r. This contradicts $f \in A^{-n}$, and so Γ_0 is a set of uniqueness for A^{-n}. Since Γ_0 was arbitrary, we conclude by the first observation that Γ is a set of sampling for A^{-n}. □

6 Proof of Theorem 1.2

We first note the following.

Lemma 6.1 *Every set of interpolation for A^{-n} is uniformly discrete.*

Proof. Let $\{a_j\}$ be a sequence with $|a_j| \leqq 1$, and $|a_k| - |a_m| = 1$ for some given k and m. Then the inequality

$$1 = |a_k - a_m| = |S(z_k) - S(z_m)| \leqq M(\{z_j\})u(\rho(z_k, z_m)),$$

deduced from Lemma 2.1, yields the result. □

In proving the necessity part of the theorem, we will make use of Bruna and Pascua's result that the sets of interpolation for A^{-n} coincide with those for A_0^{-n}. We collect first a few auxiliary results

Lemma 6.2 *$\Gamma_j \rightharpoonup \Gamma$ implies $M_0(\Gamma) \leqq \liminf M_0(\Gamma_j)$.*

Proof. We may assume that the right-hand side is bounded, and even that $\sup |M_0(\Gamma_j)| < \infty$ by picking an appropriate subsequence. The result then follows by the compactness property. □

Lemma 6.3. *Let $\Gamma = \{z_k\}$ and $\Gamma' = \{z'_k\}$ be uniformly discrete sets such that $\rho(z_k, z'_k) \leqq h$ for each k, where $M_0(\Gamma)u(h) < 1$. Then*

$$|M_0(\Gamma)^{-1} - M_0(\Gamma')^{-1}| \leqq u(h).$$

Proof. The proof is a slight modification of the proof of Lemma 2 in [2, p. 351], where Lemma 2.1 is used instead of Bernstein's theorem. We omit the details. □

Thus, in particular, we have

$$M_0(\Gamma') \leqq \frac{M_0(\Gamma)}{1 - M_0(\Gamma)u(h)}. \tag{20}$$

We remark that the contents of Lemmas 6.1 and 6.3 were also verified in [3] (Theorems 9 and 8) by some other (less direct) arguments.

For a certain technical reason, we shall need to consider the following notion of Beurling's. For $z \in U$, let

$$\mu_0(z, \Gamma) = \sup_f (1 - |z|^2)^n |f(z)|,$$

where f ranges over those functions $f \in A_0^{-n}$ for which $f(\zeta) = 0$, $\zeta \in \Gamma$, and $\|f\| \leqq 1$. The following analogue of Lemma 3 in [2, p. 352] is in our context trivial.

Lemma 6.4 *$M_0(\Gamma) < \infty$ implies $\mu_0(z, \Gamma) > 0$ when $z \notin \Gamma$.*

Lemma 6.5 *For $z_0 \notin \Gamma$, we have*

$$M_0(\Gamma \cup \{z_0\}) \leqq \frac{1 + 2M_0(\Gamma)}{\mu_0(z_0, \Gamma)}.$$

Proof. We assume, by Möbius invariance, that $z_0 = 0$, and proceed as in the proof of Lemma 4 in [2, p. 353]. □

Lemma 6.6 *Given δ_0, l_0, and n, there exists a positive constant $C = C(\delta_0, l_0, \alpha)$ such that if $M_0(\Gamma) \leqslant l_0$ and $\rho(z, \Gamma) \geqq \delta_0$, then*

$$\mu_0(z, \Gamma) \geqq C.$$

Proof. As the proof of Lemma 8.4 in [14] (which is a slight modification Lemma 5 in [2, p. 353]; here it is crucial that we work in A_0^{-n} and not in A^{-n}). □

One part of the proof of Theorem 1.1 can now be completed.

