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Game Theory

Extension of the Perles-Maschler Solution to N-Person Bargaining Games

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Abstract: The superadditive solution for 2-person Nash bargaining games was axiomatically defined in Perles/Maschler (1981). In Perles (1982) it was shown that the axioms are incompatible even for 3-person bargaining games. In this paper we offer a generalization method of this solution concept for n-person games. In this method, the Kalai-Smorodinsky solution (1975) is revealed as the rule followed to determine the movements along the path of intermediate agreements.

1 Introduction

Here we consider the Bargaining Problem as was specified by Nash (1950). In it the question of selecting an agreement is resolved by means of the construction of a *solution*, which is a function defined in the considered domain of bargaining games, and the achieved proposal is justified by means of an axiomatic characterization.

Alternative proposals for solutions have subsequently appeared in the literature. Two of the most relevant are the Kalai-Smorodinsky solution (1975) and the superadditive solution by Perles-Maschler (1981). Although they all were defined for the case of two players, Nash's solution, like that of Kalai-Smorodinsky, did not present problems for their extension to the case of three or more players. The Nash solution is uniquely generalized in a natural way (see Roth (1979)); while the Kalai-Smorodinsky solution can be generalized in several ways on different possible domains (see Thomson (1980), Peters and Tijs (1984) and Imai (1983)).

Nevertheless, this has not been possible for the superadditive solution. In Perles (1982) a negative result for the case of three players is obtained, which shows the impossibility of finding a solution that satisfies the same set of axioms which characterize it in the case of two players, in a sufficiently basic domain of problems.

There still remains, however, the interesting problem of finding a satisfactory generalization of this solution concept for the n-person case, i.e., discover a well-defined function for an arbitrary number of players, that coincides with the superadditive solution for the particular case of $n = 2$.

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In this work we present a procedure for such a generalization, inspired by the second method followed by Perles and Maschler for the calculation of the solution, through which they designated a path of intermediate agreements or also a Status Quo Set. It will be seen that this path can be reinterpreted as the trajectory which would be originated if the followed rule for moving along it, in each infinitesimal time interval, were the same as that which is followed, in only one step, in the Kalai-Smorodinsky solution.

The organization of the present work is as follows. In section 2 the Perles-Maschler solution is defined and the alternative definition is explained. In section 3 the equivalence of both solutions for the case $n = 2$ is proven. The main contribution is presented in section 4, where, for sufficiently smooth games, the existence and uniqueness of the outcome produced by the new procedure for $n \geq 3$ is proven, and, in section 5, where the solution is also defined on a subclass of polygonal games.

2 The PM Solution for n = 2

Consider 2-person bargaining games of the form of $S \equiv (S, 0)$, where $0 \equiv (0, 0)$ is the conflict point and S is the feasible set in the utility space of the players. We shall impose the following requirements on S:

H1: S is a nonempty, convex and compact set
H2: $S \subset \mathbb{R}^2_+$, and $\exists x \in S$ such that $x \gg 0$
H3: S is comprehensive, i.e., if $u \in S$ and $0 \leq v \leq u$ then $v \in S$

We shall denote by $\mathcal{B}^2$ the family of all 2-person bargaining games $S \equiv (S, 0)$ which verify H1, H2 and H3. Perles and Maschler axiomatically defined the concept of superadditive solution (see Perles/Maschler (1981)).

Definition 2.1: A superadditive solution is a function $\phi : \mathcal{B}^2 \to \mathbb{R}^2_+$ which satisfies the following axioms (written ϕ^S for $\phi(S)$):

Axiom 1: (Pareto Optimality). For each S in $\mathcal{B}^2$, $\phi^S \in OP(S)$, where $OP(S) := \{x \in S : [y \geq x, y \neq x] \Rightarrow y \notin S\}$.

Axiom 2: (Scale Covariance). For each S in $\mathcal{B}^2$ and $a \in \mathbb{R}^2_{++}$ then $\phi^{a*S} = a * \phi^S$, where $a * S := \{a * x : x \in S\}$ and $(a_1, a_2) * (x_1, x_2) := (a_1x_1, a_2x_2)$.

Axiom 3: (Symmetry). If S in $\mathcal{B}^2$ is symmetric, i.e., $(u_1, u_2) \in S \Leftrightarrow (u_2, u_1) \in S$, then $\phi^S_1 = \phi^S_2$.

Axiom 4: (Superadditivity). For all R and S in $\mathcal{B}^2$, $\phi^{R+S} \geq \phi^R + \phi^S$, where $R + S := \{x + y : x \in R, y \in S\}$.

Axiom 5: (Continuity). The restriction of ϕ to $\mathcal{B}_0^2$ is continuous, where the topology in $\mathcal{B}_0^2$ is induced by the Hausdorff metric, and $\mathcal{B}_0^2$ is the subfamily of $\mathcal{B}^2$ that satisfies $\partial S = OP(S)$ (∂S is the north-east boundary of S).

Axioms 1, 2, and 3 are standards and appear in many other solution concepts. For a brief motivation of axiom 4, remember that in Nash's theory, the points in S can represent expectations on uncertain events. These events can themselves be bargaining problems. For example, the players are able to know that tomorrow the feasible set will be, either R with probability p, or S with probability $1 - p$; for which the set of feasible outcomes will be $H = pR + (1 - p)S$. Axiom 4 lets us avoid the possibility that one player prefers reaching an immediate agreement (for example, $\phi_1^H > p\phi_1^R + (1 - p)\phi_1^S$) while the other prefers to wait in order to know which of the two, R or S, is the effective game ($\phi_2^H < p\phi_2^R + (1 - p)\phi_2^S$).

Both authors proved that this solution exists in $\mathcal{B}^2$ and is unique. They also gave an explicit formula for ϕ which we have defined further on.

Given a $S \in \mathcal{B}^2$, we denote p^S and q^S to be the points of intersection of ∂S with the u_2-axis and the u_1-axis of $\mathbb{R}_+^2$ respectively.

Definition 2.2: The Perles-Maschler solution of $S \in \mathcal{B}^2$ is defined as that point $u^S = PM(S) \in \partial S$ which satisfies:

$$\int_{p^S}^{u^S} \sqrt{-du_1 du_2} = \int_{u^S}^{q^S} \sqrt{-du_1 du_2}, \quad u^S \text{ Pareto optimal} \tag{2.1}$$

where the integrals are taken along arcs of ∂S.

Remark 1: It is straightforward to extend the theory to games with disagreement point other than $(0, 0)$. It is sufficient to replace the utilities by the utility gains in the definition 2.2, and extending the axiom 2 to include *Translation Covariance*: for all $b \in \mathbb{R}^2$, $\phi(S + b) = \phi(S) + b$.

For the question of whether a similar solution for games with more players exists, unfortunately a negative answer is obtained. In Perles (1982) it is proven that a solution $u : \mathcal{B}^3 \to \mathbb{R}_+^3$ which simultaneously satisfies axioms 1, 2, 3 and 4 does not exist. (For the proof of non-existence the subfamily of games which are finite sum of non-degenerate simplices is used).

Although an extension for n players using the axiom of superadditivity cannot be carried out, it is possible, nevertheless, to extend the rule which is underlying in the determination of u for $\mathcal{B}^n$, with $n \geq 3$.

To illustrate the intuitive meaning of the rule which we will propose, let us first consider the interpretations of the PM solution in an example like that in figure 1.

Perles and Maschler propose two intuitive procedures which give rise to their solution.

