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Compactification of Kähler manifolds with negative Ricci curvature

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Compactification is an important step in understanding the properties of a non-compact complex manifold. Once a complex manifold can be compactified, the global properties of algebraic geometry can be applied to understand the structure of the non-compact one. In general, suitable conditions have to be imposed. In this article, we prove the following theorem, which has been around as a conjecture for quite a long time.

Theorem 1 *Let X be a complex manifold endowed with a complete Kähler metric of finite volume, bounded Riemannian sectional curvature and negative Ricci curvature. Then X is biholomorphic to a Zariski-open subset of a projective-algebraic variety.*

The interest in compactification can be traced at least back to the classical result of Satake [Sa], Baily-Borel [BB] and Ash-Mumford-Rappoport-Tai [AMRT] on arithmetic quotients of bounded symmetric domains. The methods employed are algebraic in nature. Along another direction, Andreotti-Grauert [AG], Siu-Yau [SY], Nadel-Tsuji [NT] and Nadel [N] obtained interesting results using essentially the analytic properties of pseudoconcavity. In [M2], Mok initiated a new scheme of attacking the problem and results in the work of Mok-Zhong [MZ]. The present work can be considered as a continuation and completion of this scheme. The hypotheses of the theorem are satisfied by Kähler-Einstein manifolds which include in particular quotients of bounded symmetric domains mentioned before. For dimension two, the result is proved in [Y].

The conditions required in the theorem are rather reasonable. The examples of Ballmann-Gromov-Schroeder [BGS] show that the Kähler condition is necessary. Finiteness in volume is also required by considering the example of the unit disc in C with Poincare metric. In view of the recent examples of Anderson-Kronheimer-LeBrun [AKL], negativity in the Ricci curvature is reasonable. A question that one would like to ask is whether the boundedness in the Riemannian sectional curvature is necessary, which in general seems to be difficult to verify. We remark that it is possible to replace this condition by integral bounds on the sectional curvature over the manifold. Moreover, we succeeded in generalizing the above results of compactification to V-manifolds. Details will appear elsewhere.

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Contents

Notations

$\bar{Z}$: compact Kähler manifold obtained by partial embedding
F: birational mapping from X to $\bar{Z}$
ω: the Kähler form on X
$\omega_{\bar{Z}}$: the Kähler form on $\bar{Z}$
dV_M: the volume form of M
h: the coefficient of volume form associated to ω in local coordinates
h_0: the coefficient of volume form associated to $\omega_{\bar{Z}}$ in local coordinates
W: the union of the branching locus and the base locus of F
W_1: an irreducible component of W
D: divisors in $\bar{Z}$ complementary to $F(X)$
K: the canonical line bundle of X
$\Gamma^\alpha(X, K^p)$: the space of L^α holomorphic sections of K^p
$\Gamma^0(X, K^p) := \bigcup_{\alpha>0} \Gamma^\alpha(X, K^p)$
c, c_1, c_2: some positive constants

1 Preliminaries

1.1

We are going to collect known results from [M2, MZ] and [Y] in this section.

1.2

In [MZ], Mok and Zhong obtained the following results.

Proposition 1 (Mok-Zhong) *Let $F: X \to Z$ be a birational embedding into a non-singular projective-algebraic variety defined by pluricanonical sections of class L^0. Let W be the set of base locus and the branching locus of F. Then $\Omega = F(X - W)$ is a Zariski open subset of Z.*

In particular, if X is of finite topological type, the proposition implies the result of our main theorem. There are essentially two steps in the proof of the above proposition. First they show that the field of rational functions M arised from taking the quotients of the pluricanonical sections of $\Gamma^0(X, K^p)$ for $p > 0$ satisfies Siegel's Theorem. This enables them to construct a mapping into an algebraic manifold $\bar{Z}$. The mapping is birational in the sense that M is isomorphic to the function field on $\bar{Z}$. Then they show that the image of the mapping is Zariski-open in the manifold constructed by using Bezout's type estimates.

We also recall the following standard result from L^2-estimates.

