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Automorphic forms on $O_{s+2,2}(\mathbf{R})$ and infinite products

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The denominator function of a generalized Kac-Moody algebra is often an automorphic form for the group $O_{s+2,2}(\mathbf{R})$ which can be written as an infinite product. We study such forms and construct some infinite families of them. This has applications to the theory of generalized Kac-Moody algebras, unimodular lattices, and reflection groups. We also use these forms to write several well known modular forms, such as the elliptic modular function j and the Eisenstein series E_4 and E_6, as infinite products.

Contents

1. Introduction

The main result of this paper is a method for constructing automorphic forms on $O_{s+2,2}(\mathbf{R})^+$ as infinite products. For example, a special case of theorem 10.1

states that if $24|s$ and we define $c(n)$ by $\eta(\tau)^{-s} = q^{-s/24}\prod_{n>0}(1-q^n)^{-s} = \sum_n c(n)q^n$ and ρ is a certain vector then

$$\Phi(v) = e^{-2\pi i(\rho,v)}\prod_{r>0}(1-e^{-2\pi i(r,v)})^{c(-(r,r)/2)}$$

is an automorphic form for the discrete subgroup $O_{II_{s+2,2}}(\mathbf{Z})^+$ of $O_{II_{s+2,2}}(\mathbf{R})^+$ (or rather, its analytic continuation is an automorphic form, as the infinite product does not converge everywhere). We first describe some applications of this method, and then describe the proof.

The simplest application is a product formula for the elliptic modular function $j(\tau)$. More precisely,

$$j(\tau) = q^{-1}\prod_{n>0}(1-q^n)^{a(n^2)}$$

where the $a(n)$'s are the coefficients of a certain nearly holomorphic ("holomorphic except for poles at cusps") modular form $3q^{-3} - 744q + \cdots$ of weight 1/2 (see example 2 of section 14 for a precise description of the $a(n)$'s). There are similar product formulas for many other modular forms and functions, for example the Eisenstein series E_4, E_6, E_8, E_{10} and E_{14} and the modular function $j(\tau) - 1728$. The usual product formula $\Delta(\tau) = q\prod_{n>0}(1-q^n)^{24}$ is the simplest case of these product formulas. More generally, theorem 14.1 gives an isomorphism between a certain additive group of nearly holomorphic modular forms of weight 1/2 and a multiplicative group of meromorphic modular forms all of whose zeros and poles are either cusps or imaginary quadratic irrationals. In particular, as an immediate corollary of theorem 14.1, we find a product formula for the classical modular polynomial

$$\prod_{[\sigma]}(j(\tau) - j(\sigma)) = q^{-H(-D)}\prod_{n>0}(1-q^n)^{c_0(n^2)}$$

where σ runs through a complete set of representatives modulo $SL_2(\mathbf{Z})$ for the imaginary quadratic numbers which are roots of an equation of the form $a\sigma^2 + b\sigma + c = 0$ $(a, b, c \in \mathbf{Z})$ of some fixed discriminant $b^2 - 4ac = D < 0$ (except that when σ is a conjugate of one of the elliptic fixed points i or $(1 + i\sqrt{3})/2$ we have to replace the corresponding factor $j(\tau) - 1728$ or $j(\tau)$ by $(j(\tau) - 1728)^{1/2}$ or $j(\tau)^{1/3}$). The exponents $c_0(n^2)$ are coefficients of the unique nearly holomorphic weight 1/2 modular form for $\Gamma_0(4)$ whose power series $\sum_{n\in\mathbf{Z}} c_0(n)q^n$ is of the form $q^D + O(q)$ and whose coefficients $c_0(n)$ vanish unless $n \equiv 0, 1 \bmod 4$ (Kohnen's "plus space" condition). The product on the left, as a function of $j(\tau)$, is just the classical modular polynomial for discriminant D, whose degree is the Hurwitz class number $H(-D)$.

This formula can be compared with the Gross-Zagier formula ([G-Z], formula 1.2 and theorem 1.3)

$$\prod_{[\tau_1],[\tau_2]}(j(\tau_1) - j(\tau_2))^{4/w_1w_2} = \pm\prod_{x\in Z,n,n'>0,x^2+4nn'=d_1d_2} n^{\varepsilon(n')}$$

where the first product is over representatives of equivalence classes of imaginary quadratic irrationals of discriminants d_1, d_2, w_1 and w_2 are the number of roots of 1 in the orders of discriminants d_1, d_2, and $\varepsilon(n') = \pm 1$ is defined in [G-Z]. It is also related to the denominator formula

$$j(\sigma) - j(\tau) = p^{-1} \prod_{m>0, n\in\mathbf{Z}} (1 - p^m q^n)^{c(mn)}$$

of the monster Lie algebra (where $p = e^{2\pi i\sigma}$, $q = e^{2\pi i\tau}$, and $j(\tau) - 744 = \sum_n c(n)q^n = q^{-1} + 196884q + \cdots$). These 3 product formulas for $\prod(j(\sigma) - j(\tau))$ cover the cases when both, one, or neither of σ and τ run over representatives of imaginary quadratic numbers of fixed discriminant, while the others can be arbitrary complex numbers with large imaginary part. In spite of the similarity of the left hand sides, there does not seem to be any obvious way to deduce any of these 3 formulas from the others.

There are several strange results about Niemeier lattices (even 24-dimensional unimodular lattices) and the Leech lattice, which were proved by Niemeier, Venkov, and Conway [C-S]. For example, every Niemeier lattice either has no roots or the root system has rank 24, the number of roots is divisible by 24, and the Leech lattice is the Dynkin diagram of $II_{25,1}$. We will find generalizations of these results for all $24n$-dimensional even unimodular lattices K in section 12. For example, K either has no vectors of norm at most $2n$ (i.e., it is extremal) or the vectors of norm at most $2n$ span the vector space $K \otimes \mathbf{R}$, and if $\theta_K(\tau)$ is the theta function of K, then the constant term of $\theta(\tau)/\Delta(\tau)^n$ is divisible by 24. We also use properties of one automorphic form to give a very short proof of the existence and uniqueness of the Leech lattice.

Many examples of automorphic forms on $O_{s+2,2}(\mathbf{R})$ which are modular products are the denominator formulas of generalized Kac-Moody algebras. The simplest example is the product formula for the denominator function $j(\sigma) - j(\tau)$ of the monster Lie algebra given above. This function obviously transforms under a group of the form $(SL_2(\mathbf{Z}) \times SL_2(\mathbf{Z})).(\mathbf{Z}/2\mathbf{Z})$, which is isomorphic to the congruence subgroup $O_{II_{2,2}}(\mathbf{Z})^+$ of $O_{II_{2,2}}(\mathbf{R})$. (Strictly speaking, this is an automorphic function rather than an automorphic form because it has weight 0 and is not holomorphic at the cusps.) A second example is the denominator formula for the fake monster Lie algebra

$$\Phi(v) = \sum_{w\in W} \sum_{n>0} \det(w)\tau(n)e^{-2\pi i n(w(\rho),v)} = e^{-2\pi i(\rho,v)} \prod_{r>0} (1 - e^{-2\pi i(r,v)})^{p_{24}(1-r^2/2)}.$$

This function is obviously antiinvariant under the group $O_{II_{25,1}}(\mathbf{Z})^+$, but also turns out to be an automorphic form of weight 12 under the group $O_{II_{26,2}}(\mathbf{Z})^+$. This is equivalent to saying that Φ satisfies the functional equation

$$\Phi(2v/(v,v)) = -((v,v)/2)^{12}\Phi(v).$$

We can construct many new examples of generalized Kac-Moody algebras from automorphic forms on $O_{s+2,2}(\mathbf{R})$, and conversely we can find many examples of such automorphic forms using generalized Kac-Moody algebras.

There are close connections between automorphic forms on $O_{s+2,2}(\mathbf{R})$ and hyperbolic reflection groups. For any such automorphic form with a modular product we will define its "Weyl vectors". These often turn out to be the Weyl vectors for some hyperbolic reflection group. One example given in section 16 corresponds to the reflection group of the even sublattice of $I_{21,1}$; this is the largest dimension in which the reflection group of a hyperbolic lattice has finite index in the automorphism group. Similarly all the reflection groups of the lattices $I_{n,1}$ for $n \leqq 19$ that were investigated by Vinberg have automorphic forms associated with them.

We now discuss how to construct automorphic forms as infinite products. This construction depends on 3 results, given in sections 4, 5, and 6. The first result (theorem 5.1) states that under mild conditions a modular product can be analytically continued as a meromorphic function to the whole of the Hermitian symmetric space H of $O_{s+2,2}(\mathbf{R})$, and its poles and zeros can only lie on certain special divisors, called quadratic divisors. (A modular product is, roughly speaking, an infinite product whose exponents are given by the coefficients of some nearly holomorphic modular forms; see section 5.) The proof of this uses the Hardy-Ramanujan-Rademacher asymptotic series for the coefficients of a nearly holomorphic modular form. The second result (theorem 6.5) is a generalization of the Macdonald identities from affine root systems to "affine vector systems". This generalization states (roughly) that an infinite product over the vectors of an affine vector system is a Jacobi form. (For affine root systems the usual Macdonald identities follow easily from this using the fact that any Jacobi form can be written as a finite sum of products of theta functions and modular forms.) The third result we use (section 4) is a description of Hecke operators V_ℓ for certain parabolic subgroups ("Jacobi subgroups") of discrete subgroups of $O_{s+2,2}(\mathbf{R})$.

If we put these three results together, we sometimes find that an expression of the form $\exp(\rho+\sum_{\ell\geqq 0}\phi|V_\ell)$, where ϕ is a nearly holomorphic Jacobi form, is an automorphic form on $O_{s+2,2}(\mathbf{R})$. We prove this by showing that it transforms like an automorphic form under 2 parabolic subgroups $J(\mathbf{Z})^+$ and $F(\mathbf{Z})^+$, and then checking (in theorem 8.1) that these two subgroups generate a discrete subgroup of $O_{s+2,2}(\mathbf{R})^+$ of finite covolume. The invariance under the Jacobi group $J(\mathbf{Z})^+$ follows from the results on Hecke operators on Jacobi forms that we recall in sections 2 to 4, and the invariance under the Fourier group $F(\mathbf{Z})^+$ follows by calculating the Fourier coefficients explicitly and checking that these are invariant under $F(\mathbf{Z})^+$.