Procedure 1: Suppose two points moving towards each other along ∂S, starting simultaneously from p and from q. Each point moves so that the product of the com-

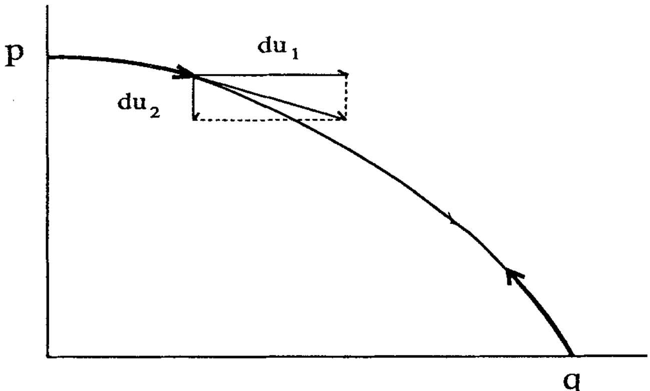

Fig. 1.

ponents of its velocity vector in the time interval dt is a constant, for example, -1, i.e., $du_1 du_2 = -1$. Then, these points will meet at u^S. The total time which each one will need to traverse ∂S is:

$$t^S = \int_{\partial S} \sqrt{-du_1 du_2}$$

and, thus, condition (2.1) means:

$$(1/2)t^S = \int_p^{u^S} \sqrt{-du_1 du_2} = \int_{u^S}^q \sqrt{-du_1 du_2}$$

Procedure 2: Suppose that the players leave from 0 along a continuous path which takes them until u^S. Each point of the path can be thought of as an intermediate agreement that preserves the balance of power in the following sense: if s is on the path and we consider the residual game (S, s), the solution in it should supply the same result u^S. Furthermore, for each $t \in [0, (1/2)t^S]$ the respective point $s(t)$ of the path is defined by $s(t) = (v_1(t), w_2(t))$ where $v(t)$ and $w(t)$ are the points of ∂S which verify:

$$\int_p^{v(t)} \sqrt{-du_1 du_2} = \int_{w(t)}^q \sqrt{-du_1 du_2} = t$$

In this way, we have defined the Status Quo Set by means of the graph of a function $s_2 = \varphi(s_1)$ which joins the origin with u^S. Later on we will see the value which the derivative φ' takes at each point. In our example, ∂S can be expressed as the graph of two functions, one the inverse of the other, $u_2 = f(u_1)$ or else $u_1 = g(u_2)$ (where for each $(u_1, u_2) \in \partial S$ one has $f'(u_1) = 1/g'(u_2)$). Imagine that the two players I and II begin negotiations in q and p respectively. At the beginning, each player demands the highest possible payoff for himself; since $(q_1, p_2) \notin S$, (which is the utopia point of the game), those demands are incompatible. Then, in order to achieve an agreement, the players make gradual concessions by moving towards each other along ∂S. If at each instant t, they move in such a way that during the infinitesimal time interval

dt, the changes of the utilities due to further concessions satisfy $dv_1 dv_2 = dw_1 dw_2$, where w and v are the positions of I and II at time t (see figure 2), then we have:

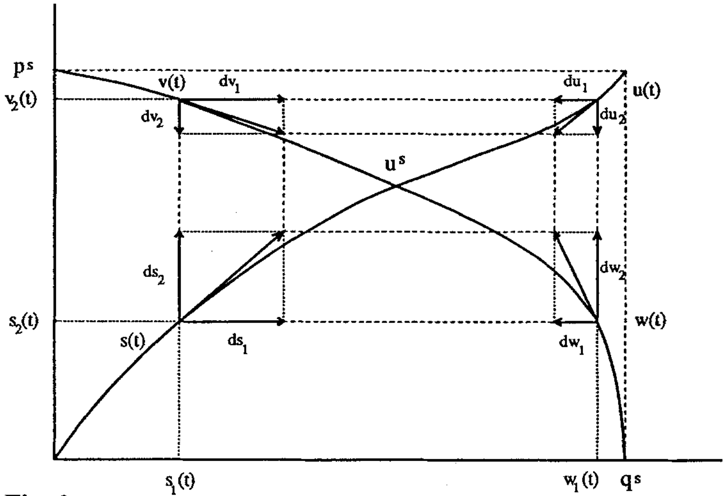

Fig. 2.

$$\varphi' = \frac{ds_2}{ds_1} = \frac{dw_2}{dv_1} = \frac{f'(w_1)dw_1}{g'(v_2)dv_2} = \frac{f'(w_1)dv_1}{g'(v_2)dw_2} = \frac{f'(w_1)}{g'(v_2)} \cdot \frac{1}{\varphi'}$$

which means:

$$\forall t \in (0, (1/2)t^S] : \varphi'(s_1(t)) = \sqrt{\frac{f'(w_1(t))}{g'(v_2(t))}} = \sqrt{f'(w_1(t)) \cdot f'(v_1(t))}$$

The slopes in $v(t)$ and $w(t)$ determine the exchange rates for the utilities: one util of I for $-f'(v_1(t))[-f'(w_1(t))]$ utils of II. This means that when the players move along the status quo set, the ratio of the rates of their gains is the *geometric mean of the above exchange rates.*

Although this condition does not have an easy intuitive meaning, as the very same authors emphasize (P/M (1981), end of section 5[2]) the condition $dv_1(t)dv_2(t) = dw_1(t)dw_2(t)$ can be rewritten as follows:

$$\frac{dv_1(t)}{dw_1(t)} = \frac{dw_2(t)}{dv_2(t)} \tag{2.2}$$

which has a more attractive interpretation in terms of mutual concessions:

$$\frac{\text{the utils that } I \text{ is offered}}{\text{the utils that } I \text{ yields}} = \frac{\text{the utils that } II \text{ is offered}}{\text{the utils that } II \text{ yields}}$$

[2] Also see the footnote 13, in which they thank Martin Beckmann for showing them this aspect.

And it can also be thought of as an equitable rule of concessions until finally reaching the agreement u^S.

It is this condition which will open us the way for tackling bargaining problems with three or more players. To be able to adequately interpret (2.2) we will need to define beforehand the *Utopia Points path.* At each point $s(t)$ of the status quo path there will be associated the corresponding utopia point of the residual game $(S, s(t))$, defined by $u(t) = (w_1(t), v_2(t))$, $\forall t \in [0, (1/2)t^S]$. And so, the Status Quo path automatically defines the Utopia Points path, and this will be represented by the graph of a function $u_2(t) = \psi(u_1(t))$, which joins u^S with the utopia point of the game (q_1, p_2), (see figure 2). If we do $du_1(t) = dw_1(t), du_2(t) = dv_2(t), ds_1(t) = dv_1(t)$ and $ds_2(t) = dw_2(t)$, then (2.2) is converted into:

$$\frac{ds_2(t)}{ds_1(t)} = \frac{du_2(t)}{du_1(t)}$$

which, in terms of the functions φ and ψ, is the condition:

$$\forall s_1 \in (0, u_1^S) : \varphi'(s_1) = \psi'(u_1), \quad \text{where } u_1 = g(\varphi(s_1))$$

Since moving along a path of intermediate agreements $(s_1, \varphi(s_1))$ implies making concessions in the utopia points $(u_1, \psi(u_1))$; then, in order to reach the point u^S the rule the players need to follow is that *the ratio of the rates of their gains – φ' – at each point has to coincide with the ratio of the rates of their losses – ψ'.*

Remark 2: It is interesting to point out the connection that exists between the Perles-Maschler solution and the Kalai-Smorodinsky solution, which is made evident from the previously stated rule. For the game like that which appears in figure 2, given the point of disagreement (0, 0) and the utopia point (q_1, p_2), the outcome that the KS solution produces is that point on the boundary, $v \in \partial S$, such that $v_1 \div v_2 = q_1 \div p_2$, (see Kalai-Smorodinsky (1975)). Said in another way, the proportion between the gains which the players have obtained in v must coincide with the proportion between the maximum aspirations which they had at the start before any type of agreement was materialized. However, it is straightforward that this condition can always be rewritten as $v_1 \div v_2 = (q_1 - v_1) \div (p_2 - v_2)$. That is to say, in the agreement, the proportion between obtained gains with respect to the conflict point is the same as the proportion of the losses with respect to the utopia point.

With certain analogy, in the PM solution, when the players move in a continuous way along the path of intermediate agreements from 0 to u^S, their corresponding maximum aspirations decrease with continuity from (q_1, p_2) to u^S, in such a way that at each instant t and at each infinitesimal time interval dt, the proportion between the gains passing from $s(t)$ to $s(t + dt)$ is equal to the proportion between the losses passing from $u(t)$ to $u(t + dt)$. Thus, KS is *revealed as the rule followed to determine the movements along the path of intermediate agreements, at each infinitesimal instant of time dt.*

3 Equivalence

In this section we will give an alternative definition of the Perles-Maschler solution, based on the last interpretation presented in section 2, and we will demonstrate that both definitions are equivalent.