Lemma 1 *Let $x \in X$. There exists a positive integer m such that the pluricanonical map defined by $\Gamma^0(X, K^m)$ provides an embedding of neighbourhood of x to its image.*

In this article, the sections from the previous lemma are always composed with Segre's mapping from $P^a \times P^b$ to P^{ab+a+b} given by

$$([x_i]_{0 \leqq i \leqq a}, [y_j]_{1 \leqq j \leqq b}) \to [x_i y_j]_{0 \leqq i \leqq a, 0 \leqq j \leqq b} \,.$$

1.3

We need the following definition.

Definition 1 Suppose we fix a birational embedding F satisfying the conditions of Proposition 1. Let $W = \bigcup_{i \in I} W_i$ be the decomposition of W into irreducible components of a subvariety W of X, here I is an index set. Then a family of curves $\{c_i, i \in I\}$ is said *to satisfied condition* C *with respect to* W if the image of C_i by F in $\bar{Z}$ is a branch of a set defined by polynomials of bounded degree, and the curves C_i intersect W_i only at isolated points of $W_i - \bigcup_{j \neq i} W_j$.

As remarked in [Y], following essentially the argument of [M1, pp. 249–352], we obtain the following result.

Proposition 2 *Suppose there exists a family of irreducible analytic curves C_i, $i \in K$ satisfying condition* C *with respect to $W = \bigcup_{i \in I} W_i$ of Proposition* 1. *Then we can adjoin a finite number of sections in $\Gamma^0(X, K^m)$ to F to embed W.*

1.4

The image of the birational mapping F is $\bar{Z} - V$. We have the following estimates of asymptotic volume form from [Y].

Proposition 3 ([Y], Proposition 2.3) *The coefficient of the volume form on $\bar{Z} - V$ induced from ω on X by F can be expressed as*

$$h = \prod_i \|s_i\|^{-2\alpha_i} \cdot e^{\varphi} \cdot h_0$$

where

(a) $\alpha_i \leqq 1$, $i = 1, \ldots, N$;

(b) *φ is bounded from above and* $\sqrt{-1}\partial\bar{\partial}_{\varphi} \geqq -c\omega_{\bar{Z}}$; (c) $\liminf_{y \to x} \dfrac{\varphi(y)}{\log\|s_i(y)\|} = 0$ *as y tends to a generic point $x \in V_i$.*

Moreover, if $x \in V_i$ and $\alpha_i = 1$, we have $\lim_{y \to x} \varphi(x) = -\infty$.

2 Scheme of proof

2.1

In this subsection, we are going to lay out the scheme of proof of our main theorem.

2.2

As mentioned in §1, we obtain a birational mapping $F = (f_1, \ldots, f_{n+1}, g)$ from [MZ]. If we can remove the bad set W which consists of the branching locus and base locus by attaching a finite number of sections, the theorem will be proved. Let $f_{ij} = f_i + c_{ij} f_j$, $1 \leqq i < j \leqq n+1$, where c_{ij} are going to be specified later. Instead of considering the set of sections $S_1 = \{f_i, i = 1, \ldots, n+1\}$, we are going to consider the set $S_2 = \{f_i, f_{jk}, 1 \leqq i \leqq n+1, 1 \leqq j < k \leqq n+1\}$ with suitable constants c_{ij}. The main theorem is reduced to the following proposition, keeping in mind the results of §1.3.

Proposition 4 *We can find $g_1, \ldots, g_{n+1}$ from the set S_2 such that for the birational mapping G obtained from g and g_i, $i = 1, \ldots, n+1$, we can find a family of curves $\{c_i, i \in I\}$ which satisfied condition* C *with respect to the set of base locus and branching locus of G.*

Notice that for F, the set W of the union of branching locus and base locus has only countably many irreducible components. Hence we can choose the numbers c_{ij}, $1 \leqq i, j \leqq n+1$ such that any $n+1$ sections $g_1, \ldots, g_{n+1}$ among S_2 would have the same rank as $f_1, \ldots, f_{n+1}$ on any of the irreducible component of W. There are l different choices of $n+1$ sections from S_2, corresponding to different choices of birational embeddings G_i, $1 \leqq l$, where

$$l = \binom{\frac{1}{2}(n+1)(n+2)}{n+1}.$$

2.3

We would try to express Condition C in §1.3 to a form that can be handled more easily. First of all, we try to understand the geometry of the situation. We recall the following observation from [Y]. For details, please refer to §3.2 of [Y].