When the Jacobi form ϕ is holomorphic, this is similar to a method for constructing automorphic forms on $Sp_4(\mathbf{R})$ found by Maass [M], and generalized to $O_{s+2,2}(\mathbf{R})$ by Gritsenko [G], who showed that $\sum_{\ell\geqq 0}\phi|V_\ell$ was an automorphic form. The two main extra complications we have to deal with when ϕ is not holomorphic are firstly that this sum no longer converges everywhere and so has to be analytically continued, and secondly that the "Weyl vector" ρ has to be chosen correctly.

Notation (in roughly alphabetical order)

$^+$ If G is a subgroup of a real orthogonal group then G^+ means the elements of G of positive spinor norm.

$'$ If L is a lattice then L' means the dual of L.

$\perp$ If u is a vector (or sublattice) of a lattice then $u^\perp$ means the orthogonal complement of u.

$\bar{}$ If λ is a vector in a lattice then $\bar{\lambda}$ is the orthogonal projection of λ into some sublattice.

α A coordinate of a vector in M.

a An entry of a matrix $\binom{ab}{cd}$ in $SL_2(\mathbf{Z})$.

$A(v)$ A Fourier coefficient of an automorphic form on $O_M(\mathbf{R})^+$.

β A coordinate of a vector in M.

b An entry of a matrix $\binom{ab}{cd}$ in $SL_2(\mathbf{Z})$.

B_k A Bernoulli number: $\sum_{k\in\mathbf{Z}} B_k t^k/k! = t/(e^t - 1)$.

γ A coordinate of a vector in M.

$\Gamma_0(N)$ $\{\binom{ab}{cd} \in SL_2(\mathbf{Z}) | c \equiv 0 \bmod N\}$

$\Gamma(z)$ Euler's gamma function.

c An entry of a matrix $\binom{ab}{cd}$ in $SL_2(\mathbf{Z})$.

$c(v)$ The multiplicity of a vector in a vector system, or a Fourier coefficient of a modular form or Jacobi form, or an exponent of a modular product.

$\mathbf{C}$ The complex numbers.

C The positive cone in a Lorentzian lattice.

δ A coordinate of a vector in M.

δ_n^m 1 if $m = n$, 0 otherwise.

Δ The delta function, $\Delta(\tau) = q\prod_{n>0}(1 - q^n)$.

d The number of elements of a vector system, an entry of a matrix $\binom{ab}{cd}$ in $SL_2(\mathbf{Z})$.

D The discriminant of a quadratic divisor or an imaginary quadratic irrational or an imaginary quadratic field.

$e^{\pm z}$ means e^z if $\Re(z) < 0$, e^{-z} if $\Re(z) > 0$.

e_n The Dynkin diagrams or lattices of e_8, e_{10}, and so on.

E_k An Eisenstein series of weight k, equal to $1-(2k/B_k)\sum_{n>0}\sigma_{k-1}(n)q^n$.

ζ $e^{2\pi iz}$. $\zeta^y = e^{2\pi i(y,z)}$.

f A function.

F A Fourier group. See section 2.

$F(\tau)$ $= \sum_{n>0}\sigma_1(2n-1)q^{2n-1}$.

${}_2F_1$ The hypergeometric function.

g An element of the group G, or a function.

G A group.

GL A general linear group.

GO A general orthogonal group.

$\eta(\tau)$ $= q^{1/24}\prod_{n>0}(1 - q^n)$.

h The height of a vector. See section 13.

$h(\tau)$ $h(\tau) = h_0(4\tau) + h_1(4\tau)$ is a modular form of weight 1/2.
H, H_u Hermitian symmetric spaces of $O_{n+2,2}(\mathbf{R})$.
$H(n)$ The Hurwitz class number of n. See section 13.
$H_{N,j}(n)$ A generalization of the Hurwitz class number of n. See section 13.
$\mathbf{H}_{N,j}(\tau)$ A function with coefficients $H_{N,j}(n)$. See section 13.
θ, θ_K Theta functions of lattices or cosets of lattices. See section 3.
I_ν A modified Bessel function.
$II_{m,n}$ The even unimodular Lorentzian lattice of dimension $m+n$ and signature $m-n$.
$\Im$ The imaginary part of a complex number.
j The elliptic modular function $j(\tau) = q^{-1} + 744 + 196884q + \cdots$.
J A Jacobi group. See section 2.
J_ℓ A double coset of $J(\mathbf{Z})^+$.
κ A vector of K.
k The weight of an automorphic form or Jacobi form.
K An even positive definite lattice of dimension s.
K_μ A modified Bessel function.
λ An element of K.
Λ The Leech lattice. See [C-S].
ℓ An integer, usually indexing Hecke operators.
L An even Lorentzian lattice of dimension $s+2$, sometimes equal to $K \oplus II_{1,1}$.
μ An element of K, or a real number.
m The index of a Jacobi form or vector system. See section 3 or 6.
M An even lattice of dimension $s+4$ and signature s, sometimes equal to $L \oplus II_{1,1}$.
$M[U]$ The lattice generated by U and all vectors of M having integral inner product with everything in U.
ν A real number.
n An integer, often indexing the coefficients of a modular form.
N The level of a modular form.
O An orthogonal group.
$O(q^n)$ A sum of terms of order at most q^n.
p $e^{2\pi i\sigma}$
$p_m(n)$ The number of partitions of n into parts of m colors.
P A principle $\mathbf{C}^*$ bundle over H.
$\wp^{(n)}$ $\wp^{(n)}(z,\tau) = \frac{d}{dz}^n \wp(z,\tau)$, where $\wp$ is the Weierstrass function. See section 7.
q $e^{2\pi i\tau}$
Q A quadratic form; $Q(v) = (v,v)/2$.
$\mathbf{Q}$ The rational numbers.
ρ A Weyl vector. See section 6.
R A commutative ring, usually $\mathbf{Z}$, $\mathbf{Q}$, $\mathbf{R}$, or $\mathbf{C}$.
$\Re$ The real part of a complex number.
r An element of the ring R, or a vector of K.
$\mathbf{R}$ The real numbers.

σ A complex number with positive imaginary part.
$\sigma_{k-1}(n)$ $= \sum_{d|n} d^{k-1}$ if $n > 0$, $-B_k/2k$ if $n = 0$.
s The signature of a lattice or Jacobi form.
$sl_2(\mathbf{R})$ The set of 2 by 2 real matrices of trace 0.
SL A special linear group.
SO A special orthogonal group.
τ A complex number with positive imaginary part, or Ramanujan's function $\tau(n)$.
$t_{u,v}, t_r$ Automorphisms of a lattice. See section 2.
T_ℓ A Hecke operator.
$T_{\ell,\ell}$ A Hecke operator.
u A norm zero vector in a lattice, often contained in U.
U A 2-dimensional null lattice, usually a sublattice of M.
U_ℓ A Hecke operator acting on Jacobi forms.
ϕ, ϕ_u Jacobi forms. See sections 2 and 3.
Φ An automorphic form on $O_M(\mathbf{R})^+$.
ψ A Jacobi form. See sections 2 and 3.
Ψ A meromorphic modular form with a modular product expansion.
v A vector in a vector system. See section 6.
V A vector system. See section 6.
V_ℓ A Hecke operator acting on Jacobi forms. See section 4.
$V_{\ell,\ell}$ A Hecke operator acting on Jacobi forms. See section 4.
W A Weyl chamber. See section 12.
x A real number or an element of $K \otimes \mathbf{R}$, often equal to $\Re(z)$.
y A real number or an element of $K \otimes \mathbf{R}$, often equal to $\Im(z)$.
z An element of the complexification of K, or a complex number, often equal to $x + iy$.
$\mathbf{Z}$ The integers.

Terminology.

Nearly holomorphic. Meromorphic with all poles at the cusps.

Automorphic form. See section 2.

Classical Jacobi form. A function of several variables transforming like a modular form in one of them and like a theta function in the others. See section 3.

Fourier group. A certain parabolic subgroup of O_M. See section 2.

Height. The height of a vector is the minimum inner product with a Weyl vector. See section 13.

Index. See section 6 for the index of a vector system, and sections 2 and 3 for the index of a Jacobi form.

Jacobi form. See sections 2,3.

Jacobi group. A parabolic subgroup of O_M. See section 2.

Koecher principle. An nearly holomorphic automorphic form on a simple group of rank greater than 1 is automatically holomorphic at the cusps.

Modular product. An infinite product whose exponents are the coefficients of nearly holomorphic modular forms. See section 5.

Primitive sublattice. A sublattice Z of M is primitive if Z contains any vector of M a nonzero multiple of which is in Z.
Rational quadratic divisor. The zero set of $a(y, y) + (b, y) + c$ where $a \in \mathbf{Z}, b \in L, c \in \mathbf{Z}$. See section 5.
Signature. The signature of a Jacobi form is the signature of any of the lattices associated to it, and is one less than the number of variables the Jacobi form depends on.
Singular weight. Weight $s/2$ or 0. See section 3.
Spezialschar. ("Special space.") A space of automorphic forms whose Fourier coefficients satisfy certain relations. See section 9 and [M paper I].
Spinor norm. A homomorphism from a real orthogonal group to $\mathbf{R}^*/\mathbf{R}^{*2}$ taking reflections of vectors of positive or negative norm to 1 or -1 respectively.
Theta function. A modular form or Jacobi form depending on a lattice. See section 3.
Vector system. A multiset of vectors in a lattice with some of the properties of a root system. See section 6.
Weyl chamber. A generalization of the Weyl chamber of a root system to vector systems. See section 6.
Weyl vector. A generalization of the Weyl vector of a root system to vector systems. See sections 6, 12.

2. Automorphic forms and Jacobi forms

We summarize some general facts about automorphic forms on $O_{s+2,2}(\mathbf{R})$ and set up some notation for them. General references for this section are [F] for automorphic forms and [E-Z] for Jacobi forms. The book [F] covers modular forms on symplectic groups rather than orthogonal groups, but most of the general results carry over with only minor changes. Similarly the book [E-Z] covers only Jacobi forms of signature 1, but many of the results can easily be generalized to Jacobi forms of arbitrary signature.

