Invent. math. 106, 411–431 (1991)

Inventiones mathematicae

A unification of Knizhnik-Zamolodchikov and Dunkl operators via affine Hecke algebras

Ivan Cherednik*

Research Institute for Mathematical Sciences, Kyoto 606, Japan

Oblatum 20-XII-1990 & 25-III-1991

Summary. Some generalizations of the Lusztig-Lascoux-Schützenberger operators for affine Hecke algebras are considered. As corollaries we obtain Lusztig's isomorphisms from affine Hecke algebras to their degenerate versions, a "natural" interpretation of the Dunkl operators and a new class of differential-difference operators generalizing Dunkl's ones and the Knizhnik-Zamolodchikov operators from the two dimensional conformal field theory.

Table of contents

0 Introduction 411
1 Affine Hecke algebras and the Intertwiners 413
2 The Lusztig-Lascoux-Schützenberger operators 417
3 The unification, some examples 424

Introduction

The first aim of this paper is to consider natural vector versions of the Lusztig operators [Lu3] and the Lascoux-Schützenberger operators [LS1, LS2] and calculate (in the scalar case) the representations of the corresponding affine Hecke algebras in which these operators act. The key point of this calculation is equivalent to some form of the main theorem from [Ka] (we give a new more simple proof of it.) As corollaries one obtains Lusztig's isomorphisms between affine Hecke algebras and their degenerate (graded) versions [Lu1] and a natural construction of the Dunkl differential-difference operators [Du, He1] together with their trigonometric counterparts close to Heckman's operators [He2]. The second aim is a unification of the Dunkl and the Knizhnik-Zamolodchikov operators from [Ch1, Ch 2] taking the vector analogue of the Lusztig operators as a basis. Given a root system $\Sigma \subset \mathbf{R}^n$ and a representation of the corresponding Weyl group

* On leave from A.N. Belozersky Lab., Bldg. "A", Moscow State University, Moscow 119899, USSR

$W \subset \mathrm{Aut}(\mathbf{R}^n)$, we define a new commutative family of differential-difference operators generalizing both the Knizhnik-Zamolodchikov and the Dunkl operators.

Several preliminary points on affine Hecke algebras and the intertwining operators are worth mentioning. The intertwiners play a very important role in the theory of p-adic representations of unramified principal series. The latter are directly connected with the representations of the corresponding p-adic affine Hecke algebra $\mathscr{H}$ which are induced from characters of the so-called Bernstein-Zelevinsky commutative subalgebra $\mathscr{Y} \subset \mathscr{H}$. Here $\mathscr{H}$ depends on a parameter q which is a power of prime p. However it is quite natural to assume q to be an arbitrary complex number, because the defining relations for $\mathscr{H}$ depends on q algebraically. In several papers (see [Ma, Ka, Ro]) explicit formulas were used for the intertwining operators between the representations induced from conjugated characters with respect to a natural action of W on $\mathscr{Y}$. For example, they were useful to Rogawski in making more lucid the Zelevinsky theorems on p-adic representations of GL_n [Ze]. These intertwiners can be considered as elements of $\mathscr{H}$ satisfying the Coxeter relations of W. The last fact was not formulated in the above papers, but follows directly from them (see [Lu1] and e.g. [Ch 3], where the case $W = \mathfrak{S}_{n+1}$ was considered).

As a consequence one gets an isomorphism π between $\mathscr{Y}[W]$ (the semi-direct product of $\mathscr{Y}$ and $\mathbf{C}[W]$) and $\mathscr{H}$ after some localization of $\mathscr{Y}$. This isomorphism is useless for the most interesting (special) representations of $\mathscr{H}$ because of this localization. Nevertheless, it can be applied to obtain a certain map without denominators from $\mathscr{H}$ to its degeneration $\mathscr{H}'$.

The relations for $\mathscr{H}'$ in the case $W = \mathfrak{S}_{n+1}$ were found for the first time by Murphy (see [Mu]). She defined a commutative subalgebra in $\mathbf{C}[\mathfrak{S}_{n+1}]$, closely connected with the so-called Young's bases for $\mathfrak{S}_{n+1}$, and calculated the cross-relations between its generators and the adjacent transpositions. It was shown in [Dr] (see also [Ch 4]) that her subalgebra is the image of the counterpart $\mathscr{Y}' \subset \mathscr{H}'$ of $\mathscr{Y}$ with respect to a canonical surjection $\mathscr{H}' \to \mathbf{C}[\mathfrak{S}_{n+1}]$. Drinfeld defined $\mathscr{H}'$ for $W = \mathfrak{S}_{n+1}$ as a certain limit of $\mathscr{H}$, when $q \to 1$.

Drinfeld's construction can be extended naturally to arbitrary Weyl groups W. Lusztig gave the general definition of $\mathscr{H}'$ (which he called the graded affine Hecke algebra) in papers [Lu1, Lu2]. The analogues of the above intertwiners can be easily calculated for $\mathscr{H}'$ and coincide with the formulas from the paper [Ch 5] devoted to the W-invariant quantum R-matrices. By the way, the Matsumoto-Rogawski formulas for the intertwiners are closely connected with the basic trigonometric R-matrix (in Jimbo's form).

In [Ch 3, Ch 4] and some other papers it was shown by means of the technique of intertwiners that the classification of the irreducible representations, the theory of Young bases, the character formulas and some other points are quite parallel for $\mathscr{H}'$ and $\mathscr{H}$, when $W = \mathfrak{S}_{n+1}$ and q is generic. It is now possible to explain this coincidence *a priori*.

After some localization we get an isomorphism π': $\mathscr{H}'_{\mathrm{loc}} \xrightarrow{\sim} \mathscr{Y}'_{\mathrm{loc}}[W]$ in the same manner as π. The semi-direct product $\mathscr{Y}'[W]$ of $\mathscr{Y}'$ and $\mathbf{C}[W]$ can be identified with $\mathscr{Y}[W]$ after a suitable completion of $\mathscr{Y}$ and $\mathscr{Y}'$. Then the composition map $\tilde{\pi} = \pi \circ (\pi')^{-1}$ will be an isomorphism between $\mathscr{H}$, $\mathscr{H}'$ both localized and completed. It follows from [Lu1] that the completion (without any localization) is enough to define $\tilde{\pi}$. This completion is compatible with the category of finite-dimensional representations.

Independently, analogous isomorphisms were obtained in [Ch 2] for $W = \mathfrak{S}_{n+1}$ as a result of some direct calculation of the monodromy matrices for the generalized Knizhnik-Zamolodchikov equation on $\mathscr{H}'$. It was mentioned there that the final formula (an expression of generators of $\mathscr{H}$ in terms of these of $\mathscr{H}'$) can be considered independently of its monodromy interpretation and can be directly extended to arbitrary Weyl groups W.

Summarizing, we have the following approaches: guess and check formulas for $\tilde{\pi}$: $\mathscr{H}_{\mathrm{compl}} \to \mathscr{H}'_{\mathrm{compl}}$ without any preliminary theory, use the intertwiners (as was explained above) or calculate some monodromy representation. A fourth possibility is to use the Lusztig-Lascoux-Schützenberger operators (see Sect. 2).

Another part of this paper (Sect. 3) is connected with the following construction. It was demonstrated in [Ch 1, Ch 2] that one can define some kind of Knizhnik-Zamolodchikov equations for arbitrary classical W-invariant r-matrices. The latter are certain quasi-classical limits of the quantum W-invariant R-matrices from [Ch 5] (see also [Ch 1]). These two notions for $W = \mathfrak{S}_{n+1}$ are equivalent respectively to the ordinary concepts of r-matrices and R-matrices from the Soliton theory (Faddeev, Sklyanin et al.). Some r-matrices of type D_n were introduced by Sklyanin for certain integrable equations with boundary conditions.

We consider our version of the Lusztig-Lascoux-Schützenberger operators as a certain quantum R-matrix (with a new type of dependence on the spectral parameter). The quasi-classical limit of these operators of the first kind (connected with $\mathscr{H}'$) gives the family of Dunkl operators [Du, He2]. The operators of the second kind (for $\mathscr{H}$) produce some family of "trigonometric" operators, which are close to Heckman's ones [He1] but do not coincide with them.

We note that the origins of all these operators are in the Bernstein-Gelfand-Gelfand and Demazure difference operations [BGG, De]. The Laplacians defined for the Dunkl and Heckman operators are conjugate to the Schrödinger "rational and trigonometric" operators from the quantum theory of the Calogero-Moser and Olshanetsky-Perelomov integrable systems (see e.g. [HO]). Our trigonometric operators have the above property as well. Another application is connected with Macdonald's q-analogues of the Jacobi polynomials. The corresponding property of Heckman's operators (to be self-adjoint with respect to some form) holds good for our operators as well.

The main point of this paper is a common definition for both the Knizhnik-Zamolodchikov and the Dunkl operators (together with their trigonometric analogues) on the basis of the general form of the Lusztig-Lascoux-Schützenberger operators. It gives birth to many interesting algebraic and analytical questions in the theory of Hecke algebras. Moreover, this construction should be connected with some (maybe new) quantum groups. The corresponding quasi-classical limits are expected to yield a new kind of τ-functions.

1 Affine Hecke algebras and the intertwiners

Most of the following facts are known. Nevertheless we prefer to give the proofs, because they are short and either new or useful for Sect. 2. The main references are [Ma, Ka, Ro, KL, Lu1, Lu2, Ch3].

Let $\Sigma = \{\alpha\} \subset \mathbf{R}^n$ be a root system of type $A_n, B_n, \ldots, G_2$, and let s_α be the orthogonal reflections in the hyperplanes $(\alpha, u) = 0$ with respect to the canonical euclidean form $(\,,)$ on $\mathbf{R}^n \ni u$. Later on, $\{\alpha_1, \ldots, \alpha_n\} \subset \Sigma$ will be the simple roots for some fixed Weyl chamber, Σ_+, Σ_- the sets of all positive $(\alpha > 0)$ and negative $(\alpha < 0)$ roots and W the Weyl group generated by the reflections $s_i = s_{\alpha_i}$ $(1 \leqq i \leqq n)$.

The length of $w \in W$(the length of the reduced decompositions of w with respect to $s_1, \ldots, s_n$) will be denoted by $l(w)$; $l(\mathrm{id}) = 0$. Let $\mathbf{C}[W] = \oplus_w \mathbf{C}w$ be the group algebra of W. We assume $(\,,)$ and the action of W to be extended $\mathbf{C}$-linearly to $u \in \mathbf{C}^n$.

Throughout this paper we fix arbitrary $q', q'' \in \mathbf{C}$ and $h', h'' \in \mathbf{C}$ and put $q_\alpha = q'$ or q'', $h_\alpha = h'$ or h'' respectively for short or long α. We also set $q_i = q_{\alpha_i}$, $h_i = h_{\alpha_i} (1 \leqq i \leqq n)$.

Definition 1.1 (see [Ma, Lu1]) (a) The Hecke algebra H is generated over $\mathbf{C}$ by $T_1, \ldots, T_n$ with the following homogeneous relations of degree m

$$T_i T_{i'} T_i T_{i'} \ldots = T_{i'} T_i T_{i'} T_i \ldots \text{ (m factors on both sides)}, \tag{1.1}$$

where $m = m_{ii'}$ is the order of $s_i s_{i'}$, $1 \leqq i, i' \leqq n$,

$$(T_i + 1)(T_i - q_i) = 0, \quad q_i = q'. \tag{1.2}$$

(b) The affine Hecke algebra $\mathscr{H}$ is generated by $\mathscr{H}$ and $\{Y_\alpha, \alpha \in \Sigma\}$ satisfying the following relations $(1 \leqq i \leqq n)$

$$T_i Y_\alpha - Y_{s_i(\alpha)} T_i = (q_i - 1)(Y_{s_i(\alpha)} - Y_\alpha)(Y_i - 1)^{-1}, \tag{1.3}$$

$$[Y_\alpha, Y_\beta] = 0, \quad \alpha, \beta \in \Sigma, \quad Y_{\alpha+\beta} = Y_\alpha Y_\beta, \tag{1.4}$$

where the latter holds for $\alpha + \beta \in \Sigma \cup \{0\}$, $Y_i \overset{\text{def}}{=} Y_{\alpha_i}$, $Y_0 \overset{\text{def}}{=} 1$, the r.h.s. of (1.3) is a polynomial of $\{Y_i, Y_i^{-1}\}$.