Proof of the necessity. Put $\alpha = D^+(\Gamma)$, and assume $M_0(\Gamma) < \infty$. Let $\varepsilon_j \to 0$, and pick a sequence of points z_j such that

$$D(\Gamma_{z_j}, r_j) \geqq \alpha - \varepsilon_j, \tag{21}$$

with, say, $r_j > 1 - \varepsilon_j$. For each z_j, put $\Gamma_{z_j} = \Gamma_j = \{z_k^{(j)}\}$, and construct a new sequence of points $\Gamma'_j = \{\zeta_k^{(j)}\}$ by letting

$$\zeta_k^{(j)} = \frac{|z_k^{(j)}| - \delta_0}{1 - \delta_0 |z_k^{(j)}|} \frac{z_k^{(j)}}{|z_k^{(j)}|},$$

where $\delta_0 > 0$ is chosen such that $u(\delta_0) M_0(\Gamma) < 1$. An elementary computation then shows that

$$D(\Gamma'_j, r_j) \geqq (1 + \delta_0) D(\Gamma_j, r_j) - C|\log(1 - r_j)|^{-1}. \tag{22}$$

On the other hand, by Lemma 6.6 and the choice of δ_0, we can find a function f_j, vanishing on $\Gamma'_j \cap \{|z| > \varepsilon\}$, with $f_j(0) = 1$, and

$$(1 - |z|^2)^n |f_j(z)| \leqq C,$$

C depending only on $M_0(\Gamma)$ δ_0, and ε. Jensen's formula in conjunction with (21) and (22) then yields

$$(1 + \delta_0)(\alpha - \varepsilon_j) - C|\log \varepsilon_j)|^{-1} \leqq n.$$

We let $\varepsilon_j \to 0$, and conclude that $\alpha > n$. □

Proof of the sufficiency. Put $\alpha = D^+(\Gamma)$ and $\varepsilon = \frac{1}{2}(n - \alpha)$. Since $D_K^+(\Gamma) = D^+(\Gamma)$, we may for each $z_k \in \Gamma$ construct a function g_k with the properties:

$$g_k(z_k) = (1 - |z_k|^2)^{-n+\varepsilon},$$

$$g_k(z_j) = 0, \quad j \neq k,$$

$$|g_k(z)| \leqq C(1 - |z|^2)^{-n+\varepsilon},$$

C independent of k. This is an immediate consequence of Lemma 4.1 and the Möbius invariance. The interpolation problem is then solved explicitly by the formula

$$f(z) = \sum_k a_k (1 - |z_k|^2)^{n-\varepsilon} g_k(z) \left(\frac{1 - |z_k|^2}{1 - \bar{z}_k z} \right)^s, \tag{23}$$

where $s > 1 + \varepsilon$. To see that $f \in A^{-n}$, we observe that

$$|f(z)| \leqq \sup_k \{(1 - |z_k|^2)^n |a_k|\} (1 - |z|^2)^{-n+\varepsilon} \sum_j \frac{(1 - |z_j|^2)^{s-\varepsilon}}{|1 - \bar{z}_j z|^s}$$

by the uniform bound on the functions g_k. In order to estimate this sum, we recall that for $1 < t < s$, we have

$$\int_U \frac{(1-|z|^2)^{t-2}}{|1-\bar{\zeta}z|^s} d\omega(z) \leqq C(1-|\zeta|^2)^{t-s} \tag{24}$$

(see Lemma 4.2.2 in [18, p. 53]). Thus, by (2), we have

$$\sum_j \frac{(1-|z_j|^2)^t}{|1-\bar{\zeta}z_j|^s} \leqq C(1-|\zeta|^2)^{t-s},$$

C depending only on δ. We conclude that

$$(1-|z|^2)^n |f(z)| \leqq C \sup_k \{(1-|z_k|^2)^n |a_k|\},$$

which is the desired estimate. □

7 L^2 density theorems for sampling and interpolation

We may now with some extra effort obtain similar density theorems for the weighted Bergman spaces. These results should be compared with the theory of nonharmonic Fourier series; see [17], and in particular [7]. It is remarkable that our description in terms of densities is *complete* (as it was for the Bargmann-Fock space in [14, 15]), while for the Paley-Wiener space the necessary and sufficient density conditions do not coincide, and correspondingly there is a rich theory of Riesz bases of complex exponentials.