If M is any even integral lattice (with associated quadratic form $Q(v) = (v, v)/2$) we write O_M for the algebraic group of rotations of M, so that $O_M(R)$ is the group of rotations of $M \otimes R$ preserving the quadratic form Q of $M \otimes R$. We write GL_M and SL_M for the general and special linear groups of M, SO_M for the special orthogonal group of M, and GO_M for the general orthogonal group (or conformal group) consisting of the linear transformations multiplying the quadratic form by an invertible element. We think of SO_M, O_M, GL_M, and so on as being algebraic groups defined over $\mathbf{Z}$, so for example $O_M(\mathbf{Z})$ is the group of automorphisms of the lattice M.

There is a "spinor norm" homomorphism from $O_M(R)$ to R^*/R^{*2}, which which has the property that a reflection of a vector of norm $Q(v)$ has spinor norm $Q(v) \in R^*/R^{*2}$. In this paper R^*/R^{*2} can usually be identified with the group $\{1, -1\}$ of order 2, and the reflection of a vector of positive or negative norm then has spinor norm $+1$ or -1 respectively. If G is a subgroup of $O_M(R)$ we write G^+ for the subgroup of G of elements of spinor norm $1 \in R^*/R^{*2}$. The elements of $O_M(\mathbf{R})$ with determinant 1 and positive spinor norm form the connected component $SO_M(\mathbf{R})^+$ of the identity. If M is positive definite the spinor norm on $O_M(\mathbf{R})$ is always positive, if M is negative definite it coincides with the determinant, and if M is indefinite then $SO_M(\mathbf{R})^+$ has index 4 in $O_M(\mathbf{R})$. If M is Lorentzian then the rotations of positive spinor norm are exactly those that preserve rather than interchange the two cones or negative norm vectors of $M \otimes \mathbf{R}$.

The group $O_M(R)^+$ is the image of the pin group $Pin_M(R)$ induced by the natural homomorphism from Pin_M to O_M, and similarly $SO_M(R)^+$ is the image of the spin group $Spin_M(R)$. Notice that the map from Pin_M to O_M is an epimorphism of algebraic groups, but the map from $Pin_M(R)$ to $O_M(R)$ is not necessarily an epimorphism of groups. It would really be more natural to use the pin and spin groups throughout this paper rather than the orthogonal and special orthogonal groups, but this is not (yet) essential and we will stick to O_M and SO_M to save having to describe the construction of Pin_M and $Spin_M$.

From now on we assume that M is a nonsingular even lattice of signature s and dimension $s+4$. We assume that we have chosen a "spin orientation" on M, by which we mean a choice of orientation on each 2-dimensional negative definite subspace of $M \otimes \mathbf{R}$ which varies continuously. There are 2 possible spin orientations on M, and they are interchanged by any rotation of negative spinor norm.

We construct a model for the Hermitian symmetric space of $O_M(\mathbf{R})$. We let P be the vectors $z = x + iy \in M \otimes \mathbf{C}$ such that $z^2 = 0$, $x^2 < 0$, and (x, y) is a positively oriented base of the 2-dimensional vector space spanned by x and y. This space P is acted on by $\mathbf{C}^*$ in the obvious way, and we define H to be the quotient of P by this $\mathbf{C}^*$ action. Then P and H both have natural complex structures, H is an Hermitian symmetric space, and P is a principle $\mathbf{C}^*$ bundle over H. There is a natural compactification of H which is the closure of H in the projective space of $M \otimes \mathbf{C}$.

The space P is acted on transitively by $GO_M(\mathbf{R})^+$, the group of all conformal transformations of M of positive spinor norm. The subspace of P of all vectors $x + iy$ with $x^2 = -1$ is acted on transitively by $O_M(\mathbf{R})^+$ and is a principle S^1 bundle over H.

Complex conjugation in $M \otimes \mathbf{C}$ maps P and H to their complex conjugates $\bar{P}$ and $\bar{H}$. If we identify $\bar{P}$ and $\bar{H}$ with their complex conjugates using complex conjugation (which commutes with $GO_M(\mathbf{R})$) then we get an action of $GO_M(\mathbf{R})$ on H and P such that elements of negative spinor norm act as antiholomorphic transformations. This is similar to the extension of the usual action of $GL_2(\mathbf{R})^+$ (the subgroup of elements of positive determinant) on the upper half plane extended to an action of $GL_2(\mathbf{R})$ on the complex plane with the real line removed. If we identify the upper and lower half planes using complex conjugation, then we get an action of $GL_2(\mathbf{R})$ on the upper half plane, with the elements of negative determinant acting as antiholomorphic transformations.

The group $O_M(\mathbf{Z})^+$ is a discrete subgroup of $O_M(\mathbf{R})^+$. We will say that a function Φ on P is a nearly holomorphic automorphic form of weight $k \in \mathbf{Z}$ for $O_M(\mathbf{Z})^+$ if it has the following properties.

1 Φ is holomorphic on P.

2 Φ is homogeneous of degree $-k$, i.e., $\Phi(vz) = z^{-k}\Phi(v)$ for $z \in \mathbf{C}$.

3 Φ is invariant under $O_M(\mathbf{Z})^+$, i.e., $\Phi(\gamma v) = \Phi(v)$ for $\gamma \in O_M(\mathbf{Z})^+$. More generally, we also allow $\Phi(\gamma v) = \det(\gamma)\Phi(v)$ and call such forms antiinvariant under $O_M(\mathbf{Z})^+$.

If Φ is also "holomorphic at the cusps" (see below) then we call Φ a holomorphic automorphic form, or automorphic form for short. For $s \geqq 1$ any nearly holomorphic form is automatically holomorphic by the Koecher boundedness principle, which also holds for $s = 0$ provided the lattice M does not have square determinant (by the Koecher boundedness principle for Hilbert modular forms). (The Koecher boundedness principle states that any automorphic form on an Hermitian symmetric space associated to a group of real rank greater than 1 is automatically holomorphic at all cusps if it is holomorphic on the symmetric space. See the article on pp. 296-300 by Baily in [B-M].)

Homogeneous functions of degree $-k$ on P can be identified with sections of the line bundle P^k over H, so nearly holomorphic automorphic forms of weight k are just invariant (or antiinvariant) holomorphic sections of P^k.

We can restrict Φ to the subspace of P with $x^2 = y^2 = -1$ and then lift it to a function on $O_M(\mathbf{R})^+$ (or better, to a function on $Pin_M(\mathbf{R})$). The conditions on Φ then say that this lift is left invariant under $O_M(\mathbf{Z})^+$ and transforms under right multiplication by the elements of a maximal compact subgroup according to some representation (described by k) of this compact subgroup. Hence our definition is equivalent to a special case of the usual definition of an automorphic form on a reductive Lie group. We can also define forms of half integral weight either by using the double cover of the line bundle P instead of P, or by allowing Φ to be a 2-valued holomorphic function, or by using functions on the pin group. A form of weight k on the group $O_M(\mathbf{R})^+$ becomes a form of weight $2k$ on $Pin_M(\mathbf{R})$. This is because the weight k indexes representations of an S^1 subgroup of $O_M(\mathbf{R})^+$ or $Pin_M(\mathbf{R})$, and the map between the corresponding S^1 subgroups is 2 to 1, so the integer parameterizing irreducible representations has to be doubled. Forms of half integral weight on O_M correspond to ordinary modular forms of integral weight rather than half integral weight, because the double cover $Spin_M \to SO_M$ corresponds to the double cover $SL_2 \to PGL_2$ rather than the metaplectic double cover of the special linear group. In particular if M has dimension 3 then automorphic forms on $O_M(\mathbf{R})^+$ of weight k can be identified with ordinary modular forms of weight $2k$ (rather than k). This annoying factor of 2 in the weights is unavoidable and is not just caused by a bad choice of conventions: one construction in this paper starts with an ordinary modular form for $SL_2(\mathbf{Z})$ of weight k, and ends up with an ordinary modular form of weight $2k$, and this factor of 2 is essentially caused by the doubling of weights when lifting forms on O_M to forms on Pin_M.

We now study the parabolic subgroups of O_M. The subgroup fixing a nonzero null sublattice of M is a maximal parabolic subgroup, and this null sublattice can have rank 1 or 2. If it has rank 1 we will call the corresponding parabolic subgroup a Fourier group, and if it has rank 2 we call the corresponding parabolic subgroup a Jacobi group. The reason for this terminology is that the "Fourier–Jacobi" expansion of an automorphic form with respect to a parabolic subgroup is essentially either a Fourier series expansion or an expansion in terms of Jacobi forms, depending on whether the parabolic subgroup is a Fourier group or a Jacobi group. What we call the Jacobi group is essentially a central extension of the Jacobi group in [E-Z, p. 10]; see also [E-Z Theorem 1.4] and [E-Z Theorem 6.1] for other appearances of this central extension.

Suppose that U is a 2-dimensional primitive null sublattice of M, and let J be the corresponding Jacobi subgroup of GO_M. (A sublattice U of M is called primitive if $U = M \cap U \otimes \mathbf{R}$.) There is an obvious induced action of J on the lattices U and $U^\perp/U = K$, so we get a homomorphism from J to $GO_K \times GL_U$. The connected component of the kernel of this homomorphism is a Heisenberg group whose structure we will now describe. (This is the "same" Heisenberg group that turns up regularly in the theory of theta functions.) If $u \in U$ and $v \in u^\perp/u$ then we define an automorphism $t_{u,v}$ of M by

$$t_{u,v}(w) = w + (w,u)v - ((w,v) + (w,u)(v,v)/2)u.$$

For fixed u these form a group of automorphisms of M fixing u and all elements of $u^\perp/u$. If u, v is a positively oriented basis of U and $r \in R$ then we define t_r by

$$t_r = t_{ru,v}.$$

This depends on U but not on the choice of positively oriented basis for U, and commutes with all automorphisms $t_{u,v}$ for $u \in U$, $v \in U^\perp$. The automorphisms $t_{u,v}$ for $u \in U$, $v \in U^\perp$ satisfy the relations

$$t_{u_1,v_1}t_{u_2,v_2} = t_{u_2,v_2}t_{u_1,v_1}t_r$$

where $r = ((v_1, v_2)$ times the determinant of a linear transformation taking a positively oriented basis of U to u_1, u_2. They generate a Heisenberg group of dimension $2s + 1$ whose center is the group of elements of the form t_r.