Given $S \in \mathcal{B}^2$, let $x^S, y^S \in OP(S)$ such that $x_2^S = p_2^S$ and $y_1^S = q_1^S$, and denote by $s_0 = (x_1^S, y_2^S)$, (see figure 3). Furthermore, $OP(S)$ is the graph of any of the following functions: $u_2 = f(u_1) := \max\{u_2 : u \in S\}$ and $u_1 = g(u_2) := \max\{u_1 : u \in S\}$. These functions are inverse to each other, finite concave, strictly decreasing and absolutely continuous in $[x_1^S, q_1^S]$ and in $[y_2^S, p_2^S]$, respectively. In that way, these functions are differentiable almost everywhere. For each point $x \in S$, being $x \geq s_0$, we have its associated utopia point $(g(x_2), f(x_1))$. For each function $\eta : [x_1^S, q_1^S] \rightarrow \mathbb{R}$ we have a path of points $C(S) := \{(x_1, \eta(x_1)) : x_1 \in [x_1^S, q_1^S]\}$ defined. The Perles-Maschler path is defined as:

Definition 3.1: Given $S \in \mathcal{B}^2$, we say that $C(S)$ is the Perles-Maschler path if η is an absolutely continuous and strictly increasing function verifying:

i) $\eta(x_1^S) = y_2^S$

ii) $\forall x \in C(S) \cap S$:

ii.1) $(g(x_2), f(x_1)) \in C(S)$

ii.2) $\eta'(x_1) = \eta'(g(x_2))$ almost everywhere.

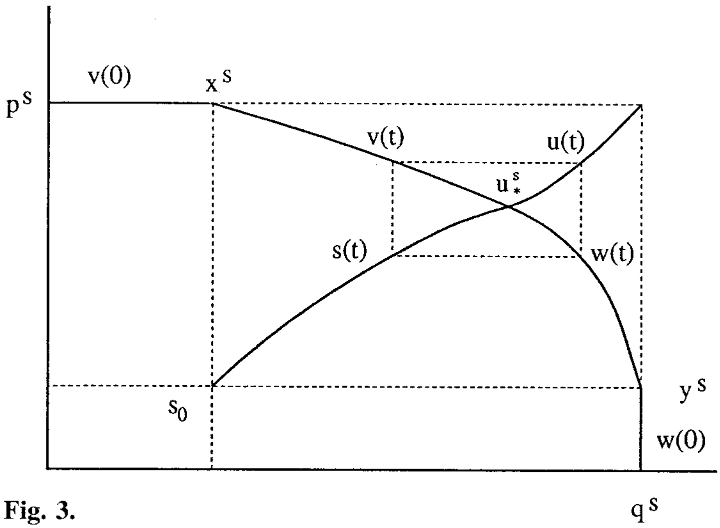

Fig. 3.

In the way in which $C(S)$ and ∂S are constructed, it is straightforward that their intersection is a unique point, which we will denote by u_*^S (i.e., $\{u_*^S\} = C(S) \cap \partial S$). Then we have the following theorem:

Theorem 1: For each $S \in \mathcal{B}^2$, $u_*^S = u^S$.

Proof: The proof is divided into two steps: in the first, we will see that the point $u^S = PM(S)$ is the intersection point of ∂S with the graph of a determined function which we will construct and which verifies all of the properties of definition 3.1 and, in the second step, we will see that u_*^S is uniquely determined and so, $u_*^S = u^S$.

STEP 1: The point u^S is defined as that point on ∂S which verifies:

$$\int_{p^S}^{u^S} \sqrt{-du_1 du_2} = \int_{u^S}^{q^S} \sqrt{-du_1 du_2} = (1/2)t^S$$

For each $t \in [0, (1/2)t^S]$, let $v(t)$ and $w(t)$ be the points on ∂S which satisfy:

$$\int_{p^S}^{v(t)} \sqrt{-du_1 du_2} = \int_{w(t)}^{q^S} \sqrt{-du_1 du_2} = t \tag{3.1}$$

For $t > 0$, the points $v(t)$ and $w(t)$ are uniquely determined; if $S \in \mathcal{B}^2 \setminus \mathcal{B}_0^2$, the points $v(0)$ or $w(0)$ (or both) constitute a line segment of ∂S which is parallel to an axis, if $S \in \mathcal{B}_0^2$, then $v(0) = p^S$ and $w(0) = q^S$. (See figure 3)

For $0 < t \leq (1/2)t^S$, let $s(t) = (v_1(t), w_2(t))$ and, for $t = 0$, let $s(0) = s_0$. For the set $\delta^S := \{s(t) : 0 \leq t \leq (1/2)t^S\}$ we will designate it to be the Status Quo set.

Without loss of generality for that which remains of the proof, we are able to suppose that $S \in \mathcal{B}_0^2$ and, therefore, $s(0) = s_0 = (0, 0)$. Thus, ∂S is the graph of any of the following functions: $u_2 = f(u_1) := \max\{u_2 : (u_1, u_2) \in S\}$ or $u_1 = g(u_2) := \max\{u_1 : (u_1, u_2) \in S\}$. These functions are inverse to each other; they are finite concave, strictly decreasing and absolutely continuous in $[0, q_1^S]$ and in $[0, p_2^S]$, respectively. In that way, these functions are differentiable almost everywhere. Because of the absolute continuity of f and g, expression (3.1) can be written as:

$$\int_0^{v_1(t)} \sqrt{-f'(u_1)}\, du_1 = \int_0^{w_2(t)} \sqrt{-g'(u_2)}\, du_2 = t \tag{3.2}$$

Since the integrands in (3.2) are positive almost everywhere, it follows that both $v_1(t)$ and $w_2(t)$ increase strictly and continuously from 0 to u_1^S and u_2^S, respectively, as t increases from 0 to $(1/2)t^S$. Thus, the status quo set is the graph of a continuous, strictly increasing function $s_2 = \varphi(s_1)$, connecting the origin to the point u^S.

It is well known that the right derivative $f'_+(u_1)$ exists and is continuous on the right in $0 \leq u_1 \leq u_1^S$. Similarly, the left derivative $f'_-(u_1)$ exists and is continuous on the left in $0 < u_1 \leq u_1^S$. Moreover, $f'_+ = f'_-$ almost everywhere. So, since $s_1(t) = v_1(t)$, it follows from (3.2):

$$\left(\frac{dt}{ds_1}\right)_- = \sqrt{-f'_-(s_1)} \quad \text{and} \quad \left(\frac{dt}{ds_1}\right)_+ = \sqrt{-f'_+(s_1)} \quad 0 < s_1 \leq u_1^S$$

(The second equality holds also at $s_1 = 0$).

Similarly, since $s_2(t) = w_2(t)$, one shows that:

$$\left(\frac{dt}{ds_2}\right)_- = \sqrt{-g'_-(s_2)} \quad \text{and} \quad \left(\frac{dt}{ds_2}\right)_+ = \sqrt{-g'_+(s_2)} \quad 0 < s_2 \le u_2^S$$

Thus, since $s = (v_1, w_2)$ on δ^S:

$$\left(\frac{d\varphi}{ds_1}\right)_+ = \sqrt{-f'_+(v_1)} \div \sqrt{-g'_+(w_2)} = \sqrt{f'_+(v_1) \cdot f'_-(w_1)} \quad 0 < s_1 \le u_1^S$$

$$\left(\frac{d\varphi}{ds_1}\right)_- = \sqrt{-f'_-(v_1)} \div \sqrt{-g'_-(w_2)} = \sqrt{f'_-(v_1) \cdot f'_+(w_1)} \quad 0 < s_1 \le u_1^S$$

hence, $\varphi'(s_1) = \sqrt{f'(v_1) \cdot f'(w_1)}$ almost everywhere $s_1 \in [0, u_1^S]$.