$$\begin{tikzcd}
\bar{Y} \arrow[d, "\sigma"'] \arrow[dr, "\tau"] & \\
\bar{Z}_1 \arrow[r] & \bar{Z} \\
X \arrow[u, "F_1"] \arrow[ur] &
\end{tikzcd}$$

Let W_1 be any irreducible component of W. Using Lemma 1 of §1, we can embed a generic point of W_1 into an algebraic manifold $\bar{Z}_1$ by adjoining sections $h_1, \ldots, h_s$ for some s to F. By Hironaka's resolution of singularity (cf. [U]), we can obtain an algebraic manifold $\bar{Y}$ such that the mapping σ is obtained from a sequence of blow-down's and τ a morphism. If W_1 that we are considering is a component of the branching locus, we just leave it like that. But if W_1 is a base locus, we blow up the subvariety and still call the resulting manifold $\bar{Y}$. In all cases, W_1 corresponds to a divisor E on $\bar{Y}$. Moreover, E is mapped into the divisor D in $\bar{Z}$ by construction. In the following discussion, actually we would only consider the component E_1 of E obtained from generic points of W_1 by σ^{-1}.

Let y be a point on E_1 which is the inverse image of σ from a generic point of W_1. Let x be its image by τ in $\bar{Z}$. Let w_i, z_i, $i = 1, \ldots, n$ be local coordinate functions on $\bar{Y}$ and $\bar{Z}$ respectively such that E is defined by $w_1 = 0$. Then the mapping τ can be represented by

$$\begin{aligned} z_1 &= w_1^{p_1}\zeta_1 \\ &\vdots \\ z_n &= w_1^{p_n}\zeta_n\,. \end{aligned}$$

Here ζ_i, $i = 1, \ldots, n$ are analytic functions which are not divisible by w_1. At least one of the integers p_i's is greater than or equal to one. We recall the following results from [Y].

Proposition 5 (Proposition 5.1, [Y]) *Suppose all the exponents p_i can be bounded uniformly by a positive constant independent of the element $W_1 \in W$. Then there exists a family of curves C_i satisfying condition* C *with respect to $W = \bigcup_{i\in I} W_i$.*

The proof of the proposition follows basically from [Y], where we constructed a curve on each $\bar{Y}$ which cuts E_1 at isolated points and the image of the curve in $\bar{Z}$ has degree bounded by a uniform constant. The only modification that we have to add is to observe that the mapping σ obtained above has degree bounded by $n - 1$ for a generic point of W_1 so that when we project the curve by σ to $\bar{Z}_1$, the image would cut W_1 at isolated points.

2.4

From the previous argument, it follows that Proposition 4 would be proved provided that we can bound p_i uniformly. By taking different projections, it is clear that we can reduce the proposition to the following statement.

Proposition 6 *For each subvariety $W_i \in W$ and each birational mapping G_j, $j = 1, \ldots, l$ considered above, at least one of p_k for the corresponding τ, which depends on both W_i and G_j, is bounded uniformly.*

We can also deduce the reduction of the proof in the following way. For notational simplicity, let us just take F as an example. Suppose for τ which corresponds to (F, g), p_1 is bounded by a uniform constant. If all the p_i's are equal, we are done. Hence we may assume that $p_2 > p_1$. Consider the birational mapping obtained from (F', g), where $F' = \{f_1, f_1 + c_{12}f_2, f_3, \ldots, f_n\}$. The corresponding morphism τ' would have the corresponding $p'_1 = p'_2 = p_1$ in local coordinates as in §1.3. The reason is that

$$w_1^{p_1}\zeta_1 + c_{12}w_1^{p_2}\zeta_2 = w_1^{p_1}(\zeta_1 + c_{12}w_1^{p_2-p_1}\zeta_2)\,.$$

2.5

It follows immediately from the previous subsection that we only have to consider the situation that the image of $\tau(E)$ are isolated points. Our strategy is to prove that Proposition 6 is true for all subvarieties W_i of W except for some components in the base locus in §3. This allows us to construct a new birational mapping without branching locus outside the base locus. The remaining components are dealt with by applying the argument to the new birational mapping again. This is done in §4 and allows us to conclude the proof of our main theorem.