If Φ is an automorphic function and J is a Jacobi group we define the Jacobi expansion of Φ as follows. We let t_r for $r \in \mathbf{R}$ be the elements of the center of the Jacobi group. We define ϕ_m for $m \in \mathbf{Z}$ by

$$\phi_m(v) = \int_{r\in\mathbf{R}/\mathbf{Z}} \Phi(t_r(v))e^{2\pi imr}dr.$$

This is well defined because $\Phi(t_r(v) = \Phi(v)$ for $r \in \mathbf{Z}$. The Jacobi expansion of Φ is then

$$\Phi = \sum_{m\in\mathbf{Z}} \phi_m,$$

and the functions ϕ_m have the following properties.

1 ϕ_m is a homogeneous function of weight k.
2 ϕ_m is holomorphic on P.
3 $\phi_m(t_r(v)) = e^{2\pi imr}\phi_m(v)$.
4 ϕ_m is invariant (or possibly antiinvariant) under the Jacobi group $J(\mathbf{Z})^+$.
5 ϕ_m is "holomorphic at the cusps" (at least if Φ is); the meaning of this is described below in the section on Fourier subgroups.

Functions with these properties are called **Jacobi forms** of index m and weight k and signature s. If the lattice M is 5 dimensional (i.e., the signature is 1) then these are more or less the same as the Jacobi forms of [E-Z] multiplied by an elementary function. The signature s is the signature of any of the 3 lattices K, L, or M associated with the Jacobi form. If ϕ_m is a Jacobi form we can analytically continue it to the space of all norm 0 vectors v in $M \otimes \mathbf{C}$ such that $\Im((v,u_1)/(v,u_2)) > 0$, where u_1 and u_2 is any oriented base of U, by saying that $\phi_m(t_r(v)) = e^{2\pi imr}\phi_m(v)$ must hold for all complex values of r.

Now suppose that u is a primitive norm 0 element of M, and let F be the corresponding Fourier group. There is an induced action of F on the lattice $L = u^\perp/u$, which gives a homomorphism from F to the group GO_L. The connected component of the kernel of this homomorphism is a unipotent abelian group containing the elements $t_{u,v}$ for $v \in L$. Suppose that Φ is either an automorphic form or a Jacobi form of a 2-dimensional lattice containing u. Then $\Phi(t_{u,v}(w)) = \Phi(w)$ for $v \in L$. We define A_m for $m \in L'$ by

$$A_m(w) = \int_{v \in L\otimes\mathbf{R}/L} \Phi(t_{u,v}(w))e^{2\pi imv}dv.$$

The Fourier expansion of Φ is then

$$\Phi = \sum_{m\in L'} A_m.$$

We will see shortly that the A_m's are elementary factors times exponential functions, so this is essentially just the usual Fourier series expansion of Φ. We say that Φ is holomorphic at F if the Fourier coefficients A_m are 0 unless m lies in the closed positive cone of L. (The vectors of nonpositive norm in $L \otimes \mathbf{R}$ form two closed cones; the positive one is defined to be the one containing a norm 0 vector v such that u, v is a positively oriented basis of the 2-dimensional space they span in M.) If ϕ is a Jacobi form corresponding to some Jacobi group J, then we say that ϕ is holomorphic (at the cusps) if the Fourier expansion of ϕ at every cusp of J is holomorphic, i.e., if the Fourier expansion of ϕ at every Fourier subgroup F such that $F \cap J$ is parabolic is holomorphic. This condition on F just means that F is the Fourier group of some 1-dimensional lattice contained in the 2-dimensional null lattice of J.

If F is a Fourier group of a norm zero vector u, we will construct another model H_u of the Hermitian space H, on which the action of F is easier to visualize. We put $L = u^\perp/u$ so that L is a Lorentzian lattice, and we write $L^1(R)$ for the space (vectors of $M \otimes R$ which have inner product 1 with u)/Ru, so that L^1 is an affine space over L. We define H_u to be the vectors in $x + iy \in L^1(\mathbf{C})$ such that y is in the positive open cone of $L \otimes \mathbf{R}$. If we write P^1 for the vectors of P having inner product 1 with u, then H is naturally isomorphic to P^1 because each fiber of P over H has a unique element in P^1. Also each element of P^1 represents an element of H_u. This maps P^1 onto H_u, because given any point $x+iy \in M \otimes \mathbf{C}$ representing a point in H_u, we can add a multiple of u to x to make the norm of x equal to that of y, and can then add a multiple of u to y to make x and y orthogonal. The point $x+iy$ then lies in P^1. Hence we have constructed isomorphisms from H to P^1 and from P^1 to H_u, so H_u can be identified with H. We identify functions of degree $-k$ on P with functions on P^1 by restriction, and we identify functions on P^1 with functions on H_u by using the isomorphism from P^1 to H_u. Hence functions of degree $-k$ on P (in particular automorphic forms of weight k) can be identified with functions on H_u.

3. Classical theory

We will show that the definitions in the previous section are equivalent to the usual definitions of Jacobi forms (at least when M has dimension 5) by working out a simple case explicitly. We choose K to be an even positive definite lattice and we let $L = K \oplus II_{1,1}$, $M = L \oplus II_{1,1}$. We can write vectors of M in the form $(\kappa, \alpha, -\delta, \gamma, \beta)$ with $\kappa \in K$, $\alpha, \beta, \gamma, \delta \in \mathbf{Z}$, and this vector has norm $\kappa^2/2 + \alpha\delta - \gamma\beta$. We put $u_1 = (0,0,1,0,0)$, $u_2 = (0,0,0,0,1)$. We define J to be the Jacobi group of $\langle u_1, u_2\rangle$, and we let F be the Fourier group of $u = u_2$. We let H_u be the Hermitian symmetric space defined in the previous section. We can identify H_u with a subspace of $L \otimes \mathbf{C}$ (because we have a canonical vector $(0,0,0,1,0)$ which has inner product -1 with u.)

It is particularly easy to describe functions on H which are invariant under $F(\mathbf{Z})$. If we consider the associated function Φ_u on H_u, then we can expand Φ_u as a Fourier series (as f is invariant under translation by L), and the Fourier coefficients have to be invariant under the natural action of F on L'.

As an example we work out the condition on Φ_u that corresponds to the weight k automorphic form f being invariant under the transformation taking (v,γ,β) to (v,β,γ) ($v \in L \otimes \mathbf{C}$). By definition, $\Phi_u(v) = f(v,1,v^2/2)$ and $f(v,\gamma,\beta) = f(v,\beta,\gamma)$, and $f(v,\gamma,\beta) = \gamma^{-k}\Phi_u(v/\gamma)$ (for $v^2 = 2mn$). From this it follows that

$$\Phi_u(-2v/(v,v)) = ((v,v)/2)^k \Phi_u(v).$$

Suppose now that ϕ is a Jacobi form of index m and weight k. We will show how to identify ϕ with a classical Jacobi form. The conditions on ϕ are

$$\begin{aligned}
\phi(z,\alpha,-\delta,\gamma,\beta) &= r^k\phi(rz, r\alpha, -r\delta, r\gamma, r\beta) \\
&= e^{-2\pi imr}\phi(z,\alpha,-\delta+r\gamma,\gamma,\beta-r\alpha) \\
&= \phi(z, a\alpha+b\gamma, -d\delta-c\beta, c\alpha+d\gamma, a\beta+b\delta) \\
&= \phi(z+\alpha\lambda, \alpha, -\delta+(z,\lambda)+\alpha\lambda^2/2, \gamma, \beta) \\
&= \phi(z+\gamma\mu, \alpha, -\delta, \gamma, \beta+(z,\mu)+\gamma\mu^2/2)
\end{aligned}$$

for $r \in \mathbf{R}$, $\lambda \in K$, $\mu \in K$, $z \in K \otimes \mathbf{C}$, $\binom{ab}{cd} \in SL_2(\mathbf{Z})$, $\alpha, \beta, \gamma, \delta \in \mathbf{R}$.

We extend ϕ so that it is defined for all norm 0 vectors such that $\Im(\alpha/\gamma) > 0$ by insisting that ϕ should satisfy $\phi(z, \alpha, -\delta + r\gamma, \gamma, \beta - r\alpha) = e^{2\pi imr}\phi(z, \alpha, -\delta, \gamma, \beta)$ for all complex values of r. If we define $\phi_u(z, \tau)$ for $\tau \in \mathbf{C}, \Im(\tau) > 0$, $z \in K \otimes \mathbf{C}$ by

$$\phi_u(z, \tau) = \phi(z, \tau, 0, 1, z^2/2)$$

then we find that ϕ_u has the following properties.

$$\phi_u(z/(c\tau + d), (a\tau + b)/(c\tau + d)) = (c\tau + d)^k e^{2\pi imc(z^2/2)/(c\tau+d)}\phi_u(z, \tau)$$

$$\phi_u(z + \lambda\tau + \mu, \tau) = e^{-2\pi im(z\lambda+\tau\lambda^2/2)}\phi_u(z, \tau) \quad (\lambda, \mu \in K).$$

Conversely, if we are given ϕ_u with these properties and we define ϕ by

$$\phi(z, \alpha, -\delta, \gamma, \beta) = \phi_u(z/\gamma, \alpha/\gamma)\gamma^{-k}e^{-2\pi im\delta/\gamma}$$

then ϕ transforms like a Jacobi form. If K is a one dimensional lattice spanned by a vector of norm 2, then the relations for ϕ_u are equivalent to the relations in [E-Z, p. 1] defining classical Jacobi forms (except for a misprint in their equation (1), where the term $2\pi imcz$ should be $2\pi imcz^2$). There is an extra factor of 2 in some of the norms in some of our formulas compared to the ones in [E-Z]; this appears because we normalize K to be an even lattice generated by an element of norm 2, while in [E-Z] K is a lattice generated by an element of norm 1.

