Invent. math. 106, 27–60 (1991)

Inventiones mathematicae

Complete Kähler manifolds with zero Ricci curvature II

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Oblatum 11-V-1989 & 26-II-1991

This is a continuation of our previous paper on settling the non-compact version of Calabi's conjecture on open manifold. In both these papers, open manifolds will mean quasiprojective manifolds M which can be written as $\bar{M}\backslash D$. We are constructing complete Kähler metrics on M with either zero Ricci curvature or non-negative Ricci curvature. As was explained in the program outlined by the second author in the Congress in Helsinki, D is related to the zeroes of a section of $K_{\bar{M}}^{-1}$. In our previous paper [TY1], we dealt with the case when the multiplicity is equal to one. In this paper, we finish the case when the multiplicity is greater than one. We also allow orbifold type singularities in all these discussions. Our constructions include practically all known examples of complete Kähler manifolds with zero Ricci curvature of finite topological type. (It should be noted that M. Anderson, P. Kronheimer and Le Brun have recently constructed such examples with infinite topological type.) Besides constructing many new examples of such manifolds which may serve as gravitational instantons, these matrices provide a bridge between metric geometry and algebraic geometry of M because we do have some understanding of complete manifolds with non-negative curvature.

Acknowledgements. We would like to express our gratitude to the referee for the time and effort spent reviewing our manuscript. His excellent suggestions have resulted in a paper which is much more readable. When we obtained these results in 1986, Peter Li has shown great interest in their applications. We wish to thank his moral support.

1 Statements of main theorems

In [TY1], the authors constructed a complete Kähler metric on a quasi-projective manifold $M = \bar{M}\backslash D$ with prescribed Ricci form representing $C_1(-K_{\bar{M}} - L_D)$. Here $\bar{M}$ is a compact Kähler manifold, D is a neat and almost ample smooth divisor in $\bar{M}$ (cf. Definition 1.1) and L_D is its associated line bundle. In fact, the whole argument in [TY1] can be generalized to the case that $\bar{M}$ is a normal Kähler orbifold and D is an admissible divisor (cf. Definition 1.1). In this paper, we will construct complete Kähler metrics on $M = \bar{M}\backslash D$ with prescribed Ricci form in

$C_1(-K_{\bar{M}} - \beta L_D)$ for $\beta > 1$ under some suitable assumptions on $\bar{M}$ and D. Let $\bar{M}$ be a compact Kähler orbifold with $\dim_{\mathbb{C}}(\mathrm{Sing}(\bar{M})) \leqq n-2$, where $n = \dim_{\mathbb{C}} \bar{M}$ and $\mathrm{Sing}(\bar{M})$ denotes the set of singular points. Note that $\mathrm{Sing}(\bar{M})$ is a subvariety of $\bar{M}$. We assume that each point of $\bar{M}$ admits a neighborhood which is the quotient of a euclidean ball in C^n by a finite group. Natural patching conditions are imposed on the overlaps of these neighborhoods. These two properties characterize complex orbifolds. A Kähler orbifold is just a complex orbifold with a Kähler orbifold metric. We refer readers to [Ba] for definition of Kähler orbifolds in detail. On a Kähler orbifold, one can also define line bundles, divisors, etc.

Definition 1.1 Let D be a divisor in the Kähler orbifold $\bar{M}$. Then

(i) D is *neat*, if no compact holomorphic curve in $\bar{M}\backslash D$ is homologous to an element in $N_1(D)$, where $N_1(D)$ denotes the abelian group generated by holomorphic curves supported in D.

(ii) D is *almost ample* if there exists an integer $m > 0$ such that a basis of $H^0(\bar{M}, mL_D)$ gives a morphism from $\bar{M}$ into some projective space CP^N which is biholomorphic in a neighborhood of D.

(iii) D is *admissible* if $\mathrm{Sing}(\bar{M}) \subset D$, D is smooth in $\bar{M}\backslash\mathrm{Sing}(\bar{M})$ and for any $x = \mathrm{Sing}(\bar{M})$, let $\pi_x\colon \tilde{U}_x \to U_x$ be its local uniformization with $\tilde{U}_x \subset C^2$, then $\pi_x^{-1}(D)$ is smooth in $\tilde{U}_x$.

Now we are ready to state our main theorem of this paper. The proof of this theorem will be given in Sects. 2, 4, and 5.

Theorem 1.1 *Let $\bar{M}$ be a compact Kähler orbifold of complex dimension n. Let D be a neat, almost ample and admissible divisor in $\bar{M}$, and L_D be the associated line bundle of D. Let Ω be any* (1.1)*-form representing the first Chern class $C_1(-K_{\bar{M}} - \beta L_D)$ with $\beta > 1$. Assuming that D admits a Kähler metric with Kähler form ω_D such that*

$$\mathrm{Ric}(\omega_D) = (\beta - 1)\omega_D + \Omega \tag{1.1}$$

then there is a complete Kähler metric g_Ω over $\bar{M}\backslash D$ whose Ricci curvature form is Ω. Moreover, if we denote by $R(g_\Omega)$ the curvature tensor of g_Ω and by $\rho(\cdot)$ the distance function on M from some fixed point with respect to g_Ω, then $R(g_\Omega)$ decays at the order of at least ρ^{-3} with respect to g_Ω-norm whenever D is biholomorphic to CP^{n-1} and $L_D|_D$ is the $\frac{n}{\beta-1}$-multiple of the hyperplane line bundle on CP^{n-1}; otherwise, $R(g_\Omega)$ decays at the order of exactly ρ^{-2} with respect to g_Ω-norm. Furthermore the metric g_Ω has euclidean volume growth.

Corollary 1.1 *Let $\bar{M}$, D be as in Theorem* 1.1. *Suppose that $-K_{\bar{M}} = \beta L_D$ and D admits a Kähler-Einstein metric with positive scalar curvature. Then $M = \bar{M}\backslash D$ has a complete Ricci-flat Kähler metric such that its curvature tensor decays as described in Theorem* 1.1.

Remarks. (1) In case that $\bar{M}$ is a smooth Kähler manifold, D is ample and $1 < \beta < n+1$, the existence part in the above corollary is also recently rediscovered by S. Bando and R. Kobayashi [BK] who made extra technical assumptions and draw less precise conclusion.

(2) It is still open whether or not D admits a Kähler-Einstein metric with positive scalar curvature. Note that $C_1(D) = C_1(\bar{M})|_D - C_1(L_D)|_D = (\beta - 1)C_1(L_D)|_D > 0$. In case that D is the Fermat hypersurface in CP^n of degree

$n-1$ or n, the first author proved the existence of Kähler-Einstein metrics on D in [T1]. When D is a complex surface other than $CP^2 \# \overline{CP^2}$ and $CP^2 \# 2\overline{CP^2}$, by the results in [TY2, T2]. D admits a Kähler-Einstein metric. Therefore, M admits a complete Ricci-flat metric if $\bar{M} = CP^n$ and either D is a smooth hypersurface of degree $n-1$ or n, or $n = \dim_{\mathbf{C}} \bar{M} = 3$.

Corollary 1.2 *Let $\bar{M}$, D be as in Theorem 1.2. Suppose that there is a semi-positive $(1,1)$-form in $C_1(-K_{\bar{M}} - \beta L_D)$ for some $\beta > 1$. Then there is a complete Kähler metric with nonnegative Ricci curvature and the curvature decay as described in Theorem 1.1. Also such a metric has euclidean volume growth.*

Proof. Define a holomorphic invariant $\alpha(D) > 0$ as follows. Take a G-invariant Kähler metric ω in $C_1(L_D)|_D$, where G is a maximal compact subgroup in $\mathrm{Aut}(D)$, define

$$P_G(D,\omega) = \left\{ \varphi \in C^2(M,R) \mid \omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi \geqq 0,\ \varphi \text{ is } G-\text{inv.},\ \sup_D \varphi = 0 \right\} \tag{1.2}$$

$$\alpha(D) = \sup\left\{ \alpha \mid \exists C > 0, \text{ s.t. } \int_D e^{-\alpha\varphi}\omega^{n-1} \leqq C \text{ for all } \varphi \in P_G(D,\omega) \right\}. \tag{1.3}$$

Then one can easily prove that $\alpha(D)$ is independent of choices of ω and G, so it is a holomorphic invariant. In [T1], it is proved that $\alpha(D) > 0$. Now choose a $\beta' < 1 + \alpha(D)$ and $\beta' \leqq \beta$. Then our assumptions imply that there is a semi-positive (1.1)-form $\Omega_{\beta'}$ in $C_1(-K_{\bar{M}} - \beta' L_D)$. The method of [T1] can be applied here to conclude the existence of a Kähler metric with Kähler form ω_D and Ricci form being $(\beta' - 1)\omega_D + \Omega_{\beta'}$.

To see it, we first choose a metric h with its Kähler form ω_h in $C_1(L_D)$. Then $(\beta' - 1)\omega_h + \Omega_{\beta'}$ represents the first Chern class $C_1(D)$. Therefore, there is a function f such that

$$\mathrm{Ric}(h) - (\beta' - 1)\omega_h - \Omega_{\beta'} = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial} f$$

and

$$\int_D e^f \omega_h^{n-1} = \int_D \omega_h^{n-1}.$$

The required ω_D will be of the form $\omega_h + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_{\beta'}$ and $\varphi_{\beta'}$ satisfies the following complex Monge-Amperé equation for $t = \beta' - 1$,

$$\left\{ \begin{array}{l} \left(\omega_h + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right)^{n-1} = e^{f - t\varphi}\omega_h^{n-1} \\ \omega_h + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi > 0 . \end{array} \right. \tag{1.4$_t$}$$

By the second author's higher-order estimates in the solution of Calabi conjecture [Y2], in order to solve $(1.4)_t$, it suffices to give an apriori C^0-estimate for the solutions. If φ_t is a solution of $(1.4)_t$ and h_t is the metric with Kähler form $\omega_h + \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\varphi_t$, then

$$\begin{aligned}\mathrm{Ric}(h_t) &= (\beta' - 1 - t)\omega_h + t\left(\omega_h + \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\varphi_t\right) + \Omega_{\beta'} \\ &\geqq t\omega_{h_t} \text{ for } t \leqq \beta' - 1 .\end{aligned}$$

Therefore, the equations in $(1.4)_t$ for $t \leqq \beta' - 1$ are exactly those treated in [T1]. In particular, there is an apriori C^0-estimate for the solutions of $(1.4)_t$ with $0 \leqq t \leqq \beta' - 1$ and $\beta' - 1 < \frac{n}{n-1}\alpha(D)$ (cf. §2 in [T1]). It implies the existence of ω_D for $\beta' < 1 + \frac{n}{n-1}\alpha(D)$. Then this corollary follows from Theorem 1.1.

Remark. If there is a positive (1.1)-form in $C_1(-K_{\bar M} - \beta L_D)$ for some $\beta > 1$, then the complete metric constructed in Corollary 1.2 has positive Ricci curvature.

Examples. For any $n > 0$ and $d < n + 1$, the complement $CP^n \backslash D$ of a hypersurface D of degree d admits a complete Kähler metric with euclidean volume growth, positive Ricci curvature and quadratic decay of the curvature tensor.

Finally, we state an application of Theorem 1.1 on the topology of projective manifolds with some ampleness conditions on its anticanonical line bundle.

Theorem 1.2 *Let $\bar M$ be a projective normal orbifold. If there is an admissible, neat and almost ample divisor D in $\bar M$ such that $C_1(-K_{\bar M} - L_D)$ admits a semi-positive (1.1)-form. Then $\bar M$ is simply-connected.*

The proof of it will be given in Sect. 6. One should also be able to draw some results on the simple-connectedness of the resolutions of $\bar M$. In case $C_1(\bar M)$ is positive, this result follows from a result of S. Kobayashi [Ko] and the solution of Calabi conjecture by the second author.

We believe that the assumption on the neatness of D is superfluous.

2 Kähler metrics with approximating properties

Let $\bar M$ be a Kähler orbifold of complex dimension n, D be an admissible divisor in $\bar M$ as defined in the last section. Then in particular, the divisor D is a Cartier divisor and induces a line bundle L_D on the orbifold $\bar M$. We further assume that the restriction of L_D to D is ample. Therefore, there is an orbifold hermitian metric on L_D such that its curvature form is positive definite along D. Let Ω to be a closed (1.1)-form in the Chern class $C_1(-K_{\bar M} - \beta L_D)$, where β is a real number and $\beta > 1$. The goal of this section is to construct a complete Kähler metric g such that

$$\mathrm{Ric}(g) - \Omega = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial f \qquad \text{on } M \tag{2.1}$$

for some functions f with sufficiently fast decay, where $\mathrm{Ric}(g)$ is the Ricci form of the metric g. In local coordinates, if g is represented by the tensor $(g_{i\bar{j}})_{1\leq i,j\leq n}$, then

$$\mathrm{Ric}(g) = -\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(\det(g_{i\bar{j}})_{1\leq i,j\leq n}) .$$

We fix an orbifold hermitian metric $\|\cdot\|$ on L_D such that its curvature form is a given Kähler form ω_D on D when restricted to the infinity D. This latter form ω_D on D will be specified in the following discussion. Denote by $\|\cdot\|_\varphi$ the new hermitian metric $\|\cdot\|\cdot e^{-\varphi/2}$ on L_D for any smooth function φ defined on $\bar{M}$. Let S be the defining section of D and define

$$\omega_\varphi = \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\left(\|S\|_\varphi^{-\frac{2(\beta-1)}{n}}\right) . \tag{2.2}$$

Then a simple computation shows

$$\omega_\varphi = \frac{1}{n}(\beta-1)\|S\|_\varphi^{-\frac{2(\beta-1)}{n}}\tilde{\omega}_\varphi + \frac{\sqrt{-1}}{2\pi}\frac{1}{n^2}(\beta-1)^2\|S\|_\varphi^{-\frac{2(\beta-1)}{n}}\frac{D_\varphi S\wedge\overline{D_\varphi S}}{|S|^2} \tag{2.3}$$

where $\tilde{\omega}_\varphi$ is the curvature form of the hermitian metric $\|\cdot\|_\varphi$ of L_D and D_φ is the covariant derivative with respect to $\|\cdot\|_\varphi$. It follows that ω_φ is positive definite near D as long as the closed (1.1) form $\tilde{\omega}_\varphi$ is positive definite along D. In fact, we shall only be interested in those functions φ which are constant along D. Therefore, the (1.1)-form $\tilde{\omega}_\varphi$ is always positive definite along D. Now we determine ω_D on D. Put $\alpha = \dfrac{\beta-1}{n}$, then

$$\omega_\varphi^n = \alpha^n\|S\|_\varphi^{-2\alpha n}\tilde{\omega}_\varphi^{n-1}\wedge\left(\tilde{\omega}_\varphi + \frac{\alpha n\sqrt{-1}}{2\pi}\frac{D_\varphi S\wedge\overline{D_\varphi S}}{|S|^2}\right) . \tag{2.4}$$

For a given Kähler metric g' with Kähler form ω' on $\bar{M}$, there is a function ψ unique up to constant such that

$$\Omega = \mathrm{Ric}(g') - \beta\tilde{\omega} + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\psi \tag{2.5}$$

where $\tilde{\omega}$ is the curvature form of $\|\cdot\|$ on L_D, i.e., $\tilde{\omega} = \tilde{\omega}_0$. Note that $\tilde{\omega}|_D = \omega_D$. We define a smooth function f_φ near D by

$$f_\varphi(x) = -\beta\log(\|S\|^2)(x) - \log\left(\frac{\omega_\varphi^n}{(\omega')^n}\right)(x) - \psi(x) \tag{2.6}$$

for x in the set where ω_φ is positive definite.