(c) The degenerate affine Hecke algebra $\mathscr{H}'$ is obtained by adding pairwise commuting $\{y_\alpha, \alpha \in \Sigma\}$ with the relations

$$s_i y_\alpha - y_{s_i(\alpha)} s_i = h_i (y_{s_i(\alpha)} - y_\alpha)/y_i = -2h_i(\alpha, \alpha_i)/(\alpha_i, \alpha_i), \tag{1.5}$$

where $y_{\alpha+\beta} = y_\alpha + y_\beta$ for $\alpha + \beta \in \Sigma \cup \{0\}$, $y_i = y_{\alpha_i}$, $y_0 = 0$, to $H' \overset{\text{def}}{=} \mathbf{C}[W]$.

Let $\mathbf{C}(Y)$, $\mathbf{C}(y)$ be the quotient fields of the ring of polynomials $\mathbf{C}[Y] \overset{\text{def}}{=} \mathbf{C}[Y_\alpha, \alpha \in \Sigma]$, $\mathbf{C}[y] \overset{\text{def}}{=} \mathbf{C}[y_\alpha, \alpha \in \Sigma]$, and $\bar{\mathscr{H}}$, $\bar{\mathscr{H}}'$ be generated by H and $\mathbf{C}(Y)$ or respectively by H' and $\mathbf{C}(y)$. The algebras $\bar{\mathscr{H}}$, $\bar{\mathscr{H}}'$ are well-defined and contain $\mathscr{H}$, $\mathscr{H}'$ because of the decomposition

$$\mathscr{H} = \oplus_{w \in W} C[Y] T_w = \oplus_{w \in W} T_w C[Y], \tag{1.6}$$

and the analogous decomposition for $\mathscr{H}'$ with w instead of T_w (see [Lu1, Proposition 3.7]). Here

$$T_w = T_{i_l} \ldots T_{i_1} \quad \text{if} \quad w = s_{i_l} \ldots s_{i_1} \quad \text{for} \quad l = l(w) \tag{1.7}$$

depends only on w owing to (1.1). We note that (1.6) and its counterpart for $\mathscr{H}'$ can be deduced directly from formulas (2.9)–(2.12) below. But one has to prove independently that (2.9)–(2.12) give a representation of $\mathscr{H}$ or $\mathscr{H}'$ in this way. This is not difficult. Another way is to use Φ_i and φ_i below. For some questions (e.g., for $q'q''=0$) it is convenient to replace $\mathbf{C}[Y]$ by $\mathbf{C}[Y_\alpha, \alpha\in\Sigma_-]$ in the definitions. To do this we multiply (1.3) by Y_i^{-1}.

There is the action

$$w(Y_\alpha)=Y_{w(\alpha)},\ w(y_\alpha)=y_{w(\alpha)},\quad \alpha\in\Sigma\,,$$

of the Weyl group $W\ni w$ on $\mathbf{C}[Y]$ and $\mathbf{C}[y]$. Note that $\oplus_{i=1}^n \mathbf{C}y_i$ can be identified with our basic $\mathbf{C}^n$ in a natural way. Then $\mathbf{C}[y]$ is nothing else but the symmetric algebra of $\mathbf{C}^n$ with the usual induced action of W.

Proposition 1.2 (a) *The elements*

$$\Phi_i=T_i+(q_i-1)/(Y_i-1)\in\bar{\mathscr{H}},\quad 1\leqq i\leqq n\,, \tag{1.8}$$

satisfy the relations

$$\Phi_i Y_\alpha = Y_{s_i(\alpha)}\Phi_i,\quad \alpha\in\Sigma\,. \tag{1.9}$$

(b) *The elements* $\Phi_w=\Phi_{i_l}\dots\Phi_{i_1}$, *where* $w=s_{i_l}\dots s_{i_1}$, $l(w)=l$, *have the following properties*

$$\Phi_w Y_\alpha = Y_{w(\alpha)}\Phi_w,\ \alpha\in\Sigma\,, \tag{1.10}$$

$$\bar{\mathscr{H}}=\oplus_{w\in W}\mathbf{C}(Y)\Phi_w=\oplus_{w\in W}\Phi_w\mathbf{C}(Y)\,. \tag{1.11}$$

(c) *The element* Φ_w *does not depend on the choice of reduced decomposition of* w:

$$\Phi_w\Phi_{w'}=\Phi_{ww'}\quad \textit{if}\quad l(ww')=l(w)+l(w')\,. \tag{1.12}$$

(d) *Properties* (a), (b), (c) *hold good for* $\mathscr{H}'$ *with* Y_i *replaced by* y_i *and* Φ_i *by*

$$\phi_i=s_i+h_i y_i^{-1},\quad 1\leqq i\leqq n. \tag{1.13}$$

The proof can be found in [Lu1, Sect. 5]. The particular case of A_n (to be more precise, of the root system for GL_{n+1}) was considered in [Ch 3, Sect. 3]. Statements (a), (b), (c) are close to the corresponding (slightly weaker) properties of the intertwiners from [Ma, (4.3.2)] and [Ro]. For the sake of completeness we will prove the proposition briefly (the proof from [Lu1] is somewhat different).

Formulas (1.8), (1.9) are nothing else but relations (1.3). Let us introduce the set $\{\Phi_w\}$ for some choice of the corresponding reduced decompositions of elements $w\in W$ (now we do not worry about the uniqueness). Then (1.10) is quite clear. Assertion (1.11) can be easily deduced from (1.6) by induction. The only thing we need for this is the set of decompositions

$$\Phi_w=T_w+\sum_{w'}P_{w'}T_{w'}, \tag{1.14}$$

where $P_{w'}\in\mathbf{C}(Y)$, $w, w'\in W$, $l(w')<l(w)$. To prove (c) we will use the following

Lemma 1.3 (a) *The algebra* $\mathbf{C}[Y]$ *coincides with its centralizer* $Z\mathbf{C}[Y]=\{Z\in\mathscr{H}\,|\,[Z,\mathbf{C}[Y]]=0\}$ *(see* [Ch3] *for* GL_{n+1}*).*

(b) *The centre of* $\mathscr{H}$ *is the algebra* $\mathbf{C}[Y]^W$ *of* W*-invariant polynomials in* $\mathbf{C}[Y]$ *(due to Bernstein).*

(c) *The above statements hold good for* $\mathbf{C}(Y)\subset\bar{\mathscr{H}}$ *and for* $\mathscr{H}'$, $\bar{\mathscr{H}}'$.

Proof. Let $Z = \sum_{w \in W} Z_w \Phi_w \subset Z\mathbf{C}(Y) \subset \bar{\mathscr{H}}$ and $Z_w \neq 0$ for some $w \neq \mathrm{id}$ (see (1.11)). Then $PZ - ZP \neq 0$ for $P \in \mathbf{C}(Y)$ such that $w(P) \neq P$, because of (1.10) and (1.11). The proof of (b), (c) is of the same length (see [Lu 1] for (b)). □

It results from (1.10) that $\Phi_w \Phi_{w'} = \Phi_{ww'} P$ for some $P \in \mathbf{C}(Y)$. However $\Phi_w \Phi'_w$ and $\Phi_{ww'}$ for $l(ww') = l(w) + l(w')$ belong to the same affine space $T_{ww'} + (\oplus_{s \in W} T_s \mathbf{C}(Y))$, where $l(s) < l(ww')$ by virtue of (1.14) together with definition (1.7). Hence $P = 1$ because of (1.6). Thus, we have got (1.12). As for $\mathscr{H}'$, the proof of (a), (b), (c) is the same. □

Let us fix a pair $M \supset m$ of maximal ideals $M \subset \mathbf{C}[Y]$, $m \subset \mathbf{C}[Y]^W$ ($M \in \mathrm{Specm}\, \mathbf{C}[Y]$, $m \in \mathrm{Specm}\, \mathbf{C}[Y]^W$, where Specm means *Spec maximal*). The corresponding homomorphisms $\mathbf{C}[Y] \to \mathbf{C}[Y]/M \to \mathbf{C}$, $\mathbf{C}[y]^W \to \mathbf{C}[y]^W/m \to \mathbf{C}$ will be denoted by χ_M, χ_m. For all elements from $\mathscr{H}$ and almost all elements $Z \in \bar{\mathscr{H}}$ it is possible to define "*the right value*" of Z at M.

$$Z(M) = \sum_w T_x \chi_M(Z_w), \text{ where } Z = \sum_w T_w Z_w\,, \tag{1.15}$$

$Z_w \in \mathbf{C}(Y)$, $w \in W$, $Z(M) \in H$. The set of all M where Z is well-defined is open in Specm $\mathbf{C}[Y]$. The analogous pairs $M \supset m$ and the same definition (1.15) will be used for $\mathscr{H}'$.

Definition 1.4 Let $I_M = \mathrm{Ind}_{\mathbf{C}[Y]}^{\mathscr{H}} \chi_M$ be the universal $\mathscr{H}$-module generated by the representation χ_M of $\mathbf{C}[Y]$ (see e.g., [Ma, Ka, Ro]). As a H-module it is canonically isomorphic to H with the left regular action ($A(B) = AB$ for A, $B \in H$). The $\mathscr{H}$-module structure of I_M can be uniquely determined by the relations

$$Y_\alpha(1) = \chi_M(Y_\alpha) \overset{\mathrm{def}}{=} Y_\alpha(M),\ \alpha \in \Sigma,\ 1 \in H \simeq I_M\,.$$

The analogous definition can be given for $\mathscr{H}'$ and $H' = \mathbf{C}[W]$.

The Weyl group W acts on M and $\chi = \chi_M$ in the following way:

$$Y_\alpha(M_w) = \chi_{M_w}(Y_\alpha) = (w^{-1}(\chi))(Y_\alpha) = \chi(Y_{w(\alpha)}) = Y_{w(\alpha)}(M)\,, \tag{1.16}$$

where $\alpha \in \Sigma$, $w \in W$. Note that $M_{ww'} = (M_w)_{w'}$ for w, $w' \in W$, $(ww')\,(\chi) = w(w'(\chi))$. The ideals M_w, $w \in W$, constitute the set of all maximal ideals over m (i.e., containing m). The same holds for $\mathscr{H}'$. We will fix M and denote I_{M_w} by I_w ($I_{\mathrm{id}} = I_M$).

Proposition 1.5 (See e.g. [Ro]) (a) *The action of the center* $\mathbf{C}[Y]^W \subset \mathscr{H}$ *on each* I_w *is scalar and induces the above homomorphism* χ_m.

(b) *Every irreducible representation* U *of* $\mathscr{H}$ *with the action of* $\mathbf{C}[Y]^W$ *(it should be scalar) via* χ_m *is a quotient-module of some* I_w.

(c) *Given* I_w, $I_{w'}$, *there is a non-trivial* $\mathscr{H}$*-homomorphism* $\mu: I_w \to I_{w'}$.

(d) *All the irreducible constituents of each* I_w *with the corresponding multiplicities (i.e. the composition factors) coincide with those of* $I_{\mathrm{id}} = I_M$.

(e) *These statements hold good for* $\mathscr{H}'$.