We now summarize some facts about Jacobi forms, which are straightforward extensions of standard results about Siegel modular forms and Jacobi forms of signature 1. We say that a Jacobi form of signature s has singular weight if its weight is 0 or $s/2$. We say that an automorphic form on $O_M^+(\mathbf{R})$ has singular weight if its weight is 0 or $s/2$. We define a theta function of weight $k = s/2$ and index $m \in \mathbf{Z}$ to be a linear combination of functions of the form

$$\theta_{K+r}(z, \tau) = \sum_{\lambda \in K+r} q^{\lambda^2/2}\zeta^{m\lambda}$$

where K is some positive definite rational lattice of dimension s and $r \in K \otimes \mathbf{Q}$ ($q = e^{2\pi i\tau}$, $\zeta^\lambda = e^{2\pi i(z,\lambda)}$). Any theta function is a holomorphic Jacobi form of singular weight.

Theorem 3.1. *Any (nearly) holomorphic Jacobi form ϕ of positive index can be written as a sum of products of theta functions and (nearly) holomorphic modular forms (though these theta functions and modular forms may have higher level than ϕ).*

For the case of Jacobi forms of signature 1 this is theorem 5.1 of [E-Z]. The proof for higher signatures is essentially the same.

Corollary 3.2. *Any holomorphic Jacobi form of weight 0 is constant, and there are no nonconstant Jacobi forms of weight less than the singular weight $s/2$. Any holomorphic Jacobi form of singular weight $s/2$ is a sum of theta functions.*

Proof. This follows from theorem 3.1 and the fact that any theta function has weight $s/2$.

Corollary 3.3. *Any holomorphic automorphic form either has weight 0 in which case it is constant, or has weight at least $s/2$. If it has singular weight $s/2$ then all the Fourier coefficients corresponding to vectors of nonzero norm vanish.*

Proof. If f can be written as a sum of Jacobi forms (which is the only case we will use in this paper and is always true if $s \geqq 2$) then this follows from the previous corollary. This is the analogue of the second proof in [F, appendix IV]. In general the corollary can be proved by using the Laplacian operator, as in the first proof given in [F, appendix IV].

In particular there is a gap between 0 and $s/2$, such that there are no modular forms with weights in this gap. This phenomenon does not occur for Siegel modular forms because it just happens that all half integers less than the largest singular weight are also singular weights. (In both cases the number of singular weights is equal to the real rank of the corresponding Lie group.) Similarly the gap between weights 0 and $s/2$ of holomorphic Jacobi forms of signature s is not really noticeable in [E-Z] because $s/2$ is equal to $1/2$.

4. Hecke operators for Jacobi groups

Suppose that M is an even lattice of dimension $s+4$ and signature s and that U is a 2-dimensional primitive null sublattice and J the corresponding Jacobi group. We will assume that we are in the simplest ("level 1") case, so we assume that the map from M to U' is onto, and we assume that the discrete group we are working with is the full group $J(\mathbf{Z})^+$ (rather than some congruence subgroup).

Suppose that Y is a 2-dimensional lattice containing U. We define $M[Y]$ to be the lattice generated by Y and the vectors of M that have integral inner product with all vectors of Y. The fact that M maps onto U' implies M can be written as $K \oplus II_{2,2}$ where U is contained in $II_{2,2}$, and this implies that $M[Y]$ is isomorphic to M under some isomorphism mapping Y to U. (In the higher level case this is not always true, and we have to restrict ourselves to lattices Y having this property.) We define J_ℓ to be the set of all isomorphisms of positive spinor norm from some lattice of the form $M[Y]$ with $[Y:U]=\ell$ to M. This is a union of double cosets of $J(\mathbf{Z})^+ = J_1$ because $J(\mathbf{Z})^+$ acts on the set of lattices Y with $[Y:U]=\ell$. Two elements a and b of J_ℓ are in the same right J-coset if and only if ab^{-1} is in J_1, which happens if and only if a^{-1} and b^{-1} both map U to the same lattice Y. Hence the right cosets of $J(\mathbf{Z})^+$ in J_ℓ correspond exactly to the lattices Y with $[Y:U]=\ell$, and in particular there are only a finite number of such right cosets.

If ϕ is a Jacobi form for $J(\mathbf{Z})^+$ we define the Hecke operator V_ℓ by

$$(\phi|V_\ell)(v) = (1/\ell)\sum_{g \in J_1\backslash J_\ell} \phi(gv).$$

This operator maps Jacobi forms of weight k and index m for J_1 to Jacobi forms of weight k and index $m\ell$. (The index gets multiplied by ℓ because the elements of J_ℓ act on $\Lambda^2(U)$ and hence on the center of the nilradical of J_1 as multiplication by ℓ.) We define the operator $V_{\ell,\ell}$ similarly except that we restrict to the coset corresponding to lattices Y such that $Y/U = (\mathbf{Z}/\ell\mathbf{Z})^2$, so that

$$(\phi|V_{\ell,\ell})(v) = (1/\ell^2)\phi(gv)$$

where g maps $(1/\ell)U$ to U. The operator $V_{\ell,\ell}$ maps Jacobi forms of weight k and index m to Jacobi forms of weight k and index $m\ell^2$. The operator $\ell^{2-k}V_{\ell,\ell}$ is denoted by U_ℓ in [E-Z].

We define an action of GL_2 on M by

$$\begin{pmatrix} ab \\ cd \end{pmatrix}(z, \alpha, -\delta, \gamma, \beta) = (z, (a\alpha + b\gamma)/\ell, -d\delta - c\beta, (c\alpha + d\gamma)/\ell, b\delta + a\beta)$$

where $\ell = ad - bc$. A set of right coset representatives of $J_1\backslash J_\ell$ is then given by the usual set of matrices $\binom{ab}{0d}$ with $0 \leqq b < d$, $ad = \ell$. Using this set of representatives we can calculate the relations between the Hecke operators and the Fourier expansion of $f|T_\ell$ in the same way as for ordinary modular forms, and we get the following results.

Theorem 4.1.

1 *If ℓ and ℓ' are coprime then $V_\ell V_{\ell'} = V_{\ell\ell'}$.*

2 $V_{\ell,\ell}V_{\ell',\ell'} = V_{\ell\ell',\ell\ell'}$.

3 *If p is prime then $V_pV_{p^n} = V_{p^{n+1}} + pV_{p,p}V_{p^{n-1}}$. (More generally, $V_\ell V_{\ell'} = \sum_{d|(\ell,\ell')} dV_{d,d}V_{\ell\ell'/d^2}$.)*

4 *The operators of the form V_ℓ and $V_{\ell,\ell}$ for $\ell \geqq 1$ all commute with each other.*

Proof. For signature 1 see [E-Z Theorem 4.2]; the proof in the general case is similar. (There is a misprint in equation (10) of [E-Z, Theorem 4.2]; the term $V_\ell \circ U_{\ell'}$ should be $V_\ell \circ V_{\ell'}$.)

If we choose a norm 0 vector u and let ϕ_u be the classical Jacobi form associated to ϕ and u as in the previous section then we can calculate the effect of V_ℓ on the Fourier coefficients of ϕ_u as follows.

Theorem 4.2. *If $\phi_u(z,\tau) = \sum c(r,n)q^n\zeta^r$ then*

$$\phi_u|V_{\ell,\ell} = \ell^{k-2}\sum_{r,n} c(r/\ell, n)q^n\zeta^r$$

and

$$\phi_u|V_\ell = \sum_{r,n} q^n\zeta^r \sum_{a|(r,\ell,n)} a^{k-1}c(r/a, n\ell/a^2).$$

Proof. This is a straightforward calculation using the standard set of coset representatives. Details for the case of signature 1 are given in the proof of [E-Z, Theorem 4.2], and the proof in the general case is similar.

5. Analytic continuation

In this section we prove that certain infinite sums and products on $L\otimes\mathbf{C}$ can be analytically continued to multivalued functions whose singularities and branch points (and zeros in the case of products) are known explicitly.

We let L be an $(s+2)$-dimensional Lorentzian lattice (of signature s), and we let C be one of the two cones of negative norm vectors in $L\otimes\mathbf{R}$. We choose some vector in $-C$ that is not orthogonal to any nonzero vector of L, and the expression $x>0$ means that x has positive inner product with this vector.

We define a **modular product** to be an infinite product of the form

$$\Phi(y)=e^{-2\pi i(\rho,y)}\prod_{x\in L,x>0}(1-e^{-2\pi i(x,y)})^{c(x)}$$

where $y\in L\otimes\mathbf{C}$, $\Im(y)\in C$, $c(x)$ is the coefficient of $q^{-(x,x)/2}$ of some nearly holomorphic modular form f_x of weight $-s/2$, and the modular forms f_{x_1} and f_{x_2} are equal whenever x_1-x_2 lies in NL for some fixed integer N.

We define a **rational quadratic divisor** to be the set of points y with $\Im(y)\in C$ such that $a(y,y)+(b,y)+c=0$ for some $a\in\mathbf{Z}$, $b\in L$, $c\in\mathbf{Z}$ with $(b,b)-4ac>0$. If $M=L\oplus II_{1,1}$, then the points y is some rational quadratic divisor are just the points $(y,1,y^2/2)\in M\otimes\mathbf{C}$ that are orthogonal to the norm b^2-4ac vector $(b,-2a,-c)$ of M. Hence rational quadratic divisors correspond to equivalence classes of positive norm vectors of M, where two vectors are equivalent if they are rational multiples of each other.

For example, if L is a 1-dimensional lattice then a rational quadratic divisor is just an imaginary quadratic irrational number in the upper half plane.

Theorem 5.1. *Any modular product $\Phi(y)$ converges to a holomorphic function whenever $\Im(y)$ is in C and has sufficiently large norm. This function can be analytically continued to a multivalued meromorphic function for all y with $\Im(y)\in C$ all of whose singularities and zeros lie on rational quadratic divisors.*

We will see later that along any rational quadratic divisor $a(y,y)+(b,y)+c=0$ the function $\Phi(y)$ is locally of the form $(a(y,y)+(b,y)+c)^s$ times a holomorphic function for some complex number s (except of course where the rational quadratic divisor meets other singularities or zeros of Φ). The complex number s is called the multiplicity of the zero of Φ along this rational quadratic divisor. The function Φ is holomorphic if and only if the multiplicity of every rational quadratic divisor is a nonnegative integer.