Lemma 2.1 *The following two statements are equivalent.*
(1) *$f_0(x)$ converges to a constant uniformly as x tends to D.*
(2) *The induced metric g_D satisfies the equation*

$$\mathrm{Ric}(g_D) = (\beta-1)\omega_D + \Omega|_D \text{ on } D . \tag{2.7}$$

Proof. Choose local coordinates $(z_1, \ldots, z_n)$ at a point in D such that $z_n = S = 0$ defines D locally, and $z' = (z_1, \ldots, z_{n-1})$ defines a coordinate system along D. Let $\tilde{\omega}$, g', $\|\cdot\|$ be locally represented by $(h_{i\bar{j}})_{1 \leqq i,j \leqq n}$, $(g'_{i\bar{j}})_{1 \leqq i,j \leqq n}$ and a positive function a, respectively. Then by (2.6),

$$f_0(x) = -\log(a \det(h_{i\bar{j}})_{1 \leqq i,j \leqq n-1}/\det(g'_{i\bar{j}})_{1 \leqq i,j \leqq n})(x) - \psi(x) + O(\|S(x)\|)$$

where x is near D. Note that $a^{-1} \det(g'_{i\bar{j}})_{1 \leqq i,j \leqq n}|_D$ is a well-defined volume form on D. Write $x = (z', z_n)$, then

$$f_0(x) = -\log\left(\frac{a \det(h_{i\bar{j}})_{1 \leqq i,\bar{j} \leqq n-1} e^{\psi}}{\det(g'_{i\bar{j}})_{1 \leqq i,j \leqq n}}\right)(z', 0) + O(\|S(x)\|)\,.$$

Therefore $\lim_{x \to D} f_0(x) = \text{const.}$ if and only if $\dfrac{a \det(h_{i\bar{j}})_{1 \leqq i,\bar{j} \leqq n-1} e^{\psi}}{\det(g'_{i\bar{j}})_{1 \leqq i,j \leqq n}}(z', 0)$ is constant in the local coordinates $(z_1, \ldots, z_{n-1}) = z'$ of D. By (2.5), (2.7), this latter statement is exactly the one in (2). The lemma is proved.

Remark. Equation (2.7) is equivalent to the following complex Monge-Ampère equation

$$\begin{cases} \left(\omega + \dfrac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi\right)^{n-1} = e^{h-(\beta-1)\varphi}\omega^{n-1} \text{ on } D \\ \omega + \dfrac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi > 0 \end{cases} \tag{2.8}$$

where ω is a given Kähler form on D representing $C_1(L_D)$ and h is a given function on D determined by Ω and ω. In case $-K_{\bar{M}} = \beta D$ and $\Omega = 0$, it is the equation involved in constructing Kähler-Einstein metric with positive scalar curvature. While the general existence is not known yet, we have some positive results (cf. [T1, TY2, T2]). Let $\alpha(D)$ be the invariant defined in (1.3). Then the method in [T1] can be applied to conclude the existence of ω_D for $\beta < 1 + \alpha(D)$ (cf. the proof of Corollary 1.2).

From now on, we assume that ω_D is a Kähler form on D such that (2.7) holds. Then by choosing ψ in (2.5) properly, $f_0(x)$ converges to zero uniformly as x tends to the infinity D. On the other hand, we remark that

$$\|S\|^2 \omega_\varphi^{n-1} \wedge \left(\omega_\varphi + \frac{\alpha n\sqrt{-1}}{2\pi} \frac{D_\varphi S \wedge \overline{D_\varphi S}}{|S|^2}\right)$$

is a smooth $2n$-form on M, so that f_0 can be smoothly extended to $\bar{M}$ by defining $f_0(x) = 0$ for $x \in D$. Therefore, there is a $\delta_0 > 0$ such that in the neighborhood $V_0 = \{x \in M \mid \|S(x)\| < \delta_0\}$ of D, we have

$$f_0 = S \cdot u_1 + \overline{S \cdot u_1} \tag{2.9}$$

where u_1 is a C^∞-local section in $\Gamma(V_0, L_{\bar{D}}^{-1})$.

We would like to choose φ_1 of the form $S \cdot \theta_1 + \overline{S \cdot \theta_1}$ with $\theta_1 \in \Gamma(V_0, L_{\bar{D}}^{-1})$ such that $f_1 = f_{\varphi_1}$ vanishes along D at the order of two. The obstruction to the existence of such a φ_1 lies in the kernel $\ker(\Box - 2)$ on D, where $\Box = \mathrm{tr}_{\omega_D}(D\bar{D})$ is the laplacian of L_D^{-1} on D. To overcome this difficulty, one must introduce the term $(-\log \|S\|^2)$ in φ_1. It resembles the case of constructing approximated Kähler-Einstein metrics on strongly pseudoconvex domains in C^n considered by C. Fefferman (cf. [Fef, CY2]). In the following, we will construct by induction a sequence of hermitian metrics $\{\|\cdot\|_m\}_{m\geqq 0}$ of L_D defined on $\bar{M}$ such that for any $m \geqq 0$, there is a $\delta_m > 0$ satisfying:

(i) The associated Kähler metric ω_m of $\|\cdot\|_m$ defined in (2.2) or (2.3) is positive definite in the neighborhood $V_m = \{\|S(x)\| < \delta_m\}$ of D.

(ii) The function f_m defined by (2.6) has an expansion

$$\sum_{k\geqq m+1} \sum_{\ell=0}^{\ell_k} u_{k\ell}(-\log \|S\|_m^2)^\ell \text{ in } V_m \tag{2.10}$$

where $u_{k\ell}$ are smooth C^∞-functions defined on the closure $\bar{V}_m$ and $u_{k\ell} = O(\|S\|^{m+1})$, ℓ_k are nonnegative integers.

Let $\|\cdot\|_0 = \|\cdot\|$. Then by (2.9) and the definition of $\|\cdot\|$, both (i) and (ii) hold for this hermitian metric. Suppose now that we have found $\|\cdot\|_m$. We then go on to construct $\|\cdot\|_{m+1}$.

Lemma 2.2 *Let φ be a smooth function defined on V_m which can be written $\sum_{i+j=m+1}(S^i\bar{S}^j\theta_{ij} + \bar{S}^iS^j\bar{\theta}_{ij})(-\log\|S\|_m^2)^k$, and let f_φ be defined by (2.5) with $\omega_\varphi = \frac{\sqrt{-1}}{2\pi}\partial\bar\partial(e^\varphi\|S\|_m^{-2})^{\frac{\beta-1}{n}}$. Then*

$$\begin{aligned} f_\varphi = f_m - (-\log\|S\|_m^2)^k \sum_{i+j=m+1}\Bigg[&(S^i\bar{S}^j\theta_{ij} + \bar{S}^iS^j\bar{\theta}_{ij}) \,. \\ &\cdot\left(\frac{ij}{\alpha} - (m+2) - j(n-1) + (S^i\bar{S}^j\Box_m\theta_{ij} + \bar{S}^iS^j\overline{\Box_m\theta_{ij}})\right)\Bigg] - \frac{k\varphi}{\log\|S\|_m^2}\left(\frac{m+1}{\alpha} - 2\right) \\ &- \frac{k(k-1)\varphi}{\alpha(\log\|S\|_m^2)^2} + \sum_{k'\geqq m+2}\sum_{\ell=0}^{\ell_{k'}} u_{k'\ell}(-\log\|S\|_m^2)^\ell \end{aligned} \tag{2.11}$$

where $\Box_m$ is the laplacian $\mathrm{tr}_{\omega_D}(D_m\bar{D}_m)$ of the bundle $L_D^{-i}\otimes\bar{L}_D^{-j}$ on D with respect to the hermitian metric $\|\cdot\|_m$ and ω_D, and D_m is the covariant derivative with respect to $\|\cdot\|_m$.

Proof. First we remark that $\theta_{ij} = \bar{\theta}_{ji}$, since φ is real valued. By the definition (2.6), we have

$$\begin{aligned} f_\varphi &= -\beta\log\|S\|_m^2 - \log\left(\frac{\omega_\varphi^n}{(\omega')^n}\right) - \psi \\ &= f_m + \beta\varphi - \log\left(\frac{\omega_\varphi^n}{\omega_m^n}\right). \end{aligned} \tag{2.12}$$

Therefore, it suffices to compute the ratio $\frac{\omega_\varphi^n}{\omega_m^n}$. Note that the covariant derivatives D_φ and D_m are related to each other by the equation

$$D_\varphi S = D_m S - S\partial\varphi \,.$$

Moreover, if we denote by $\tilde{\omega}_m$ the curvature form of the hermitian metric $\|\cdot\|_m$, then

$$\tilde{\omega}_\varphi = \tilde{\omega}_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi .$$

Using (2.4), we obtain

$$\begin{aligned}
\omega_m^n &= \alpha\|S\|_m^{-2\alpha n}\tilde{\omega}_m^{n-1}\wedge\left(\tilde{\omega}_m + \frac{\alpha n\sqrt{-1}}{2\pi}\frac{D_mS\wedge\overline{D_mS}}{|S|^2}\right)\\
&= \alpha^n\|S\|_m^{-2\alpha n-2}(\|S\|_m^2 + \alpha\|D_mS_m\|^2)\tilde{\omega}_m^n\\
\omega_\varphi^n &= \alpha^n\|S\|_\varphi^{-2\alpha n}\tilde{\omega}_\varphi^{n-1}\wedge\left(\tilde{\omega}_\varphi + \frac{\alpha n\sqrt{-1}}{2\pi}\frac{D_\varphi S\wedge\overline{D_\varphi S}}{|S|^2}\right)\\
&= \alpha^n\|S\|_m^{-2\alpha n}e^{\alpha n\varphi}\left(\tilde{\omega}_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right)^{n-1}\wedge\left\{\left(\tilde{\omega}_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi + \right.\right.\\
&\quad\left.\left. + \frac{\alpha n\sqrt{-1}}{2\pi}\left(\frac{D_mS\wedge\overline{D_mS}}{|S|^2}\right) - \frac{D_mS}{S}\wedge\bar{\partial}\varphi - \partial\varphi\wedge\frac{\overline{D_mS}}{\bar{S}} + \partial\varphi\wedge\bar{\partial}\varphi\right)\right\}.
\end{aligned}$$

From the definition of φ one can compute

$$\begin{aligned}
\partial\varphi = \sum_{i+j=m+1}&\left\{(D_mS^i\bar{S}^j\theta_{ij} + \bar{S}^iD_mS^j\bar{\theta}_{ij} + S^i\bar{S}^jD_m\theta_{ij}\right.\\
&\left. + \bar{S}^iS^jD_m\bar{\theta}_{ij})(-\log\|S\|_m^2)^k + k(-\log\|S\|_m)^{k-1}\cdot(S^i\bar{S}^j\theta_{ij} + \bar{S}^iS^j\bar{\theta}_{ij})\left(-\frac{D_mS}{S}\right)\right\},
\end{aligned}$$

hence,

$$\begin{aligned}
\partial\varphi\wedge\frac{\overline{D_mS}}{\bar{S}} = \sum_{i+j=m+1}&\left\{(iS^i\bar{S}^j\theta_{ij} + j\bar{S}^iS^j\bar{\theta}_{ij})(-\log\|S\|_m^2)^k\cdot\right.\\
&\left.\cdot\frac{D_mS\wedge\overline{D_mS}}{|S|^2} + (-\log\|S\|_m^2)^k(S^i\bar{S}^jD_m\theta_{ij} + \bar{S}^iS^jD_m\bar{\theta}_{ij})\wedge\frac{\overline{D_mS}}{S}\right\}\\
&+ \frac{k\varphi}{\log\|S\|_m^2}\frac{D_mS\wedge\overline{D_mS}}{|S|^2}\partial\varphi\wedge\bar{\partial}\varphi + O(2m+2)\frac{D_mS\wedge\overline{D_mS}}{|S|^2}\\
&+ \sum_{i+j=m+1}\left(D_m\theta_{ij}\wedge\frac{\overline{DS}}{\bar{S}}O(2m+2) + \frac{D_mS}{S}\wedge\overline{D_m\theta_{ij}}O(2m+2)\right.\\
&+ D_m\bar{\theta}_{ij}\wedge\frac{\overline{D_mS}}{\bar{S}}O(2m+2) + \frac{D_mS}{S}\wedge\bar{D}_m\theta_{ij}O(2m+2)\\
&+ D_m\theta_{ij}\wedge\bar{D}_m\theta_{ij}O(2m+2) + D_m\theta_{ij}\wedge\overline{D_m\theta_{ij}}O(2m+2) +\\
&\left.+ D_m\bar{\theta}_{ij}\wedge\bar{D}_m\theta_{ij}O(2m+2) + D_m\bar{\theta}_{ij}\wedge\overline{D_m\theta_{ij}}O(2m+2)\right).
\end{aligned}$$

Here we use $O(2m+2)$ to denote those functions of form

$$\sum_{s=\ell}^{p_\ell} \sum_{t=0}^{q_s} v_{st}(-\log \| v \|_m^2)^t$$

where $p_\ell, q_\ell, \ldots, q_{p_\ell}$ are positive integers, v_{st} are smooth functions in a neighborhood of D and $v_{st} = O(\| S \|^{s,t})$ near D. We further compute the complex Hessian of φ.

$$\begin{aligned}
\partial\bar\partial\varphi = (-\log \| S \|_m^2)^k \sum_{i+j=m+1} \Big\{ & ij(S^i \bar S^j \theta_{ij} + \bar S^i S^j \bar\theta_{ij}) \frac{D_m S \wedge \overline{D_m S}}{|S|^2} \\
& + iS^i\bar S^j \frac{D_m S}{S} \wedge \bar D_m \theta_{ij} + jS^i \bar S^j D_m \theta_{ij} \wedge \frac{\overline{D_m S}}{\bar S} + i\bar S^i S^j D_m \bar\theta_{ij} \wedge \frac{\overline{D_m S}}{S} \\
& + j\bar S^i S^j \frac{D_m S}{S} \wedge \bar D_m \theta_{ij} + S^i D_m \bar D_m \bar S^j \theta_{ij} + D_m \bar D_m \bar S^i S^j \bar\theta_{ij} \\
& + S^i \bar S^j D_m \bar D_m \theta_{ij} + \bar S^i S^j D_m \bar D_m \bar\theta_{ij} \Big\} + \frac{k(m+1)}{\log \| S \|_m^2} \varphi \frac{D_m S \wedge \overline{D_m S}}{|S|^2} \\
& - k(-\log \| S \|_m^2)^{k-1} \Big\{ \frac{D_m S}{S} \wedge \sum_{i+j=m+1} (S^i \bar S^j \bar D_m \theta_{ij} + \bar S^i S^j \bar D_m \bar\theta_{ij}) \\
& + \sum_{i+j=m+1} (S^i \bar S^j D_m \theta_{ij} + \bar S^i S^j D_m \bar\theta_{ij}) \wedge \frac{\overline{D_m S}}{\bar S} \Big\} + \frac{k(k-1)\varphi}{(\log \| S \|_m^2)^2} \frac{D_m S \wedge \overline{D_m S}}{|S|^2} .
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\bar D_m D_m S^j &= -D_m \bar D_m S^j + jS^j \tilde\omega_m = jS^j \tilde\omega_m \\
\bar D_m D_m S^i &= -D_m \bar D_m S^i + iS^i \tilde\omega_m = iS^i \tilde\omega_m \\
\bar D_m D_m \theta_{ij} &= -D_m \bar D_m \theta_{ij} - (i-j)\theta_{ij} \tilde\omega_m .
\end{aligned}$$

In particular, these imply that

$$\| S \|_m^2 (\partial\bar\partial\varphi)^\ell \wedge \tilde\omega_m^{n-\ell} = \tilde\omega_m^n O(2m+2) \quad \text{for} \quad \ell \geqq 2 .$$