However, the status quo set δ^S determines another set Γ^S of utopia points defined by $\Gamma^S := \{u(t) : 0 \le t \le (1/2)t^S\}$ where $u(t) := (w_1(t), v_2(t))$. Furthermore, $u(t) = (g(s_2(t)), f(s_1(t)))$, $\forall t \in [0, (1/2)t^S]$. (See figure 2)

Since $v_1(t)$ and $w_2(t)$ increase strictly and continuously from 0 to u_1^S and u_2^S, respectively, and since $\partial S = OP(S)$, then $w_1(t)$ and $v_2(t)$ decrease strictly and continuously as t increases from 0 to $(1/2)t^S$. Thus, Γ^S is the graph of a continuous and strictly increasing function $u_2 = \psi(u_1)$, connecting u^S to the point (q_1^S, p_2^S).

Note that (3.2) can be rewritten as:

$$\int_{v_2(t)}^{p_2^S} \sqrt{-g'(u_2)}\, du_2 = \int_{w_1(t)}^{q_1^S} \sqrt{-f'(u_1)}\, du_1 = t$$

Since $u_1(t) = w_1(t)$ and $u_2(t) = v_2(t)$, we have:

$$\left(\frac{dt}{du_2}\right)_- = -\sqrt{-g'_-(u_2)} \quad \text{and} \quad \left(\frac{dt}{du_2}\right)_+ = -\sqrt{-g'_+(u_2)} \quad u_2^S \le u_2 < p_2^S$$

$$\left(\frac{dt}{du_1}\right)_- = -\sqrt{-f'_-(u_1)} \quad \text{and} \quad \left(\frac{dt}{du_1}\right)_+ = -\sqrt{-f'_+(u_1)} \quad u_1^S \le u_1 < q_1^S$$

Then, since $u = (w_1, v_2)$ on Γ^S:

$$\left(\frac{d\psi}{du_1}\right)_+ = \sqrt{-f'_+(w_1)} \div \sqrt{-g'_+(v_2)} = \sqrt{f'_+(w_1) \cdot f'_-(v_1)} \quad u_1^S \le u_1 < q_1^S$$

$$\left(\frac{d\psi}{du_1}\right)_- = \sqrt{-f'_-(w_1)} \div \sqrt{-g'_-(v_2)} = \sqrt{f'_-(w_1) \cdot f'_+(v_1)} \quad u_1^S \le u_1 < q_1^S$$

from where, $\psi'(u_1) = \sqrt{f'(w_1) \cdot f'(v_1)}$ almost everywhere $u_1 \in [u_1^S, q_1^S]$, and furthermore:

$$\left(\frac{d\psi}{du_1}\right)_+ = \left(\frac{d\varphi}{ds_1}\right)_- \quad \text{and} \quad \left(\frac{d\psi}{du_1}\right)_- = \left(\frac{d\varphi}{ds_1}\right)_+$$

hence: $\varphi'(s_1) = \psi'(u_1)$ almost everywhere $s_1 \in [0, u_1^S]$, $u_1 \in [u_1^S, q_1^S]$, being $u_1 = g(s_2)$ and $s_2 = \varphi(s_1)$.

Moreover, φ and ψ are absolutely continuous functions in $[0, u_1^S]$ and $[u_1^S, q_1^S]$ respectively, because the directional derivatives of f are bounded in every closed subinterval of $[0, q_1^S)$.

It follows that, joining the graphs of φ and ψ, we get the graph of an absolutely continuous and strictly increasing function $x_2 = \eta(x_1)$, connecting the origin to the point (q_1^S, p_2^S) and with the property ii) of definition 3.1. Furthermore, the intersection point of the graph of this function with ∂S is precisely u^S.

STEP 2: Now we will see that the point u_*^S is uniquely determined. In order to do this we will demonstrate that the graph of any absolutely continuous and strictly increasing function $x_2 = \eta(x_1)$, connecting the origin to the point (q_1^S, p_2^S) and with the property ii), intersects ∂S always at the same point: u^S. In effect, by the property ii.1), if $x_2 = \eta(x_1)$, then, $f(x_1) = \eta(g(x_2))$ and, therefore, $f(x_1) = \eta(g(\eta(x_1)))$. By deriving this expression, we get:

$$f'(x_1) = \eta'(g(\eta(x_1))) \cdot g'(\eta(x_1)) \cdot \eta'(x_1) \quad \text{almost everywhere} \quad x_1 \in [0, q_1^S]$$

Since by the property ii.2), $\eta'(x_1) = \eta'(g(\eta(x_1)))$ almost everywhere, we get:

$$f'(x_1) = (\eta'(x_1))^2 \cdot g'(\eta(x_1)) \quad \text{almost everywhere} \quad x_1 \in [0, q_1^S] \Rightarrow$$

$$\sqrt{-f'(x_1)} = \eta'(x_1) \cdot \sqrt{-g'(\eta(x_1))} \quad \text{almost everywhere} \quad x_1 \in [0, q_1^S] \Rightarrow$$

$$\int_0^{x_1} \sqrt{-f'(u_1)}\, du_1 = \int_0^{x_1} \eta'(u_1) \cdot \sqrt{-g'(\eta(u_1))}\, du_1 \Rightarrow$$

$$\int_0^{x_1} \sqrt{-f'(u_1)}\, du_1 = \int_0^{\eta(x_1)} \sqrt{-g'(u_2)}\, du_2$$

Therefore, the intersection of the graph of η and S coincides with the status quo set of S, thus, the graph of η intersects ∂S at u^S. □

4 Definition of the PM Solution for n Players

In this section we are going to extend this equivalent definition of the PM solution to the case of n players. Let us consider the family of n-person bargaining games $S \equiv (S, 0)$ which verifies the following assumptions:

G1: $S \subset \mathbb{R}^n$, S is a nonempty, compact, convex and comprehensive set and $0 \in \text{int}(S)$

(comprehensive : $[\forall x, y \in S : x \leq y] \Rightarrow [\forall z : x \leq z \leq y \Rightarrow z \in S]$)

G2: $\exists k \ll 0$, $k \in S$ such that $\partial S_k = OP(S_k)$, where $S_k := \{x \in S : x \geq k\}$

From G2 it follows that ∂S_k is the graph of any of the following functions:

$$x_i = g^i(x_{-i}) := \max\{u_i \in \mathbb{R} : (x_{-i} \mid u_i) \in S_k\}, i = 1, ..., n$$

where:

$$x_{-i} := (x_1, ..., x_{i-1}, x_{i+1}, ..., x_n)$$
$$(x_{-i} \mid u_i) := (x_1, ..., x_{i-1}, u_i, x_{i+1}, ..., x_n)$$

Furthermore, these functions g^i are well defined, they are continuous and strictly decreasing in S_i^k, where $S_i^k := \{x_{-i} \in \mathbb{R}^{n-1} : x \in S_k\}$, $i = 1, ..., n$.

We shall impose the following additional assumption:

G3: $\forall i = 1, ..., n : g^i$ is a C^2 class function in $\text{int}(S_k^i)$

which implies that $\forall x \in I(S_k)$, the partial derivatives of g^i are negative, $g_j^i(x_{-i}) < 0$, $\forall i, j = 1, ..., n$, $i \neq j$, where $I(S_k) := \{x \in S : x \gg k\}$. Moreover, since the negotiation set $S^+ := S \cap \mathbb{R}_+^n \subset I(S_k)$, g^i is a C^2 class function in S^+, $\forall i = 1, ..., n$.

We denote $\mathcal{B}_+^n$ to be the family of n-person bargaining games which verify G1, G2 and G3.

Assumption G1 is merely another way of rewriting H1, H2 and H3; the only difference is that negative payments for players are admitted. This does not pose any kind of conflict since the extension of the solution is created in terms of the negotiation rule which underlies in the status quo sets, and not with regard to the axiom of superadditivity (see section 6 of P/M (1981)). G2 and G3 are of a technical character and are concerned with the fact that the solution is obtained as a resolution result of a system of differential equations; in particular, G2 guarantees that the initial condition is in the interior of the domain and G3 provides sufficient conditions of smoothness of the problem.

As in section 2, for each disagreement point $x \in S_k$, its corresponding utopia point $g(x) := (g^1(x_{-1}), ..., g^n(x_{-n}))$ is associated with it and, by means of $n-1$ functions $\eta_i : D \subset [k_1, r_1] \to \mathbb{R}$, $i = 2, ..., n$, (where $r_1 := \max\{x_1 : x \in S_k\}$ and D is an open set such that $[0, g^1(0)] \subset D$), one can define a path in $\mathbb{R}^n$, $C(S) := \{x = (x_1, \eta_2(x_1), ..., \eta_n(x_1)) : x_1 \in D\}$.