Define for each $n > 0$ the weighted Bergman space,

$$A^{-n,2} = \{f \text{ analytic in } U\colon \int_U |f(z)|^2 (1-|z|^2)^{2n-1} d\omega(z) < \infty\}$$

(we use this somewhat unusual notation to obtain a natural correspondence to Korenblum's work). We say that a sequence of distinct points $\{z_j\}$ of U is a *set of sampling* for $A^{-n,2}$ if there exist positive constants K_1 and K_2 such that

$$K_1 \int_U |f(z)|^2 (1-|z|^2)^{2n-1} d\omega(z) \leqq \sum_j |f(z_j)|^2 (1-|z_j|^2)^{2n+1} \leqq K_2 \int_U |f(z)|^2 (1-|z|^2)^{2n-1} d\omega(z) \tag{25}$$

for every $f \in A^{-n,2}$. $\{z_j\}$ is a *set of interpolation* for $A^{-n,2}$ if for every sequence $\{a_j\}$ for which $\{(1-|z_j|^2)^{n+\frac{1}{2}} a_j\} \in l^2$, there exists a function $f \in A^{-n,2}$ such that $f(z_j) = a_j$ for all j.

Our theorems are the following.

Theorem 7.1 *A sequence Γ of distinct points in U is a set of sampling for $A^{-n,2}$ if and only if it can be expressed as a finite union of uniformly discrete sets and it contains a uniformly discrete subsequence Γ' for which $D^-(\Gamma') > n$.*

Theorem 7.2 *A sequence Γ of distinct points in U is a set of interpolation for $A^{-n,2}$ if and only if Γ is uniformly discrete and $D^+(\Gamma) < n$.*

The main difficulty consists in proving the sufficiency of the condition in Theorem 7.1. This proof, which rests on an application of Theorem 1.1, will be given in detail below, but we start by indicating how the other statements follow from previous results and the arguments used above. Note that the "Möbius invariance" of $A^{-n,2}$ is provided by the isometric transformations

$$f(z) \mapsto (\Phi'(z))^{n+\frac{1}{2}} f(\Phi(z)),$$

which for $A^{-n,2}$ play the role that the transformations T^n_Φ do for A^{-n}.

The necessity part of Theorem 7.2 The fact that a set of interpolation is uniformly discrete is well known (see [12]). The rest of the proof follows along lines very similar to those of the proof of the necessity part of Theorem 1.2; we omit the details of making the necessary modification.

The sufficiency part of Theorem 7.2 It is relatively easy to see that (23) solves the interpolation problem for $A^{-n,2}$ as well; the technique of estimation is a variant of that used in A^{-n} case and in fact the same as that of the proof of Theorem 1.2 in [13], where the details can be found.

The necessity part of Theorem 7.1 It is easy to see that a set of sampling can be expressed as a finite union of uniformly discrete sets; see Lemma 7.1 of [14]. It is likewise easy to adopt the proof technique of Lemma 7.2 of [14] in order to see that a set of sampling will contain a uniformly discrete set that is also a set of sampling. The rest of the proof can be performed essentially as the proof of the necessity part of Theorem 1.1; again, we omit the easy details.

The sufficiency part of Theorem 7.1 The basic ingredient in the proof is a formula which we deduce from Theorem 1.1. It is an analogue of Beurling's *linear balayage operator* [2, p. 348–350].

The difficulty consists in verifying the left inequality in (25), since the right inequality holds trivially by the assumption that Γ is a finite union of uniformly discrete sets. We may assume that Γ is uniformly discrete and that $\alpha = D^-(\Gamma) > n$. Then by Theorem 1.1, Γ is a set of sampling for $A_0^{-(n+\varepsilon)}$, where, say $\varepsilon = (\alpha - n)/2$. This means that the linear transformation

$$Tf = \{(1 - |z_j|^2)^{n+\varepsilon} f(z_j)\}_{z_j \in \Gamma}$$

is a bounded invertible mapping from $A_0^{-(n+\varepsilon)}$ onto a closed subspace of the sequence space c_0 (i.e., the closed subspace of l^∞ of sequences $\{w_j\}$ for which $w_j \to 0$). Denote this subspace by a_0. Then any bounded linear functional ϕ on $A_0^{-(n+\varepsilon)}$ induces a bounded linear functional on a_0 by

$$\tilde{\phi}(\xi) = \phi(T^{-1}\xi),$$

with $\|\tilde{\phi}\| \leqq K\|\phi\|$. For each $\zeta \in U$, let ϕ_ζ denote the normalized functional of point evaluation at ζ, i.e.,