Now using the fact that $\| D_m S \|_m$ is nonvanishing along D and the above identities, we can compute the ratio of the two volume forms as follows,

$$\begin{aligned}
\frac{\omega_\varphi^n}{\omega_m^n} = & \frac{\| S \|_m^2 e^{\alpha n \varphi}}{(\| S \|_m^2 + \alpha \| D_m S \|_m^2)\tilde\omega_m^n} \left(\omega_m + \frac{\sqrt{-1}}{2\pi} \partial\bar\partial\varphi \right)^{n-1} \wedge \Big\{ \left(\tilde\omega_m + \frac{\sqrt{-1}}{2\pi} \partial\bar\partial\varphi \right) \\
& + \frac{\alpha n \sqrt{-1}}{2\pi} \left(\frac{D_m S \wedge \overline{D_m S}}{|S|^2} - \frac{D_m S}{S} \wedge \bar\partial\varphi - \partial\varphi \wedge \frac{\overline{D_m S}}{\bar S} + \partial\varphi \wedge \bar\partial\varphi \right) \Big\}
\end{aligned}$$

$$
\begin{aligned}
&= \frac{\|S\|_m^2 e^{\alpha n\varphi}}{(\|S\|_m^2 + \alpha\|D_mS\|_m^2)\tilde{\omega}_m^n}\left\{\tilde{\omega}_m^{n-1} \wedge \left[\left(\tilde{\omega}_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right)\right.\right.\\
&\quad \left. + \frac{\alpha n\sqrt{-1}}{2\pi}\left(1 - \left(m + 1 + \frac{2k}{\log\|S\|_m^2}\right)\varphi\right)\frac{D_mS \wedge \overline{D_mS}}{|S|^2}\right]\\
&\quad \left. + (n-1)\tilde{\omega}_m^{n-2} \wedge \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi \wedge \left(\tilde{\omega}_m + \frac{\alpha n\sqrt{-1}}{2\pi}\frac{D_mS \wedge \overline{D_mS}}{|S|^2}\right)\right\} + O(m+2)\\
&= e^{\alpha n\varphi} + \frac{\|S\|_m^2 e^{\alpha n\varphi}}{(\|S\|_m^2 + \alpha\|D_mS\|_m^2)\tilde{\omega}_m^n}\left\{\left(n\tilde{\omega}_m^{n-1} \wedge \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right) - \right.\\
&\quad \left. - \alpha n\left(m + 1 + \frac{2k}{\log\|S\|_m^2}\right)\varphi\tilde{\omega}_m^{n+1} \wedge \frac{\sqrt{-1}}{2\pi}\frac{D_mS \wedge \overline{D_mS}}{|S|^2}\right]\\
&\quad \left. + \alpha n(n-1)\tilde{\omega}_m^{n-2} \wedge \left(\frac{\sqrt{-1}}{2\pi}\frac{D_mS \wedge \overline{D_mS}}{|S|^2}\right)\right\} + O(m+2)\\
&= 1 + \alpha n\varphi - \left(m + 1 + \frac{2k}{\log\|S\|_m^2}\right)\varphi + \frac{\|S\|_m^2}{(\|S\|_m^2 + \alpha\|D_mS\|_m^2)\tilde{\omega}_m^n}\\
&\quad \left\{\alpha n(n-1)\tilde{\omega}_m^{n-2} \wedge \left(\frac{\sqrt{-1}}{2\pi}\sum_{i+j=m+1} S^i\bar{S}^j D_m\bar{D}_m\theta_{ij} + \bar{S}^i S^j \overline{D\bar{D}\theta_{ij}}\right.\right.\\
&\quad \left. + S^i D_m\bar{D}_m\bar{S}^j\theta_{ij} + D_m\bar{D}_m\bar{S}^i S^j\bar{\theta}_{ij}\right)\\
&\quad \wedge \frac{\sqrt{-1}}{2\pi}\frac{D_mS \wedge \overline{D_mS}}{|S|^2} + n(-\log\|S\|_m^2)^k\\
&\quad \times \sum_{i+j=m+1}(ij(S^i\bar{S}^j\theta_{ij} + \bar{S}^i S^j\bar{\theta}_{ij})\tilde{\omega}_m^{n-1} \wedge \frac{\sqrt{-1}}{2\pi}\frac{D_mS \wedge \overline{D_mS}}{|S|^2}\Bigg)\\
&\quad \left. + \frac{nk\varphi}{\log\|S\|_m^2}\left(m + 1 + \frac{k-1}{\log\|S\|_m^2}\right)\right\} + O(m+2)\\
&= 1 + \alpha n\varphi - \left(m + 1 + \frac{2k}{\log\|S\|_m^2}\right)\varphi + \sum_{i+j=m+1}(S^i\bar{S}^j\Box_m\theta_{ij} + \bar{S}^i S^j\overline{\Box_m\theta_{ij}})\\
&\quad - \sum_{i+j=m+1}(S^i\bar{S}^j\theta_{ij} + \bar{S}^i S^j\bar{\theta}_{ij})\left((n-1)j - \frac{ij}{\alpha}\right)(-\log\|S\|_m^2)^k\\
&\quad + \frac{k\varphi}{\alpha\log\|S\|_m^2}\left(m + 1 + \frac{k-1}{\log\|S\|_m^2}\right) + O(m+2)\,.
\end{aligned}
$$

Then the lemma follows.

Now we apply this lemma to the construction of $\|\cdot\|_{m+1}$, that is, finding a function φ such that $\|\cdot\|_{m+1}^2 = e^{-\varphi}\|\cdot\|_m^2$ satisfies (i), (ii) given above. The condition (i) will be automatically true as long as φ is constant along D which is always fulfilled in our choice of φ through Lemma 1.2. Therefore, it suffices to eliminate the terms $\sum_{\ell=0}^{\ell_{m+1}} u_{m+1\ell}(-\log \|S\|_m^2)^\ell$ from f_m in (2.10). It can be done successively as follows.

Let f_m be given by (2.10). Write

$$u_{m+1,\ell_{m+1}} = \sum_{i+j=m+1} S^i\bar{S}^j(v_{ij} + v'_{ij}) + \bar{S}^i S^j(\bar{v}_{ij} + \bar{v}'_{ij}) \tag{2.13}$$

where $v_{ij}|_D$ are perpendicular to $\ker((n-1)\Box_m + \frac{ij}{\alpha} - m - 2 - j(n-1))$ and $v'_{ij}|_D$ are in the above kernel.

If there are some i, j with $i + j = m + 1$ such that $v'_{ij}|_D \neq 0$, applying Lemma 2.2 with $k = \ell_{m+1} + 1$ and $\theta_{ij} = k^{-1}\left(\frac{m}{\alpha} - 2\right)^{-1} v'_{ij}$, we have

$$f_\varphi = \sum_{i+j=m+1} S^i\bar{S}^j v_{ij} + \bar{S}^i S^j \bar{v}_{ij} + \text{lower order terms} .$$

Now one can solve the equations for $\theta_{ij} \in \Gamma(D, L_D^{-i} \otimes \bar{L}_D^{-j})$.

$$\Box_m\theta_{ij} + \left(\frac{ij}{\alpha} - m - 2 - j(n-1)\right)\theta_{ij} = v_{ij}|_D \quad \text{on } D . \tag{2.14}$$

Extend θ_{ij} to $\bar{M}$, then we apply Lemma 2.2 with $k = \ell_{m+1}$ and θ_{ij} given above and conclude f_φ is of order $\|S\|_m^{m+1}(-\log \|S\|_m^2)^{\ell_m - 1}$. Replace f_m in (2.10) by this f_φ and repeat the above process. After finite steps, we eventually eliminate $\sum_{\ell=0}^{\ell_{m+1}} u_{m+1,\ell}(-\log \|S\|_m^2)^\ell$ from f_m. Let φ_m be the sum of those φ in Lemma 2.2 in the above finite steps. Define $\|\cdot\|_{m+1}^2 = e^{-\varphi_m}\|\cdot\|_m^2$. Then the hermitian metric $\|\cdot\|_{m+1}$ satisfies (i), (ii) as we want.

Let ω_m be the (1.1)-form on M defined by (2.2) with $\|\cdot\|_\varphi$ replaced by $\|\cdot\|_m$. Then for $\delta_n > 0$ small, ω_m is positive definite in $V_m = \{\|S(x)\| \leq \delta_n\}$ and defines a Kähler metric g_m on the manifold V_m with the associated Kähler form ω_m.

Lemma 2.3 *The Kähler manifolds with boundary $(V_m, \partial V_m, g_m)$ are all complete, equivalent to each other near D and have euclidean volume growth. Furthermore, for each m, the function $\|S\|_m^{-\alpha}$ is equivalent to any distance function from a fixed point in V_m near D.*

Proof. Fix $m > 0$. Put $\psi = \|S\|_m^{-\infty}$. Then

$$|\nabla_m\psi|_{g_m}^2 = \frac{\sqrt{-1}}{2\pi}\frac{\partial\psi \wedge \bar{\partial}\psi \wedge \omega_m^{n-1}}{\omega_m^n}$$

where ∇_m denotes the gradient with respect to g_m. By (2.3) and (2.4) with ω_φ replaced by ω_m, we have

$$|\nabla_m\psi|_{g_m}^2 = \frac{\sqrt{-1}}{2\pi}\frac{D_mS \wedge \overline{D_mS} \wedge \omega_m^{n-1}}{(\|S\|_m^2 + \alpha\|D_mS\|_m^2)\tilde{\omega}_m^n}\alpha^{-n+2}\|S\|_m^{2\alpha(n-1)}$$

$$= \frac{\sqrt{-1}}{2\pi}\alpha\frac{D_mS \wedge \overline{D_mS} \wedge \omega_m^{n-1}}{(\|S\|_m^2 + \alpha\|D_mS\|_m^2)\tilde{\omega}_m^n} = \frac{\alpha\|D_mS\|_m^2}{n(\|S\|_m^2 + \alpha\|D_mS\|_m^2)} .$$

Since $\|D_m S\|_m$ is nonvanishing near D. $|\nabla_m \psi|^2(x)$ converge to $\frac{1}{n}$ as x approaches to D. Therefore, ψ is equivalent to the distance function from the boundary ∂V_m near D. In particular, it implies that each $(V_m, \partial V_m, g_m)$ is complete, since ψ goes to infinity near D. To estimate volume growth of $(V_m, \partial V_m, g_m)$, we first remark that ω_m^n is equivalent to $\|S\|_m^{-2\alpha n-2}\tilde{\omega}_m^n$ is the same as:

$$\int_{\|S\|_m(x)\leqq \Gamma^{\frac{1}{\alpha}}} \|S\|_m^{-2\alpha n-2}\tilde{\omega}_m^n$$

and is of order Γ^{2n}. Therefore, $(V_m, \partial V_m, g_m)$ has the euclidean volume growth.

The equivalence of these metrics g_m near D follows from (2.3).

Next, we compute the curvature tensors of these metrics g_m near D.

Lemma 2.4 *Let $(V_m, \partial V_m, g_m)$ be a complete Kähler manifold with boundary defined as above. Denote by $R(g_m)$ the curvature tensor of the metric g_m. Then the norm of $R(g_m)$ with respect to g_m decays at the order at least $\|S\|^{2\alpha}$ near D, moreover, the integral $\int_{V_m}|R(g_m)|_{g_m}^n\omega_m^n$ is finite if and only if D is biholomorphic to CP^{n-1} and g_D is the $\frac{1}{\alpha}$-multiple of the standard Fubini-Study metric on CP^{n-1}, where g_D is the Kähler metric with Kähler form ω_D, and the Kähler form of the Fubini-Study metric is given by $\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\log(\sum_{j=0}^{n-1}|w_j|^2)$ in homogeneous coordinates.*

In fact, we have the following expansion of $R(g_m)$ along D. There is a finite covering $\{U_t\}$ of $\underline{D}$ in $\tilde{M}$ satisfying: for each t, there is a local uniformization $(\tilde{U}_t, \pi_t)$ of M_t with $\pi_t\colon \tilde{U}_t \to U_t$ such that $\pi_t^{-1}(D)$ is smooth in $\tilde{U}_t$, and for some local coordinate system $(z_1, \ldots, z_n)$ in $\tilde{U}_t$ with $z_n = S$ and $z' = (z_1, \ldots, z_{n-1})$ tangent to D along D, one has

$$\begin{aligned}&\sum_{i,j,k,\ell=1}^{n} R(\pi_t^* g_m)_{i\bar{j}k\bar{\ell}}(z', z_n)\xi^i\bar{\xi}^j\xi^k\bar{\xi}^\ell\\&= \alpha\|S\|^{-2\alpha}(\pi_t(z', z_n))\sum_{i,j,k,\ell=1}^{n-1}(R(\pi_t^* g_D|_{\pi_t^{-1}(D)})_{i\bar{j}k\bar{\ell}}\\&\quad - \alpha(h_{i\bar{j}}h_{k\bar{\ell}} + h_{i\bar{\ell}}h_{k\bar{j}}))(z', 0)\cdot\xi^i\bar{\xi}^j\xi^k\bar{\xi}^\ell + O(\|S\|^{2\alpha+1}(\pi_t(z', z_n)) \qquad (2.15)\end{aligned}$$

for any g_m-unit tangent vector $(\xi^1, \ldots, \xi^n)$, where $(h_{i\bar{j}})$ is the curvature tensor of the hermitian metric $\|\cdot\|_m$ in local coordinates $(z_1, \ldots, z_n)$.

Proof. It suffices to prove (2.15). Without losing generality, we may assume that $U_t \cap \tilde{M}$ is smooth. Given any point x in $U_t \cap M$, choose coordinates $(z_1, \ldots, z_n)$ such that z_n is the local representation of S in U_t and $(z_1, \ldots, z_{n-1}) = z'$ is tangent to D along D satisfying:

$$\begin{aligned}&h_{i\bar{j}}(z_1(x), \ldots, z_{n-1}(x), 0) = \delta_{ij} \quad \text{for } i, j \leqq n-1\\&\frac{\partial h_{i\bar{j}}}{\partial z_k}(z_1(x), \ldots, z_{n-1}(x), 0) = 0 \quad \text{for } i, j, k \leqq n-1 \qquad (2.16)\end{aligned}$$

where $x = (z_1(x), \ldots, z_n(x))$.