Definition 4.1: Given a game $S \in \mathcal{B}_+^n$, we say the curve $C(S)$ is the Perles-Maschler path if, $\forall i = 2, ..., n$, η_i is a continuous, strictly increasing and differentiable function ($\forall x_1 \in D : \eta'(x_1) > 0$) verifying:

i) $\eta_i(0) = 0, \forall i = 2, ..., n$

ii) $\forall x \in C(S) \cap S_k$:

ii.1) $g(x) \in C(S)$

ii.2) $\eta_i'(x_1) = \eta_i'(g^1(x_{-1})), \forall i = 2, ..., n$.

Because $C(S)$ is a strictly increasing curve, its intersection with ∂S_k provides one unique point, which allows us to have the following definition of the Perles-Maschler solution in $\mathcal{B}_+^n$.

Definition 4.2: The Perles-Maschler solution is defined as that function $PM : \mathcal{B}^n_+ \to \mathbb{R}^n$ that verifies $PM(S) := C(S) \cap \partial S_k$, for all $S \in \mathcal{B}^n_+$.

Theorem 2: For each game $S \in \mathcal{B}^n_+$, the point $PM(S)$ is well defined.

Proof: Given a game $S \in \mathcal{B}^n_+$, in order to see that $PM(S)$ is well defined, we will demonstrate that there exists a unique Perles-Maschler path. To do this, we will divide the proof into three steps: in the first step, we will show that, supposing that such a curve existed, then its tangent vectors at each point are a solution of a determined eigenvectors problem; in the second step, we will see that the aforementioned problem has a unique solution at each point of $I(S_k)$ and, finally, in the third step, we will demonstrate that there exists a unique curve passing through the origin whose tangent vectors form a part of this field of eigenvectors.

STEP 1: Suppose that the path $C(S)$ exists, (which we will denote by C when there is no place for confusion); then, by property ii.1), if $x_i = \eta_i(x_1)$, for $i = 2, ..., n$, we have:

$$g^i(x_1, \eta_{-i}(x_1)) = \eta_i(g^1(\eta(x_1))), \quad i = 2, ..., n \tag{4.1}$$

where:

$$\eta(x_1) := (\eta_2(x_1), ..., \eta_n(x_1))$$
$$\eta_{-i}(x_1) := (\eta_2(x_1), ..., \eta_{i-1}(x_1), \eta_{i+1}(x_1), ..., \eta_n(x_1)).$$

By deriving the expression (4.1), we obtain for each $i = 2, ..., n$:

$$\left.\begin{aligned} & g^i_1(x_1, \eta_{-i}(x_1)) + \sum_{\substack{k=2 \\ k \neq i}}^{n} g^i_k(x_1, \eta_{-i}(x_1)) \cdot \eta'_k(x_1) = \\ & = \eta'_i(g^1(\eta(x_1))) \cdot \left[\sum_{k=2}^{n} g^1_k(\eta(x_1)) \cdot \eta'_k(x_1)\right] \\ & \forall x_1 : (x_1, \eta(x_1)) \in I(S_k) \end{aligned}\right\} \tag{4.2}$$

Let $h \in \mathbb{R}^n$, $h \gg 0$, be the tangent vector to C at the point $(x_1, \eta(x_1))$, then, $h = (1, \eta'_2(x_1), ..., \eta'_n(x_1))$, where vector h depends on the point $(x_1, \eta(x_1))$; then, by property ii.2), $\eta'_i(g^1(\eta(x_1))) = h_i$, $\forall i = 2, ..., n$. Therefore, it is deduced from (4.2) that:

$$\left.\begin{aligned} & g^i_1(x_1, \eta_{-i}(x_1)) + \sum_{\substack{k=2 \\ k \neq i}}^{n} g^i_k(x_1, \eta_{-i}(x_1)) \cdot h_k = h_i \cdot \left[\sum_{k=2}^{n} g^1_k(\eta(x_1)) \cdot h_k\right] \\ & \forall x_1 : (x_1, \eta(x_1)) \in I(S_k), \forall i = 2, ..., n \end{aligned}\right\} \tag{4.3}$$

If we call $\lambda = \sum_{k=2}^{n} g_k^1(\eta(x_1)) \cdot h_k$, ($\lambda$ also depends on $(x_1, \eta(x_1))$), we obtain from (4.3) that:

$$g_1^i(x_1, \eta_{-i}(x_1)) + \sum_{\substack{k=2 \\ k \neq i}}^{n} g_k^i(x_1, \eta_{-i}(x_1)) \cdot h_k = \lambda \cdot h_i \qquad , \forall x_1 : (x_1, \eta(x_1)) \in I(S_k) \quad \forall i = 2, \ldots, n$$

Thus, λ and h satisfy the following eigenvalues and eigenvectors problem:

$$\lambda h = G(x_1, \eta(x_1)) \cdot h, \quad \forall x_1 : (x_1, \eta(x_1)) \in I(S_k)$$

where:

$$G(x_1, \eta(x_1)) = \begin{bmatrix} 0 & g_2^1(\eta(x_1)) & \cdots & g_n^1(\eta(x_1)) \\ g_1^2(x_1, \eta_{-2}(x_1)) & 0 & \cdots & g_n^2(x_1, \eta_{-2}(x_1)) \\ \vdots & \vdots & & \vdots \\ g_1^n(x_1, \eta_{-n}(x_1)) & g_2^n(x_1, \eta_{-n}(x_1)) & \cdots & 0 \end{bmatrix}$$

Therefore, if path C exists, then, for each point $(x_1, \eta(x_1)) \in C \cap I(S_k)$, there exists a $\lambda = \lambda(x_1, \eta(x_1)) \in \mathbb{R}$ such that the tangent vector to the curve C at $(x_1, \eta(x_1))$, $h = h(x_1, \eta(x_1)) \gg 0$, satisfies $\lambda h = G(x_1, \eta(x_1))h$.

Moreover, since $G(x_1, \eta(x_1)) \cdot h = dg(x_1, \eta(x_1))(h) \leq 0$, then, $\lambda \leq 0$ (see figure 4).

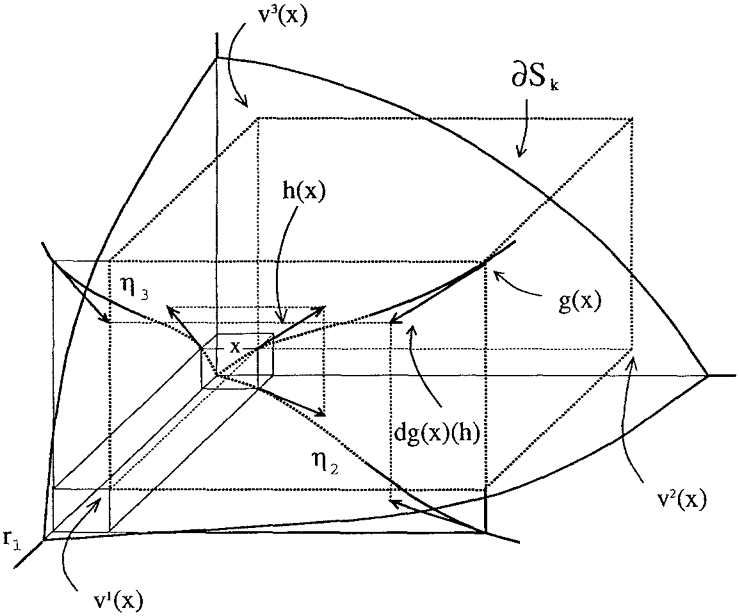

Fig. 4. h(x) collineal to dg(x)(h)

where $v^i(x) := (g^i(x_{-i}), x_{-i}), \forall x \in C \cap S_k, \forall i \in N$.