3 Estimates of degree

3.1

In this section, we are going to generalize the results of [Y] to handle the situation that the image by τ of the irreducible component E_1, which consists of the inverse image of a generic point of W_1 from σ^{-1}, is a single point.

3.2

The idea of the proof involves estimates on the union of the branching locus and the base locus. We first try to take care of the set of base locus in this section. Notice that for an irreducible component in the base locus, the corresponding $\tau(E_1)$ is contained in the divisor D on $\bar{Z}$. From the results of §2, we know that the set of all W_1 with $\dim(\tau(E_1)) > 0$ can be removed by adjoining a finite number of pluricanonical sections in $\Gamma^0(X, K^m)$ for some m. Hence we may assume that $\dim(\tau(E_1)) = 0$. In such a case, the mapping F can obviously be extended continuously across W_1. Riemann's Extension Theorem implies that F can be holomorphically extended over W_1. In this case, W_1 behaves as if it is contained in an irreducible component of the branching locus. If $\dim(E_1) = n - 1$, Proposition 7 of §3.8 can be applied to deal with the situation. The other cases would be taken care of in §4.

3.3

From this point on, we assume that W_1 is an irreducible component of the branching locus with the property that $\tau(E_1)$ is a single point on D. Then $\dim(W_1) = n - 1$ since it corresponds to the vanishing of the Jacobian of our birational mapping. We recall that the divisor D in $\bar{Z}$ can be written as $\bigcup_{i=1}^{N} D_i$, where D_i are divisors in normal crossing. We would show in this subsection the following result.

Lemma 2 *If the image $\tau(E_1)$ does not lie in the intersection point of n components D_i, at least one of the exponents p_i is bounded uniformly.*

Proof. Let z be a generic point of $\tau(E_1)$ and U a small neighbourhood of y. We choose a holomorphic coordinate system in U such that locally $\tau(E_1)$ lies in the intersection of i divisors, $i < n$, and the divisors are given by $z_j = 0, j = 1, \ldots, i$. From Proposition 3, §1, we can write the volume on $\bar{Z} - D$ induced from ω locally on U as

$$dV_z = \prod_{j=1}^{i} |z_j|^{2\alpha_j} \cdot |\eta|^2 \cdot |dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n|^2$$

for some holomorphic nonvanishing function η. It follows that the volume form on $\bar{Y}$ obtained from ω on X by F_1 and σ can be expressed in the following way in a local coordinates chosen as in §1.3.

$$dV_{\bar{Y}} = |w_1|^{2\sum_{j=1}^{i} p_j(1-\alpha_j) + 2\sum_{j=1+1}^{n} p_j - 2} \cdot |\zeta|^2 \cdot |dw_1 \wedge dw_2 \wedge \cdots \wedge dw_n|^2 .$$

Here ζ is a holomorphic function in the neighbourhood. On the other hand, we know that $dV_{\bar{Y}}$ at a generic point of E_1 is non-degenerate as the same thing happens for W_1. It follows that each p_j for $j > i$ is bounded from above by 1. We can then apply the arguments in §1.4 to obtain our result.

Remark. It is quite clear that the proof of the lemma implies that p_i would be bounded if $\alpha_i < 1$. Hence in particular, we assume that $\alpha_i = 1$ in what follows.

3.4

Let E_1 be an irreducible component of E. From the previous subsection, it is clear that we only have to consider the case that the image $\tau(E_1)$ is given by $(0, 0, \ldots, 0) \in \bigcap_{i=1}^{n} D_i$. First of all, we make the following observation.

Lemma 3 *There are only a finite number of irreducible components W_1 of W which would give rise to E_1 having empty intersection with other component of E.*

Proof. In such case, E_1 obtained from such component would be equal to one of the connected component of $\tau^{-1}(D)$. But D has only a finite number of connected components. By using the fact that F^{-1} is biholomorphic to its image on $\bar{Z} - D$, it is quite clear that there would only be a finite number of such W_1.