We may also assume that $\|\cdot\|_m$ is represented by a positive function a in U_t such that $a(x)=1$, $da(x)=0$, $\dfrac{\partial^2 a}{\partial z_j \partial z_j}(x)=0$ for $i,j \leqq n$. Then one obtains by computations

$$g_m^{i\bar{j}}(x) = \begin{cases} \alpha^{-1}|z_n|^{2\alpha}(1+O(|z_n|)) & \text{if } i,j=n \\ \alpha^{-2}|z_n|^{2\alpha+2}(1+O(|z_n|)) & \text{if } i=j=n \\ O(|z_n|^{2\alpha+1}) & \text{if } i,j<n,\ i\neq j \\ O(|z_n|^{2\alpha+2}) & \text{if } i \text{ or } j=n,\ i\neq j \end{cases} \tag{2.17}$$

$$\frac{\partial g_{mi\bar{j}}}{\partial z_k}(x) = \alpha|z_n|^{-2\alpha}\left(\frac{\partial h_{i\bar{j}}}{\partial z_k} - \frac{\alpha}{z_n}(\delta_{ni}h_{k\bar{j}} + \delta_{nk}h_{i\bar{j}}) - \frac{\alpha(1+\alpha)\delta_{ni}\delta_{nk}\delta_{nj}}{|z_n|^2 z_n}\right) \tag{2.18}$$

$$\begin{aligned} \frac{\partial^2 g_{mi\bar{j}}}{\partial z_k \partial \bar{z}_\ell}(x) = \alpha|z_n|^{-2\alpha}\cdot &\left\{\left[\frac{\partial^2 h_{i\bar{j}}}{\partial z_k \partial \bar{z}_\ell} + \alpha(h_{i\bar{j}}h_{k\bar{\ell}} + h_{i\bar{\ell}}h_{k\bar{j}})\right]\right. \\ &- \alpha\bar{z}_n^{-1}\left(\delta_{n\ell}\frac{\partial h_{i\bar{j}}}{\partial z_k} + \delta_{nj}\frac{\partial h_{i\bar{\ell}}}{\partial z_k}\right) - \alpha z_n^{-1}\left(\delta_{ni}\frac{\partial h_{k\bar{j}}}{\partial \bar{z}_\ell} + \delta_{nk}\frac{\partial h_{i\bar{j}}}{\partial \bar{z}_\ell}\right) \\ &+ \frac{\alpha^2}{|z_n|^2}(\delta_{ni}\delta_{nj}h_{k\bar{\ell}} + \delta_{nk}\delta_{nj}h_{i\bar{\ell}} + \delta_{nk}\delta_{n\ell}h_{i\bar{j}} + \delta_{ni}\delta_{n\ell}h_{k\bar{j}}) \\ &\left. + \alpha(\alpha+1)^2\frac{\delta_{ni}\delta_{nj}\delta_{nk}\delta_{n\ell}}{|z_n|^4}\right\}. \end{aligned} \tag{2.19}$$

Given any g_m-unit tangent vector $(\xi', \ldots, \xi^n)$ at x, one derives

$$\begin{aligned} |\xi^i| &\leqq C|z_n|^\alpha(x),\ i=1,2,\ldots,n-1 \\ |\xi^n| &\leqq \frac{1}{\alpha}|z_n|^{\alpha+1}(x), \end{aligned} \tag{2.20}$$

where C is a uniform constant independent of $(\xi^1, \ldots, \xi^n)$ and x near D. Now using (2.16)–(2.20), one has

$$\begin{aligned} R(g_m)_{i\bar{j}k\bar{\ell}}(x)\xi^i\bar{\xi}^j\xi^k\bar{\xi}^\ell &= \left(-\frac{\partial^2 g_{mi\bar{j}}}{\partial z_k \partial \bar{z}_\ell}(x) + \sum_{u,v=1}^n g_m^{u\bar{v}}(x)\frac{\partial g_{mi\bar{v}}}{\partial z_k}(x)\frac{\partial g_{mu\bar{j}}}{\partial \bar{z}_\ell}(x)\right)\xi^i\bar{\xi}^j\xi^k\bar{\xi}^\ell \\ &= O(|z_n|^{2\alpha+1}(x)) - \alpha|z_n|^{-2\alpha}(x) \\ &\quad \times\left(\frac{\partial^2 h_{i\bar{j}}}{\partial z_k \partial \bar{z}_j} + \alpha(h_{i\bar{j}}h_{k\bar{\ell}} + h_{i\bar{\ell}}h_{k\bar{j}})\right)(x)\xi^i\bar{\xi}^j\xi^k\bar{\xi}^\ell \\ &\quad -4\alpha^3|z_n|^{-2\alpha-2}(x)|\xi^n|^2 h_{i\bar{j}}(x)\xi^i\bar{\xi}^j - \alpha^2(\alpha+1)^2|z_n|^{-2\alpha-4}|\xi^n|^4 \\ &\quad + \alpha^4|z_n|^{-4\alpha-2}(x)|\xi^n|^2\sum_{u,v=1}^n g_m^{u\bar{v}}(x)\left(2\xi^i h_{i\bar{v}}(x)\right. \\ &\quad \left.+ \frac{(1+\alpha)\xi^n\delta_{nv}}{|z_n|^2(x)}\right)\cdot \\ &\quad \cdot\left(2\bar{\xi}^j h_{u\bar{j}}(x) + \frac{(1+\alpha)\bar{\xi}^n\delta_{nu}}{|x_n|^2(x)}\right) \end{aligned}$$

$$
\begin{aligned}
&= O(|z_n|^{2\alpha+1}(x)) - \alpha|z_n|^{-2\alpha}(x) \\
&\times\left(\frac{\partial^2 h_{i\bar{j}}}{\partial z_k\partial \bar{z}_\ell} + \alpha(h_{i\bar{j}}h_{k\bar{\ell}} + h_{i\bar{\ell}}h_{k\bar{j}})\right)(x)\xi^i\bar{\xi}^j\xi^k\bar{\xi}^\ell \\
&- 4\alpha^3|z_n|^{-2\alpha-2}(x)|\xi^n|^2\sum_{i=1}^{n-1}|\xi^1|^2 - \alpha^2(\alpha+1)^2|z_n|^{-2\alpha-4}|\xi^n|^4 + \\
&+ \alpha^4|z_n|^{-4\alpha-2}(x)|\xi^n|^2\left(4\sum_{i=1}^{n-1}\left(\frac{|z_n|^{2\alpha}(x)}{\alpha}|\xi^i|^2\right)\right. \\
&\left.+\frac{(1+\alpha)^2|\xi^n|^2}{\alpha^2|z_n|^2}\right) \\
&= O(|z_n|^{2\alpha+1}(x)) - \alpha|z_n|^{-2\alpha}(x) \\
&\times\left(\frac{\partial^2 h_{i\bar{j}}}{\partial z_k\partial \bar{z}_\ell}(x) + \alpha(h_{i\bar{j}}h_{k\bar{\ell}} + h_{i\bar{\ell}}h_{k\bar{j}})\right)(x)\xi^i\bar{\xi}^j\xi^k\bar{\xi}^\ell .
\end{aligned}
$$

Then (2.15) follows easily from it.

Corollary 2.1 *The norm of the curvature tensor $R(g_m)$ with respect to g_m decays exactly at the order ρ_m^{-2} near D unless D is biholomorphic to CP^{n-1} and g_D is the $\frac{1}{\alpha}$-multiple of the Fubini-Study metric on CP^{n-1}, where ρ_m is the distance function from a fixed point in V_m with respect to g_m. In the later case, $\|R(g_m)\|_{g_m}$ decays at the order at least ρ_m^{-3}.*

Proof. It follows from Lemma 2.3 and 2.4.

Next, we study the asymptotic behavior of the covariant derivatives of $R(g_m)$ near D.

Lemma 2.5 *Let $(V_m, \partial V_m, g_m)$ be the complete Kähler manifold as in Lemma 2.4. Then*

$$\|\nabla^k R(g_m)\|_{g_m}(x) = O(\rho_m(x)^{-(k+2)}) \tag{2.21}$$

or equivalently

$$\|\nabla_m^k R(g_m)\|_{g_m}(x) = O(\|S\|^{(k+2)\alpha}(x)) \tag{2.22}$$

where ∇_m denotes the covariant derivative with respect to g_m.

Proof. We will sketch a proof of this lemma in the different spirit from that of Lemma 2.4. This proof will be simpler, but less informative than (2.15).

Fix an m. Choose $\delta > 0$ such that $V_\delta = \{x \mid \|S\|(x) < \delta\}$ is contained in V_m. Clearly, it suffices to show (2.21) for those x in V_δ. Since the hermitian metric $\|\cdot\|$ of L_D is smoothly defined on $\bar{M}$, the admissibility of D implies that the total space of the unit sphere bundle of $L_D|_D$ with respect to $\|\cdot\|$ is a smooth manifold of real dimension $2n+1$. We denote it by M_1. Furthermore, since L_D is just the normal bundle of D in $\bar{M}$, there is a diffeomorphism Ψ from $M_1 \times (0, \delta)$ induced by the exponential map of $(\bar{M}, h)$ along D with respect to a fixed orbifold metric h.

The Kähler metric g_m is given by its associated form

$$\omega_m = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}(e^{\alpha\varphi_m} \| S \|^{-2\alpha}) . \tag{2.23}$$

Here φ_m is a function of form $\sum_{k=1}^{N_m} \sum_{\ell=0}^{\ell_k} u_{k\ell}(-\log \| S \|^2)^\ell$ with $u_{k\ell}$ being smooth in $\bar{M}$ and of order $O(\| S \|^k)$ near D. Therefore, the pull-back metric $\Psi^* g_m$ on $M_1 \times (0, \delta)$ is of the form

$$\Psi^* g_m = \| S \|^{-2\alpha} h(\| S \|, \| S \| \log \| S \|) + \| S \|^{-\alpha} d \| S \|^{-\alpha} U(\| S \|, \| S \| \log \| S \|)$$
$$+ U(\| S \|, \| S \| \log \| S \|)(d \| S \|^{-\alpha})^3$$

where $H(t_1, t_2)$, $v(t_1, t_2)$, $u(t_1, t_2)$ are C^∞-smooth families of metrics, 1-tensors, functions on M_1, respectively. They also satisfy: for any integer $\ell > 0$, there is a uniform constant C_ℓ such that all up to order ℓ covariant derivatives of h, v, u with respect to a fixed metric $\tilde{h}$ in M_1 are bounded by C_ℓ for $0 < t_1 < \delta$, $0 < t_2 < \delta \log \delta$.

Writing Γ for $\| S \|^{-\alpha}$, we have

$$\Psi^* g_m = \Gamma^2 h(\Gamma^{-\frac{1}{\alpha}}, \Gamma^{-\frac{1}{\alpha}} \log \Gamma^{-\frac{1}{\alpha}}) + \Gamma d\Gamma v(\Gamma^{-\frac{1}{\alpha}}, \Gamma^{-\frac{1}{\alpha}} \log \Gamma^{-\frac{1}{\alpha}})$$
$$+ u(\Gamma^{-\frac{1}{\alpha}}, \Gamma^{-\frac{1}{\alpha}} \log \Gamma^{-\frac{1}{\alpha}}) d\Gamma^2 \tag{2.24}$$

where $\delta^{-\alpha} < \Gamma < +\infty$. So we may regard $\Psi^* g_m$ as a metric defined on $M_1 \times (\delta^{-\alpha}, \infty)$.

For any fixed x in V_δ, $\Psi^{-1}(x)$ is in $M_1 \times (\delta^{-\alpha}, \infty)$. Put $\Gamma_x = \Gamma(\Psi^{-1}(x)) = \| S \|^{-\alpha}(x)$. By Lemma 2.3, this Γ_x is just the distance $\rho_m(x)$ of x from a fixed point in V_m with respect to g_m. Therefore, (2.21) is equivalent to the following

$$\| \tilde{\nabla}^k R(\Gamma_x^{-2} \bar{\Psi}^* g_m) \|_{\Gamma_x^{-2} \Psi^* g_m}(x) = O_k(1) \tag{2.25}$$

where $O_k(1)$ denotes a quantity bounded by a constant depending only on k, and $\tilde{\nabla}$ is the covariant derivative of $\Gamma_x^{-2} \Psi^* g_m$.

On the other hand, (2.25) follows easily from the expression (2.24) of $\Psi^* g_m$ and the boundedness on the derivatives of h, v, u in (2.24). Hence, the lemma is proved.

To obtain the approximated Kähler metric on M, we first assume for simplicity that the divisor D is ample in $\bar{M}$. Then there is a hermitian orbifold metric $\| \cdot \|'$ on L_D with curvature form $\tilde{\omega}' > 0$ on $\bar{M}$. Define

$$\omega_g = \omega_3 + C_\varepsilon \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}(-(\| S \|')^{2\varepsilon}),\ \varepsilon > 0,\ C_\varepsilon > 0 \tag{2.26}$$

then by some direct computations, one can easily prove that ω_g is positive definite on M. So ω_g gives a Kähler metric g. In general, we assume that D is neat and almost ample in $\bar{M}$. There is a hermitian metric $\| \cdot \|'$ on L_D with its curvature from $\tilde{\omega}' \geqq 0$. By the same arguments as in the proof of Theorem 5.1 in [TY1], one can find a (1.1)-form ω_E with $\omega_E|_D = 0$, $\varepsilon > 0$, $C_\varepsilon > 0$ such that

$$\omega_g = \omega_3 + C_\varepsilon \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}(-(\| S \|')^{2\varepsilon}) + \omega_E > 0 . \tag{2.27}$$

Note that $\omega_E = 0$ in case that D is ample.The metric g with Kähler form ω_g is our approximated metric. For the reader's convenience, we summarize the above discussions by the following proposition.

Proposition 2.1 *Let $\bar{M}$, D be given as in the beginning of this section, $\Omega \in C_1(-K_{\bar{M}} - \beta L_D)$. Put $\alpha = \frac{\beta - 1}{n}$. Then there are sequences of neighborhoods $\{V_m\}_{m \geq 1}$ of D, complete Kähler metrics ω_m on $(V_m \backslash D, \partial(V_m \backslash D))$ defined by (2.4) such that*

$$\mathrm{Ric}(\omega_m) - \Omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} f_m \text{ on } V_m \backslash D \tag{2.28}$$

where f_m are smooth functions on M satisfying: $f_m = O(\| S \|^{m+\frac{1}{2}})$, and $|\nabla^k f_m|_{g_m} = O(\| S \|^{m+\frac{1}{2}+\alpha k})$ for $k \geq 1$. The symbol ∇ denotes the covariant derivative with respect to g_1. If we further assume that D is neat and almost ample in $\bar{M}$, then (2.27) defines a complete Kähler metric g with Kähler form ω_g such that

$$\mathrm{Ric}(\omega_g) - \Omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} f ,$$

where f is smooth on M, $f = O(\| S \|^{2\alpha + 2\varepsilon})$ and $\sup_{1 \leq k \leq 2} |\nabla^k f|_g < \infty$. Moreover, the curvature tensors $R(g_m)$ and $R(g)$ decay at the order $O(\| S \|^{2\alpha})$ and at the order $O(\| S \|^{2\alpha + 2\varepsilon})$ for $\varepsilon \leqq \frac{1}{2}$ or $O(\| S \|^{2\alpha + 1})$ for $\varepsilon \geqq \frac{1}{2}$ iff $D \cong CP^{n-1}$ and L_D is the $\frac{1}{\alpha}$-hyperplane line bundle on CP^{n-1}. Also, the covariant derivatives of the scalar curvatures of g_m, g are bounded.

Proof. We adopt the notations in the proof of Lemma 2.5. The estimate $|\nabla^k f_m|_{g_m}(x) = O(\| S \|^{m+\frac{1}{2}+\alpha k}(x))$ is the same as

$$|\tilde{\nabla}^k \Psi^* f_m|_{\Gamma_x^{-2} \Psi^* g_m}(\Psi^{-1}(x)) = O_k(1) \Gamma_x^{-\frac{1}{\alpha}(m+\frac{1}{2})} . \tag{2.29}$$

This latter one (2.29) follows from (2.1) and (2.25). The estimates on $R(g_m)$ come from Lemma 2.4 and 2.5.

By choosing δ smaller if necessary, we may assume that ω_E vanishes in V_δ. Then

$$\omega_g = \omega_3 + C_\varepsilon \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}(-\| S \|')^{2\varepsilon} \qquad \text{in } V_\delta$$

and ω_g is uniformly equivalent to ω_s in V_δ, i.e., there is a constant $\tilde{C}_\delta$ such that

$$\tilde{C}_\delta^{-1} \omega_3 \leqq \omega_g \leqq \tilde{C}_\delta \omega_3 . \tag{2.30}$$

Also

$$f = f_3 - \log(\omega_g^n / \omega_3^n) . \tag{2.31}$$

Therefore, in order to have the required estimates on f, $|\nabla^k f|_g$ $(k = 1, 2)$, $\| R(g) \|_g$ and $\| \nabla_g R(g) \|_g$, we only need to show for $\ell = 0, 1, \ldots, 5$,

$$|\nabla^\ell (\| S \|')^{2\varepsilon}|_{g_3}(x) = O(\| S \|^{2\varepsilon + \alpha\ell}(x)), \qquad x \in V_\delta \tag{2.32}$$

where ∇_g is the covariant derivative of g.