STEP 2: Thus, for each $x = (x_1, ..., x_n) \in I(S_k)$, we set the following eigenvalues and eigenvectors problem:

$$\lambda(x) \cdot h(x) = G(x) \cdot h(x), \quad h(x) \geq 0, \quad \lambda(x) \leq 0 \tag{4.4}$$

Call $A(x) = -G(x)$ and $\mu(x) = -\lambda(x)$, then, problem (4.4) is equivalent to the problem:

$$\mu(x) \cdot h(x) = A(x) \cdot h(x), \quad h(x) \geq 0, \quad \mu(x) \geq 0 \tag{4.5}$$

The matrix $A(x) \equiv (a_{ij}(x))$ verifies that $a_{ii}(x) = 0, \forall i = 1, ..., n$, and $a_{ij}(x) = -g_j^i(x_{-i}) > 0, \forall i, j = 1, ..., n, \; i \neq j$. Therefore, $A(x)$ is an irreducible matrix, $\forall x \in I(S_k)$. Then, by the Perron-Frobenius theorem (see Gantmacher (1959)), for each $x \in I(S_k)$, problem (4.5) has one unique solution $\mu^*(x) > 0$, (the Frobenius root), and there exists an eigenvector associated with it, $h(x) \gg 0$, and all real eigenvectors are uniquely determined, up to the product by a scalar. Furthermore, the Frobenius root is a simple root of the characteristic equation: $\det(\mu I - A) = 0$. Then, for each $x \in I(S_k)$, we take that eigenvector $h(x)$ associated to $\mu^*(x)$ such that $h_1(x) = 1$, i.e., $h(x) = (1, h_2(x), ..., h_n(x))$, and we define the following function:

$$f : I(S_k) \to \mathbb{R}^{n-1}_{++}$$
$$x \to f(x) = (h_2(x), ..., h_n(x))$$

Because of the foregoing, f is well defined. Let us see that f is a C^1 class function in $I(S_k)$. For each $x \in I(S_k)$, the Frobenius root $\mu^*(x)$ of problem (4.5) is the only positive root of the characteristic equation $p(x, \mu) := \det(\mu(x)I - A(x)) = 0$, and is a simple root of such an equation, (therefore, $p_\mu(x, \mu^*) \neq 0$). As $\det(\mu(x)I - A(x))$ is a polynomial of the nth degree whose coefficients are C^1 class functions in $I(S_k)$, because g^i is a C^2 function in $I(S_k)$, $\forall i = 1, ..., n$, then we have the conditions to apply the implicit function theorem, which yields that $\mu^*(x)$ is a C^1 function in $I(S_k)$.

Given $x \in I(S_k)$, and its Frobenius root, $\mu^*(x)$, any of the associated eigenvector, $h = h(x)$, is a solution to the system:

$$(\mu^*(x)I - A(x)) \cdot h(x) = 0 \tag{4.6}$$

This system is a system of n linear equations whose coefficients are C^1 class functions in $I(S_k)$, and its solution is a 1-dimensional vectorial subspace, so, the rank of the matrix $(\mu^*(x)I - A(x))$ is $n - 1$. If we denote for $B(x) := (\mu^*(x)I - A(x))$, there exists a submatrix of order $n - 1$ whose determinant is different from zero; suppose without loss of generality, that this submatrix is formed by the first $n - 1$ rows and $n - 1$ columns of $B(x)$. Then, it is easy to see that the solution to system (4.6) is of the form:

$$\{\alpha(B_1(x), B_2(x), ..., B_n(x)) : \alpha \in \mathbb{R}\}$$

where:

$$B_i(x) = (-1)^{i+1}\begin{vmatrix} b_{11}(x) & \dots & b_{1,i-1}(x) & b_{1,i+1}(x) & \dots & b_{1n}(x) \\ \vdots & & \vdots & \vdots & & \vdots \\ b_{n-1,1}(x) & \dots & b_{n-1,i-1}(x) & b_{n-1,i+1}(x) & \dots & b_{n-1,n}(x) \end{vmatrix}$$

Thus, $B_i(x)$ is a $\mathcal{C}^1$ class function in $I(S_k)$, $\forall i = 1, ..., n$. Furthermore, it is known that there exists a strictly positive eigenvector associated to $\mu^*(x)$. So, there exists an $\alpha \in \mathbb{R}$ such that:

$$\alpha(B_1(x), B_2(x), ..., B_n(x)) = (1, h_2(x), ..., h_n(x))$$

with $h_1(x) > 0$ and $h_i(x)$ a $\mathcal{C}^1$ class function in $I(S_k)$, $\forall i = 2, ..., n$. Hence, it follows that function f is a $\mathcal{C}^1$ function in $I(S_k)$.

STEP 3: Let us now see that there exists a unique curve passing through the origin whose tangent vectors form part of this field of eigenvectors defined by function f. For this, we set the following problem of ordinary differential equations:

$$\begin{cases} \eta'(x_1) = f(x_1, \eta(x_1)) \\ \eta(0) = 0 \end{cases} \tag{4.7}$$

i.e.:

$$\begin{cases} \eta_i'(x_1) = h_i(x_1, \eta(x_1)) & ,i = 2, ..., n \\ \eta_i(0) = 0 & ,i = 2, ..., n \end{cases}$$

As f is a $\mathcal{C}^1$ function in $I(S_k)$ and $0 \in \text{int}(S_k)$, there exists $k' \in \mathbb{R}^n$ such that $k \ll k' \ll 0$. Then, f is a $\mathcal{C}^1$ function in $S_{k'}$, where $S_{k'} := \{x \in S : x \geq k'\}$ is compact and $0 \in \text{int}(S_{k'})$, therefore, problem (4.7) has a unique solution $x_i = \eta_i(x_1)$, $i = 2, ..., n$, defined in an interval of the form $[-\varepsilon, \varepsilon]$ and, furthermore, this solution is extendable to the right until reaching ∂S^+, i.e., problem (4.7) has a unique solution $x_i = \eta_i(x_1)$, $i = 2, ..., n$, defined at an interval $[-\varepsilon, \alpha]$ such that $(\alpha, \eta(\alpha)) \in \partial S^+$. By definition, this point $(\alpha, \eta(\alpha))$ is the Perles-Maschler solution for the game $S \in \mathcal{B}_+^n$. □

Remark 3: A fact to take into account with respect to theorem 2 is that the convexity of S throughout the theorem did not play a role, so, from the point of view of the definition of the solution, convexity is not a necessary domain restriction.

Remark 4: Concessions Path.

It is easy to prove that a condition equivalent to that of section 2, $dv_1 dv_2 = dw_1 dw_2$, is verified. For this, beginning from path C, the corresponding *Concessions Paths* C^i are defined, which are the points in ∂S associated with each player i that converge in $PM(S)$:

$$C^i(S) := \{v^i(x) = (g^i(x_{-i}), x_{-i}) \in \partial S_k : x \in C(S) \cap S_k\}$$

If at each point $x \in C(S) \cap S_k$ the tangent vector to the curve is $h(x)$, then the corresponding tangent vector to the curve $C^i(S)$ at the point $v^i(x)$ is $dv^i(x) = (dg^i(x_{-i})(h_{-i}(x)), h_{-i}(x))$ (see figure 5).

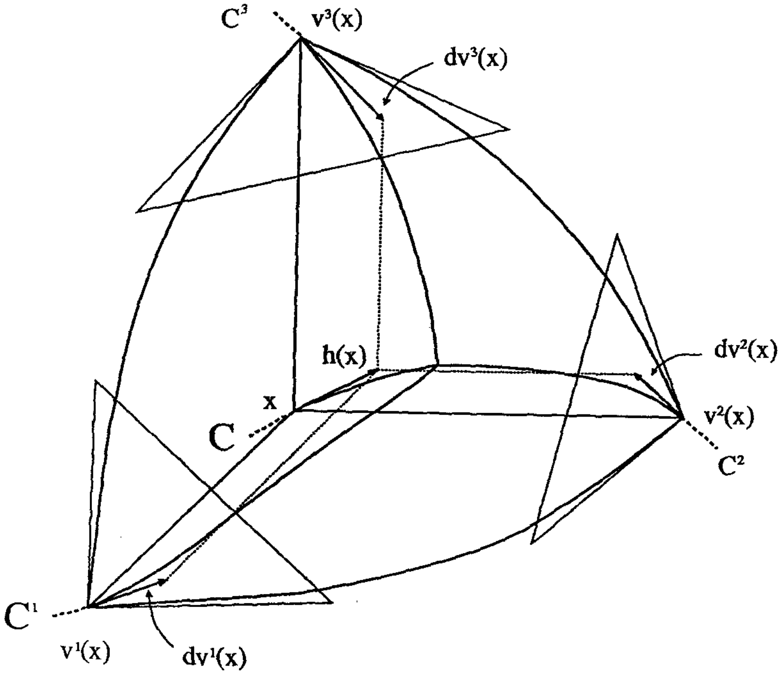

Fig. 5.