3.5

From this point on, we assume that E_1 has non-empty intersection with some other component E_2 of E. Moreover, we may assume that E_1 and E_2 are in normal crossing at the very beginning by Hironaka's resolution of singularities. Let $y \in E_1$ and V be a coordinate neighbourhood of y in $\bar{Y}$ which is mapped into U in $\bar{Z}$ by τ. Recall that we can express τ as follows in suitable coordinates.

$$z_1 = w_1^{p_1}\zeta_1$$

$$\vdots$$

$$z_n = w_1^{p_n}\zeta_n$$

From this we have

$$\frac{z_i^{p_j}}{z_j^{p_i}} = \frac{\zeta_i^{p_j}}{\zeta_j^{p_i}}, \quad 1 \leqq i, j \leqq n .$$

As the left hand side is a well-defined meromorphic function in a neighbourhood of E_1 in $\bar{Y}$, so is the right hand side. We call such a function θ_{ij}. In particular, they are well-defined on E_1.

Lemma 4 *θ_{j1} has at least a zero and a pole on E_1 for each $j \geqq 2$. Moreover, in a coordinate patch with W_1 given as $w_1 = 0$,*

$$dw_1 \wedge d\theta_{21} \wedge \cdots \wedge d\theta_{n1}$$

is non-degenerate at a generic point of W_1.

Proof. We prove this by contradiction. Suppose on the contrary that θ_{n1} does not have a zero or pole when restricted to E_1. Then we can write

$$\theta_{n1}(w_1, w') = c_{n1} + w_1 f_{n1}(w_1, w')$$

for some constant $c_{n1} \neq 0$, here we represent $(w_2, \ldots, w_n)$ by w'. On the other hand, from

$$\frac{z_i^{p_j}}{z_j^{p_i}} = \theta_{ij}\,,$$

we have for $2 \leqq i \leqq n$,

$$p_1 \frac{dz_i}{z_i} = p_i \frac{dz_1}{z_1} + \frac{d\theta_{i1}}{\theta_{i1}}\,.$$

We are going to apply the result of §1.4. Notice that we may assume that all the α_i in Proposition 3 of §1 are equal to 1, as observed in §3.2. Restricting to a smaller neighbourhood if necessary, we have on V,

$$\begin{aligned} dV_{\bar{Y}} &= e^{\varphi(z_1, \ldots, z_n)} \prod_{i=1}^{n} |z_i|^{-2} \cdot |dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n|^2 \\ &= c \cdot e^{\varphi(z_1, \ldots, z_n)} |z_1|^{-2} \prod_{i=2}^{n} |\theta_{i1}|^{-2} \cdot |dz_1 \wedge d\theta_{21} \wedge \cdots \wedge d\theta_{n1}|^2 \\ &= e^{\varphi(w_1^{p_1}\zeta_1, \ldots, w_1^{p_n}\zeta_n)} |w_1|^{-2p_1} |\eta|^2 \cdot |(w_1^{p_1} d\zeta_1 + p_1 w_1^{p_1 - 1} \zeta_1 dw_1) \\ &\quad \wedge d\theta_{21} \wedge \cdots \wedge d\theta_{(n-1)1} \wedge (f_{n1} dw_1 + w_1 df_{n1})|^2 \\ &= e^{\varphi(w_1^{p_1}\zeta_1, \ldots, w_1^{p_n}\zeta_n)} |w_1|^{-2(p_1 - 1 + 1 - p_1)} |\zeta|^2 |dw_1 \wedge \cdots \wedge dw_n|^2 \\ &= e^{\varphi(w_1^{p_1}\zeta_1, \ldots, w_1^{p_n}\zeta_n)} |\zeta|^2 |dw_1 \wedge \cdots \wedge dw_n|^2\,. \end{aligned}$$

Here η and ζ are holomorphic functions in such a neighbourhood. The factor e^{φ} would be zero in such case at a generic point of E_1 and hence at the preimage of a generic point of W_1 by σ, from Proposition 3 in §1. This implies that the volume form obtained from ω is degenerate on W_1, which is absurd. By the same reason, θ_{j1} has a zero or pole for other values of j. The other statement of the lemma follows from the same argument.

Remark. The above lemma allows us to use $w_1, \theta_{21}, \ldots, \theta_{n1}$ as local coordinate functions at a generic point of W_1.