Denote by θ the function $\Psi^*(\|S\|')^{2\varepsilon}$ defined on $M_1 \times (\delta^{-\frac{1}{\alpha}}, \infty)$. Then (2.32) is equivalent to

$$|\tilde{\nabla}^{\ell}\theta|_{\Gamma_x^{-2}\Psi^* g_3}(\Psi^{-1}(x)) = O(\Gamma_x^{-\frac{2\varepsilon}{\alpha}}) . \tag{2.33}$$

On the other hand, on $M_1 \times (\delta^{-\frac{1}{\alpha}}, \infty)$, the function θ is written of form $e^{\tilde{\theta}}(\cdot, \Gamma^{-\frac{1}{\alpha}})\Gamma^{-\frac{2\varepsilon}{\alpha}}$, where $\tilde{\theta}(\cdot, \cdot)$ is a C^∞-smooth function in $M_1 \times (S^{-\frac{1}{\alpha}}, \infty)$ with all its derivatives bounded in terms of a fixed product metric. As in the proof of Lemma 2.5, put $\tilde{\Gamma} = \Gamma/\Gamma_x$, then $\theta = e^{\tilde{\theta}(\cdot, \Gamma_x^{-\frac{1}{\alpha}})\tilde{\Gamma}^{-\frac{1}{\alpha}})}\Gamma_x^{-2\frac{\varepsilon}{\alpha}}\tilde{\Gamma}^{-\frac{2\varepsilon}{\alpha}}$, so (2.33) follows from the boundedness of the curvature tensor of $\gamma_x^{-2}\Psi^* g_m$ near $\Psi^{-1}(x)$.

Remark. By formula (2.6) which defines f_m, f, one can derive the following equations on $V_m \backslash D$ for $m \geqq 1$.

$$e^{f_m}\omega_m^n = e^f \omega_g^n . \tag{2.34}$$

3 Sobolev inequalities

In this section, we derive Sobolev inequalities on complete Kähler manifolds (M, g) described in previous sections. Precisely, we prove

Proposition 3.1 *Let $M = \bar{M} \backslash D$, where $\bar{M}$ is a Kähler orbifold of complex dimension $n \geqq 2$ and D is a neat, almost ample, admissible divisor in $\bar{M}$ (cf. Sect. 1). Let g be the Kähler metric on M given by Proposition 2.1. Then there is a constant $C > 0$ such that for any smooth function h in $C^\infty(M, R)$ with compact support, one has*

$$\left(\int_M |h|^{\frac{2n}{n-1}} dV_g \right)^{\frac{n-1}{n}} \leqq C \int_M |\nabla h|^2 dV_g \tag{3.1}$$

where ∇h denotes the gradient of h with respect to g.

We would like to point out that the Sobolev inequality (3.1) seems to be unknown in general for complete Riemannian manifolds with bounded curvature and euclidean volume growth. Our proof here for Proposition 3.1 strongly uses the asymptotic cone structure of (M, g) and cannot be applied to the general case. Such a Sobolev inequality is one of crucial difficult parts in the proof of the existence of g_Ω in Theorem 1.1. By the way, in [BK], the authors neither showed the validity of Sobolev inequality nor studied the asymptotic decay of the curvature tensors for those Kähler manifolds they required. They took the Sobolev inequality for granted in the process of their proof of the main theorem.

The rest of this section is devoted to the proof of Proposition 3.1.

First we study the asymptotic properties of the complete Kähler manifold (M, g). Let S be the defining section of D and $\|\cdot\|$ be the hermitian orbifold metric on L_D as in (2.5) and (2.6). Define for $\delta > 0$,

$$V_\delta = \{x \in M \mid \|S\|(x) \leqq \delta\} \tag{3.2}$$

then $(V_\delta, \partial V_\delta, g|_{V_\delta})$ is a complete Kähler manifold with boundary ∂V_δ. By the assumption that D is admissible, one can easily show that ∂V_δ is smooth for δ sufficiently small. The following lemma is elementary.

Lemma 3.1 *The above manifold $(V_\delta, \partial V_\delta)$ is diffeomorphic to $(\tilde{M}_1 \times (0, \delta], \tilde{M}_1 \times \delta)$, where $\tilde{M}_1$ is an orientable riemannian manifold of real dimension $2n - 1$ and is a finite unramified covering of a minimal submanifold M_1 in some unit sphere S^{2k+1}. Moreover, under the diffeomorphism, the metric g is equivalent to the one of form*

$$ds^2 = r^{-2\alpha}(ds_1^2 + r^{-2}dr^2) \tag{3.3}$$

where ds_1^2 is the pull-back of the standard metric on S^{2k+1} under the covering map $\pi: \tilde{M}_1 \to M_1 \subset S^{2k+1}$, and r is the euclidean distance from the origin in R.

Proof. The restriction $L_D|_D$ is just the normal bundle of D in $\bar{M}$. By the admissibility of D, one can easily check that for any hermitian metric $\|\cdot\|'$ on L_D, the unit sphere bundle S_D^1 of $L_D|_D$ with respect to $\|\cdot\|'$ is a smooth manifold. Note that

$$S_D^1 = \{x \in L_D|_D \mid \|x\|' = 1\} . \tag{3.4}$$

We choose the metric $\|\cdot\|'$ as follows. Since D is almost ample, there is a holomorphic map $\psi: \bar{M} \to CP^{k+1}$ for some large k, such that ψ is one-to-one in a neighborhood of D in $\bar{M}$ and $H_{\bar{M}} = mL_D$ for the hyperplane line bundle H on CP^{k+1}. We take the metric $\|\cdot\|'$ to be the restriction of $\frac{1}{m}$-multiple of the standard metric on H. Now we define $\tilde{M}_1$ to be S_D^1 with the chosen metric $\|\cdot\|'$.

Now S^m is a global section of mL_D. Let $CP^k \subset CP^{k+1}$ be the hyperplane defined by this section. It is a well-known fact that the unit sphere bundle $S_H^1(CP^k) = \{x \in H|_{CP^k} \mid \|x\|' = 1\}$ is just S^{2k+1} and the natural bundle projection $p: S^{2k+1} \to CP^k$ is the Hopf map, so $M_1 = p^{-1}(D) \subset S^{2k+1}$ is a minimal submanifold. It is easy to see that $\tilde{M}_1$ is a finitely unramified covering of M_1 of degree m.

The diffeomorphism from $(V_\delta, \partial V_\delta)$ onto $\tilde{M}_1 \times [0, \delta]$ is induced by the exponential map of $(\bar{M}, h)$ along D with respect to some fixed Kähler orbifold metric h on $\bar{M}$. The equivalence between g and the metric ds^2 in (3.4) follows from the definition of g and the standard expansion formula for the exponential map. The lemma is proved.

Let $R_\delta = \{t \in R \mid t \geqq \delta^{-\frac{1}{\alpha}}\}$, we see that the manifold $(\tilde{M}_1 \times (0, \delta), ds^2)$ is equivalent to $(\tilde{M}_1 \times R_\delta, \rho^2 ds_1^2 + d\rho^2)$.

Proposition 3.1 will follow from Lemma 3.1 and the following Proposition 3.2. To see it, we first extend the metric ds^2 on $\hat{M}_1 \times R_\delta \cong V_\delta$ to the whole manifold M, still denoted by ds^2. Then g and ds^2 are uniformly equivalent. Now (M, ds^2) satisfies the assumptions in Proposition 3.2 below, so by taking $f = h^{\frac{2(n-1)}{n-2}}$ in (3.5), we have

$$\left(\int_M |h|^{\frac{2n}{n-2}} dV\right)^{\frac{n-1}{n}} \leqq \frac{2(n-1)}{n-2} C \int_M h^{\frac{n}{n-2}} |\nabla h|_{ds^2} dV .$$

Applying Schwarz inequality to the integral on the right, we obtain

$$\left(\int_M |h|^{\frac{2n}{n-2}} dV\right)^{\frac{n-1}{n}} \leqq \frac{2(n-1)}{n} C \left(\int_M |h|^{\frac{2n}{n-2}} dV\right)^{1/2} \left(\int_M |\nabla h|_{ds^2}^2 dV\right)^{1/2} .$$

It yields

$$\left(\int_M |h|^{\frac{2n}{n-2}} dV\right)^{\frac{n-2}{n}} \leqq \frac{4(n-1)^2}{n^2} C^2 \int_M |\nabla h|^2_{ds^2} dV .$$

Since both g and ds^2 are uniformly equivalent in M, the inequality (3.1) follows from the above. Note that C in (3.1) may be different from that in (3.5).

Proposition 3.2 *Let (X, ds^2) be a complete riemannian manifold of real dimension n, $U \subset X$ be compact subset such that $(X\backslash U, ds^2)$ is equal to $(\tilde{Y} \times R_1, \rho^2 ds_1^2 + d\rho^2)$, where $\tilde{Y}$ is a finite covering of a compact minimal submanifold Y in S^k and ρ is the euclidean distance on R_+ and ds_1^2 is any riemannian metric on $\tilde{Y}$. Then there is a constant C depending only on U, $\tilde{Y}$, Y and ds_1^2 on $\tilde{Y}$ such that for any smooth function f with compact support, we have*

$$\left(\int_M |f|^{\frac{n}{n-1}} dV\right)^{\frac{n-1}{n}} \leqq C \int_M |\nabla f|_{ds^2} dV . \tag{3.5}$$

The rest of this section is devoted to the proof of this section. Without losing generality, we may assume that ds_1^2 is the pull-back metric of the standard one on S^k under the covering map π: $\tilde{Y} \to Y \subset S^k$. The map π extends to a map, still denoted by π, from the cone $\tilde{Y} \times R$ onto the minimal cone $Y \times R_+$ in R^{k+1} such that $\pi|_{\tilde{Y} \times R_+}$ is a finite covering. Moreover, by the previous choice of ds_1^2, we have

$$\rho^2 ds_1^2 + d\rho^2 = \pi^*(ds_e^2) \tag{3.6}$$

where ds_e^2 is the euclidean metric on R^{k+1}. Thus, the following lemma is essentially due to L. Simon, etc. (cf. [Si]).

Lemma 3.2 *There is a constant $C_1 = C_1(n, m)$ depending only on the dimension n and the degree $m = \deg(\pi)$, such that for any smooth function f on $\tilde{Y} \times R_+$ with compact support in $\tilde{Y} \times R_+$, we have*

$$\left(\int_{\tilde{Y} \times R_+} |f|^{\frac{n}{n-1}} dV\right)^{\frac{n-1}{n}} \leqq C_1 \int_{\tilde{Y} \times R_+} |\nabla f| dV \tag{3.7}$$

where dV, ∇ are the volume form and the gradient of the metric $\rho^2 ds^2 + d\rho^2$.

Proof. We may assume that f is nonnegative. Define $\pi_* f$ on $Y \times R_+$ by

$$\pi_* f(x) = \sum_{y \in \pi^{-1}(x)} f(y) . \tag{3.8}$$

Then $\pi_* f$ is a smooth function on $Y \times R_+$. Since $Y \times R_+$ is a minimal cone in R^{k+1}, we have, by [Si], the following

$$\left(\int_{Y \times R_+} |\pi_* f|^{\frac{n}{n-1}} \pi_*(dV)\right)^{\frac{n-1}{n}} \leqq C' \int_{Y \times R_+} |\nabla f| \pi_* dV \tag{3.9}$$

where C' is a constant depending only on n. Now (3.7) follows from (3.8) with $C_1 = mC'$.

It is now well known (cf. [Y2, Si], etc.) that the Sobolev inequality in Proposition 3.2 is equivalent to the following Isoperimetric inequality: for any compact smooth hypersurface $\partial\Omega$ in X bounding a domain Ω,

$$(\mathrm{Vol}_{ds^2}(\Omega))^{\frac{n-1}{n}} \leqq C\,\mathrm{Vol}_{ds^2}(\partial\Omega) \tag{3.10}$$

where the constant C is the same as that in (3.5).

Lemma 3.2 says that if $\Omega \subset X\backslash U$, then (3.10) holds for $C = C_1$ independent of Ω. Let B_r be the domain bounded by the compact hypersurface $\tilde{Y}\times\{r\} \subset \tilde{Y}\times R_1$ for $r \geqq 2$. Then $U \subset\subset B_2$. By the choice of ds^2, the function ρ is a convex on in $X\backslash U$. The boundary ∂B_r is defined by $P = \Gamma$, so is convex for $r \geqq 2$. Put

$$a = \max_{2\leqq r\leqq 3} \{\mathrm{Vol}_{ds^2}(B_r)\}, \quad b = \max_{2\leqq r\leqq 3} \{\mathrm{Vol}_{ds^2}(\partial B_r)\}\,. \tag{3.11}$$

By Sard's theorem, for almost all $r > 2$, the intersection $\Omega \cap \partial B_r$ is a union of smooth connected domains. In particular, $\partial(\Omega \cap \partial B_r)$ is homologous to zero in ∂B_r. Let r be any one of those values. Then there exists an area minimizing two-sided hypersurface Σ_r in B_r with boundary $\partial(\Omega \cap \partial B_r)$ (cf. [Fed]). Note that the convexity of ∂B_r gives a barrier such that Σ_r lies inside B_r. This Σ_r may not be smooth everywhere if $n \geqq 7$, but $\mathrm{Sing}(\Sigma_r)$ has Hausdorff codimension $\geqq 6$. By area-minimality of Σ_r, we have

$$\mathrm{Vol}_{ds^2}(\Sigma_r) \leqq \mathrm{Vol}_{ds^2}(\partial\Omega \cap B_r)\,. \tag{3.12}$$

Now we can finish the proof of Proposition 3.2. First we assume that $\mathrm{Vol}_{ds^2}(\Omega) \geqq 3a + (3C_1 b)^{\frac{n}{n-1}}$, where C_1 is given in Lemma 3.2. Choose r between 2 and 3 such that $\Omega \cap B_r$ is smooth. Let Ω_1 be the domain enclosed by $\Omega \cap \partial B_r$ and $\partial\Omega \cap (X\backslash B_r)$. Then $\Omega_1 \subset\subset X\backslash U$, and

$$\mathrm{Vol}_{ds^2}(\Omega_1) \geqq \frac{2}{3}\mathrm{Vol}(\Omega)$$

$$\mathrm{Vol}_{ds^2}(\partial\Omega_1) \leqq \mathrm{Vol}_{ds^2}(\partial\Omega) + b\,.$$

By Lemma 3.2, we have

$$\begin{aligned} \frac{2}{3}\mathrm{Vol}_{ds^2}(\Omega)^{\frac{n-1}{n}} \leqq \mathrm{Vol}_{ds^2}(\Omega_1)^{\frac{n-1}{n}} &\leqq C_1\,\mathrm{Vol}_{ds^2}(\partial\Omega_1) \\ &\leqq C_1\,\mathrm{Vol}_{ds^2}(\partial\Omega) + C_1 b \\ &\leqq C_1\,\mathrm{Vol}_{ds^2}(\partial\Omega) + \frac{1}{3}\mathrm{Vol}_{ds^2}(\Omega)^{\frac{n-1}{n}}\,. \end{aligned} \tag{3.13}$$