As $dg^i(x_{-i})(h_{-i}(x)) = \sum_{\substack{k=1 \\ k \neq i}}^{n} g^i_k(x_i) \cdot h_k(x) = \lambda(x) \cdot h_i(x)$, if we take the product of the components of the vector $dv^i(x)$, we have that $\Pi(dv^i(x)) = \lambda(x)\Pi(h(x)), \forall i = 1, ..., n$.

Remark 5: Comparison with the Raiffa procedure

We can compare our procedure with that suggested by Raiffa (1953), in its continuous version. In the Raiffa procedure, starting at the status quo point, the negotiation model effects step by step improvements in the player's positions until a Pareto optimal point is reached. The rule for the construction of this negotiation path is the following: the slope at each point is the same as that of the straight line joining that point and its corresponding $g(x)$. Formally, we define this path, in the same way as definition 4.1, as follows:

Definition 4.3: Given a game $S \in \mathcal{B}^n_+$, we say the curve $R(S)$ is the Raiffa path if, $\forall i = 2, ..., n$, η_i is a continuous, strictly increasing and differentiable function verifying:

i) $\eta_i(0) = 0, \forall i = 2, ..., n$
ii) $\forall x \in R(S) \cap S_k$:
   ii.1) $g(x) \in R(S)$
   ii.2) $\eta'_i(x_1) = \dfrac{g^i(x_{-i}) - x_i}{g^1(x_{-1}) - x_1}, \forall i = 2, ..., n.$

Equivalently, the tangent vector to the curve $R(S)$ at the point x, for all $x \in R(S) \cap S_k$, is collineal to the vector $g(x) - x$. It is easy to see that $R(S)$ is uniquely determined by this system of differential equations.

5 Domain Extension

An interesting discussion appears when one wants to define the path $C(S)$ in games in which the functions g^i are C^2 class almost everywhere in S_k^i, as happens, for example, in polygonal games (i.e., games whose ∂S_k is the union of pieces of hyperplanes).

Conceptually, this should not involve problems to define the path $C(S)$ starting from the functions η_i of definition 4.1, it is sufficient to add "almost everywhere" to condition ii.2); that is to say, it must be verified that in almost everywhere $x \in C(S) \cap S_k$ and in its corresponding $g(x)$, the tangent vector $h(x)$ to the path is collineal to its corresponding $dg(x)(h(x))$.

In the points $x \in I(S_k)$ in which all the functions g^i, $\forall i = 1, ..., n$, are C^2 class in a neighbourhood, there exists a unique hyperplane $H^i(x)$ tangent to ∂S_k in its corresponding points $v^i(x)$. If we denote $\lambda^i(x)$ to be the corresponding normal vector to $H^i(x)$, then the corresponding ith row of the matrix $G(x)$ is uniquely determined by $g_{ij}(x) = -\lambda_j^i(x)/\lambda_i^i(x)$, $\forall j \neq i$, y $g_{ii}(x) = 0$; which allows us to determine the vector $h(x)$ that solves (4.4). Now then, in the points $x \in I(S_k)$ in which some function g^i is not differentiable, in their corresponding points $v^i(x)$ there are more than one (possibly infinite) tangent hyperplanes to be considered. We shall then have as many matrices $G(x)$ as possible combinations of associated hyperplanes.

To illustrate this fact, we consider the most simple case in which problems may appear, like that in figure 6.

Here, the point $v^3(x)$ on ∂S_k is simultaneously determined by means of two different functions: $v_3^3(x) = g^{31}(x_{-3}) = g^{32}(x_{-3})$. Now, in order to find $h(x)$, we can determine the third row of $G(x)$ by means of the normal vector either to $H_1^3(x)$, or to $H_2^3(x)$. From among the two possible solutions which appear in the figure, only $h^1(x)$ is feasible, because $h^2(x)$ marks an increase direction of $C^3(S)$, starting from $v^3(x)$, in a region of ∂S_k that is *not* determined by g^{32}.

A priori, the possibility of finding games in which, constructing $C(S)$, there appear branches in the path, may not be dismissed, because of the fact that there are several feasible solutions $h(x)$, or else interruptions, due to the non-existence of these. In the following, we shall show sufficient conditions which guarantee the existence and the uniqueness of the vector $h(x)$ at those points $x \in I(S_k)$ in which some function g^i is not differentiable.

Fix the point x in what remains. The tangent vector $h(x)$ only shows the direction of the path at this point, so, without loss of generality, we can normalize it supposing that it belongs to $\Delta := \{h \in \mathbb{R}_+^n | \sum_{i=1}^n h_i = 1\}$. Suppose that for each $h \in \Delta$ and for each $i \in \{1, ..., n\}$, there exists the following directional derivative of the function g^i at the point x in the direction of the vector h_{-i}, which we shall denote by $d_i(h)$:

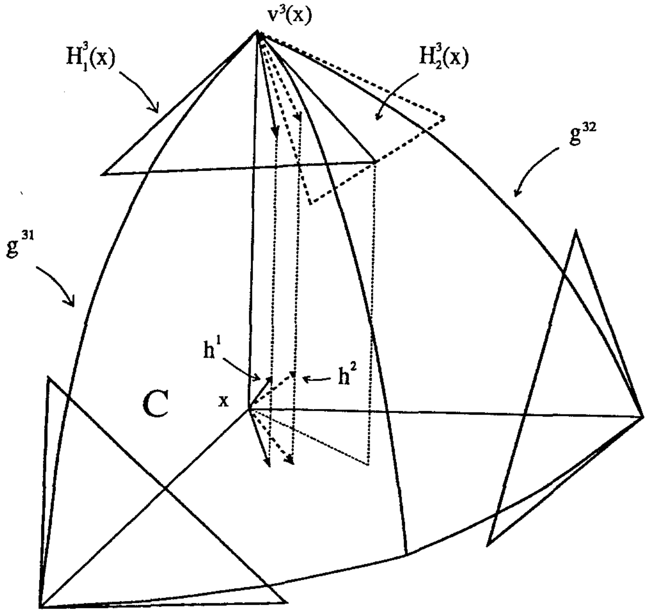

Fig. 6.

$$d_i(h) := \lim_{\substack{t \to 0 \\ t > 0}} \frac{g^i(x_{-i} + th_{-i}) - g^i(x_{-i})}{t}$$

Then, we can define the following function:[3]

$$d : \Delta \to \mathbb{R}^n$$
$$h \to d(h) := (d_i(h))_{i=1}^n$$

There is a direction starting from $v^i(x)$, $\forall i = 1, ..., n$, associated with each vector $h \in \Delta$. This direction is the following: (see figure 7)

$$dv^i(x)(h) = (d_i(h), h_{-i})$$

Therefore, the vector $d(h)$ represents the decrease direction of the players' utilities starting from $g(x)$ when the utilities are increased in the direction of the vector h starting from x. Hence, we are looking for a vector $h \in \Delta$ which is collineal to the vector $d(h)$.

Suppose that $d(h) \neq 0$, $\forall h \in \Delta$, then, proving the existence of such a vector h is equivalent to finding a fixed point of the function $F : \Delta \to \Delta$ defined as:

$$F(h) := \frac{d(h)}{\sum_{i=1}^{n} d_i(h)}$$

[3] Note that the functions d_i are homogeneous, i.e., $d_i(\alpha h) = \alpha \cdot d_i(h)$, $\forall \alpha > 0$.