If we allow the modular forms f_r in the definition of a modular product to have poles in the upper half plane, then their coefficients $c(n)$ increase exponentially fast which implies that the product defining Φ does not converge anywhere. On the other hand, if we insist that the modular forms f_r should be holomorphic, then their coefficients $c(n)$ have polynomial growth, which implies that the infinite product for Φ converges whenever $\Im(y)\in C$, so that

Φ is holomorphic and nonzero in this region. An example of this case is $f(\tau) = 12\sum_{n\in\mathbf{Z}} q^{n^2}$ and $\Phi(\tau) = q\prod_{n>0}(1-q^n)^{24}$.

Theorem 5.2. *Suppose that k is a positive integer and*

$$\Phi(y) = \sum_{x\in L, x>0} A(x)e^{-2\pi i(x,v)}$$

where $A(x) = \sum_{d|x} d^{k-1}c_x(-(x,x)/2d^2)$ *and* $c_x(n)$ *is the coefficient of* q^n *of some nearly holomorphic modular form* f_x *of weight* $k - s/2$, *such that* f_x *depends only on* $x \bmod N$ *for some fixed integer* N. *Then the sum for* Φ *converges whenever* $\Im(y) \in C$ *and* $-(\Im(y), \Im(y))$ *is sufficiently large, and can be analytically continued to a meromorphic function defined for all* y *with* $\Im(y) \in C$, *all of whose singularities are poles of order* k *lying on rational quadratic divisors.*

The proof of theorem 5.2 is similar to that of theorem 5.1 and slightly simpler, so we will omit it. If $k = 0$ then the sum in theorem 5.2 is, up to some elementary factors, the logarithm of the product in theorem 5.1, so the main change in the proof is that we do not first need to take logarithms. Lemmas 5.3 and 5.4 are sufficiently general for the extension of the proof to theorem 5.2.

The idea of the proof of theorem 5.1 is that $\log(\Phi(y))$ is given by a Fourier series whose coefficients depend on the coefficients of modular forms. The singularities of any periodic function are closely related to the asymptotic behavior of its Fourier coefficients, and we know the asymptotic behavior of the coefficients of modular forms because of the Hardy-Ramanujan-Rademacher series. Hence we can find all the singularities of $\log(\Phi(y))$, which gives us all the singularities and zeros of $\Phi(y)$. Before giving the proof of theorem 5.1 we prove two preliminary lemmas.

Lemma 5.3. *Suppose that* $f(\tau) = \sum_{n\in\mathbf{Z}} c(n)q^n$ *is a nearly holomorphic modular form which has half integral weight k (which may be positive or negative or zero). Suppose that its expansion at the cusp* a/c ($c > 0$, $ad - bc = 1$) *is*

$$(c\tau + d)^{-k} f((a\tau + b)/(c\tau + d)) = \sum_{n\in\mathbf{Q}} c_{a/c}(n)e^{2\pi i n\tau}.$$

Then for any positive number ε *we can find a finite sum of terms of the series*

$$\sum_{m>0}\sum_{c>0}\sum_{0\leqq a<c, 0\leqq d<c, c|(ad-1)} 2\pi c_{a/c}(-m)e^{2\pi i(an-md)/c} I_{1-k}(4\pi\sqrt{mn}/c)(m/n)^{(1-k)/2}/c$$

which differs from $c(n)$ *by at most* $O(e^{\varepsilon\sqrt{n}})$. ($I_{1-k}(z)$ *is the modified Bessel function of the first kind; see [E 7.2.2].)*

Proof. This is the Hardy-Ramanujan-Rademacher series for the coefficients of nearly holomorphic modular forms. The case when the form f has negative weight and level 1 is proved in [R] (and in this case the series converges absolutely to $c(n)$). The case when f has level greater than 1 is an easy generalization of the case when f has level 1. If f has weight 0, then Rademacher showed that the series still converges to $c(n)$ provided they are added in the right order and $n \neq 0$, which again implies lemma 5.3. If f has positive weight then a similar argument shows that the series we get is an asymptotic series for $c(n)$ with the stated error term. (In this case the series does not usually converge to $c(n)$.) This proves lemma 5.3.

Lemma 5.4. *If m is positive, k is real and $s \geqq -1$ then the integral*

$$I = 2\pi \int_{x\in C} e^{-2\pi i(x,y)} I_{1-k}(4\pi\sqrt{m(-x,x)/2})((-x,x)/2)^{(k-1)/2} d^{s+2}x$$

converges if $\Im(y) \in C$ and $-(\Im(y),\Im(y))$ is sufficiently large, and can be analytically continued to a multivalued holomorphic function in the region of all $y \in L\otimes \mathbf{C}$ with $(y,y)\neq 0$ and $(y,y)\neq 2m$.

In other words, the integral can be extended to a function which is (multivalued) holomorphic for $\Im(y) \in C$ apart from a singularity along a rational quadratic divisor.

Proof. We first evaluate the integral

$$\int_{x\in C} e^{-2\pi i(x,y)} f(\sqrt{(-x,x)}) d^{s+2}x$$

where $y \in iC$, (y,y) is sufficiently large, and f is any continuous function defined for non negative real numbers which does not increase more than exponentially fast at infinity.

This integral is equal to

$$\frac{2\pi^{(s+1)/2}}{\Gamma((s+1)/2)} \int_{x=0}^{\infty}\int_{r=0}^{x} e^{-2\pi x\sqrt{(y,y)}} f(\sqrt{x^2-r^2}) r^s dr dx.$$

(The factor in front is the area of a sphere of radius 1 in $s+1$-dimensional space.) If we put $t^2 = x^2 - r^2$ and then change x to tx, we find that this integral is equal to

$$\frac{2\pi^{(s+1)/2}}{\Gamma((s+1)/2)} \int_{t=0}^{\infty}\int_{x=t}^{\infty} e^{-2\pi x\sqrt{(y,y)}} f(t)(x^2-t^2)^{(s-1)/2} t dx dt$$

$$= \frac{2\pi^{(s+1)/2}}{\Gamma((s+1)/2)} \int_{t=0}^{\infty}\int_{x=1}^{\infty} e^{-2\pi xt\sqrt{(y,y)}} f(t)(x^2-1)^{(s-1)/2} t^{s+1} dx dt.$$

The integral over x can carried out explicitly using Gubler's formula ([E vol. 2, 7.3.4, formula 15])

$$\Gamma(\mu+1/2)K_\mu(z) = \pi^{1/2}(z/2)^\mu \int_1^\infty e^{-zx}(x^2-1)^{\mu-1/2} dx$$

which is valid for $\mathscr{R}(\mu) > -1/2$, $\mathscr{R}(z) > 0$, with $z = 2\pi t\sqrt{(y,y)}$, $\mu = s/2$. (K_μ is the modified Bessel function of the third kind; [E vol. 2, 7.2.2].) We find that

$$\int_{x\in C} e^{-2\pi i(x,y)} f(\sqrt{(-x,x)}) d^{s+2}x = 2\int_{t=0}^{\infty} f(t) K_{s/2}(2\pi t\sqrt{(y,y)}) t^{s/2+1} (y,y)^{-s/4} dt.$$

We now substitute $f(t) = I_{1-k}(4\pi t\sqrt{m/2})t^{k-1}$ into this and find that

$$I = 4\pi 2^{(1-k)/2}\int_{t=0}^{\infty} I_{1-k}(4\pi t\sqrt{m/2})K_{s/2}(2\pi t\sqrt{(y,y)})(y,y)^{-s/4}t^{s/2+k}dt.$$

By [E vol. 2, 7.14.2, formula 35]

$$\begin{aligned}&2^{\rho+1}\alpha^{\nu+1-\rho}\Gamma(\nu+1)\int_0^{\infty} K_\mu(\alpha t)I_\nu(\beta t)t^{-\rho}dt\\ =&\beta^\nu\Gamma(\nu/2-\rho/2+\mu/2+1/2)\Gamma(\nu/2-\rho/2-\mu/2+1/2)\times\\ &\times {}_2F_1(\nu/2-\rho/2+\mu/2+1/2,\nu/2-\rho/2-\mu/2+1/2;\nu+1;\beta^2/\alpha^2)\end{aligned}$$

whenever $\alpha > \beta$, $\mathcal{R}(\nu-\rho+1\pm\mu) > 0$. (${}_2F_1$ is the hypergeometric function [E volume 1, chapter II].) If we set $\mu = s/2$, $\nu = 1-k$, $\beta = 4\pi\sqrt{m/2}$, $\alpha = 2\pi\sqrt{(y,y)}$, $\rho = -k-s/2$ in this we find that

$$\begin{aligned}&2^{1-k-s/2}(2\pi\sqrt{(y,y)})^{s/2+2}\Gamma(2-k)\int_0^{\infty} K_{s/2}(2\pi t\sqrt{(y,y)})I_{1-k}(4\pi t\sqrt{m/2})t^{k+s/2}dt\\ =&(4\pi\sqrt{m/2})^{1-k}\Gamma(1+s/2){}_2F_1(1+s/2,1;2-k;2m/(y,y))\end{aligned}$$

so that

$$\begin{aligned}&2\pi\int_0^{\infty} K_{s/2}(2\pi t\sqrt{(y,y)})I_{1-k}(4\pi t\sqrt{m/2})t^{k+s/2}dt\\ &= m^{(1-k)/2}(y,y)^{-1-s/4}2^{-(k+1)/2}\times\\ &\quad\times {}_2F_1(1+s/2,1;2-k;2m/(y,y))\Gamma(1+s/2)/\Gamma(2-k)\end{aligned}$$

We find that

$$\begin{aligned}I =& 2^{1-k}m^{(1-k)/2}(y,y)^{-1-s/2}\times\\ &\times {}_2F_1(1+s/2,1;2-k;2m/(y,y))\Gamma(1+s/2)/\Gamma(2-k)\end{aligned}$$

The hypergeometric function ${}_2F_1(a,b;c;z)$ can be analytically continued whenever z is not 0, 1, or ∞, so the function I can be analytically continued whenever (y,y) is not 0 or $2m$. (If k is a positive integer at least 2 then the hypergeometric function has a pole in k, but this cancels out with the pole of $\Gamma(2-k)$, so I is still a well defined analytic function.) This proves lemma 5.4.