Proof. The action of $\mathbf{C}[Y]^W$ on $1 \in H \sim I_w$ is via χ_m. By definition, 1 generates I_w as H-module. Hence, the centre acts on I_w by χ_m. As for (b), U is to be finite-dimensional. Indeed, elements of $\mathbf{C}[Y]^W$ act on U as some scalars, since U is irreducible and $\mathscr{H}$ is finite-dimensional over $\mathbf{C}[Y]^W$. Therefore there is at least one eigenvector u_0 of $\mathbf{C}[Y]$ in U, corresponding to some character χ_{M_w}. The map $1 \to u_0$ gives the required homomorphism $I_w \to U$. We have checked (a)–(b). Statement (c) results from

Lemma 1.6 *Given M, w, there is a one-dimensional family $M(v)$ of maximal ideals in $\mathbf{C}[Y]$, analytically depending on a parameter $v \in \mathbf{C}$, which is close to 0, such that $M(0) = M$ and the function $v^k \Phi_w(M(v))$ (see (1.15)) has a non-zero value $\hat{\Phi}_w \in H$ for some $k \in \mathbf{Z}$. Then the map*

$$I_{M'} \simeq H \ni A \to A\hat{\Phi}_w \in H \simeq I_M, \; M' = M_{w^{-1}}, \tag{1.17}$$

is a $\mathscr{H}$-homomorphism. It is true for H' as well.

Proof. It is enough to prove (1.17) for $M' = M(v)$, $\hat{\Phi}_w = \Phi_w(M(v))$, $v \neq 0$. Concerning the H-action, the statement is clear. Therefore one has to check the following relation only:

$$Y_\alpha(\hat{\Phi}_w) = \chi_{M'}(Y_\alpha)\hat{\Phi}_w \text{ in } I_M . \tag{1.18}$$

Let us use (1.10) and (1.16):

$$\begin{aligned} Y_\alpha(\hat{\Phi}_w) &= (Y_\alpha \Phi_w)(M) = (\Phi_w Y_{w^{-1}(\alpha)})(M) = \Phi_w(Y_{w^{-1}(\alpha)}(M)) \\ &= \Phi_w(M) Y_\alpha(M') = Y_\alpha(M')\Phi_w(M) = Y_\alpha(M')\hat{\Phi}_{w}. = \chi_{M'}(Y_\alpha)\hat{\Phi}_w . \end{aligned}$$

The existence of $M(v)$ is evident because Φ_w is well-defined in some open subset of Specm $\mathbf{C}[Y]$. □

It is sufficient to prove (d) for $w = s_i (1 \leqq i \leqq n)$. There are three possibilities for Φ_{s_i} from Lemma 1.6. It is invertible and therefore has to give an isomorphism between I_{id} and I_{s_i} if $Y_i(M) \neq q_i^{\pm 1}$ or $q_i = 1$. Otherwise $\Phi_{s_i}(M)$ is to be proportional to

$$C_i^+ = T_i + 1 \text{ or } C_i^- = T_i - q_i . \tag{1.19}$$

Then $\Phi_{s_i}(s_i(M))$ is proportional to C_i^- or C_i^+ respectively. The corresponding $\mathscr{H}$-homomorphisms $I_{\mathrm{id}} \xrightarrow{\mu} I_{s_i} \xrightarrow{\mu'} I_{\mathrm{id}}$ satisfy the following relations:

$$\ker(\mu) = \mathrm{im}(\mu'), \; \ker(\mu') = \mathrm{im}(\mu) .$$

Really, $\{A \in H, AC_\pm^i = 0\} = HC_\mp^i$. Thus, $\ker(\mu) \oplus (I_{\mathrm{id}}/\ker(\mu)) \simeq \ker(\mu') \oplus (I_{s_i}/\ker(\mu'))$. It completes the proof for $\mathscr{H}$. The case of $\mathscr{H}'$ is quite analogous. □

2 The Lusztig-Lascoux-Schützenberger operators

We preserve the notations of Sect. 1. Let us fix a representation $\nu: H \to \mathrm{End}\, V$ for some $\mathbf{C}$-space V or respectively $\nu: H' = \mathbf{C}[W] \to \mathrm{End}\, V$ and introduce the space $V^0 = \mathbf{C}[Y] \otimes_{\mathbf{C}} V$ (or $V^0 = \mathbf{C}[y] \otimes_{\mathbf{C}} V$). We will denote $P \otimes v$ by Pv for $P \in \mathbf{C}[Y]$, $\mathbf{C}[y]$, $v \in V$. The algebra $\mathbf{C}[W] \times H$ acts on V^0 in a natural way (W on $\mathbf{C}[Y]$ and H on V via ν). We will identify $\nu(A) \times 1$ with $\nu(A)$ for $A \in H$ and $1 \times w$ with $w \in W$.

Similarly, V^0 for $\mathbf{C}[y]$ can be considered as a $\mathbf{C}[W] \times H'$-module. We will keep the same notations in this case.

Theorem 2.1 (a) *The representation* $\mathrm{Ind}_H^{\mathscr{H}} V$ *(the universal* $\mathscr{H}$*-module, generated by the above* H*-module* V*) is isomorphic to* V^0 *as a* $\mathbf{C}$*-vector space and can be uniquely determined by the formulas* (*cf.* [Lu1, 3.12])

$$T_i^0 = s_i v(T_i) + (q_i - 1)(Y_i^0 - 1)^{-1}(s_i - 1), \quad 1 \leqq i \leqq n, \tag{2.1}$$

$$Y_\alpha^0(P) = Y_\alpha P, \text{ where } P \in V^0, \quad \alpha \in \Sigma, \tag{2.2}$$

and X^0 *is the image of* $X \in \mathscr{H}$ *in* $\mathrm{End}\, V^0$.

(b) *The analogous* $\mathscr{H}'$*-module* $\mathrm{Ind}_{H'}^{\mathscr{H}'} V$ *is isomorphic to* V^0 *as a* $\mathbf{C}$*-space and satisfies the following defining relations* (*cf.* [Lu1, 4.4])

$$s_i^0 = s_i v(s_i) + h_i(y_i^0)^{-1}(s_i - 1), \quad 1 \leqq i \leqq n, \tag{2.3}$$

$$y_\alpha^0(p) = y_\alpha p, \quad \text{for} \quad p \in V^0, \alpha \in \Sigma. \tag{2.4}$$

(c) *The homomorphism* $\mathscr{H} \ni X \to X^\circ \in \mathrm{End}\, V^0$ *and the corresponding one for* H *both are injective.*

Proof. Formulas (2.2), (2.4) are valid by definition. It results from (1.3) that the relations

$$[T_i^0 s_i, Y_\alpha] = (q_i - 1)(Y_i^0 - 1)^{-1}(Y_\alpha - s_i(Y_\alpha))^0 s_i$$

hold good in $\mathrm{End}\, V^0$. The operators

$$\Delta_i : P^0 \to [T_i^0 s_i, P^0], \ \hat{\Delta}_i = P^0 \to ((1 - s_i)(P))^0 s_i,$$

acting from $\mathbf{C}[Y]^0 \ni P^0$ to $\mathbf{C}[Y]^0 \cdot W \subset \mathrm{End}\, V^0$, both satisfy the main property of derivatives

$$\Delta(P^0 Q^0) = P^0 \Delta(Q^0) + \Delta(P^0) Q^0, P, Q \in \mathbf{C}[Y].$$

Hence,

$$\Delta_i = (q_i - 1)\,(Y_i^0 - 1)^{-1}\,\hat{\Delta}_i$$

and

$$\begin{aligned} T_i^0(Pv) &= (T_i^0 s_i)(s_i(P)v) \\ &= [T_i^0 s_i, s_i(P)^0](v) + s_i(P)^0(v(T_i)v) \\ &= \{s_i v(T_i) + (q_i - 1)(Y_i^0 - 1)^{-1}(s_i - 1)\}(Pv) \end{aligned}$$

for arbitrary $P \in \mathbf{C}[Y]$, $v \in V$. The proof of (b) is the same.

Lemma 2.2 *The elements* Φ_i *and* $\phi_i (1 \leqq i \leqq n)$ *from* (1.8) *and* (1.13) *act on* V^0 *as the operators*

$$\Phi_i^0 = (v(T_i) + (q_i - 1)(Y_i^0 - 1)^{-1})s_i, \tag{2.5}$$

$$\varphi_i^0 = (v(s_i) + h_i(y_i^0)^{-1})s_i. \tag{2.6}$$

It results from the lemma that $\Phi_w^0 = F_w w$ for $w \in W$, where $F_w \neq 0$ belong to $v(H) \cdot \mathbf{C}(Y)^0$. Hence, Φ_w^0 are commuting with any elements from $W = 1 \times W \subset \mathrm{End}\, V^0$. Since w are linearly independent over $v(H) \cdot \mathbf{C}(Y)^0$, the same holds good for $\{\Phi_w^0\}$. Hence, the homomorphism $\mathscr{H} \ni X \to X^0$ is an embedding (see (1.11)). □

Corollary 2.3 *The operators given by formulas* (2.1)–(2.4) *take* V^0 *into* V^0 *and satisfy relations* (1.1)–(1.4) *and the corresponding relations in* $\mathscr{H}'$.

This corollary was proven directly in the case of one-dimensional V by Lusztig [Lu3] and for A_n by Lascoux-Schützenberger (see [LS1, LS2], where the root system of GL_{n+1} without the affine relation was considered). As far as I know, Kato was the first to explain the formulas from [Lu3] by means of the construction of Theorem 2.1 for one-dimensional V (see also [KL]).

Till the end of this section we will consider only one-dimensional $V_{\pm}$, though several subsequent statements can be extended to general V. By definition, for $1 \leqq i \leqq n$

$$\nu_+(T_i) = q_i \quad \text{or} \quad \nu_+(s_i) = 1\,, \tag{2.7}$$

$$\nu_-(T_i) = -1 \quad \text{or} \quad \nu_-(s_i) = -1\,. \tag{2.8}$$

The formulas from Theorem 2.1 respectively for $\mathscr{H}$ and $\mathscr{H}'$ look as follows

$$T_i^0 + 1 = (q_i - Y_i^0)(Y_i^0 - 1)^{-1}(s_i - 1)\,, \tag{2.9}$$

$$-(T_i^0 - q_i) = (s_i + 1)(q_i Y_i^0 - 1)(Y_i^0 - 1)^{-1} \quad \text{for} \quad \nu_-\,, \tag{2.9a}$$

$$T_i^0 - q_i = (q_i Y_i^0 - 1)(Y_i^0 - 1)^{-1}(s_i - 1)\,, \tag{2.10}$$

$$-(T_i^0 + 1) = (s_i + 1)(q_i - Y_i^0)(Y_i^0 - 1)^{-1} \quad \text{for } \nu_+ \tag{2.10a}$$

$$s_i^0 + 1 = (h_i - y_i^0)(y_i^0)^{-1}(s_i - 1)\,, \tag{2.11}$$

$$-(s_i^0 - 1) = (s_i + 1)(h_i + y_i^0)(y_i^0)^{-1} \quad \text{for} \quad \nu_-\,, \tag{2.11a}$$

$$s_i^0 - 1 = (h_i + y_i^0)(y_i^0)^{-1}(s_i - 1)\,, \tag{2.12}$$

$$-(s_i^0 + 1) = (s_i + 1)(h_i - y_i^0)\,(y_i^0)^{-1} \text{ for } \nu_+\,, \tag{2.12a}$$

where (2.9) is equivalent to (2.9a) and so on, $1 \leqq i \leqq n$.

Let $h' = 1 = h''$, $q' - 1 = \delta'$, $q'' - 1 = \delta''$, $\mathbf{C}[\tilde{y}] \stackrel{\text{def}}{=} \mathbf{C}[y] \otimes_{\mathbf{C}} \mathbf{C}[[\delta]]$ the algebra of formal power series of δ', δ'' with the coefficients in $\mathbf{C}[y]$, and let $\tilde{\mathscr{H}}'$ be generated by $\mathscr{H}'$ and $\mathbf{C}[[\delta]]$. Due to Drinfeld [Dr] one should substitute $q_i^{y_i}$ for Y_i to obtain $\mathscr{H}'$ from $\mathscr{H}$.