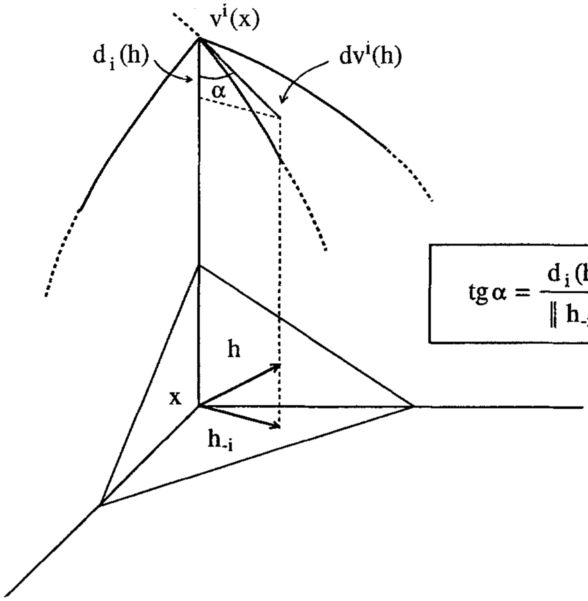

Fig. 7.

because, when $F(h^*) = h^*$, we have that $\lambda^* h^* = d(h^*)$, being $\lambda^* = \sum_{i=1}^{n} d_i(h^*)$, which is the general condition equivalent to that in (4.4), that we had for the case in which all the functions g^i are C^2 class (since, in this case, $d(h) = dg(x)(h)$).

Theorem 3: If S verifies G2 and the function d is well-defined and continuous in Δ, then the function F has an unique fixed point.

Proof:

Existence: The function F is well-defined because $d_i(h) \leq 0, \forall i = 1, ..., n$, and $\sum_{i=1}^{n} d_i(h) \neq 0$, since the functions g^i are strictly decreasing. Therefore, $F_j(h) := \frac{d_j(h)}{\sum_{i=1}^{n} d_i(h)} \geq 0$ and $\sum_{i=1}^{n} F_i(h) = 1$.

Moreover, F is a continuous function because d is continuous, and Δ is a compact and convex set, so, by the Brower's fixed point theorem, there exists $h^* \in \Delta$ such that $F(h^*) = h^*$ and, by construction, this means that there exists $h^* \in \Delta$ and $\lambda^* \leq 0$ such that $\lambda^* h^* = d(h^*)$.

Uniqueness: Suppose there exist two fixed points of F, i.e., there exist $h^*, k^* \in \Delta$ and $\lambda_1^*, \lambda_2^* \leq 0$ such that:

$$\lambda_1^* h^* = d(h^*), \quad \lambda_2^* k^* = d(k^*) \tag{5.1}$$

Let us see, firstly, that, for each $i \in \{1, ..., n\}$, there exists a vector $z^i \in \mathbb{R}^n$, $z^i \gg 0$ which is perpendicular simultaneously to both directions $dv^i(x)(h^*)$ and $dv^i(x)(k^*)$. In effect, Denote by A_i to the following matrix of two rows and n columns:

$$A_i := \begin{bmatrix} dv^i(x)(h^*) \\ dv^i(x)(k^*) \end{bmatrix}$$

We look for a vector $z^i \in \mathbb{R}^n$, $z^i \gg 0$ such that $A_i z^i = 0$. By means of one theorem of the alternative, we have that, either $A_i z^i = 0$, $z^i \gg 0$ has a solution, or $zA_i \gg 0$ has a solution. Let us now see that the second alternative is not possible. Suppose that it were possible, i.e., there exists $z \in \mathbb{R}^2$ such that $zA_i \gg 0$, then:

$$\begin{cases} z_1 h_j^* + z_2 k_j^* > 0, \forall j \neq i \\ z_1 d_i(h^*) + z_2 d_i(k^*) > 0 \end{cases}$$

Taking into account that $h_j^*, k_j^* \geq 0$, $\forall j$, and $d_i(h^*), d_i(k^*) \leq 0$, then z_1 and z_2 are two non-zero numbers and of different sign. Suppose, without loss of generality, that $z_1 > 0$ and $z_2 < 0$. Therefore, we have:

$$h_{-i}^* \gg \alpha k_{-i}^*, \quad d_i(h^*) > \alpha d_i(k^*), \quad \text{being } \alpha = -z_2/z_1 > 0$$

Then, since g^i is strictly decreasing, we deduce that:

$$d_i(k^*) = \lim_{\substack{t \to 0 \\ t > 0}} \frac{g^i(x_{-i} + t\alpha k_{-i}^*) - g^i(x_{-i})}{t\alpha} \geq \lim_{\substack{t \to 0 \\ t > 0}} \frac{g^i(x_{-i} + th_{-i}^*) - g^i(x_{-i})}{t\alpha} = \frac{d_i(h^*)}{\alpha}$$

so, $\alpha d_i(k^*) \geq d_i(h^*)$, which is a contradiction. Therefore, we have shown that, for each $i \in \{1, ..., n\}$, there exists a vector $z^i \in \mathbb{R}^n$, $z^i \gg 0$ which is perpendicular to both $dv^i(x)(h^*)$ and $dv^i(x)(k^*)$, this implies:

$$\begin{cases} d_i(h^*) \cdot z_i^i + \sum_{\substack{j=1 \\ j \neq i}}^{n} z_j^i \cdot h_j^* = 0, & \forall i = 1, ..., n \\ d_i(k^*) \cdot z_i^i + \sum_{\substack{j=1 \\ j \neq i}}^{n} z_j^i \cdot k_j^* = 0, & \forall i = 1, ..., n \end{cases}$$

$$\Rightarrow \begin{cases} d_i(h^*) = -\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{z_j^i}{z_i^i} \cdot h_j^*, & \forall i = 1, ..., n \\ d_i(k^*) = -\sum_{\substack{j=1 \\ j \neq i}}^{n} \frac{z_j^i}{z_i^i} \cdot k_j^*, & \forall i = 1, ..., n \end{cases}$$

Then, from (5.1) we conclude that h^* and k^* are solutions to the following eigenvector problem:

$$\lambda h = A \cdot h, \quad \lambda \leq 0, \quad h \geq 0 \tag{5.2}$$

were the elements a_{ij} of the matrix A are given by:

$$a_{ij} = -\frac{z_j^i}{z_i^i}, i \neq j, \quad a_{ii} = 0, \quad \forall i, j \in \{1, ..., n\}$$

Calling $\mu = -\lambda$ and $B = -A \geq 0$, problem (5.2) is equivalent to the following:

$$\mu h = B \cdot h, \quad \mu \geq 0, \quad h \geq 0 \tag{5.3}$$

The matrix B is irreducible, so, by the Perron-Frobenius theorem, the problem (5.3) has one unique solution up to the product by a scalar; therefore, $h^* = k^*$. □

The sufficient conditions of theorem 3 which guarantee the existence and the uniqueness of the vector $h(x)$ at each point $x \in I(S_k)$ are satisfied in the case in which S is a non-empty and compact set, verifies G2 and whose ∂S_k is the finite union of pieces of hyperplanes. Denote by $\mathcal{P}_0$ the family of such a games[4].

In this case, each hyperplane determines, for each $i \in \{1, ..., n\}$, a region of $I(S_k)$ in which the function g^i is defined through the equation of that hyperplane. By means of the intersection of all these regions, we obtain a finite partition of $I(S_k)$ so that the vector $h(x)$ is constant in the interior of each region of this partition. Moreover, the vector $h(x)$ is well-behaved in the following sense: if the path arrives at the boundary between two regions, is easy to see that the vector $h(x)$ is constant up to the boundary with another region. So, the path we look for is uniquely determined and it is piecewise affine. Thus, we can state the following corollary:

Corollary: For each $S \in \mathcal{P}_0$, the Perles-Maschler path is uniquely determined and, therefore, $PM(S)$ is well-defined.

Remark 6: We believe that it is true the next conjecture: the solution PM is continuous on $\mathcal{P}_0$ with the following norm:[5]

$$\|f\| := \|f\|_\infty + \sum_{i=1}^{n} \|f_i\|_\infty$$

If this was true, by means of the limit, we could define it on the closure of $\mathcal{P}_0$.

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[4] These conditions are also satisfied in the case in which S is a non-empty, *convex* and compact set, and verifies G2 (see Rockafellar: *Convex Analysis*, chapter 5 (1970))

[5] For a function $f : X \to \mathbb{R}$, we define $\|f\|_\infty := \max_{x \in X} |f(x)|$.

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