We can now prove theorem 5.1. We can ignore the factor $e^{-2\pi i(\rho,y)}$, and can assume that the product is taken only over those values of x in some coset $v+NL$ of NL, so that $c(x)$ is equal to the coefficient $c(-(x,x)/2)$ of $q^{-(x,x)}$ of some fixed nearly holomorphic modular form $f(\tau)$. If we expand the coefficients using lemma 1 as a finite sum of Bessel functions plus a remainder term, then the estimate $O(e^{\varepsilon\sqrt{n}})$ for the remainder shows that the product using the remainder terms can be made to converge whenever $\Im(y)$ has norm at least δ for any given positive δ. It is also easy to check that the infinite product over the vectors x of negative or zero norm converges whenever $\Im(y) \in C$, so we can ignore these terms in the product. (In fact, the same is also true for the terms where r has norm less than any given constant.) Therefore it is sufficient to prove theorem 5.1 for the infinite product

$$\prod_{x\in L+v, x\in C}(1-e^{-2\pi i(x,y)})^{I_{1-k}(4\pi\sqrt{m(-x,x)/2})((-x,x)/2)^{(k-1)/2}}$$

(after replacing L by NL).

The logarithm of this is

$$-\sum_{x\in L+v, x\in C}\sum_{n>0} e^{-2\pi i n(x,y)} I_{1-k}(4\pi\sqrt{m(-x,x)/2})((-x,x)/2)^{(k-1)/2}/n.$$

The sum of all terms with n large converges whenever the norm of $\Im(y)$ is at least δ for any given positive constant δ, so it is sufficient to prove that the sum of the terms for any fixed n can be analytically continued with at most logarithmic singularities along rational quadratic divisors.

If we replace L by nL we find that we have to show that the function

$$\sum_{x\in L+v, x\in C} e^{-2\pi i(x,y)} I_{1-k}(4\pi\sqrt{m(-x,x)/2})((-x,x)/2)^{(k-1)/2}$$

has only logarithmic singularities along rational quadratic divisors. This is a finite linear combination of sums of the form

$$\sum_{x\in L, x\in C} e^{-2\pi i(x,y+r)} I_{1-k}(4\pi\sqrt{m(-x,x)/2})((-x,x)/2)^{(1-k)/2}$$

for some larger lattice L and some rational vectors $r\in L\otimes\mathbf{Q}$. If we apply the Poisson summation formula to the integral in lemma 5.4 we can evaluate this sum explicitly, and by lemma 5.4 all its singularities lie on quadratic divisors of the form $(y+r+v, y+r+v)=2m$ for some vectors v in the dual of L. (More precisely, the singularities are of the form $-m^{(k-1)/2}\log(1-2m/(y+r+v, y+r+v))$.) If we exponentiate this we find that all singularities and zeros of the product in theorem 1 lie on rational quadratic divisors because b is rational and $r+v\in L\otimes\mathbf{Q}$, which proves theorem 5.1.

6. Vector systems and the Macdonald identities

In this section we show that certain infinite products parameterized by vectors of a lattice are Jacobi forms. The Macdonald identities for root systems are more or less a special case of this result.

We first define vector systems in a lattice, which are a generalization of indecomposable root systems. Suppose that K is a positive definite integral lattice, and that we are given nonnegative integers $c(v)$ for $v\in K$ which are zero for all but a finite number of vectors of K. We say that the function c is a **vector system** if it has the following 2 properties.

1. $c(v)=c(-v)$.
2. The function taking λ to $\sum_{v\in K}c(v)(\lambda,v)^2$ is constant on the sphere of norm 1 vectors $\lambda\in K\otimes\mathbf{R}$.

We will write V for the "multiset" of vectors in a vector system, so we think of V as containing $c(v)$ copies of each vector $v \in K$, and we write $\sum_{v\in V} f(v)$ instead of $\sum_{v\in K} c(v)f(v)$ (and similarly for products over V). The second axiom for a vector system says that the directions of the vectors in it are evenly distributed over the unit sphere in some weak sense. We say the vector system V is trivial if it only contains vectors of zero norm. A decomposable root system is not usually a vector system.

Lemma 6.1. *If G is any group acting on the lattice K that acts irreducibly on $K \otimes \mathbf{R}$ and contains -1 then any orbit V of G, or any finite union of orbits of G, is a vector system. In particular if the automorphism group of K acts irreducibly on $K \otimes \mathbf{R}$ then the set V of vectors of any fixed norm is a vector system, and any finite multiset of vectors of V invariant under the automorphism group of K is a vector system.*

Proof. If $\sum_{v\in V}(\lambda, v)^2$ were not constant on the unit sphere, then the points at which it took its maximum value would span a proper subspace of $K \otimes \mathbf{R}$ invariant under the G, contradicting the fact that G acts irreducibly on $K \otimes \mathbf{R}$. This proves lemma 6.1.

We define the **index** m of a vector system by

$$m = \sum_{v\in V} \frac{(v, v)}{2\dim(K)}.$$

Lemma 6.2. *If V is a vector system and λ and μ are any vectors of K then*

$$\sum_{v\in V}(v, \lambda)(v, \mu) = 2m(\lambda, \mu)$$
$$\sum_{v\in V} v(v, \lambda) = 2m\lambda.$$

Proof. If $\lambda = \mu$ the first line follows from axiom 2 by integrating λ over the unit sphere. The case for arbitrary λ and μ follows from the case for $\lambda = \mu$ by polarization. The second identity follows from the first because both sides are vectors having the same inner product with all vectors μ. This proves lemma 6.2.

Lemma 6.3. *The index m of a vector system V is a nonnegative integer, and is 0 if and only if the vector system is trivial.*

Proof. The only nontrivial fact to prove is that m is integral. Suppose that n is the highest common factor of all the integers (λ, μ) for $\lambda \in K$, $\mu \in K$. The sum on the left of lemma 6.2 is divisible by $2n^2$ for any $\lambda, \mu \in K$ (the factor of 2 comes from the fact that if $v \in V$ then $-v \in V$), so if we let λ and μ run through all vectors of K we see from lemma 6.2 that m is divisible by $2n^2/2n = n$ and is therefore integral. This proves lemma 6.3.

The index m is closely related to the dual Coxeter number of a root system, and can be thought of as measuring the "average amount of norm per dimension" of the vector system. If the vector system is an indecomposable root system with roots of maximal length 2, then its index is equal to the dual Coxeter number.

The hyperplanes orthogonal to the vectors of a vector system V divide $K \otimes \mathbf{R}$ into cones that we call the **Weyl chambers** of the vector system. (Warning: unlike the case of root systems, the Weyl chambers need not be all the same shape.) If we choose a fixed Weyl chamber W then we can define the positive and negative vectors of the vector system by saying that v is positive or negative ($v > 0$ or $v < 0$) if v has positive or negative inner product with some vector in the interior of the Weyl chamber. This does not depend on which vector in the Weyl chamber we choose, and every vector of the vector system is either positive or negative.

We define the **Weyl vector** $\rho = \rho_W$ of W by

$$\rho = \frac{1}{2} \sum_{v \in V, v > 0} v.$$

Lemma 6.4. *If λ is in the dual of K, then $2(\rho, \lambda) \equiv m(\lambda, \lambda) \bmod 2$, and in particular $m(\lambda, \lambda)$ is integral. (Notice that the Weyl vectors for different Weyl chambers differ by elements of K, so that $2(\rho, \lambda)$ is well defined mod 2 independently of the choice of Weyl chamber.)*

Proof.

$$\begin{aligned}(2\rho, \lambda) &= \sum_{v>0} (v, \lambda) \\ &\equiv \sum_{v>0} (v, \lambda)^2 \bmod 2 \\ &= m(\lambda, \lambda),\end{aligned}$$

which proves lemma 6.4.

For example, if K is an even unimodular lattice, then this lemma shows that $\rho \in K$ because it has integral inner product with every element of the dual of K. This does not imply that the Weyl vector of the root system of K lies in K because the root system of K is not always a vector system.

Finally we define d to be the number of vectors in V (counted with multiplicities), and we define the weight k to be half the number of zero vectors in V (so $k = c(0)/2$). For example, if V is the weights of some representation of a simple finite dimensional Lie algebra, then d is the dimension of this representation.

If V is a vector system in K we define the (untwisted) **affine vector system** of V to be the multiset of vectors $(v, n) \in K \oplus \mathbf{Z}$ with $v \in V$. We say that (v, n) is positive if either $n > 0$ or $n = 0, v > 0$.

We select a Weyl chamber W with its corresponding Weyl vector ρ and positive vectors, and define

$$\psi(z, \tau) = q^{d/24} \zeta^{-\rho} \prod_{v \in V, n \in \mathbf{Z}, (v,n) > 0} (1 - q^n \zeta^v)$$

where $q^a = e^{2\pi i a \tau}$, $\zeta^v = e^{2\pi i (z, v)}$. When V is the vector system of a finite dimensional or affine Kac-Moody algebra this is essentially the denominator of the Weyl-Kac character formula.

The main aim of this section is to prove the following generalization of the Macdonald identities for untwisted affine root systems.

Theorem 6.5. *The function ψ is a nearly holomorphic Jacobi form of weight k and index m. More precisely,*

$$\begin{aligned}\psi(z,\tau+1) &= e^{2\pi i d/24}\psi(z,\tau)\\ \psi(z/\tau,-1/\tau) &= (-i)^{d/2-k}(\tau/i)^k e^{2\pi i m(z,z)/2\tau}\psi(z,\tau)\\ \psi(z+\mu,\tau) &= (-1)^{2(\rho,\mu)}\psi(z,\tau)\\ \psi(z+\lambda\tau,\tau) &= (-1)^{2(\rho,\lambda)}q^{-m(\lambda,\lambda)/2}\zeta^{-m\lambda}\psi(z,\tau)\end{aligned}$$

for any $\lambda,\mu\in K'$. The function ψ can be written as a finite sum of theta functions times nearly holomorphic modular forms.

For example, if V is an indecomposable root system of rank n together with $c(0)$ copies of the zero vector, then the product is just the product occurring in the Macdonald identity of the untwisted affine root system of V. Moreover this product is a holomorphic Jacobi form of singular weight, so can be written as a finite sum of theta functions. This turns out to be a sum over the (finite) Weyl group of theta functions, and this sum can be written as a sum over the affine Weyl group. Hence we recover the usual Macdonald identities. I do not know of any cases other than the Macdonald identities where the sum of theta functions times modular forms has been worked out explicitly.