$$\phi_\zeta(f) = (1 - |\zeta|^2)^{n+\varepsilon} f(\zeta).$$

Trivially, $\|\phi_\zeta\| = 1$. By the above reasoning, and since the dual space of c_0 is l^1, there exists for each ζ a sequence of numbers $\{g_j(\zeta)\}$ such that

$$(1 - |\zeta|^2)^{n+\varepsilon} f(\zeta) = \sum_{z_j \in \Gamma} (1 - |z_j|^2)^{n+\varepsilon} f(z_j) g_j(\zeta) \tag{26}$$

with

$$\sum_j |g_j(\zeta)| \leqq K. \tag{27}$$

In fact, if we apply (26) to the function $f(z)((1-|\zeta|^2)/(1-\bar{\zeta}z))^s$, s an arbitrary real number, we obtain the more general formula

$$(1-|\zeta|^2)^{n+\varepsilon} f(\zeta) = \sum_j (1-|z_j|^2)^{n+\varepsilon} f(z_j) \left(\frac{1-|\zeta|^2}{1-\bar{\zeta}z_j}\right)^s g_j(\zeta). \tag{28}$$

It is possible to improve the convergence of (28) even more. Fix an arbitrary positive number c, and define for each ζ,

$$\Lambda_\zeta = \left\{ z_j \in \Gamma\colon\ \rho(z_j, \zeta) > \frac{1}{2};\ |g_j(\zeta)| > c\frac{(1-|\zeta|^2)(1-|z_j|^2)}{|1-\bar{z}_j\zeta|^2} \right\}.$$

It is clear that Λ_ζ satisfies the Blaschke condition so that we may apply (28) to $B_\zeta(z)f(z)$, where $B_\zeta(z)$ is the Blaschke product associated with Λ_ζ, and $f \in A_0^{-(n+\varepsilon)}$. Thus

$$(1-|\zeta|^2)^{n+\varepsilon} B_\zeta(\zeta) f(\zeta) = \sum_j (1-|z_j|^2)^{n+\varepsilon} f(z_j) \left(\frac{1-|\zeta|^2}{1-\bar{\zeta}z_j}\right)^s \tilde{g}_j(\zeta)$$

where

$$\tilde{g}_j(\zeta) = B_\zeta(z_j) g_j(\zeta).$$

Note that for $\rho(z_j, \zeta) \leqq \frac{1}{2}$, we have

$$|\tilde{g}_j(\zeta)| \leqq K \leqq \frac{4}{3} K \frac{(1-|\zeta|^2)(1-|z_j|^2)}{|1-\bar{z}_j\zeta|^2},$$

since by (1),

$$\left|\frac{\zeta - z_j}{1-\bar{z}_j\zeta}\right|^2 = 1 - \frac{(1-|\zeta|^2)(1-|z_j|^2)}{|1-\bar{z}_j\zeta|^2} \tag{29}$$

So for *all* j we have

$$|\tilde{g}_j(\zeta)| \leqq C \frac{(1-|\zeta|^2)(1-|z_j|^2)}{|1-\bar{z}_j\zeta|^2},$$

C independent of ζ. Observe also that by (29), the definition of Λ_ζ, and (27),

$$|B_\zeta(\zeta)| \geqq C,$$

C independent of ζ. We have therefore

$$(1-|\zeta|^2)^{n+\varepsilon} f(\zeta) = \sum_j (1-|z_j|^2)^{n+\varepsilon} f(z_j) \left(\frac{1-|\zeta|^2}{1-\bar{\zeta}z_j}\right)^s h_j(\zeta), \tag{30}$$

where $h_j(\zeta) = \tilde{g}_j(\zeta)/B_\zeta(\zeta)$, and the following estimates:

$$\sum_i |h_j(\zeta)| \leqq C, \tag{31}$$

and

$$|h_j(\zeta)| \leqq C \frac{(1-|\zeta|^2)(1-|z_j|^2)}{|1-\bar{z}_j\zeta|^2}. \tag{32}$$

(30) holds for functions in $A_0^{-(n+\varepsilon)}$. However, since $A_0^{-(n+\varepsilon)} \cap A^{-n,2}$ is dense in $A^{-n,2}$, it is now easy to see that (30) is in fact valid for all $f \in A^{-n,2}$.