Corollary 2.4 (see [Lu1, Sect. 9]) *After the formal substitution* $Y_i = q_i^{y_i} = 1 + \delta_i y_i + \ldots \in \mathbf{C}[\tilde{y}]$, *where* $\delta_i = \delta'$ *or* δ'' *for short or long* $\alpha_i (1 \leqq i \leqq n)$, *the pairwise equivalent formulas*

$$T_i + 1 = (q_i - q_i^{y_i})(1 - y_i)^{-1}(q_i^{y_i} - 1)^{-1}\, y_i(s_i^0 + 1)\,, \tag{2.13}$$

$$T_i - q_i = (s_i^0 - 1)(q_i^{y_i+1} - 1)(y_i + 1)^{-1}(q_i^{y_i} - 1)^{-1}\, y_i\,, \tag{2.13a}$$

$$T_i - q_i = (q_i^{y_i+1} - 1)(y_i + 1)^{-1}(q_i^{y_i} - 1)^{-1}\, y_i(s_i^0 - 1)\,, \tag{2.14}$$

$$T_i + 1 = (s_i^0 + 1)(q_i - q_i^{y_i})(1 - y_i)^{-1}(q_i^{y_i} - 1)^{-1}\, y_i\,, \tag{2.14a}$$

where s_i^0 *are respectively from* (2.11)–(2.12a), *give two homomorphisms from* $\mathscr{H}$ *with undetermined* q', q'' *into* $\tilde{\mathscr{H}}'$. *If* q', q'' *are not roots of unity they induce an equivalence of the categories of finite-dimensional representations of* $\mathscr{H}$ *and* $\mathscr{H}'$.

The proof is in combining (2.9)–(2.12). □

Formula (2.14a) is in fact from [Lu1] (see 5.1). An analogue of (2.13), (2.14) (with more complicated multipliers) was obtained in [Ch 2], in terms of Φ_i^0, ϕ_i^0 from (2.5), (2.6). Without going into detail we will give another application of (2.9)–(2.12).

Corollary 2.5 *Let us introduce one more affine Hecke algebra* $\hat{\mathscr{H}}$ *with generators* $\{\hat{T}_i, \hat{Y}_i, 1 \leqq i \leqq n\}$ *for* $\hat{q}' = (q')^{k'}$, $\hat{q}'' = (q'')^{k''}$, *where* $k', k'' \in \mathbf{C}$. *Then the formal substitution* $\hat{Y}_i = Y_i^{k_i}$ $(k_i = k'$ *or* k'' *for short or long* $\alpha_i)$ *together with one of the following four sets of formulas*

$$\hat{T}_i + 1 = (q_i^{k_i} - Y_i^{k_i})(q_i - Y_i)^{-1}(Y_i^{k_i} - 1)^{-1}(Y_i - 1)(T_i + 1)\,, \qquad (2.15)$$

$$\hat{T}_i - q_i^{k_i} = (T_i - q_i)(q^{k_i} Y_i^{k_i} - 1)(q_i Y_i - 1)^{-1}\,(Y_i^{k_i} - 1)^{-1}\,(Y_i - 1)\,, \qquad (2.15a)$$

$$\hat{T}_i - q_i^{k_i} = (q^{k_i} Y_i^{k_i} - 1)(q_i Y_i - 1)^{-1}\,(y_i^{k_i} - 1)^{-1}\,(Y_i - 1)(T_i - q_i)\,, \qquad (2.16)$$

$$\hat{T}_i + 1 = (T_i + 1)(q_i^{k_i} - Y_i^{k_i})(q_i - Y_i)^{-1}(Y_i^{k_i} - 1)^{-1}(Y_i - 1) \qquad (2.16a)$$

induces a homomorphism from $\hat{\mathscr{H}}$ *to some extension of* $\mathscr{H}$ *or its representation in which the functions of* Y *from the r.h.s. of* (2.15) *and* (2.16) *are well-defined. In particular, it is so if either* q', q'' *are not roots of unity or* $Y_1, \ldots, Y_n$ *act as some operators with* Spec $Y_i \not\ni q_i$ *for all* i *(resp.* Spec $Y_i \not\ni q_i^{-1}$*). For* $k', k'' \in \mathbf{N}$ *the localization of* $\mathscr{H}$ *by the elements* $(Y_i^{k_i} - 1)(Y_i - 1)^{-1}$, $1 \leqq i \leqq n$, *is the required extension. In all these cases* (2.15) *is equivalent to* (2.15a) *and* (2.16) *to* (2.16a).

Now we will start to calculate the $\mathscr{H}$-module V^0 for $V = V_\varepsilon$, $\varepsilon = \pm$, $v = v_\varepsilon$ (see (2.7), (2.8)). Put

$$Q_+ = \sum_{w \in W} T_w,\; Q_- = \sum_{w \in W} (-1)^{l(w_0) - l(w)} q_{w_0} q_w^{-1} T_w\,,$$

$$q_w = (q')^{l'(w)} (q'')^{l''(w)}\,,$$

where w_0 is the element of the maximal length, $l'(w)$ (or $l''(w)$) is the number of s_{i_k} for short α_{i_k} (or respectively for long α_{i_k}) in the reduced decompositions $w = s_{i_l} \ldots s_{i_1}$ $(l = l(w) = l'(w) + l''(w))$. The well-known defining properties of $Q_\pm$ are as follows:

$$C_i^\mp Q_\pm = Q_\pm C_i^\mp = 0,\; C_i^\pm Q_\pm = Q_\pm C_i^\pm = (q_i + 1) Q_\pm\,, \qquad (2.17)$$

where $1 \leqq i \leqq n$, $C_i^\pm$ are from (1.19).

Definition 2.6 (a) A $\mathscr{H}$-module J is called *v-special of type* $\varepsilon = \pm$ if $\dim_{\mathbf{C}}(Q_\varepsilon J) = 1$ and J is generated by ("the space of v_ε-spherical vectors") $Q_\varepsilon J$ as a $\mathscr{H}$-module or, equivalently, as a $\mathbf{C}[Y]$-module. The same definition is for $\mathscr{H}'$ and $Q_\pm = \sum_{w \in W} (\pm 1)^{l(w_0) - l(w)} w$.

(b) A character $\chi : \mathbf{C}[Y] \to \mathbf{C}$ is called *v_ε-special* if, firstly, $W(\chi) \overset{\text{def}}{=} \{w \in W, w(\chi) = \chi\}$ is generated by elements in $\{s_1, \ldots, s_n\} \cap W(\chi)$ (i.e. χ is special in the sense of [Ro, Sect. 4]). Secondly, for $q_\alpha \neq 1$

$\chi(Y_\alpha)$ is forbidden to be equal to q_α^ε, where we have identified $\pm$ and ± 1.

As for $\mathscr{H}'$, the last condition has the form $\chi(y_\alpha) \neq \varepsilon$ for $\alpha > 0$.

Lemma 2.7 *A v_ε-special J has only one non-zero irreducible quotient-module. The latter is also v_ε-special and is of multiplicity one in J as an irreducible constituent.*

Proof. If $\sigma : J \to U \neq \{0\}$ is a surjective $\mathscr{H}$-homomorphism and U is irreducible, then U is generated by $\sigma(Q_\varepsilon J) = Q_\varepsilon U$. Hence, U is v_ε-special.

Let $N = \ker\sigma$, $N \not\supset N' \subset J$ be a $\mathscr{H}$-submodule. Then $N + N' = J(U$ is irreducible) and $N'/(N \cap N') \simeq U$. Therefore $Q_\varepsilon N' \neq \{0\}$ and $Q_\varepsilon J \subset N'$ because of $\dim_{\mathbf{C}}(Q_\varepsilon J) = 1$. But $Q_\varepsilon J$ generates J. We see that $N' \subset N$. If there is another v_ε-special irreducible constituent in J, then it has to be in N and $Q_\varepsilon N \neq \{0\}$. This is impossible. □

Lemma 2.8 *Suppose a $\mathscr{H}$-module J to contain an eigenvector u_χ of $\mathbf{C}[Y]$ (or $\mathbf{C}[y]$ for $\mathscr{H}'$) with a v_ε-special character χ. If $\psi = w(\chi)$ is v_ε-special for some $w \in W$, then there exists an eigenvector $u_\psi \in J$ for ψ generating the same $\mathscr{H}$-submodule of J as u_χ.*

Proof. It results from (1.18) that

$$Y_\alpha(\hat{\Phi}_w(M)u_\chi) = \chi_{M'}(Y_\alpha)(\hat{\Phi}_w(M)u_\chi)\,, \tag{2.18}$$

where $\alpha \in \Sigma$, $\chi = \chi_M$, $M' = M_{w^{-1}}$, $\hat{\Phi}_w$ is from Lemma 1.6 for $w \in W$. Following the proof of Lemma 1.6 one can easily show that for every v_ε-special ψ which is conjugated to χ there is $w \in W$ such that $\psi = w(\chi) = \chi_{M'}$ and $\hat{\Phi}_w$ is invertible. Really, we can find $w = s_{i_1} \ldots s_{i_l}$, where all $\Phi_{i_1}(M)$, $\Phi_{i_2}(M_{s_{i_1}})$, $\Phi_{i_3}(M_{s_{i_1} s_{i_2}})$ and so on are invertible. □

Let V^0 be defined for $V = V_\varepsilon$, $\varepsilon = \pm$, m be a maximal ideal of $\mathbf{C}[Y]^W$ ($m \in \operatorname{Specm} \mathbf{C}[Y]^W$). Our aim is to calculate $V_m^0 = V^0/mV^0$. The latter is a $\mathscr{H}$-module because of Lemma 1.3(b).

Proposition 2.9 *The module V_m^0 is v_ε-special. The set of its characters coincides with that of arbitrary I_M, $M \supset m$, and is equal to $\{w(\chi_M), w \in W\}$.*

Proof. Let us construct a family $m(v) \in \operatorname{Specm} \mathbf{C}[Y]^W$, depending on small $v \in \mathbf{C}$, such that $m(0) = m$ and $m(v)$ are in a general position. The latter means that $W(\chi) = \{\mathrm{id}\}$ for each $\chi = \chi_M$, where $m \in M \in \operatorname{Specm} \mathbf{C}[Y]$, and $\chi(Y_\alpha) \neq q_\alpha^{\pm 1}$ for arbitrary $\alpha \in \Sigma$ (see Lemma 1.6). Then every $V_{m(v)}^0$ for $v \neq 0$ is irreducible. Indeed, it is linearly generated by its eigenvectors, which all are simple. Moreover, the set of the corresponding characters is W-invariant. Therefore the existence of eigenvectors results directly from the following well-known

Lemma 2.10 *Let $C \subset \operatorname{End}_{\mathbf{C}} K$, $K = \mathbf{C}^N$, be a commutative subalgebra. Then $K = \oplus K_\chi^\infty$, where χ runs over the set of all characters of C,*

$$K_\chi^p = (x \in K, (c - \chi(c))^p x = 0 \quad \textit{for all} \quad c \in C\}$$

Here $K_\chi^\infty = K_\chi^N$. If $C = C(0)$, where subalgebras $C(v) \in \operatorname{End}_{\mathbf{C}} K$ are commutative and depend continuously on small $v \in \mathbf{C}$, then $K_\chi^\infty = \lim_{v \to 0}(\oplus K_{\chi(v)})$, when $\chi(v)$ runs over all the continuous characters of $C(v)$ with $\chi(0) = \chi$.