Hence Proposition 3.2 is proved under the assumption $\mathrm{Vol}_{ds^2}(\Omega) \geqq 3a + (3C_1 b)^{\frac{n}{n-1}}$. From now on, we assume the reverse inequality. By scaling, we assume also $3a + (3C_1 b)^{\frac{n}{n-1}} = 1$. Choose $r_1 > 0$ with $\mathrm{Vol}_{ds^2}(B_{r_1}) \geqq 1$. Fix a $r_0 > r_1 > 0$, which will be determined later. The hypersurface ∂B_{r_0} cuts Ω into two domains Ω_1 and Ω_2 where we still denote $\Omega \cap B_{r_0}$ by Ω_1. The distance function ρ on

R_+ is also defined on X and has the property $|\nabla\rho| = 1$ with respect to the metric ds^2. By the Co-area formula (cf. [Si, Fed]), there is an r between r_0 and $r_0 + 2$ such that $\partial B_r \cap \Omega$ is a smooth domain and

$$\mathrm{Vol}_{ds^2}(\partial B_r \cap \Omega) \leqq \mathrm{Vol}_{ds^2}(\Omega_2) . \tag{3.14}$$

Let Ω'_2 be the domain bounded by $B_r \cap \partial\Omega$ and $\partial B_r \cap \Omega$, then $\Omega_2 \subset \Omega'_2$. Let C_2 be the isoperimetric constant for domains which are subsets of B_{r_0+2} (note that C_2 depends on r_0). Then by (3.14)

$$\begin{aligned}\mathrm{Vol}_{ds^2}(\Omega_1) \leqq \mathrm{Vol}_{ds^2}(\Omega_1)^{\frac{n-1}{n}} &\leqq (\mathrm{Vol}_{ds^2}(\Omega'_2))^{\frac{n-1}{n}} \leqq C_2 \mathrm{Vol}_{ds^2}(\partial\Omega'_2) \\ &\leqq C_2 \mathrm{Vol}_{ds^2}(\partial\Omega) + C_2 \mathrm{Vol}_{ds^2}(\Omega_2) .\end{aligned} \tag{3.15}$$

If $\mathrm{Vol}_{ds^2}(\Omega_2) \leqq \frac{1}{2 + 3C_2} \mathrm{Vol}_{ds^2}(\Omega)$, then

$$(\mathrm{Vol}_{ds^2}(\Omega))^{\frac{n-1}{n}} \leqq 4C_2 \mathrm{Vol}_{ds^2}(\partial\Omega) . \tag{3.16}$$

Thus we may assume that $\mathrm{Vol}_{ds^2}(\Omega_2) \geqq \frac{1}{2 + 3C_2} \mathrm{Vol}_{ds^2}(\Omega)$ and $\mathrm{Vol}_{ds^2}(\partial\Omega) \leqq \mathrm{Vol}_{ds^2}(\Omega)^{\frac{n-1}{n}} \leqq 1$.

Let Ω' be the domain enclosed by Σ_r and $\partial\Omega \cap (X \backslash B_r)$.

Lemma 3.3 $\Sigma_r \cap B_{r_{0/2}} = \phi$ *if* r_0 *is sufficiently large.*

Proof. By (3.12) and the fact that $\mathrm{Vol}_{ds^2}(\partial\Omega) \leqq 1$, $\mathrm{Vol}_{ds^2}(\Sigma_r) \leqq 1$. If $\Sigma_r \cap B_{r_{0/2}} \neq \phi$, then there is an $x_0 \in \Sigma_r \cap \partial B_{r_{0/2}}$. Let $B_\ell(x_0, ds^2)$ be the geodesic ball with center at x_0 and radius $\ell > 0$, assume $\ell \ll r_0$. Note that the curvature tensor of ds^2 is bounded by $\frac{C_3}{r_0^2}$ in $B_\ell(x_0, ds^2)$. Then by the same arguments as in those for monotonicity formula [Si], one can prove the Monotonicity formula,

$$e^{\varepsilon(r_0)\ell^2} \frac{\mathrm{Vol}_{ds^2}(\Sigma_r \cap B_\ell(x_0, ds^2))}{\ell^{n-1}} \geqq e^{\varepsilon(r_0)\ell'^2} \frac{\mathrm{Vol}_{ds^2}(\Sigma_r \cap B_{\ell'}(x_0, ds^2))}{(\ell')^{n-1}} \tag{3.17}$$

for $\ell' \leqq \ell$, where $\lim_{r_0 \to \infty} \varepsilon(r_0) = 0$. In particular, by taking ℓ' to be zero, we get

$$\mathrm{Vol}_{ds^2}(\Sigma_r \cap B_\ell(x_0, ds^2)) \geqq C_4 e^{-\varepsilon(r_0)\ell^2} \ell^{n-1} \tag{3.18}$$

where C_4 is a constant depending only on the dimension n. Choose ℓ and r_0 such that $r_0 \gg \ell$ and $C_4 e^{-\varepsilon(r_0)\ell^2} \ell^{n-2} \geqq 2$. Then we get a contradiction if $x_0 \in \Sigma_r \cap \partial B_{r_{0/2}}$ exists. The lemma is proved.

Now we choose r_0 such that $r_0 > 2r_1$ and $B_{r_{0/2}} \cap \Sigma_r = \phi$. If $\Omega' \cap B_{r_1} \neq \phi$, then by Lemma 3.3, $B_{r_1} \subset \Omega'$, so $\mathrm{Vol}_{ds^2}(\Omega') \geqq 1$. It follows from (3.13) that

$$\begin{aligned}\mathrm{Vol}_{ds^2}(\Omega)^{\frac{n-1}{n}} \leqq 1 \leqq \mathrm{Vol}_{ds^2}(\Omega')^{\frac{n-1}{n}} &\leqq 3C_1 \mathrm{Vol}_{ds^2}(\partial\Omega') \\ &\leqq 3C_1 \mathrm{Vol}_{ds^2}(\partial\Omega) .\end{aligned} \tag{3.19}$$

Therefore, we may assume that $\Omega' \cap B_{r_1} = \phi$, i.e., $\Omega' \subset\subset X \backslash U$. By Lemma 3.2, we have

$$\mathrm{Vol}_{ds^2}(\Omega)^{\frac{n-1}{n}} \leqq (2 + 3C_2)^{\frac{n-1}{n}} \mathrm{Vol}_{ds^2}(\Omega')^{\frac{n-1}{n}}$$

$$\leqq C_1(2 + 3C_2)^{\frac{n-1}{n}} \mathrm{Vol}_{ds^2}(\partial\Omega) .$$

Put $C = \max\{3C_1, 4C_2, C_1(2 + 3C_2)^{\frac{n-1}{n}} + 1\}$, then the above discussions imply that (3.10) holds for any compact domain Ω in X. Proposition 3.2 is proved.

4 Existence of complete Kähler metrics with prescribed Ricci curvature

In this section, we prove the part of existence of complete Ricci-flat Kähler metrics in our main theorem stated in section one. Let $\bar{M}$ be a Kähler orbifold of complex dimension n, D be a neat, almost ample and admissible divisor on $\bar{M}$, and $M = \bar{M}\backslash D$. Let g be the complete Kähler metric on M constructed in Proposition 2.1. Then there is a smooth function of satisfying

$$\mathrm{Ric}(g) - \Omega = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial} f \tag{4.1}$$

$$f = O(\|S\|^{2\alpha + 2\varepsilon}), \quad \sup_{1 \leqq k \leqq 2} |\nabla^k f|_g < \infty \tag{4.2}$$

where Ω is a (1.1)-form in $C_1(-K_{\bar{M}} - \beta L_D)$ with $\beta > 1$, $\alpha = \frac{\beta - 1}{n}$, and S is the defining section of D in $\bar{M}$, L_D is the line bundle induced by D and $\|\cdot\|$ is a hermitian metric on L_D.

Proposition 4.1 *With $\bar{M}$, D, M, Ω, g, etc. as above. Then there is a unique solution φ of the following complex Monge-Ampère equation*

$$\begin{cases} \left(\omega_g + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi\right)^n = e^f \omega_g^n \text{ on } M \\ \omega_g + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi > 0 \end{cases} \tag{4.3}$$

such that $\varphi(x)$ converges uniformly to zero as x goes to infinity and $\omega_g + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi$ is bounded from below by a positive constant multiple of ω. In particular, it follows that there is a complete Kähler metric with Kähler form $\omega_g + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi$ and its Ricci curvature form being Ω.

We will prove this proposition in the rest of this section. First we note that for any $\delta > 0$, the following perturbed equation is always solvable.

$$\begin{cases} \left(\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right)^n = e^{f+\delta\varphi}\omega_g^n \\ \left(\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi\right) > 0 \qquad \text{on } M \end{cases} \tag{(4.4)$_\delta$}$$

(cf. [CY2], or Lemma 3.2 in [TY1]). Here we have already used the boundedness of the curvature tensor $R(g)$ and covariant derivatives of the scalar curvature of g. Let φ_δ be the unique solution of $(4.4)_\delta$. We want to prove that φ_δ converge to the required solution φ of (4.3) as δ goes to zero.

Lemma 4.1 *For any constants $\delta > 0$, $p \geqq n$, we have*

$$\int_M |\varphi_\delta|^p \omega_g^n < \infty \tag{4.5}$$

where $\rho(x)$ is the distance function from some fixed point x_0 in M with respect to the metric g.

Proof. Let η be a cut-off function defined on R^1, $\eta(t) \equiv 1$ for $t \leqq 1$; $\eta(t) \equiv 0$ for $t \geqq 2$, $-1 \leqq \eta'(t) \leqq 0$ for all t.

Multiplying $\eta^2\left(\frac{\rho}{j}\right)(1+\rho)^{\tilde{q}}\varphi_\delta|\varphi_\delta|^{p-2}$ to $(4.4)_\delta$ and then integrating, we obtain

$$\int_M \left(\left(\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_\delta\right)^n - \omega_g^n\right)\eta^2(1+\rho)^{\tilde{q}}\varphi_\delta|\varphi_\delta|^{p-2}$$
$$= \int_M (e^{f+\delta\varphi_\delta} - 1)\eta^2(1+\rho)^{\tilde{q}}\varphi_\delta|\varphi_\delta|^{p-2}\omega_g^n .$$

Before we proceed further, we remark that the solution φ_δ is bounded and the metric $\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_\delta$ is equivalent to ω_g on M. The bound and equivalence may depend on δ. Now we derive by integration by parts.

$$\int_M (e^{f+\delta\varphi_\delta} - 1)\eta^2(1+\rho)^{\tilde{q}}\varphi_\delta|\varphi_\delta|^{p-2}\omega_g^n$$
$$= \frac{-\sqrt{-1}}{2\pi}\int_M \partial\varphi_\delta \wedge \bar{\partial}(\eta^2(1+\rho)^{\tilde{q}}\varphi_\delta|\varphi_\delta|^{p-2})$$
$$\wedge\left(\omega_g^{n-1} + \cdots + \left(\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_\delta\right)^{n-1}\right)$$
$$= \frac{-\sqrt{-1}}{2\pi}\int_M \partial\varphi_\delta \wedge \left(\left(\frac{2\eta\eta^1}{i} + \frac{\tilde{q}\eta^2}{1+\rho}\bar{\partial}\rho\varphi_\delta\right) + (p-1)\eta^2\bar{\partial}\varphi_\delta\right)(1+\rho)^{\tilde{q}}$$
$$\cdot|\varphi_\delta|^{p-2}\wedge\left(\omega_g^{n-1} + \cdots + \left(\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_\delta\right)^{n-1}\right)$$
$$\leqq C_{\delta,p,\tilde{q}}\int_M (1+\rho)^{\tilde{q}-2}|\varphi_\delta|^p\omega_g^n ,$$

where $C_{\delta,p,\tilde{q}}$ denotes a constant depending only on δ, $\tilde{p}$, $\tilde{q}$.

On the other hand, it is easy to check that

$$(e^{\delta\varphi_\delta} - 1)\varphi_\delta \geqq \frac{\delta}{2} e^{-\delta \sup_M |\varphi_\delta|} |\varphi_\delta|^2 \qquad \text{on } M ;$$

therefore,

$$\int_M (e^{f+\delta\varphi_\delta} - 1)\eta^2(1+\rho)^{\tilde{q}}\varphi_\delta|\varphi_\delta|^{p-2}\omega_g^n \geqq \frac{\delta}{2} e^{-\delta\sup_M|\varphi_\delta| + \inf_M f} \int_M \eta^2(1+\rho)^{\tilde{q}}|\varphi_\delta|^p\omega_g^n - \int_M |e^f - 1|\eta^2(1+\rho)^{\tilde{q}}\eta^2|\varphi_\delta|^{p-1}\omega_g^n .$$

Since $f + O(\rho^{-2-\frac{2\varepsilon}{\alpha}})$ and $p \geqq n$, we have

$$\int_M \eta^2|e^f - 1|(1+\rho)^{\tilde{q}}\eta^2|\varphi_\delta|^{p-1}\omega_g^n \leqq C \int_M \eta^2(1+\rho)^{\tilde{q}-2-\frac{2\varepsilon}{\alpha}}|\varphi_\delta|^{p-1}\omega_g^n$$
$$\leqq C\left\{\int_M \eta^2(1+\rho)^{\tilde{q}-\frac{2\varepsilon}{\alpha}}|\varphi_\delta|^p\omega_g^n + \int_M \eta^2(1+\rho)^{\tilde{q}-\frac{2\varepsilon}{\alpha}-2p}\omega_g^n\right\} .$$

Lemma 2.3 states that the volume growth of (M, g) is like that of R^{2n}, therefore, for $p \geqq n$, $\tilde{q} \leqq 0$, we have

$$\int_M \eta^2(1+\rho)^{\tilde{q}}|\varphi_\delta|^p\omega_g^n \leqq C_{\delta,p,\tilde{q}}\left\{\int_M (1+\rho)^{\tilde{q}-\frac{2\varepsilon}{\alpha}}|\varphi_\delta|^p\omega_g^n + 1\right\}$$

where $C_{\delta,p,\tilde{q}}$ is still a constant depending only on δ, p, $\tilde{q}$, but may be different from the previous one. Let j go to infinity, we obtain for $p \geqq n$, $\tilde{q} \leqq 0$,

$$\int_M (1+\rho)^{\tilde{q}}|\varphi_\delta|^p\omega_g^n \leqq C_{\delta,p,\tilde{q}}\left\{\int_M (1+\rho)^{\tilde{q}-\frac{2\varepsilon}{\alpha}}|\varphi_\delta|^p\omega_g^n + 1\right\} .$$

Since φ_δ is bounded, by Lemma 2.3, the integral $\int_M (1+\rho)^{\tilde{q}}|\varphi_\delta|^p\omega_g^n$ will be finite if q is sufficiently negative. Then our lemma follows from an iteration of using the above inequality.