(30) seems interesting in its own right. It is an alternative to an expansion based on the theory of frames [7, 6]. Note that in [13] we obtained a similar but more explicit formula by the calculus of residues. For our present purpose we could have made it with $s = 0$, but we find the formula interesting enough to write down this general version.

We apply the Cauchy-Schwarz inequality to (30) with $f \in A^{-n,2}$, yielding

$$(1-|\zeta|^2)^{2(n+\varepsilon)}|f(\zeta)|^2 \leqq C \sum_j (1-|z_j|^2)^{2(n+\varepsilon)}|f(z_j)|^2 \left|\frac{1-|\zeta|^2}{1-\bar{\zeta}z_j}\right|^{2s} |h_j(\zeta)| \sum_k |h_k(\zeta)|.$$

For $h_j(\zeta)$ in the first sum we use the estimate (32). Hence, in view of (31), we obtain

$$(1-|\zeta|^2)^{2n-1}|f(\zeta)|^2 \leqq C \sum_j (1-|z_j|^2)^{2n+1+2\varepsilon}|f(z_j)|^2 \frac{(1-|\zeta|^2)^{2s-2\varepsilon}}{|1-\bar{\zeta}z_j|^{2s+2}}.$$

When now integrating over U, we make use of (24). Thus we choose s so that $2s - 2\varepsilon > -1$ to obtain the desired norm estimate.

Acknowledgement. The author wishes to thank the referee for a careful review of the paper, in particular for valuable comments concerning the readability of the paper and for the detection of a couple of miscalculations and several misprints.

References

1. Amar, E.: Suites d'interpolation pour les classes de Bergman de la boule et du polydisque de $\mathbf{C}^n$. Can. J. Math. **30**, 711–737 (1978)
2. Beurling, A.: In: Carleson, L., Malliavin, P., Neuberger, J., Wermer, J. The collected Works of Arne Beurling, vol. 2, Harmonic Analysis, pp. 341–365. Boston: Birkhäuser 1989
3. Bruna, J., Pascuas, D.: Interpolation in $A^{-\infty}$. J. Lond. Math. Soc. **40** 452–466, (1989)
4. Carleson, L. An interpolation problem for bounded analytic functions. Am. J. Math. **80**, 921–930 (1958)
5. Coifman, R.R., Rochberg, R.: Representation theorems for holomorphic and harmonic functions in L^p. Astérisque **77**, 11–66 (1980)
6. Daubechies, I.: The wavelet transform, time-frequency localization and signal analysis. IEEE Trans. Inf. Theory **36**, 961–1005 (1990)
7. Duffin, R.J., Schaeffer, A.C.: A class of nonharmonic Fourier series. Trans. Am. Math. Soc. **72**, 341–366 (1952)
8. Hoffman, K.: Banach Spaces of Analytic Functions. Englewood Cliffs: Prentice-Hall: 1962
9. Korenblum, B.: An extension of the Nevanlinna theory. Acta Math. **135**, 187–219 (1975)
10. Landau, H.J.: Necessary density conditions for sampling and interpolation of certain entire functions. Acta Math. **117**, 37–52 (1967)
11. Luecking, D.: Closed range restriction operators on weighted Bergman spaces. Pac. J. Math. **110**, 145–160 (1984)
12. Rochberg, R.: Interpolation by functions in Bergman spaces. Mich. Math. J. **28**, 229–237, (1982)

13. Seip, K.: Regular sets of sampling and interpolation for weighted Bergman spaces. Proc. Am. Math. Soc. **117**, 213–220 (1993)
14. Seip, K.: Density theorems for sampling and interpolation in the Bargmann-Fock space. I. J. Reine Angew. Math. **429**, 91–106 (1992)
15. Seip, K., Wallstén, R.: Density theorems for sampling and interpolation in the Bargmann-Fock space. II. J. Reine Angew. Math. **429**, 107–113 (1992)
16. Shields, A.L., Williams, D.L., Bounded projections, duality, and multipliers in spaces of analytic functions. Trans. Am. Math. Soc. **162**, 287–302 (1971)
17. Young, R.M.: An Introduction to the Theory of Nonharmonic Fourier Series. New York: Academic Press 1980
18. Zhu, K.: Operator Theory in Function Spaces. New York: Marcel Dekker 1990