3.6

We are now in a position to generalize the results of [Y] to bound the smallest exponents p_i in the definition of τ. We recall that the Lelong number of a curve C on $\bar{Z}$ is defined to be

$$\nu_C = \liminf_{z \to 0,\, z \in C} \frac{\varphi(z)}{\log\left(\sum_{i=1}^n |z_i|^2\right)} .$$

As a notation, we define for $\theta_i \in [0, 2\pi)$, $i = 2, \ldots, n$ and $p_j \in R$, $j = 1, \ldots, n$ a curve

$$C^{(p_1, \ldots, p_n)}_{(\theta_2, \ldots, \theta_n)} = \left\{ z \in U \subset \bar{Z} \,\middle|\, \frac{z_1^{p_1}}{z_1^{p_i}} = e^{\sqrt{-1}\theta_i}\varepsilon_i,\ i = 2, \ldots, n \right\}$$

for generic ε_i, $i = 2, \ldots, n$. We fix such a set of values ε from this point onward. Lemma 4 allows us to obtain such family of curves whose preimage by τ cuts W_1 at isolated points for all $\theta_i \in [0, 2\pi)$ for suitable choice of ε_i, $i = 1, \ldots, n$. Moreover, we write the Lelong number of such a curve as $\nu^{(p_1, \ldots, p_n)}_{(\theta_2, \ldots, \theta_n)}$. The induced volume form we have on $\bar{Y}$ can be written as

$$dV_{\bar{Y}} = |w_1|^{-2} e^{\varphi(w_1^{p_1}\zeta_1, \ldots, w_1^{p_n}\zeta_n)} \cdot |dw_1 \wedge \cdots \wedge dw_n|^2 .$$

Hence we have,

$$\log(|w_1|^2) + c_1 \leqq \varphi(w_1^{p_1}\zeta_1, \ldots, w_1^{p_n}\zeta_n) \leqq \log(|w_1|^2) + c_2 .$$

From this, we deduce that

$$\begin{aligned} \nu^{(p_1, \ldots, p_n)}_{(\theta_2, \ldots, \theta_n)} &= \nu^{(p_1, \ldots, p_n)}_{(0, \ldots, 0)} \\ &= \lim_{z \to 0,\, z \in C} \frac{\varphi(z)}{\log\left(\sum_{i=1}^n |z_i|^2\right)} . \end{aligned}$$

Notice that we have replaced the inferior limit by strict limit.

Let p_1 and p_n be the largest and the smallest numbers among $p_1, \ldots, p_n$ respectively. The following lemma is the generalization of Lemma 4.4 in [Y].

Lemma 5 *There exist a positive constant c_1 depending on p_i's such that*

$$\nu^{(1, \ldots, 1)}_{(\theta_2, \ldots, \theta_n)} \geqq c_1 \nu^{(p_1, \ldots, p_n)}_{(\theta_2, \ldots, \theta_n)} .$$

Suppose W'_1 is another irreducible component of W such that the corresponding E'_1 in Y is mapped to $(0, \ldots, 0)$ which is the image of E_1 in the above lemma. We can express the corresponding τ' as

$$z_i = u_1^{p'_i}\zeta'_i, \quad i = 1, \ldots, n ,$$

where W'_1 is given $u_1 = 0$ locally. Then we have the following generalization of Lemma 4.5 in [Y].

Lemma 6 *There exists a positive constant c_2 depending on p_i's such that*

$$\nu^{(1, \ldots, 1)}_{(\theta_2, \ldots, \theta_n)} \leqq c_2 \nu^{(p'_1, \ldots, p'_n)}_{(\theta_2, \ldots, \theta_n)} .$$

The proof of the above two lemmas will be given in the next subsection.

3.7

Proof of Lemma 5 From the previous argument, we may assume that $\theta_i = 0$ for $i = 2, \ldots, n$ without loss of generality. Let $z_0 = (z_1^0, \ldots, z_n^0)$ be a point on $C_{(0,\ldots,0)}^{(p_1,\ldots,p_n)}$. Consider the S^1-family of curves given by $C_{\theta_2} = C_{(\theta_2,0,\ldots,0)}$ for $\theta_2 \in [0, 2\pi)$. The picture is as follows, where $z' = (z_1, z_3, \ldots, z_n)$.