For generic $m(v)$, $v \neq 0$, $V_{m(v)}^0$ is isomorphic to H with the left regular action as a H-module because of the irreducibility of $V_{m(v)}$. Hence, $\dim_{\mathbf{C}}(Q_\varepsilon V_{m(v)}^0) = 1$ for $v \neq 0$. It is clear, that $\dim_{\mathbf{C}}(Q_\varepsilon V_m^0)$ has to be equal to 1 or less. But V_m^0 is generated by $Q_\varepsilon V^0$, since it is so for V^0. Hence, V_m^0 is v_ε-special. □

The only non-trivial part of our calculation is

Proposition 2.11 *Every induced representation I_M with special $\chi = \chi_M$ has only one eigenvector with respect to χ and possesses only one non-zero irreducible quotient-*

module. If χ is v_ε-special, then I_M is v_ε-special as well. The analogous holds true for $\mathscr{H}'$.

Proof. As for $\mathscr{H}$, the second statement is from Theorem 2.4 [Ka] (the p-adic case was considered by Muller). Close results can be found in papers by Casselman, Kato, Steinberg et al. on cyclic vectors and spherical functions in p-adic representations (see [Ka] for some references). We will give a short proof "without calculations" based on the Lusztig-Lascoux-Schützenberger operators. The reduction of the first statement to Lemma 2.12 below is due to [Ro].

Given special $\chi = \chi_M$, let us include M into some family $M(v)$ from Lemma 1.6 in a general position (see above). By Lemma 2.10 we obtain that $(I_M)_\chi^\infty = H_\chi \stackrel{\text{def}}{=} \oplus_w \mathbf{C}T_w$, $w \in W(\chi)$. Note that H_χ is generated by its subalgebras H_χ^p, where H_χ^p is the algebraic span of T_i for α_i from the p-th connected component σ_p of the suset $\{\alpha_i, s_i \in W(\chi)\}$ in the Dynkin diagram. Moreover $H_\chi \simeq \oplus_p H_\chi^p$. Let us suppose, that $\dim_{\mathbf{C}}(I_M)_\chi^1 \geqq 2$. Then one can find an element $\mathbf{C} \not\ni u^p \in H_\chi^p$ such that $Y_k(u^p) = u^p$ for $\alpha_k \in \sigma_p$ and some p. Indeed, if $\mathbf{C} \not\ni u \in (I_M)_\chi^1$, then there exists a component $u_j \notin \mathbf{C}$ (for some p) in the decomposition $u = \sum_j u_j \hat{u}_j$, where $u_j \in H_\chi^p$ and the elements $\hat{u}_j \in \prod_{p' \neq p} H_\chi^{p'}$ are linearly independent. We have $Y_k(u) = \sum_j Y_k(u_j)\hat{u}_j$ for $\alpha_k \in \sigma_p$. Hence, $u^p = u_j$ is the required element because of the independence of $\{\hat{u}_j\}$. The existence of such u^p contradicts the following key

Lemma 2.12 *Let $q' \neq 1 \neq q''$, $\chi = \chi_{M_1} = \chi_1$, where $\chi_1(Y_i) = Y_i(M_1) = 1$ for any $1 \leqq i \leqq n$. Then the induced $\mathscr{H}$-module $I_1 = I_{M_1}$ is irreducible and $\dim_{\mathbf{C}}(I_1)_{\chi_1}^1 = 1$. The same is true for $\mathscr{H}'$ if $\chi(y_i) = 0$ for $1 \leqq i \leqq n$.*

Proof. Let us first consider $\mathscr{H}'$ and $V_m^0 = V^0/mV^0$ for the ideal $m = m_0$ generated by all homogeneous elements from $\mathbf{C}[y]^W$ of deg > 0 (by definition deg $y_\alpha = 1$, deg $1 = 0$, $\alpha \in \Sigma$). This m_0 corresponds to $\chi = \chi_0 = 0$. Given homogeneous $u \in V^0$, we put $\deg \bar{u} = \min\{\deg(u + mV^0)\}$ for the image $\bar{u}$ of u in V_m^0. The following four subspaces in V_m^0 are coinciding:

(a) $\mathbf{C}\bar{d}$, where $d = \prod_{\alpha \in \Sigma_+} y_\alpha$;
(b) $\{\bar{u} \in V_m^0,\ s_i(\bar{u}) = -\bar{u},\ \ 1 \leqq i \leqq n\}$;
(c) $\{\bar{u} \in V_m^0,\ \ y_i\bar{u} = 0,\ \ 1 \leqq i \leqq n\}$;
(d) $\{$homogeneous $\bar{u}$ of $\deg \bar{u} = \max\{\deg V_m^0\}\}$.

The coincidence of (a) and (b) is clear, since $s_i(d) = -d$ for $1 \leqq i \leqq n$ and V_m^0, considered as a W-module in a natural way, is isomorphic to $\mathbf{C}[W]$ with the left regular action of W (use Lemma 2.10 and the semi-simplicity of $\mathbf{C}[W]$ or see [Bo]). Let the image $\bar{u}$ for some homogeneous $u \in V^0$ be from (c). Then $\bar{u}_i$ for $u_i = s_i u + u$ are from the same space, $1 \leqq i \leqq n$. One has: $u_i = (y_i u_i - s_i(y_i u_i))\, y_i^{-1} = \sum_k (s_i(f_k) - f_k)\, y_i^{-1} g_k$, where $y_i u_i = \sum_k f_k g_k$, $g_k \in \mathbf{C}[y]^W$, $f_k \in \mathbf{C}[y]$, deg $g_k > 0$. Hence, $\bar{u}_i = 0$ and $\bar{u} \in \mathbf{C}\bar{d}$ because $(s_i(f_k) - f_k)y_i^{-1} \in \mathbf{C}[y]$. The inclusion ($d$) $\subset$ (c) is evident ($\deg(y_i\bar{u}) = \deg \bar{u} + 1$ if $y_i\bar{u} \neq 0$). However there is at least one non-zero element in (d). It belongs to (c) $\subset$ (b) $\subset$ (a) (see above) and therefore is proportional to $\bar{d}$. Thus, (a) $\subset$ (d). Q.E.D.

As a corollary we obtain that V_m^0 is linearly generated by $\bar{d}$ and its images with respect to successive applications of the operations $p \to (s_i(p) - p)\, y_i^{-1}$ for $1 \leqq i \leqq n$. Hence, $V_m^0 = \mathbf{C}[W^0]\,\bar{d}$, where W^0 is generated by s_i^0, $1 \leqq i \leqq n$ (see (2.11), (2.12)). Here we have used the above representation for $\bar{u}_i$, where u is homogeneous.

As for I_0(with $\chi = 0$), there is a $\mathscr{H}'$-homomorphism $\sigma: I_0 \to V_m^0$ taking 1 to d. It is surjective because of $V_m^0 = \mathbf{C}[W_0]\bar{d}$. Hence, σ is an isomorphism ($\dim_{\mathbf{C}} I_0 = \dim_{\mathbf{C}} V_m^0$) and I_0 has only one eigenvector 1. If $\{0\} \neq J \subset I_0$ is a $\mathscr{H}'$-submodule, then J contains an χ_0-eigenvector $u \neq 0$ (see Proposition 2.9 and Lemma 2.10). But u is to be proportional to 1. It gives the irreducibility.

By means of Corollary 2.4 one has that the lemma is true for $\mathscr{H}$ with generic q, q'. Therefore it holds good for any $q', q'' \in \mathbf{C}$ apart from some algebraic curve $F(q', q'') = 0$. We will consider here only the case $q'q'' \neq 0$. Let us apply Corollary 2.5 for arbitrary $k', k'' \in \mathbf{Q}$, $k'k'' \neq 0$. We obtain a certain homomorphism $\kappa: \hat{\mathscr{H}} \to \mathrm{End}_{\mathbf{C}} I_1$ via some extension $\mathscr{H}_{\mathrm{ext}}$ of $\mathscr{H}$. This trick is quite normal, since $\kappa(Y_i - 1)$ in $\mathrm{End}_{\mathbf{C}} I_1$ are nilpotent and we may use the expansions $\kappa(Y_i)^k = 1 + k(Y_i - 1) + \dfrac{k(k-1)}{2}(Y_i - 1)^2 + \ldots$ in (2.15)–(2.16a) and the ordinary expansions for $(q_i - Y_i)^{-1}$ (here $k = k_i$). We see that $\mathscr{H}_{\mathrm{ext}} = \hat{\mathscr{H}}$ in $\mathrm{End}_{\mathbf{C}} I_1$.

One has $\kappa(\hat{Y}_i)(1) = Y_i^{k_i}(1) = 1$ for $1 \leq i \leq n$. Hence, there is a $\hat{\mathscr{H}}$-homomorphism $\gamma: \hat{I}_1 \to I_1(\gamma(1) = 1)$, where $\hat{I}_1$ is the analogous induced module with the character $\hat{\chi}_1$ for $\hat{\mathscr{H}}$. Given q', q'', we can find rational k', k'' to get $\hat{q}', \hat{q}''$ out of the above curve $F = 0$. Then $\hat{I}_1$ is irreducible and γ is an embedding. Moreover, it has to be surjective, since $\dim_{\mathbf{C}} \hat{I}_1 = \dim_{\mathbf{C}} I_1$. By construction, the natural image of $\mathscr{H}$ in $\mathrm{End}_{\mathbf{C}} I_1$ contains $\kappa(\hat{\mathscr{H}})$. Hence I_1 is irreducible with only one eigenvector. □

Let us prove that the subspace $Q_\varepsilon I_M$ generates I_M (i.e. I_M is v_ε-special).

Lemma 2.13 *Any v_ε-special $\mathscr{H}$-module J contains an eigenvector u_ψ with v_ε-special ψ, if its characters are W-conjugated to v_ε-special ones.*

Proof. Let U be the unique irreducible quotient-module of J (see Lemma 2.7). First let us prove this statement for U (it is v_ε-special as well). There is at least one χ with an eigenvector $u_\chi \in U_\chi^1$. By means of (2.18) one can find a chain of eigenvectors from $u = u_\chi$ to u_ψ with a v_ε-special ψ:

$$u' = \Phi_{i_1}(M)u,\ u'' = \Phi_{i_2}(M_{s_{i_1}})u',\ u''' = \Phi_{i_3}(M_{s_{i_1}s_{i_2}})u'' \text{ and so on for}$$

$$M' = M_{s_{i_1}},\ M'' = M_{s_{i_1}s_{i_2}}\ M''' = M_{s_{i_1}s_{i_2}s_{i_3}} \text{ etc.}$$

It is easy to show that we can use in this chain either invertible $\Phi_{i_k}(M^{(k-1)})$ or the pairs $(s_{i_k}, M^{(k-1)})$ with $Y_{i_k}(M^{(k-1)}) = q_i^\varepsilon$. The values $Y_{i_k}(M^{(k-1)}) = q_{i_k}^{-\varepsilon}$ may be considered as forbidden.

Let us check that all $u', u'', \ldots, u^{(k)}, \ldots$ are non-zero. If $u^{(k-1)} \neq 0$ and $\Phi_{i_k}(u^{(k-1)})$ is invertible, it is clear. Otherwise, $Y_{i_k}(M^{(k-1)}) = q_{i_k}^\varepsilon$ and $\Phi_{i_k}(M^{(k-1)}) = C_{i_k}^\varepsilon$ (see (1.19)). If $C_{i_k}^\varepsilon u^{(k-1)} = 0$, then $Q_\varepsilon C_{i_k}^\varepsilon u^{(k-1)} = (q+1)Q_\varepsilon u^{(k-1)} = 0$ (see (2.17)) and $Q_\varepsilon u^{(k-1)} = 0$. But $u^{(k-1)}$ is an eigenvector and generates U as a H-module (U is irreducible). Hence, $U = Hu^{(k-1)}$, $Q_\varepsilon U = Q_\varepsilon Hu^{(k-1)} = HQ_\varepsilon u^{(k-1)} = 0$. This contradiction proves the lemma for irreducible U.