By the definition (2.19) or (2.20) of the metric g, we see that the distance function ρ is equivalent to $\|S\|^{-\alpha}$. Thus by (4.2), $f = O(\rho^{-2-\frac{2\varepsilon}{\alpha}})$. Also one can prove that

$$\mathrm{Vol}_g(B_R(x_0)) \leqq C_5 R^{2n} \tag{4.6}$$

where C_5 is a constant independent of R, $B_R(x_0)$ is the geodesic ball with center at x_0. Choose a $p_0 > n$ such that

$$\frac{p_0 + 1}{n + p_0}\left(2 + \frac{2\varepsilon}{\alpha}\right) > 2 . \tag{4.7}$$

Then by (4.6) and $f = O(\rho^{-2-\frac{2\varepsilon}{\alpha}})$, we have

$$C_6 = \left(\int_M |e^f - 1|^{\frac{n(p_0+1)}{n+p_0}}\omega_g^n\right)^{\frac{n+p_0}{n(p_0+1)}} < +\infty .$$

Let η be a cut-off function on R^1, $\eta(t) \equiv 1$ for $t < 1$, $\eta(t) \equiv 0$ for $t \geqq 2$ and $|\eta'(t)| \leqq 1$. Rewrite $(4.4)_\delta$ as

$$-\left(\frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_\delta\right) \wedge \left(\omega_g^{n-1} + \omega_g^{n-2} \wedge \left(\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_\delta\right) + \cdots + \left(\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_\delta\right)^{n-1}\right) = (1 - e^{f+\delta\varphi_\delta})\omega_g^n . \qquad ((4.8)_\delta)$$

Multiplying $\eta^2\left(\frac{\rho(x)}{R}\right)\varphi_\delta^p$ with $p \geqq p_0$ to both sides of the Eq. $(4.8)_\delta$ and integrating by parts, we obtain

$$\int_M \left|\nabla\left(\eta\left(\frac{\rho}{R}\right)\varphi_\delta^{\frac{p+1}{2}}\right)\right|^2 \omega_g^n \leqq C_p\left\{\int_M \eta^2 |\varphi_\delta|^p |e^f - 1|\omega_g^n + \int_M \eta^2 |\varphi_\delta|^{p-1}\varphi_\delta(e^{\delta\varphi_\delta} - 1)e^f\omega_g^n + \frac{1}{R^2}\int_M |\eta'|^2 |\varphi_\delta|^{p+1}\frac{\sqrt{-1}}{2\pi}\partial\rho \wedge \bar{\partial}\rho \wedge \left(\omega_g^{n-1} + \cdots + \left(\omega_g + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_\delta\right)^{n-1}\right)\right\} . \qquad ((4.9)_\delta)$$

By Lemma 4.1, the last term in the above inequality tends to zero as $R \to +\infty$. (Note that $\omega + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_\delta$ is equivalent to ω with the constants depending on δ). Applying Sobolev inequality (3.1) to the right-handed side of $(4.9)_\delta$ and then letting $R \to \infty$, we have

$$\left(\int_M |\varphi_\delta|^{(p+1)\frac{n}{n-1}}\omega_g^n\right)^{\frac{n-1}{n}} \leqq C_p \int_M |\varphi_\delta|^p |e^f - 1|\omega_g^n . \qquad ((4.10)_\delta)$$

Note that $\varphi_\delta(e^{\delta\varphi_\delta} - 1) \geqq 0$ on M and C always denotes a constant independent of δ. In particular, by Hölder inequality, it follows from $(4.8)_\delta$ that

$$\left(\int_M |\varphi_\delta|^{(p_0+1)\frac{n}{n-1}}\omega_g^n\right) \leqq C . \qquad ((4.11)_\delta)$$

Put $p_{k+1} = (p_k + 1)\frac{n}{n-1} - 1$ for $k \geqq 0$. Then it follows from the inequalities in $(4.10)_\delta$

$$\left(\left(\int_M |\varphi|^{p_{k+1}+1}\omega_g^n\right)^{\frac{1}{p_{k+1}+1}} + 1\right) \leqq (Cp_k)^{\frac{1}{p_k+1}}\left(\left(\int_M |\varphi_\delta|^{p_k+1}\omega_g^n\right)^{\frac{1}{p_k+1}} + 1\right) . \qquad ((4.12)_\delta)$$

Letting k go to infinity, we conclude from $(4.11)_\delta$ and $(4.12)_\delta$,

$$\sup_M |\varphi_\delta| \leqq C , \qquad ((4.13)_\delta)$$

i.e., φ_δ are uniformly bounded. Note that C in $(4.13)_\delta$ may be different from previous ones.

Lemma 4.2 (cf. [Y2, TY3]) *Let φ_δ be the solution of $(4.4)_\delta$. Then*

(i) *there are constants C_7, C_8 independent of δ such that*

$$0 \leqq n + \Delta_g \varphi_\delta \leqq C_7 e^{C_8(\varphi_\delta - \inf_M \varphi_\delta)} \qquad ((4.14)_\delta)$$

where Δ_g denotes the laplacian of the metric g.

(ii) *There is an a priori estimate of the derivatives $\nabla^3 \varphi_\delta(x)$ in terms of (M, g) and $\sup_M \{|\varphi_\delta|, |\Delta_g \varphi_\delta|\}$ and $\sup_{B_1(x,g)} \{f, |\nabla f|, |\nabla^2 f|, |\nabla^3 f|\}$.*

By $(4.13)_\delta$ and Lemma 4.2 (i), (ii) and the standard elliptic theory (cf. [GT]), there is a subsequence $\{\delta_1\}$ of $\{\delta\}$ such that φ_{δ_i} converge to a solution φ of (4.3) in $C^{2,\frac{1}{2}}$-norms. Moreover, by $(4.11)_\delta$, $(4.13)_\delta$ and $(4.14)_\delta$, we have

$$\int_M |\varphi|^{(p_0+1)\frac{n}{n-1}} \omega_g^n \leqq C < \infty \qquad (4.15)$$

$$\sup_M |\varphi| \leqq C \qquad (4.16)$$

$$0 \leqq n + \Delta_g \varphi \leqq C\,. \qquad (4.17)$$

Lemma 4.3 *Let φ be as above. Then $\varphi(x)$ converges uniformly to zero as x goes to infinity.*

Proof. Since Sobolev inequality holds for smooth functions on M with compact support, one can use the standard iteration (cf. [GT, Chap. 8]) to Eq. (4.17) and conclude the mean value inequality

$$|\varphi(x)| \leqq C_9 \left(\int_{B_1(x)} |\varphi|^{(p_0+1)\frac{n}{n-1}} \omega_g^n \right)^{\frac{1}{p_0+1} \frac{n-1}{n}} \qquad (4.18)$$

where C_q is a uniform constant independent of x. Then the lemma follows from (4.15) and (4.18).

Therefore, the solution φ we constructed above is what we want in Proposition 4.1. The uniqueness of such a φ follows directly from maximum principle.

Now let g_Ω be the Kähler metric with Kähler form $\omega_g + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi$, then by the second order estimate in (4.17), g_Ω is equivalent to g and so it is complete. By Eq. (4.3) and the definition of f, we have

$$\mathrm{Ric}(g_\Omega) = \Omega\,.$$

The proposition is proved.

5 Completion of the proof of main theorem

We still adopt the notations used in Sects. 2 and 4. Given $\beta > 1$, $\alpha = \frac{\beta - 1}{n}$ with $n = \dim_{\mathbb{C}} M$ and a (1.1)-form Ω in the cohomology class $C_1(-K_{\bar{M}} - \beta L_D)$, we constructed in Proposition 4.1 a complete Kähler metric with Ω as its Ricci form.

The goal of this section is to study the asymptotic behavior of this constructed metric, and then the proof of Theorem 1.1 is finished. Without losing generality, we assume that $\beta \leqq n + 1$, i.e., $\alpha \leqq 1$.

Denote by g_Ω and ω_Ω the Kähler metric constructed in Proposition 4.1 and its Kähler form, respectively. Then

$$\omega_\Omega = \omega_g + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi \qquad \text{on } M \tag{5.1}$$

where φ is a smooth function which converges uniformly to zero as x tends to infinity D of $M = \bar{M}\backslash D$. We may assume that ω_E vanishes in a neighborhood of D (cf. the proof of Theorem 5.1 in [TY1]). Therefore, by shrinking V_3 if necessary, we have, by (2.27),

$$\omega_g = \omega_3 + C_\varepsilon \partial\bar{\partial}(-\|S\|')^{2\varepsilon} \qquad \text{on } V_3\backslash D \tag{5.2}$$

where for $m \geqq 3$, ω_m are the Kähler metrics on the truncated neighborhood $V_m\backslash D$ constructed in Proposition 2.1, S is the defining section of D and $\|\cdot\|'$ is a hermitian metric on L_D. By the definition of ω_m and the assumption $\alpha \leqq 1$, one can easily see that for $m \geqq 3$,

$$\omega_m = \omega_3 + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\psi_m \qquad \text{on } V_m\backslash D \subset V_3\backslash D \tag{5.3}$$

with $\psi_m(x)$ converging uniformly to zero as x goes to D. From (5.1), (5.2) and (5.3), we can write the Kähler form ω_Ω in the truncated neighborhood $V_m\backslash D$ as follows,

$$\omega_\Omega = \omega_m + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi_m \tag{5.4}$$

where φ_m is a smooth function on $V_m\backslash D$ and $\varphi_m(x)$ converges uniformly to zero as x goes to D. On the other hand, if f_m is the smooth function given in (2.28), then $|\nabla^k f_m|_{g_m} = O(\|S\|_m^{m+\frac{1}{2}+\alpha k})$ and by (2.24), (4.1), (5.4), we have

$$\left(\omega_m + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\varphi_m\right)^n = e^{f_m}\omega_m^n \qquad \text{on } V_m\backslash D \tag{(5.5)$_m$}$$

where $\|\cdot\|_m$ is the hermitian metric on L_D in defining ω_m. Recall that

$$\omega_m = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}(\|S\|_m^{-2\alpha}) . \tag{5.6}$$

Lemma 5.1 *Define a function ρ_m on $V_m\backslash D$ by $\rho_m(x) = \|S\|_m^{-\alpha}(x)$. Then for $\delta > 0$, we have*

(i)

$$\begin{aligned}\left(\omega_m + \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}(K\rho_m^{-2\delta})\right)^n = \Bigg(1 &- (n-1-\delta)K\delta\rho_m^{-2(\delta+1)} - \frac{K\delta(\delta+1)\rho_m^{-2\delta-2-\frac{2}{\alpha}}}{\|S\|_m^2 + \alpha\|D_m S\|_m^2} \\ &+ \frac{(n-1)(n-2\delta-2)}{2} K^2\delta^2\rho_m^{-4\delta-4} \\ &+ O(\rho_m^{-4(1+\delta)-\frac{2}{\alpha}})\Bigg)\omega_m^n . \end{aligned} \tag{5.7}$$

(ii) *for* $\alpha \leqq \frac{1}{n}$,

$$\left(\omega_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}(K\rho_m^{-2n+2}(-\log\rho_m)^\delta)\right)^n$$

$$= (1 - (n-1)K\delta\rho_m^{-2n}(-\log\rho_m^2)^{\delta-1}(1 + o(1))\omega_m^n) \qquad (5.8)$$

where K is a constant, and D_m *is the covariant derivative of* $\|\cdot\|_m$.

Proof. (i) By (2.4), we have

$$\omega_m^n = \alpha^n \|S\|_m^{-2\alpha n}\tilde{\omega}_m^{n-1} \wedge \left(\tilde{\omega}_m + \frac{\alpha n\sqrt{-1}}{2\pi}\frac{P_m S \wedge \overline{D_m S}}{|S|^2}\right)$$

$$= \alpha^n \|S\|_m^{-2\alpha n-2}(\|S\|_m^2 + \alpha\|D_m S\|_m^2)\tilde{\omega}_m^n\,,$$

where $\tilde{\omega}$ is the curvature form of the hermitian metric $\|\cdot\|_m$ on L_D. Because of the logarithmic terms in the definition of $\|\cdot\|_m$, the (1, 1)-form $\tilde{\omega}_m$ may not be defined on D. However, $\|D_m S\|_m$ is well-defined on D and nonvanishing there. In fact, $\|D_m S\|_m$ coincides with $\|DS\|$ along D.

On the other hand, we compute

$$\left(\omega_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}(KP_m^{-2\delta})\right)^n \not= \left(\omega_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}(K\|S\|_m^{2\alpha\delta})\right)^n$$

(use) (2.3)

$$= \alpha^n\|S\|_m^{-2\alpha n}\Bigg[(1 - K\delta\|S\|_m^{2\alpha(1+\delta)})\tilde{\omega}_m$$

$$+ \frac{\alpha\sqrt{-1}}{2\pi}(1 + K\delta^2\|S\|_m^{2\alpha(1+\delta)})\frac{D_m S \wedge \overline{D_m S}}{|S|^2}\Bigg]^n$$

$$= \alpha^n\|S\|_m^{-2\alpha n-2}[(1 - K\delta\|S\|_m^{2\alpha(1+\delta)_n})\|S\|_m^2$$

$$+ \alpha(1 + K\delta^2\|S\|_m^{2\alpha(1+\delta)})(1 - K\delta^2\|S\|_m^{2\alpha(1+\delta)})^{n-1}\cdot\|D_m S\|_m^2]\cdot\tilde{\omega}_m^n$$

$$= \frac{(1 - K\delta^2\|S\|_m^{2\alpha(1+\delta)})^{n-1}(\|S\|_m^2(1 - K\delta\|S\|_m^{2\alpha(1+\delta)}) + \alpha(1 + K\delta^2\|S\|_m^{2\alpha(1+\delta)})\|D_m S\|_m^2)}{\|S\|_m^2 + \alpha\|D_m S\|^2}$$

$$= \Bigg\{1 - (n-1-\delta)K\delta\|S\|_m^{2\alpha(1+\delta)} - \frac{K\delta(\delta+1)\|S\|_m^{2\alpha(1+\delta)+2}}{\|S\|_m^2 + \alpha\|D_m S\|_m^2} +$$

$$+ \frac{(n-1)(n-2\delta-2)}{2}K^2\delta^2\|S\|_m^{4\alpha(1+\delta)} + O(\|S\|_m^{4\alpha(1+\delta)+2})\Bigg\}\,\omega_m^n\,.$$

Now (5.7) follows from the above equation and $P_m = \|S\|_m^{-\alpha}$.

(ii) It suffices to compute

$$\left(\omega_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}(K\|S\|_m^{2\alpha(n-1)}(-\log\|S\|_m^2)^\delta)\right)^n$$

$$= \Bigg[\omega_m - K\left(\alpha(n-1) - \frac{\delta}{\log\|S\|_m^2}\right)^\delta \tilde{\omega}_m +$$

$$+ K\left(\alpha^2(n-1)^2 - \frac{2\delta\alpha(n-1)}{\log\|S\|_m^2} + \frac{\delta^2}{(\log\|S\|_m^2)^2}\right)$$

$$\times \|S\|_m^{2\alpha(n-1)}(\log\|S\|_m^2)^\delta \frac{D_mS \wedge \overline{D_mS}}{|S|^2}\Bigg]^n$$

$$= \frac{\omega_m^n}{\|S\|_m^2 + \alpha\|D_mS\|_m^2}\left\{(1 - K\left((n-1) - \frac{\delta}{\alpha\log\|S\|_m^2}\right)\|S\|_m^{2\alpha n}(\log\|S\|_m^2)^\delta\right\}^{n-1} \cdot$$

$$\cdot\Bigg\{\|S\|_m^2 - K\left((n-1) - \frac{\delta}{\alpha\log\|S\|_m^2}\right)\|S\|_m^{2\alpha n+2}(\log\|S\|_m^2)^\delta$$

$$+ \alpha\|D_mS\|^2 + K\left(\alpha(n-1)^2 - \frac{2\delta(n-1)}{\log\|S\|_m^2}\right)$$

$$\times \|S\|_m^{2\alpha n}(\log\|S\|_m^2)^\delta\|D_mS\|_m^2 + O\left(\frac{1}{(\log\|S\|_m^2)^2}\right)\Bigg\}$$

$$= \left\{1 - \frac{K(n-1)\delta}{\alpha}\|S\|_m^{2\alpha n}(-\log\|S\|_m^2)^{\delta-1}(1 + o(1))\right\}\omega_m^n .$$

Then (5.8) follows.