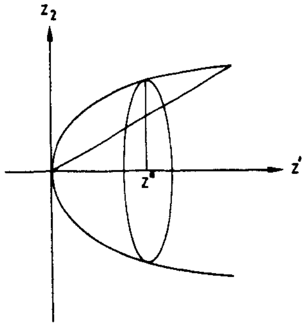

Fig. 1

As φ is subharmonic on the disc $\{z \,\|\, z_2| \leqq |\varepsilon_2|, z_i = z_i^0, i \neq 2\}$, we have

$$\varphi(z_1^0, z_1^0, z_3^0, \ldots, z_n^0) \leqq \sup_{\theta_2 \in [0, 2\pi)} \varphi(z_1^0, e^{\sqrt{-1}\theta_2} z_2^0, \ldots, z_n^0)$$

$$\leqq v_{(0,\ldots,0)}^{(p_1,\ldots,p_n)} \log\left(\sum_{i=1}^{n} |z_i^0|^2\right) + c$$

$$\leqq v_{(0,\ldots,0)}^{(p_1,\ldots,p_n)} \log(n|z_n|^2) + c$$

$$= v_{(0,\ldots,0)}^{(p_1,\ldots,p_n)} \left[\log(n) + \frac{p_n}{p_1} \log(|z_1|^2)\right] + c\,,$$

here we may assume that $|z_i| \leqq |z_j|$ for $i < j$ by taking a sufficiently small neighbourhood of $(0, \ldots, 0)$. Hence we have for the curve $C = C_{(0,\ldots,0)}^{(p_1,p_1,p_3,\ldots,p_n)}$,

$$v_{(0,\ldots,0)}^{(p_1,p_1,p_3,\ldots,p_n)} = \lim_{z \in C} \frac{\varphi(z)}{\log(|z|^2)}$$

$$\geqq v_{(0,\ldots,0)}^{(p_1,p_2,\ldots,p_n)} \frac{p_n}{p_1}\,.$$

It is clear that the same argument gives

$$v_{(\theta_2,\ldots,\theta_n)}^{(p_1,p_1,p_3,\ldots,p_n)} \geqq v_{(\theta_2,\ldots,\theta_n)}^{(p_1,p_2,\ldots,p_n)} \frac{p_n}{p_1}\,.$$

By applying this argument inductively, we have

$$
\begin{aligned}
v_{(\theta_2, \ldots, \theta_n)}^{(p_1, p_2, \ldots, p_n)} &\leqq \frac{p_1}{p_n} v_{(\theta_2, \ldots, \theta_n)}^{(p_1, p_1, p_3, \ldots, p_n)} \\
&\ \ \vdots \\
&\leqq \left(\frac{p_1}{p_n}\right)^{(n-1)} v_{(\theta_2, \ldots, \theta_n)}^{(p_1, p_1, p_1, \ldots, p_1)} .
\end{aligned}
$$

As $v_{(\theta_2, \ldots, \theta_n)}^{(p_1, p_1, \ldots, p_1)} = v_{(\theta_2, \ldots, \theta_n)}^{(1, \ldots, 1)}$, this implies that

$$
v_{(\theta_2, \ldots, \theta_n)}^{(1, \ldots, 1)} \geqq \left(\frac{p_n}{p_1}\right)^{(n-1)} v_{(\theta_2, \ldots, \theta_n)}^{(p_1, \ldots, p_n)} .
$$

Proof of Lemma 6 We just rewind the proof of the last lemma. Let $z_0 = (z_1^0, \ldots, z_n^0)$ be a point on $C_{(0, \ldots, 0)}^{(p_1', \ldots, p_n')}$. Here we assume in contrast to the choice of p_i that $p_i' \leqq p_j'$ for $i > j$ so that $|z_i| \geqq |z_j|$ for $i < j$ for a sufficiently small neighbourhood of 0 on $\bar{Z}$. Using the properties of the plurisubharmonic function again, we have

$$
\begin{aligned}
\varphi(z_1^0, z_2^0, z_3^0, \ldots, z_n^0) &\leqq \sup_{\theta_2 \in [0, 2\pi)} \varphi(z_1^0, e^{\sqrt{-1}\theta_2} z_1^0, \ldots, z_n^0) \\
&\leqq v_{(0, \ldots, 0)}^{(p_1', \ldots, p_n')} \log\left(\sum_{i=1}^n |z_i^0|^2\right) + c \\
&\leqq v_{(0, \ldots, 0)}^{(p_1', \ldots, p_n')} \log(n|z_1|^2) + c \\
&= v_{(0, \ldots, 0)}^{(p_1, \ldots, p_n)} \left[\log(n) + \log\left(\sum_{i=1}^n |z_i|^2\right)\right] + c .
\end{aligned}
$$