Thanks to Lemma 2.10, $J_\psi^\infty \neq \{0\}$ if $U_\psi^1 \neq 0$. Therefore $J_\psi^1 \neq \{0\}$ and J contains an eigenvector with some v_ε-special character. □

Now let $\chi = \chi_M$ be v_ε-special. The module $J = Q_\varepsilon I_M \subset I_M$ is v_ε-special ($\dim_{\mathbf{C}}(Q_\varepsilon H) = 1$). Hence, J contains an eigenvector with v_ε-special ψ. It results from

Lemma 2.8 that J has χ as the character of some eigenvector. However, I_M is to have the only eigenvector (namely, 1) up to a scalar factor, which corresponds to χ. We see that $1 \in J$ and $J = I_M$. $\square$

Now we are in a position to describe the $\mathscr{H}$-module V^0 for $V = V_\varepsilon$. We fix $\varepsilon = \pm$.

Theorem 2.14 (a) *Each module $V_m^0 = V^0/mV^0$ for $m \in \operatorname{Specm} \mathbf{C}[Y]^W$ is v_ε-special and has only one irreducible quotient-module $U_m \neq \{0\}$, which is also v_ε-special and is of multiplicity 1 in V_m^0 as a composition factor. These $\{U_m\}$ are pairwise non-isomorphic and constitute the set of all irreducible quotient-modules of V^0 or (that is equivalent) the set of all irreducible $\mathscr{H}$-modules U generated by $Q_\varepsilon U$. Moreover, there is only one homomorphism $V^0 \to U_m$ for each m (up to a factor of proportionality).*

(b) *The module V_m^0 is isomorphic to I_M, where $M \in \operatorname{Specm} \mathbf{C}[Y]$ is over m ($M \supset m$) and the corresponding character χ_M is v_ε-special. The modules I_M for the same m and different v_ε-special χ_M are isomorphic to each other. Up to an isomorphism there is only one v_ε-special I_M among all $M \supset m$.*

(c) *The same statements hold good for $\mathscr{H}'$.*

Proof. The modules V_m^0, U_m are v_ε-special and U_m has the multiplicity one because of Proposition 2.9 and Lemma 2.7. An arbitrary irreducible $\mathscr{H}$-module U generated by $Q_\varepsilon U$ is to be some quotient module of V^0 by definition. The action of $\mathbf{C}[Y]^W$ on U is scalar (Proposition 1.5). Hence U has to be a quotient-module of an appropriate V_m^0. Part (a) is proven.

Let u_χ be an eigenvector of V_m^0 with some v_ε-special $\chi = \chi_M$. Then $M \supset m$ (Proposition 1.5) and one can define a homomorphism $\gamma: I_M \to V_m^0$ taking 1 to u_χ. The image $\gamma(I_M)$ is v_ε-special because I_M is v_ε-special (Proposition 2.11). Hence, $Q_\varepsilon \gamma(I_M) \neq \{0\}$ and it generates the whole V_m^0 (the latter is v_ε-special). One has: $\dim_{\mathbf{C}} H = \dim_{\mathbf{C}} I_M = \dim_{\mathbf{C}} V_m^0$. Therefore γ is an isomorphism. The equivalence of different v_ε-special I_M for the same m follows from Lemma 2.8. As for the uniqueness of v_ε-special I_M, use Proposition 1.5(c).

We mention without going into detail that some points in the proof of Proposition 2.11 and the proof of the corresponding statement from [Ka] are parallel. However the reduction to $\mathscr{H}'$ and the utilization of the Lusztig-Lascoux-Schützenberger operators are new.

3 The unification, some examples

We keep to the notations of the beginning of Sect. 1. Let A be some $\mathbf{C}$-algebra equipped with a homomorphism $v_0: W \to A^*$, $\{r_\alpha, \alpha \in \Sigma\} \subset A$. Let us define the elements

$$D_u = \sum_{\alpha > 0} (u, \alpha) r_\alpha, \; u \in \mathbf{C}^n . \tag{3.1}$$

One has $D_{\zeta u + \xi v} = \zeta D_u + \xi D_v$, $u, v \in \mathbf{C}^n$, $\zeta, \xi \in \mathbf{C}$. In particular,

$$D_i \stackrel{\text{def}}{=} D_{\alpha_i^*} = \sum_{\alpha > 0} \mu_\alpha^i r_\alpha, \; D_u = \sum_i (u, \alpha_i) D_i, \; 1 \leqq i \leqq n , \tag{3.2}$$

where $\{\alpha_i^*\}$ are dual weights ($(\alpha_i, \alpha_j^*) = \delta_{ij}$ for the Kronecker symbol δ_{ij}), μ_α^i is the multiplicity of α_i in α. We put $D_u^\lambda = \sum_\alpha (u, \alpha) r_\alpha$, where $\alpha \in \lambda \cap \Sigma_+$, λ is a subset in $\mathbf{R}^n$.

Definition 3.1 The set $\{r_\alpha\}$ is a *classical A-valued r-matrix of type* $A_n, \ldots, G_2$, *if*

$$r_{w(\alpha)} = v_0(w) r_\alpha v_0(w)^{-1} \quad \text{for} \quad \alpha \in \Sigma, \quad w \in W, \tag{3.3}$$

$$[D_i^\lambda, D_j^\lambda] = 0, \quad \lambda = \mathbf{R}\alpha_i + \mathbf{R}\alpha_j. \tag{3.4}$$

Here $\{\alpha_i, \alpha_j\}$ runs over all classes of pairs of simple roots modulo the action of W on the latter (one pair for A_2, B_2, C_2, G_2, two for A_n, D_n, E_{6-8}, three for B_3, C_3, four for B_n, C_n, F_4, $n > 2$).

Proposition 3.2 (see [Ch 1, Ch 2]) *Elements* $\{D_u, u \in \mathbf{C}^n\}$ *are pairwise commutative for* $\{r_\alpha\}$ *from Definition* 3.1.

Proof. It is sufficient to consider the elements $D_1, \ldots, D_n$. One has:

$$[D_i, D_j] = \sum_\lambda d_\lambda, \quad d_\lambda = \sum_{\alpha, \beta \in \lambda_+} \mu_\alpha^i \mu_\beta^j [r_\alpha, r_\beta],$$

where $\lambda_+ = \lambda \cap \Sigma_+$, λ runs over all two-dimensional subspaces in $\mathbf{R}^n$. Indeed, only trivial pairs $\{\alpha = \beta\}$ can belong to some $\lambda \cap \lambda'$ for $\lambda \neq \lambda'$. Let γ, δ be the pair of simple roots in $\lambda_+ = \lambda \cap \Sigma_+$ (if the latter is two-dimensional) and $\gamma_\alpha, \delta_\alpha$ be the multiplicities of γ, δ in $\alpha \in \lambda_+$ respectively. Then $\mu_\alpha^i = \gamma_\alpha \mu_\gamma^i + \delta_\alpha \mu_\delta^i$. Hence,

$$\begin{aligned} d_\lambda &= \sum_{\alpha, \beta \in \lambda_+} (\gamma_\alpha \mu_\gamma^i + \delta_\alpha \mu_\delta^i)(\gamma_\beta \mu_\gamma^j + \delta_\beta \mu_\delta^j)[r_\alpha, r_\beta] \\ &= \mu_\gamma^i \mu_\gamma^j [\tilde{D}_1, \tilde{D}_1] + \mu_\delta^i \mu_\delta^j [\tilde{D}_2, \tilde{D}_2] + (\mu_\gamma^i \mu_\delta^j - \mu_\gamma^j \mu_\delta^i)[\tilde{D}_1, \tilde{D}_2], \end{aligned}$$

where $\tilde{D}_1, \tilde{D}_2$ are from (3.2) but for λ_+ and $\{\gamma, \delta\}$ in place of Σ_+ and $\{\alpha_1, \ldots, \alpha_n\}$. Therefore the identity $[\tilde{D}_1, \tilde{D}_2] = 0$ is sufficient to prove the proposition.

The roots γ, δ can be included into the system of simple roots corresponding to a certain Weyl chamber (see [Bo]). Hence, $\gamma = w(\alpha_k)$, $\delta = w(\alpha_m)$ for a suitable $w \in W$ and appropriate simple α_k, α_m. We arrive at (3.4). □

Proposition 3.3 [Ch 1] *Let* $\{R_\alpha, \alpha \in \Sigma\} \subset A^*$ *be a quantum W-invariant R-matrix in the sense of* [Ch 5], *i.e. the set* $\{R_\alpha\}$ *satisfies conditions* (3.3) *and* $t_i \overset{\text{def}}{=} R_i v_0(s_i) \in A^*$ *for* $R_i = R_{\alpha_i}$ *satisfy the braid relations* (1.1). *Assume that R depend on some parameter* $h \in \mathbf{C}$ *and*

$$R_i = 1 + h r_i + o(h), \quad 1 \leqq i \leqq n, \quad r_i \in A, \tag{3.5}$$

in a neighbourhood of $h = 0$. *Let* $[R_\alpha, R_\beta] = 0$ *if* $(\alpha, \beta) = 0$. *In the case of* G_2 *we suppose additionally that* $R_\alpha R_{\alpha+\beta} R_\beta = R_\beta R_{\alpha+\beta} R_\alpha$ *for long positive roots* $\alpha, \beta, \alpha + \beta$. *Then* $\{r_i, 1 \leqq i \leqq n\}$ *can be uniquely extended to a classical r-matrix* $\{r_\alpha, \alpha \in \Sigma\}$ *by means of* (3.3).

Proof. The cases A_2, B_2, G_2 are enough to consider. Let $\alpha = \alpha_1$, $\beta = \alpha_2$. For A_2 one has the following (quantum Yang-Baxter) identity

$$R_\alpha R_{\alpha+\beta} R_\beta = R_\beta R_{\alpha+\beta} R_\alpha,$$

which gives (see (3.5)) the well-known classical "abstract" Yang-Baxter relation

$$[r_\alpha, r_{\alpha+\beta}] + [r_\alpha, r_\beta] + [r_{\alpha+\beta}, r_\beta] = 0\,. \tag{3.6}$$

For B_2 we obtain the relation

$$R_\alpha R_{\alpha+\beta} R_{\alpha+2\beta} R_\beta = R_\beta R_{\alpha+2\beta} R_{\alpha+\beta} R_\alpha\,,$$

which results in

$$[r_\alpha, r_\beta + r_{\alpha+\beta}] = [r_{\alpha+2\beta}, r_{\alpha+\beta} - r_\beta] \tag{3.7}$$

because of the orthogonality conditions $[r_\alpha, r_{\alpha+2\beta}] = 0 = [r_{\alpha+\beta}, r_\beta]$, which follow from the corresponding conditions for R. For G_2 one has (see [Bo])

$$R_\alpha R_{3\alpha+\beta} R_{2\alpha+\beta} R_{2\beta+3\alpha} R_{\alpha+\beta} R_\beta = R_\beta R_{\alpha+\beta} R_{2\beta+3\alpha} R_{2\alpha+\beta} R_{3\alpha+\beta} R_\alpha$$

and the relations

$$[r_\beta, r_\alpha + r_{\alpha+\beta}] + [r_{2\alpha+\beta}, r_{3\alpha+\beta} - r_{3\alpha+2\beta}] = [r_\alpha, r_{3\alpha+\beta}] + [r_{3\alpha+2\beta}, r_{\alpha+\beta}] \tag{3.8}$$

together with (3.6) for r_β, $r_{3\alpha+\beta}$, $r_{3\alpha+2\beta}$ and

$$[r_\beta, r_{2\alpha+\beta}] = [r_{\alpha+\beta}, r_{3\alpha+\beta}] = [r_\alpha, r_{3\alpha+2\beta}] = 0\,.$$

Relations (3.6–8) modulo the orthogonality conditions ($(\alpha, \beta) = 0 \Rightarrow [r_\alpha, r_\beta] = 0$) are equivalent to Definition 3.1. Owing to (3.3) we have only one unknown element r_α in the case of A_2 and two of them (r_α and r_β) for B_2, G_2. □

Let us consider $\hat{A} = \mathrm{End}_{\mathbf{C}}\mathbf{C}(Y) \otimes_{\mathbf{C}} A$ or $\hat{A} = \mathrm{End}_{\mathbf{C}}\mathbf{C}(y) \otimes_{\mathbf{C}} A$ instead of A, where $\mathbf{C}(Y)$ and $\mathbf{C}(y)$ are from Sect. 1. We have the homomorphism $W \times W \to \hat{A}$ taking $1 \times w$ to $v_0(w) \in 1 \otimes A$ and $w \times 1$ to the corresponding automorphism of $\mathbf{C}(Y)$ or $\mathbf{C}(y)$. By definition

$$\hat{v}_0(w) = w v_0(w), \quad w \in W\,, \tag{3.9}$$

where w, $v_0(w)$ are identified respectively with $w \times 1$ and $1 \times v_0(w)$. One has: $w v_0(w') = v_0(w') w$ for any $w, w' \in W$.