Lemma 5.2 *Let $m \geqq 2n+2$, $n = \dim_{\mathbf{C}} M$. Then*

(i) *for $\beta > 2$, i.e., $\alpha > \frac{1}{n}$, there is a constant $C(m)$, depending on m, such that*

$$|\varphi_m(x)| \leqq \frac{C(m)}{(1+\rho_m^2(x))^{n-1}}, \quad x \in V_m \backslash D . \tag{5.9}$$

(ii) *for $\beta \leqq 2$, i.e., $\alpha \leqq \frac{1}{n}$, and any $\delta > 0$, there are constants $C(m)$ and C_δ, where C_δ may depend on δ, such that*

$$-C_\alpha(1+\rho_m^2(x))^{-n+1}(-\log\rho_m)^\delta(x) \leqq \varphi_m(x) \leqq C(m)(1+\rho_m^2(x))^{-n+1} \quad x \in V_m \backslash D . \tag{5.10}$$

Proof. Fix $m \geqq 6n$. It suffices to prove (5.9), (5.10) in a neighborhood of D. First we assume that $\beta > 2$. Then by Lemma 5.1 (i) with $\delta = n - 1$, we have

$$\left(\omega_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}(K\rho_m^{-2n+2})\right)^n = \left(1 - \frac{Kn(n-1)\rho_m^{-2n-\frac{2}{\alpha}}(1+o(1))}{\|S\|_m^2 + \alpha\|DS\|_m^2}\right)\omega_m^n \text{ on } V_m\backslash D\,. \tag{5.11}$$

On the other hand,

$$e^{f_m}\omega_m^n = \left(1 + O(\rho_m^{\frac{2m+1}{2\alpha}})\right)\omega_m^n \quad \text{on } V_m\backslash D\,. \tag{5.12}$$

Since $m \geqq 2n+1, \frac{2m+1}{\alpha} > 2n + \frac{1}{\alpha}$. Note that $\alpha \leqq 1$. Let $\varepsilon > 0$, $K = \frac{K'}{\varepsilon}$, where $|K'| = \sup_{V_m\backslash D}(|\varphi_m| + 1)$, then $|K|\rho_m^{-2(n-1)}(x) > |\varphi_m(x)|$ for $\rho_m^{-2n+2}(x) = \varepsilon$. By taking ε sufficiently small, it follows from (5.11) and (5.12) that on $\{x \in V\backslash D \,|\, \rho_m^{-2n+2}(x) \leqq \varepsilon\}$,

$$\left(\omega_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}(K\rho_m^{-2n+2})\right)^n \begin{cases} \leqq e^{f_m}\omega_m^n & \text{if } K > 0 \\ \geqq e^{f_m}\omega_m^n & \text{if } K < 0\,. \end{cases} \tag{5.13}$$

That is, $K\rho_m^{-2n+2}$ can serve as upper or lower barriers of the complex Monge-Ampère Eq. $(5.5)_m$ according to $K > 0$ or $K < 0$. Then (5.9) follows from the fact that $\varphi_m(x)$ converges uniformly to zero as $x \to D$ and maximum principle with barriers $K\rho_m^{-2n-2}$. The estimate (5.10) can be similarly proved by using Lemma 5.1 (i), (ii).

Remark. On the euclidean space R^{2n}, the positive minimal Green function decays at the order $r^{-(2n-2)}$, where r is the euclidean distance on R^{2n}. In our case here, the function ρ_m is equivalent to the distance function on $V_m\backslash D$ from ∂V_m. Therefore, for $\beta > 2$, the estimate (5.9) is optimal. For $\beta \leqq 2$, we don't know whether the logarithmic term in (5.10) is necessary.

From now on, we fix a $m \geqq 2n + 2$. The second order estimate in [Y2] implies

$$0 \leqq \omega_m + \frac{\sqrt{-1}}{2\pi}\partial\bar{\partial}\varphi_m \leqq C\omega_m \quad \text{on } V_m\backslash D \tag{5.14}$$

where C is a constant independent of m and $x \in V_m\backslash D$.

Proposition 5.1 *Let φ_m be the solution of* $(5.5)_m$ *with decay as in either* (5.9) *or* (5.10). *Then for $\frac{1}{2} > \delta > 0$, there are constants $C_{\delta,k}$, depending only on δ, such that*

$$|\nabla^k\varphi|_{g_m}(x) \leqq C_{\delta,k}\rho_m(x)^{-k-2n+2+\delta} \quad x \in V_m\backslash D\,. \tag{5.15}$$

Proof. Fix $\delta > 0$ and $m \geqq 4n$. For simplicity, we will always use C to denote a constant depending only on δ, m. We remark (cf. (2.21) in Lemma 2.5) that for $k \geqq 0$,

$$|\nabla^k R(g_m)|_{g_m}(x) = O(\rho_m(x)^{-k-2}), \quad x \in V_m\backslash D\,, \tag{5.16}$$

where $R(g_m)$ denotes the curvature tensor of the metric g_m. Define a new Kähler metric $\tilde g = R^{-2} g_m$ on $B_R(x, g_m)$ with $2R = \rho_m(x)$. Then $B_1(x, \tilde g) = B_R(x, g_m)$ and one reads from (5.16),

$$\sup_{B_1(x,\tilde g)} \{ |\nabla^k R(\tilde g)|_{\tilde g} \,|\, 0 \leqq k \leqq 4m \} \leqq C\,. \tag{5.17}$$

Put $\tilde\varphi = R^{-2}\varphi_m$, then $(5.5)_m$ is equal to

$$\left(\omega_{\tilde g} + \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\tilde\varphi\right)^n = e^{f_m}\omega_{\tilde g}^n \text{ on } B_1(x,\tilde g) \tag{5.18}$$

and for any $\delta > 0$, there is a $C_\delta > 0$ such that

$$\sup_{B_1(x,\tilde g)} (|\tilde\varphi|) \leqq C_\delta R^{-2n+\delta}\,. \tag{5.19}$$

On the other hand, by Proposition 2.1, $|\nabla^k f_m|_{g_m}(y) = O(R^{-\frac{2m+1}{\alpha}-\delta_k})$ for $y \in B_R(x, g_m)$, so

$$\sup_{B_1(x,\tilde g)} (|\nabla^k f_m|_{\tilde g}) \leqq C R^{-\frac{2m+1}{\alpha}}\,. \tag{5.20}$$

Since $|\nabla^k \varphi_m|_{g_m}(x) = R^{-k+2}|\nabla^k\tilde\varphi|_{\tilde g}$, it suffices to prove

$$|\nabla^k\tilde\varphi|_{\tilde g}(x) \leqq R^{-2n+\delta} \tag{5.21}$$

to complete the proof of this proposition.

First we consider the case $k = 1$. Note that by (5.14),

$$0 \leqq \omega_{\tilde g} + \frac{\sqrt{-1}}{2\pi}\partial\bar\partial\tilde\varphi \leqq C\omega_{\tilde g} \quad \text{on } B_1(x,\tilde g)\,. \tag{5.22}$$

Multiplying $\eta^2\tilde\varphi$ to both sides of (5.18) with proper cut-off function η and integrating by parts, we can get

$$\int_{B_{\frac12}(x,\tilde g)} |\nabla\tilde\varphi|_{\tilde g}^2\,\omega_{\tilde g}^n \leqq C_\delta R^{-4n+2\delta}\,. \tag{5.23}$$

Here we need to use $m \geqq 4n$.

Taking the derivative on (5.18) with respect to $z_\ell (1 \leqq \ell \leqq n)$ and using (5.22), we have on $B_1(x,\tilde g)$

$$\tilde g^{i\bar j}\left(\frac{\partial\tilde\varphi}{\partial z_\ell}\right)_{i\bar j} = (e^{f_m}-1)\tilde g^{i\bar j}\frac{\partial\tilde g_{i\bar j}}{\partial z_\ell} + \frac{\partial f_m}{\partial z_\ell}e^{f_m} + O(\nabla^2\tilde\varphi) \tag{5.24}$$

where $O(\nabla^2\tilde\varphi)$ denotes a function bounded by $|\nabla^2\tilde\varphi|_{\tilde g}$. Now using Moser's iteration to (5.24) with $\frac{\partial\tilde\varphi}{\partial z_\ell}$ for $1 \leqq \ell \leqq n$, we can prove

$$|\nabla\tilde\varphi|_{\tilde g}(x) \leqq C\rho_m(x)^{-2n+\delta}\,. \tag{5.25}$$

Since x is an arbitrary point in $V_m \backslash D$, the estimate (5.25) holds on $V_m \backslash D$. On the other hand, by multiplying $\eta\frac{\partial\tilde\varphi}{\partial z_\ell}$ to both sides of (5.24) with proper cut-off function, integrating by parts and summing over ℓ, we obtain

$$\int_{B_{\frac{1}{2}}(x, \tilde{g})} |\nabla^2 \tilde{\varphi}|_{\tilde{g}}^2 \omega_{\tilde{g}}^n \leqq C_\delta R^{-4n+2\delta} . \tag{5.26}$$

Inductively, suppose that we have proved

$$|\nabla^j \tilde{\varphi}|_{\tilde{g}}(x) \leqq C_\varepsilon \rho_m(x)^{-2n+\delta}, \quad x \in V_m \backslash D \tag{5.27}$$

for $j \leqq k - 1 < m$, and

$$\int_{B_{\frac{1}{2}}(x, \tilde{g})} |\nabla^k \tilde{\varphi}|_{\tilde{g}}^2(x) < C_\delta R^{-4n+2\delta} + CR^{-\frac{m-k}{2\alpha}-n+\varepsilon} . \tag{5.28}$$

By taking derivatives on (5.24), we have equations for $\dfrac{\partial^k \varphi^2}{\partial z_1^{i_1} \ldots \partial \bar{z}_n^{j_n}}$ with $\sum_{j=1}^n (i_s + j_s) = k$ as follows,

$$\tilde{g}^{i\bar{j}} \left(\frac{\partial^k \tilde{\varphi}}{\partial z_1^{i_1} \ldots \partial \bar{z}_n^{j_n}} \right)_{i\bar{j}} = \frac{\partial^k f_m}{\partial z_1^{i_1} \ldots \partial \bar{z}_n^{j_n}} e^{f_m} + O(R^{-2n+\delta}) \tag{5.29}$$

on $B_{\frac{1}{2}}(x, \tilde{g})$. Then an iteration implies

$$|\nabla^k \tilde{\varphi}|_{\tilde{g}}(x) \leqq C_\delta \rho_m(x)^{-2n+\delta} . \tag{5.30}$$

Moreover, we can deduce from (5.29) the integral estimate (5.28) with k replaced by $k + 1$. Therefore, by induction, we have proved the estimate (5.30) and the proposition follows.

By Lemma 2.4 and the remark after its proof, one can easily derive the following from the above proposition.

Proposition 5.2 *Let $M = \bar{M} \backslash D$, $\Omega \in C_1(-K_{\bar{M}} - \beta L_D)$ be given as in Proposition* 4.1, *g_Ω be the complete Kähler metric with Ricci curvature Ω constructed in Proposition* 4.1. *We denote by ρ the distance function on M from some fixed point. Then the curvature tensor $R(g_\Omega)$ decays at the order of at least ρ^{-3} if D is biholomorphic to CP^{n-1} and the induced line bundle L_D by D restricts to the $\frac{1}{\alpha}$-hyperplane line bundle on $D \cong CP^{n-1}$; if either of these two conditions falls, then $R(g_\Omega)$ decays at the order of exactly ρ^{-2}. Moreover, the covariant derivatives $\nabla^k R(g_\Omega)$ decay at the order ρ^{-k-2}.*

Now our main theorem (Theorem 1.1) follows from Proposition 4.1 and Proposition 5.2.

6 The proof of Theorem 1.2

In this section, let $\bar{M}$ be a projective normal orbifold and D be a neat, admissible, and almost ample divisor. As in Theorem 1.2, we further assume that $C_1(-K_{\bar{M}} - L_D)$ admits a semi-positive (1.1) from Ω. Then it implies the following simple lemma.

Lemma 6.1 *With $\bar{M}$, D as given above. Then there is a semi-positive* (1,1)*-form in $C_1(M)$ which is actually positive near D.*

Proof. Since D is almost ample, by Definition 1.1, (ii), there is a semi-positive (1,1)-form ω_D representing $C_1(L_D)$. Moreover, this ω_D is positive near D. The required (1,1)-form is just $\Omega + \omega_D$.

As usual, we call a projective orbifold $\bar{M}$ algebraically simply-connected if $\bar{M}$ does not admit any finitely unramified covering.

Lemma 6.2 *The orbifold $\bar{M}$ is algebraically simply-connected.*

Proof. It follows from the Kodaira-Nakano vanishing theorem and an argument due to J. Serre (cf. [Ko]). In fact, if $\bar{M}$ admits a finite covering $\tilde{M}$, then

$$\chi(\tilde{M}, \mathcal{O}_{\tilde{M}}) = d\chi(\bar{M}, \mathcal{O}_{\bar{M}}) \tag{6.1}$$

where d is the degree of the covering and $\chi(\bar{M}, \mathcal{O}_{\bar{M}}) = \sum_{i=0}^{n} (-1)^i h^0(\bar{M}, \mathcal{O}_{\bar{M}})$ is the euler genus of structure sheaf $\mathcal{O}_{\bar{M}}$ on $\bar{M}$. On the other hand, by Lemma 6.1, there is a semi-positive (1,1)-form in $C_1(M)$ which is positive in an open subset. Thus the Kodaira-Nakano vanishing theorem (cf. Theorem 2.37 in [Sh]) implies that

$$h^i(\bar{M}, \mathcal{O}_{\bar{M}}) = h^{n-i}(\bar{M}, K_{\bar{M}}) = 0 \quad \text{for } i \geqq 1 \tag{6.2}$$

$$h^i(\tilde{M}, \mathcal{O}_{\tilde{M}}) = h^{n-i}(\tilde{M}, K_{\tilde{M}}) = 0 \quad \text{for } i \geqq 1 \, . \tag{6.3}$$

Note that the vanishing Theorem 2.37 was originally stated for smooth manifolds in [Sh]. However, there is no additional difficulty to generalize it to normal orbifolds. Now $h^0(\bar{M}, \mathcal{O}_{\bar{M}}) = h^0(\tilde{M}, \mathcal{O}_{\tilde{M}}) = 0$. It follows $d = 1$ and the Lemma is proved.

Lemma 6.3 *The fundamental group $\pi_1(\bar{M})$ of $\bar{M}$ is almost nilpotent, that is, a subgroup in $\pi_1(\bar{M})$ of finite index is nilpotent.*

Proof. Since D is almost ample in $\bar{M}$, the anticanonical line bundle $-K_D$ is ample, so D is simply connected (Kobayashi [Ko] proved such a manifold to be algebraically simply connected and the second author in [Y2] proved simple connectivity by constructing a metric with positive Ricci curvature). Thus by Van-Kampe theorem, the group $\pi_1(\bar{M})$ is a quotient of $\pi_1(\bar{M}\backslash D)$ by a normal subgroup. By the assumptions on $\bar{M}$, D, there is a complete Kähler metric on $\bar{M}\backslash D$ with nonnegative Ricci curvature (cf. [TY1]). In particular, it implies that $\pi_1(\bar{M}\backslash D)$ is of polynomial growth. Then a result of Gromov in [Gr] implies the almost nilpotency of $\pi_1(\bar{M}\backslash D)$. This implies that $\pi_1(\bar{M})$ is almost nilpotent.

In case there is a nonnegative form in $C_1(-K_{\bar{M}} - \beta L_D)$ for some $\beta > 1$, $\bar{M}\backslash D$ admits a complete Kähler metric g with nonnegative Ricci curvature and euclidean volume growth. Then the well known Volume Comparison Theorem implies that any unramified covering of $(\bar{M}\backslash D, g)$ has its volume growth less than that of R^{2n}. In particular, the fundamental group $\pi_1(\bar{M}\backslash D)$, so $\pi_1(\bar{M})$, is finite. So Lemma 6.2 implies that $\pi_1(\bar{M}) = \{0\}$. In general, we only need to remark that any nilpotent group admits a subgroup of finite index. So Theorem 1.2 follows from Lemma 6.2.

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