Hence

$$
v_{(\theta_2, \ldots, \theta_n)}^{(p_1', p_2', \ldots, p_n')} \geqq v_{(\theta_2, \ldots, \theta_n)}^{(p_1', p_1', p_3', \ldots, p_n')} .
$$

Inductively, we get

$$
v_{(\theta_2, \ldots, \theta_n)}^{(1, \ldots, 1)} \leqq v_{(\theta_2, \ldots, \theta_n)}^{(p_1', \ldots, p_n')} .
$$

The picture is as follows.

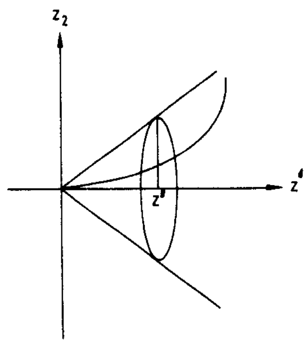

Fig. 2

Remark. In the proof of the above two lemmas, we use the fact that a subharmonic function takes smaller values in the interior of a unit disk B_1 than the superior of its value on the unit circle ∂B_1. In general, we only have uniform estimates on $\varphi(z)$ along $C^{(p_1,\ldots,p_n)}_{(\theta_2,\ldots,\theta_n)}$ for generic θ_i. However, by using Poisson's formula, we can still bound the value of φ in a smaller disk $B_{1/4}$ by its integral over ∂B_1 up to a uniform positive constant c. Then the argument still works with probably another constant c.

3.8

The previous two lemmas implies that if we fix an irreducible component W_1 of W. Then for any other component W'_1, we can find a curve cutting the corresponding E'_1 at isolated points with Lelong number bounded uniformly from below by $\frac{1}{c_1c_2}\nu^{(p_1,p_2,\ldots,p_n)}_{(\theta_2,\ldots,\theta_n)}$. In particular, it follows essentially from definition that

$$\min_{1\leqq i\leqq n}\{p'_i\}\nu^{(p'_1,\ldots,p'_n)}_{(\theta_2,\ldots,\theta_n)}=1\ ,$$

which implies that $\min_{1\leqq i\leqq n}\{p'_i\}$ is uniformly bounded from above. As the number of points in $\bar{Z}$ which are the complete intersection of n divisors D_i on $\bar{Z}$ is finite, we have

Proposition 7 *For any irreducible component of the branching locus, the minimum of the exponent p_i in the corresponding mapping τ is uniformly bounded from above.*

4 Conclusion of proof

4.1

We are going to complete the proof of the Main Theorem in this section. Let B_1 be the subset of the irreducible components of the base locus whose image in $\bar{Z}$ by the corresponding τ are not corner points. Let B_2 be the complement of B_1 in the set of base locus. Let B be the union of B_1 and the set of branching locus. Then from Proposition 7 and Proposition 5, we conclude that we can find a family of curves cutting these components at isolated points whose image in $\bar{Z}$ are curves of bounded degree. The argument in the proof of Proposition 2 allows us to find a birational mapping F' from X to another algebraic manifold $\bar{Z}'$. The new mapping is biholomorphic outside of the set B_2. We can now apply the argument for F and $\bar{Z}$ to F' and $\bar{Z}'$. The only trouble happens when a component W_1 of B_2 is mapped into a corner point in $\bar{Z}'$. In this case F' can be extended across W'_1 as in §3.2, so that if W_1 is a divisor, then Proposition 7 holds for W_1 for the mapping F'. But if $\dim(W_1)\leqq n-2$, then W_1 is contained in a branching locus of F', which is of codimension 1. This is impossible as F' is biholomorphic to its image outside of $U\cap W'_1$ for a neighbourhood U of a generic point on W. Hence by attaching a finite number of sections to F', we get a biholomorphic mapping of X into a quasi-projective variety.

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