Let $\{\rho_\alpha, \alpha \in \Sigma\}$ be a classical A-valued (i.e. "constant") r-matrix with the following "quasi-unitary" condition

$$\rho_\alpha + v_0(s_\alpha) \rho_\alpha v_0(s_\alpha) = 1 + v_0(s_\alpha) \quad \text{or} = 0 \tag{3.10}$$

respectively in the case of Y or y. We fix $\kappa', \kappa'', \delta', \delta'' \in \mathbf{C}$ and denote by κ_α either κ' or κ'' for short or long roots, $\kappa_i = \kappa_{\alpha_i}$ (the same notations hold for δ).

Theorem 3.4 *The set*

$$r_\alpha = \kappa_\alpha(\rho_\alpha + (Y_\alpha - 1)^{-1} v_0(s_\alpha)(1 - \delta_\alpha s_\alpha)),\ \alpha \in \Sigma\,, \tag{3.11}$$

$$r_\alpha = \kappa_\alpha(\rho_\alpha + y_\alpha^{-1} v_0(s_\alpha)(1 - \delta_\alpha s_\alpha))\ \textit{for}\ y \tag{3.12}$$

is a classical $\hat{A}$-valued r-matrix with respect to $\hat{v}_0$.

The proof is based on the following lemma directly resulting from formulas (2.1), (2.3).

Lemma 3.5 *In the setup of Sect. 2 let* $A = \mathrm{End}_{\mathbf{C}} V$,

$$q_i - 1 = h_i = \kappa_i h,\ h \in \mathbf{C},\ 1 \leqq i \leqq n\,. \tag{3.13}$$

We will assume that the homomorphism $v: H \to \mathrm{End}_{\mathbf{C}} V$ *from Sect.* 2 (*or its counterpart for* $H' = \mathbf{C}[W]$) *depends on* (*small*) h. *Moreover, let us suppose that for* v_0: $W \to A^* = \mathrm{Aut}_{\mathbf{C}} V$ *above*

$$v(T_i) = v_0(s_i) + h_i \rho_i v_0(s_i) + o(h) \tag{3.14}$$

or

$$v(s_i) = v_0(s_i) + h_i \rho_i v_0(s_i) + o(h) \tag{3.15}$$

and

$$v_0(w) v(T_i) v_0(w)^{-1} = v(T_j) \quad \textit{if} \quad w(\alpha_i) = \alpha_j \tag{3.16}$$

or

$$v_0(w) v(s_i) v_0(w)^{-1} = v(s_j) \quad \textit{if} \quad w(\alpha_i) = \alpha_j\,, \tag{3.17}$$

where $1 \leqq i, j \leqq n$. *Then* $\rho_{\alpha_i} = \rho_i$ *satisfy* (3.10) *because of* (1.2) *and the set*

$$R_i = T_i^0 \hat{v}_0(s_i) \quad \textit{or} \quad R_i = s_i^0 \hat{v}_0(s_i),\quad 1 \leqq i \leqq n\,, \tag{3.18}$$

for T_i^0, s_i^0 *from Theorem* 2.1 *and* $\hat{v}_0$ *from* (3.9) *can be uniquely extended to a quantum* $\hat{A}$-*valued* W-*invariant* R-*matrix with respect to* $\hat{v}_0$. *The corresponding classical* r-*matrix* (*see Proposition* 3.3) *coincides with* (3.11), (3.12) *for* $\delta' = 1 = \delta''$.

Thanks to the lemma we obtain the statement of Theorem 3.4 for $A = \mathrm{End}_{\mathbf{C}} V$ and $\delta' = 1 = \delta''$. Really,

$$R_i = T_i^0 \hat{v}_0(s_i) = 1 + h\kappa_i (Y_i - 1)^{-1}(1 - s_i) v_0(s_i) + h\kappa_i \rho_i$$

(see (2.1)). Here and further we will identify Y_i^0 and Y_i. As for (3.12), we can apply (2.3). In the case $\delta' = 0 = \delta''$ Theorem 3.4 was proven in [Ch 1, Ch 2]. The case $\delta' = \delta'' \to \infty$ is completely analogous to the previous one. A direct consideration for A_2, B_2, G_2 shows that this is enough to prove the theorem. □

Now let us first combine all ρ_α in D_u (r_α are as above), then secondly add some "scalar" terms to D_u.

Corollary 3.6 *The following elements from* $\hat{A}$ *are pairwise commutative* ($1 \leqq i \leqq n$):

$$\hat{D}_i = Y_i \partial g / \partial Y_i + v_0(x_i) + \sum_{\alpha \in \Sigma_+} \kappa_\alpha \mu_\alpha^i (Y_\alpha - 1)^{-1} v_0(s_\alpha)(1 - \delta_\alpha s_\alpha)\,, \tag{3.19}$$

$$\hat{D}_i = \partial g / \partial y_i + v_0(x_i^0) + \sum_{\alpha \in \Sigma_+} \kappa_\alpha \mu_\alpha^i\, y_\alpha^{-1} v_0(1 - \delta_\alpha s_\alpha) \tag{3.20}$$

for Y *and* y *respectively and* g *from* $\mathbf{C}(Y)^W$ or $\mathbf{C}(y)^W$. *Here* $[x_i, x_j] = 0$ *for* $1 \leqq i, j \leqq n$ *and* x_i *satisfy relation* (1.5) *for* $y_i^* = y_{\alpha_i^*}$, $h_i = -(\alpha_i, \alpha_i)\,\kappa_i/2$ ($\{\alpha_i^*\}$-*are the dual weights*), $\{x_i^0\}$ *obey the same relations but for* $h_i = 0$ ($1 \leqq i \leqq n$). *These* $\{x_i\}$ *or* $\{x_i^0\}$ *are added to* $\{s_i\}$ *and the homomorphism* v_0: $\mathbf{C}[W] \to A$ *is assumed to be extended to* $\{x_i\}$, $\{x_i^0\}$.

Proof. We will only check here that

$$v_0(x_i) \overset{\text{def}}{=} \sum_{\alpha > 0} \kappa_\alpha \mu_\alpha^i (\rho_\alpha - 1/2),\quad \alpha \in \Sigma,\quad 1 \leqq i \leqq n$$

satisfy relations (1.5) for y_i^* with the constants h_i above, where ρ_α are from Theorem 3.4. Indeed, $[\nu_0(s_j), \nu_0(x_i)] = 0$ for $j \neq i$ because $s_j(\alpha) > 0$ if $\mu_\alpha^i \neq 0$, where $\mu_\alpha^i = (\alpha, \alpha_i^*) = (s_j(\alpha), \alpha_i^*)$. Similarly,

$$\nu_0(s_i)\nu_0(x_i)\nu_0(s_i) - \kappa_i(\rho_i + \nu_0(s_i)\rho_i\nu_0(s_i) - 1) = \sum_{\alpha>0} \kappa_\alpha(\alpha, s_i(\alpha_i^*))(\rho_\alpha - 1/2) .$$

Here we have used that s_i does not change the terms $\mu_\alpha^j\alpha_j$ in α for $j \neq i$ and therefore takes $0 < \alpha \neq \alpha_i$ to some positive root. Hence, (3.10) results in

$$s_i x_i - \left(\sum_{k=1}^{n} (s_i(\alpha_i^*), \alpha_k)x_k\right)s_i = -\kappa_i, \quad 1 \leqq i \leqq n ,$$

which is (1.5) for $x_i = y_i^*$ and $h_i = -(\alpha_i, \alpha_i)\kappa_i/2$, $1 \leqq i \leqq n$.

The commutativity relations $[x_i, x_j] = 0$ are valid because the set $\{\rho_\alpha\}$ is an r-matrix. For $\{x_i^0\}$ (without g) reasoning is the same. One can deduce the general statement (without g) from this partial results, but the direct calculation of the commutativity is a more natural way and is not difficult. As for the introduction of g, it is due to the relations $\mu_\alpha^i[r_\alpha, y_j^*] + \mu_\alpha^j[y_i^*, r_\alpha] = 0$. This proves the required commutativity. □

Now we are in a position to explain the main theorem of this section. Later on, $A = \mathrm{End}_{\mathbf{C}} V$. Let us introduce partial derivatives ∂_u, $u \in \mathbf{C}^W$, on $\bar{V}^0 = \mathbf{C}(Y)\otimes_{\mathbf{C}} V$ or $= \mathbf{C}(y)\otimes_{\mathbf{C}} V$ (see Sect. 2 and Lemma 3.5) by

$$\partial_u(Y_\alpha) = (u, \alpha)Y_\alpha,\ \partial_u(y_\alpha) = (u, \alpha),\ \partial_u(V) = 0 , \tag{3.21}$$

where $\alpha \in \Sigma$. In particular, $\partial_i = \partial_{\alpha_i^*}$ $(1 \leqq i, j \leqq n)$. One has

$$\partial_{\alpha u + \beta u'} = \alpha\partial_u + \beta\partial_{u'},\ w\partial_u w^{-1} = \partial_{w(u)} \tag{3.22}$$

for $u, u' \in \mathbf{C}^n$, $\alpha, \beta \in \mathbf{C}$, $w \in W$.

Theorem 3.7 *Put*

$$\nabla_u = \partial_u + \hat{D}_u,\ \hat{D}_u = \sum_{i=1}^{n} (u, \alpha_i)\hat{D}_i,\ u \in \mathbf{C}^n , \tag{3.23}$$

where $\hat{D}_i$ are from (3.19) *or* (3.20). *Then ∇_u, $u \in \mathbf{C}^n$, form a commutative family of operators in* $\mathrm{End}_{\mathbf{C}} \bar{V}^0$ *for arbitrary $\kappa', \kappa'', \delta', \delta'', g$. For $\{\hat{D}_i\}$ of type* (3.20) *or in the case $\delta' = 0 = \delta''$ one has: $\hat{\nu}_0(w)\hat{D}_u\hat{\nu}_0(w)^{-1} = \hat{D}_{w(u)}$, where $\hat{\nu}_0$ is from* (3.9), *$u \in \mathbf{C}^n$, $w \in W$.*

Proof. Let us check the identities $[\partial_i, \mu_\alpha^j r_\alpha] = [\partial_j, \mu_\alpha^i r_\alpha]$ for $1 \leqq i, j \leqq n$, $\alpha \in \Sigma_+$. By (3.22) $\mu_\alpha^j\partial_i - \mu_\alpha^i\partial_j = \partial_u$, where $u = \mu_\alpha^j\alpha_i^* - \mu_\alpha^i\alpha_j^*$. Hence, $\partial_u(Y_\alpha - 1) = (u, \alpha)Y_\alpha = 0 = \partial_u(y_\alpha)$ and $[\partial_u, 1 - \delta_\alpha s_\alpha] = s_\alpha(\partial_u - \partial_{s_\alpha(u)}) = 0$.

We will discuss now some particular cases. Let us consider first the "rational" case of $\mathbf{C}(y)$ with $x_i^0 = 0$ for all i. If $\delta' = 0 = \delta''$, $g = 0$, then one arrives at the generalized Knizhnik-Zamolodchikov equation from [Ch 1, Ch 2]. The Dunkl operators (see [Du, He2]) can be obtained for $V = \mathbf{C}$ and $\delta' = 1 = \delta''$, $g = 0$. In this case

$$r_\alpha = \kappa_\alpha y_\alpha^{-1}(1 - s_\alpha), \quad \alpha \in \Sigma_+ . \tag{3.24}$$

The Lusztig-Lascoux-Schützenberger operators give a natural proof of the commutativity of the corresponding family (3.23).

To get the "trigonometric" operators $\{\nabla_u\}$ we need some $\{\rho_\alpha\}$ (Theorem 3.4) or, more generally, an extension of our initial representation $v_0: W \to \mathrm{End}_{\mathbf{C}} V$ to some representation of the algebra $\mathscr{H}'$, generated by $\mathbf{C}[W]$ and $x_1 \ldots x_n$ above. The quantum counterpart of the first problem is in constructing representations of H satisfying conditions (3.14) and (3.16). For example, the well-known Baxter matrices in tensor powers of vector spaces give an example for $W = \mathbf{S}_{n+1}$. Our basic representations V^0 for one-dimensional V_ε are other examples. They satisfy conditions (3.14), (3.16) and look to be very universal for many purposes.

The trigonometric $\{\nabla_u\}$ without $\{s_\alpha\}$ (i.e. for $\delta' = 0 = \delta''$) are new and generalize directly the affine system from [Ch 2] in the case of A_n, although the latter was written down in a rational form. This system is a natural candidate to obtain some interpretation of Lusztig's isomorphisms (see (2.13), (2.14) and Sect. 0) via monodromy matrices.

Till the end of this paper we will consider $\mathbf{C}[Y]$, $\delta' = 1 = \delta''$, $V = \mathbf{C}$, $g = 0$. Then both (2.9) and (2.10) modulo some multipliers and constants are equivalent and can be written as follows:

$$r_\alpha^c = \kappa_\alpha (Y_\alpha - 1)^{-1} (s_\alpha - 1) + c, \quad \alpha \in \Sigma, \quad c \in \mathbf{C}\,, \tag{3.25}$$

and $\nabla_u = \partial_u + \sum_{\alpha > 0} (u, \alpha) r_\alpha$, $u \in \mathbf{C}$. These $\{r_\alpha\}$ are analogous to

$$\hat{r}_\alpha = \tfrac{1}{2} \kappa_\alpha (Y_\alpha + 1)(Y_\alpha - 1)^{-1} (s_\alpha - 1) \tag{3.26}$$

from Heckman's paper [He1]: $\hat{r}_\alpha - r_\alpha^0 = \kappa_\alpha (s_\alpha - 1)/2$. Our $\{r_\alpha\}$ are not "unitary" ($r_\alpha^0 + s_\alpha r_\alpha^0 s_\alpha = \kappa_\alpha (1 - s_\alpha) \neq 0$). Hence, $\nabla_{w(u)} \neq w \nabla_u w^{-1}$ for some w, u. However Theorem 3.7 is valid for them. Heckman's $\hat{\nabla}_u = \partial_u + \sum_{\alpha > 0} (u, \alpha) \hat{r}_\alpha$ are W-invariant in the above sense, but do not form a commutative family. Let us show that ∇_u have the two main properties of Heckman's $\hat{\nabla}_u$. Namely, they are self-adjoint with respect to some bilinear form and produce the Schrödinger operator of the Calogero-Moser-Olshanetsky-Perelomov type. We follow [He2] very closely. Let

$$\Delta = \prod_{\alpha \in \Sigma_+} (Y_\alpha + Y_\alpha^{-1} - 2)^{\kappa_\alpha} \in \mathbf{C}[Y]^W,$$

$$\bar{P} = c_0 + \sum_{\alpha,, m \neq 0} c_\alpha^m Y_\alpha^{-m}, \quad \text{if} \quad P = c_0 + \textstyle\sum_{\alpha, m \neq 0} c_\alpha^m Y_\alpha^m \in \mathbf{C}[Y],$$

$$\langle P \rangle = c_0\,.$$

Proposition 3.8 (a) $(\nabla_u P, P') = (P, \nabla_u P')$, *where* $(P, P') = \langle P \bar{P}' \Delta \rangle$, $P, P' \in \mathbf{C}[Y]$, $u \in \mathbf{C}$.

(b) *The operator* $\square = \sum_{i=1}^n \nabla_{\alpha_i} \nabla_i - \frac{1}{4}$ *for* $c = -\frac{1}{2}$ *is equal to* $\sum_{i=1}^n \hat{\nabla}_{\alpha_i} \hat{\nabla}_i$ *and to*

$$\sum_{i=1}^n \partial_{\alpha_i} \partial_i - \sum_{\alpha > 0} \kappa_\alpha (Y_\alpha + 1)(Y_\alpha - 1)^{-1} \partial_\alpha$$

after the restriction to $\mathbf{C}[Y]^W$.

Proof. The adjoint ∇_u^* of ∇_u for $c = 0$ with respect to $(,)$ is equal to

$$\Delta^{-1} \left(\partial_u + \sum_{\alpha > 0} \kappa_\alpha (u, \alpha)(1 - s_\alpha)(Y_\alpha^{-1} - 1)^{-1} \right) \Delta\,.$$

For Heckman's one (which is self adjoint-see [He1]) the corresponding formula is with $(Y_\alpha^{-1}+1)(Y_\alpha^{-1}-1)^{-1}$ instead of $(Y_\alpha^{-1}-1)^{-1}$. Hence,

$$\begin{aligned}\hat{\nabla}_u^* - \nabla^* &= \frac{1}{2}\Delta^{-1}\sum_{\alpha>0}\kappa_\alpha(u,\alpha)(s_\alpha-1)\Delta\\ &= \sum_{\alpha>0}\kappa_\alpha(u,\alpha)(s_\alpha-1) = \hat{\nabla}_u - \nabla_u\,,\end{aligned}$$

and

$$\nabla_u^* = \nabla_u\,.$$

One has

$$\square = \sum_{i=1}^n \partial_{\alpha_i}\partial_i - 2\sum_{\alpha>0}\kappa_\alpha(Y_\alpha-1)^{-1}\partial_\alpha + 2c\sum_{\alpha>0}\kappa_\alpha\partial_\alpha + c^2$$

because $r_\alpha^c - c$ acts trivially on $\mathbf{C}[Y]^W$ and

$$\sum_{i=1}^n (u,\alpha_i)\partial_i = \sum_{i=1}^n (u,\alpha_i^*)\partial_{\alpha_i} = \partial_\alpha$$

(see [He1, Ex. 3.9]). □

Acknowledgements. This paper was written during my stay at RIMS (Kyoto University) and is partially based on my lectures at RIMS on Hecke algebras. I am grateful for the kind invitation and for hospitality. I should like to acknowledge my special indebtednes to T. Miwa. I thank S. Kato for a useful discussion on his and Lusztig's works, G. Heckman for the explanation of the main points of his last papers, V. Jones, who helped me understand the Lascoux-Schützenberger results and supplied me with his new notes, and A. Matsuo for many interesting talks about the Heckman, Opdam works and his own results. I express also appreciation to E. Date, Y. Ihara, M. Jimbo, M. Kashiwara, M. Sato, D. Zagier for their kind interest in my work, to the participants of my talks for encouragement, to my other colleagues at RIMS for many useful discussions and to S. Miwa for her assistance in preparing the manuscript for publication.

References

[BGG] Bernstein, I.N., Gelfand, I.M., Gelfand, S.I.: Schubret cells and the cohomology of G/P. Russ. Math. Surv. **28**, 1–26 (1973)

[Bo] Bourbaki, N.: Groupes et algébres de Lie, Chapter 6. Paris: Hermann 1969

[Ch1] Cherednik, I.V.: Generalized braid groups and local r-matrix systems. Dokl. Akad. Nauk SSSR, **307:1**, 27–34 (1989)

[Ch2] Cherednik, I.V.: Monodromy representations for generalized Knizhnik-Zamolodchikov equations and Hecke algebras. ITP-89-74E, Kiev. (Preprint 1989); Publ. Res. Inst. Math. Sci. (to appear)

[Ch3] Cherednik, I.V.: A new interpretation of Gelfand-Tzetlin bases. Duke Math. J. **54:2**, 563–577 (1987)

[Ch4] Cherednik, I.V.: On special bases of irreducible representations of the degenerate affine Hecke algebra. Funct. Anal. Appl. **20:1**, 87–89 (1986)

[Ch5] Cherednik, I.V.: Factorized particles on the half-line and root systems. Theor. Math. Phys. **61:1**, 35–43 (1984)

[De] Demazure, M.: Désingularisation des variétès de Schubert généralisés. Ann. Éc. Norm. Supér. **7**, 53–88 (1974)

[Dr] Drinfeld, V.G.: Degenerate affine Hecke algebras and Yangians. Funct. Anal. Appl. **20:1**, 69–70 (1986)

[Du] Dunkl, C.F.: Differential-difference operators associated to reflection groups. Trans. Am. Math. Soc. **311**, 167–183 (1989)

[He1] Heckman, G.J.: An elementary approach to the hypergeometric shift operators of Opdam. Invent. Math. (Submitted)

[He2] Heckman, G.J.: A remark on the Dunkl differential-difference operators. Proceedings of Bowdoin Conference on harmonic analysis in reductive groups, 1989. (To appear)

[HO] Heckman, G.J., Opdam, E.M.: Root systems and hypergeometric functions I. Compos. Math. **64**, 329–352 (1987)

[Ka] Kato, S.: Irreducibility of principal series representations for Hecke algebras of affine type. J. Fac. Sci., Univ. Tokyo, IA, **28:3**, 929–943 (1983)

[KL] Kazhdan, D., Lusztig, G. Proof of Deligne-Langlands conjecture for Hecke algebras. Invent. Math. **87**, 153–215 (1987)

[LS1] Lascoux A., Schützenberger M.-P.: Non-commutative Schubert polynomials. Funct. Anal. Appl. **23:3**, 63–64 (1989)

[LS2] Lascoux A., Schützenberger M.-P.: Symmetrization operators in polynomial rings. Funct. Anal. Appl. **21:4**, 77–78 (1987)

[Lu1] Lusztig, G.: Affine Hecke algebras and their graded version. J. Am. Math. Soc. **2:3**, 599–685 (1989)

[Lu2] Lusztig, G.: Cuspidal local systems and graded Hecke algebras I. Publ. Math., Inst. Hautes Étud. Sci. **67**, 145–202 (1988)

[Lu3] Lusztig, G.: Equivariant K-theory and representations of Hecke algebras. Proc. Am. Math. Soc. **94:2**, 337–342 (1985)

[Ma] Matsumoto, H.: Analyse harmonique dans les systems de Tits bornologiques de type affine. (Lect. Notes Math. vol. 590) Berlin Heidelberg New York: Springer 1979

[Mu] Murphy, G.E.: A new construction of Young's seminormal representation of the symmetric group. J. Algebra **69**, 287–297 (1981)

[Ro] Rogawski, J.D.: On modules over the Hecke algebra of a p-adic group. Invent. Math. **79**, 443–465 (1985)

[Ze] Zelevinsky, A.V.: Induced representations of reductive p-adic groups II. Ann. Sci. Éc. Norm. Supér. Sér. IV. Sér. **13**, 165–210 (1980)