Proof of theorem 6.5. We start with the two easy transformations. It is obvious that

$$\psi(z,\tau+1) = e^{2\pi i d/24}\psi(z,\tau)$$

because $q^{d/24}$ is the only factor which is changed by adding 1 to τ. The only factor of ψ that changes under adding μ to z is $\zeta^{-\rho}$ which gets multiplied by $(-1)^{2(\mu,\rho)}$, so

$$\psi(z+\mu) = (-1)^{2(\mu,\rho)}\psi(z).$$

For the transformation of adding $\lambda\tau$ to z we first assume that λ is in the Weyl chamber and calculate as follows.

$$\begin{aligned}\psi(z+\lambda\tau,\tau) =& q^{d/24}e^{-2\pi i(\rho,z+\lambda\tau)}\prod_{r,n}(1-q^{n+(\lambda,v)}\zeta^v)\\ =&\psi(z,\tau)e^{-2\pi i(\rho,\lambda)\tau}\prod_{v\in V,0<n\leqq-(\lambda,v)}(-\zeta^v q^{n+(\lambda,v)})\\ =&\psi(z,\tau)e^{-2\pi i(\rho,\lambda)\tau}\prod_{v<0}(-1)^{(\lambda,v)}\zeta^{-v(\lambda,v)}q^{-(\lambda,v)((\lambda,v)+1)/2}\\ =&\psi(z,\tau)\prod_{v<0}(-1)^{(\lambda,v)}\zeta^{-v(\lambda,v)}q^{-(\lambda,v)(\lambda,v)/2}\\ =&\psi(z,\tau)(-1)^{2(\rho,\lambda)}\zeta^{-m\lambda}q^{-m(\lambda,\lambda)/2}\end{aligned}$$

where in the last step we use lemma 6.2 twice. If this is true for two values of λ then it is also true for their sum and difference. The vectors λ in the

Weyl chamber generate the whole of K', so this transformation law holds for all $\lambda \in K'$.

Next we consider the function

$$\phi(z,\tau) = \psi(z/\tau, -1/\tau)e^{-2\pi i m z^2/2\tau}/\psi(z,\tau).$$

This function has no zeros or poles because the zeros of $\psi(z,\tau)$ (which are the divisors of points z with (v,z) in the lattice generated by 1 and τ) are the same as the zeros of $\psi(z/\tau, -1/\tau)$ (and have the same multiplicities). The transformations we have just proved for ψ imply that $\phi(z,\tau) = \phi(z+\lambda\tau+\mu,\tau)$ for all $\lambda, \mu \in K$. Hence for any fixed τ, $\phi(z,\tau)$ is a holomorphic abelian function of z and is therefore constant. We will now work out $\phi(z,\tau)$ by taking the limit of $\phi(z,\tau)$ as z tends to 0. (We cannot set $z = 0$ because the numerator and the denominator of $\phi(z,\tau)$ usually both vanish at $z = 0$.)

We define $\psi_0(z,\tau)$ by

$$\psi_0(z,\tau) = \psi(z,\tau)/\prod_{v>0}(1-\zeta^v).$$

Then $\psi_0(0,\tau) = \eta(\tau)^d$, so that

$$\psi_0(0,-1/\tau) = (\tau/i)^{d/2}\psi_0(0,\tau).$$

Also

$$\phi(z,\tau) = \frac{\psi_0(z/\tau,-1/\tau)\prod_{v>0}(1-e^{2\pi i(v,z/\tau)})}{\psi_0(z,\tau)\prod_{v>0}(1-e^{2\pi i(v,z)})e^{2\pi i z^2/2\tau}}$$

so if we take the limit as z tends to 0 and use the fact that $\phi(z,\tau)$ does not depend on z we find that

$$\begin{aligned}\phi(z,\tau) &= (\tau/i)^{d/2}/\prod_{v>0}\tau \\ &= i^{k-d/2}(\tau/i)^k.\end{aligned}$$

From the definition of ϕ, this is equivalent to the final transformation law for ψ. This proves theorem 6.5.

The results in this section can easily be extended to cover the analogues of twisted affine root systems. We will briefly sketch how to do this in the remainder of this section. A pure affine vector system of level N is defined to be the multiset of vectors of the form $(v, Nn+(v,\lambda)) \in K \oplus \mathbf{Z}$ as v runs through the vectors of some vector system and n runs through all integers, and λ is some fixed vector of K'. We define an affine vector system of level dividing N to be a union of pure affine vector systems of level dividing N. For each affine vector system of level dividing N we can define a function $\psi(z,\tau)$ as an infinite product over half the vectors in the affine vector system as above. The function ψ is then a nearly holomorphic Jacobi form for the congruence subgroup $\Gamma_0(N) = \{\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right) \in SL_2(\mathbf{Z}) | c \equiv 0 \bmod N\}$ of $SL_2(\mathbf{Z})$ (and can therefore be written as a sum of theta functions times nearly holomorphic modular functions). We can prove this in the same way as above: the product for each pure affine vector system of level N is a Jacobi form for the conjugate $\{\binom{ab}{cd} | ad - bc = 1, a, nb, c/n, d \in \mathbf{Z}\}$ of $SL_2(\mathbf{Z})$, so the product for the union of the pure affine vector systems of level dividing N is a Jacobi form for the intersection of these conjugates, which contains $\Gamma_0(N)$. For affine root systems we always have $1 \leqq N \leqq 4$.

7. The Weierstrass $\wp$ function

In this section we prove some identities involving the Weierstrass $\wp$ function that we will use in section 9. The results of this section and section 9 are not used elsewhere in this paper.

We recall that the Weierstrass $\wp$ function is defined for $\Im(\tau) > 0$, $z \in \mathbf{C}$ by

$$\wp(z,\tau) = \frac{1}{z^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(z - m\tau - n)^2} - \frac{1}{(m\tau + n)^2} \right)$$

and satisfies the functional equations

$$\wp(z + \lambda\tau + \mu, \tau) = \wp(z,\tau)$$
$$\wp\left(\frac{z}{c\tau + d}, \frac{a\tau + b}{c\tau + d} \right) = (c\tau + d)^2 \wp(z,\tau).$$

In other words $\wp$ is a meromorphic Jacobi form of weight 2 and index 0 and signature 1 (see [E-Z p. 2]).

We also recall the formulas for the Eisenstein series E_2

$$E_2(\tau) = \frac{3}{\pi^2} \sum_m \left(\sum_n{}' \frac{1}{(m\tau + n)^2} \right)$$
$$= 1 - 24 \sum_{n>0} \sigma_1(n) q^n$$

where $\sum_n'$ means we omit $n = 0$ if $m = 0$. This function satisfies the functional equation $E_2((a\tau + b)/(c\tau + d)) = (c\tau + d)^2 E_2(\tau) + 12c(c\tau + d)/2\pi i$ for $\binom{ab}{cd} \in SL_2(\mathbf{Z})$. The Eisenstein series $E_k(\tau)$ for k even and $k \geqq 4$ is equal to $1 - (2k/B_k) \sum_{n>0} \sigma_{k-1}(n) q^n$ and satisfies the functional equation $E_k((a\tau + b)/(c\tau + d)) = (c\tau + d)^k E_k(\tau)$ for $\binom{ab}{cd} \in SL_2(\mathbf{Z})$.

By differentiating the partial fraction decomposition

$$\frac{1}{z} + \sum_{n\neq 0} \left(\frac{1}{z - n} + \frac{1}{n} \right) = \pi \cot(\pi z) = -\pi i - 2\pi i \sum_{n>0} e^{2\pi i n z}$$

(valid for $\Im(z) > 0$) we find that

$$\sum_{n\in\mathbf{Z}} \frac{1}{(z + n)^2} = (2\pi i)^2 \sum_{n>0} n e^{\pm 2\pi i n z}$$

(valid for z not real), where $e^{\pm x}$ means e^x if $|e^x| < 1$ and e^{-x} if $|e^x| > 1$. From this we see that

$$\wp(z,\tau) = (2\pi i)^2 \sum_{m\in\mathbf{Z}}\sum_{n>0} ne^{\pm 2\pi i n(z+m\tau)} - \tfrac{\pi^2}{3}E_2(\tau)$$

whenever $\Im(z+m\tau)$ is nonzero for all integers m.

By differentiating repeatedly with respect to z we find that for positive even integers k the derivatives $\wp^{(k-2)}(z,\tau) = \frac{d}{dz}^{k-2}\wp(z,\tau)$ satisfy

$$\wp^{(k-2)}(z+\lambda\tau+\mu,\tau) = \wp^{(k-2)}(z,\tau)$$
$$\wp^{(k-2)}\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k\wp^{(k-2)}(z,\tau)$$
$$\wp^{(k-2)}(z,\tau) = (2\pi i)^k \sum_{m\in\mathbf{Z}}\sum_{n>0} n^{k-1}e^{\pm 2\pi i n(z+m\tau)} - \delta_k^2E_2(\tau)\pi^2/3$$

(where δ_m^n is 1 if $m=n$ and 0 otherwise).

Suppose that K is a positive definite even lattice of dimension s and that $c(r)$ is an integer defined for $r\in K$ such that $c(r) = c(-r)$ and $c(r) = 0$ for all but a finite number of $r\in K$. Choose a vector ρ not orthogonal to any r with $c(r)\neq 0$, and say that the pair $(r,n)\in K\oplus\mathbf{Z}$ is positive if $n>0$ or $n=0$ and $(r,\rho)>0$.

Theorem 7.1. *Suppose that k is an even positive integer and suppose that if $k=2$ then $\sum_{r\in K}c(r) = 0$. Then the function*

$$\psi(z,\tau) = -c(0)B_k/2k + \sum_{(r,n)>0}\sum_{a|(r,n)} a^{k-1}c(r/a)q^n\zeta^r$$

can be extended to a meromorphic function defined for $\Im(\tau)>0$, $z\in K\otimes\mathbf{C}$ which satisfies the functional equations

$$\psi\left(\frac{z}{c\tau+d}, \frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k\psi(z,\tau)$$
$$\psi(z+\lambda\tau+\mu,\tau) = \psi(z,\tau)$$

for $\lambda,\mu\in K'$, $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)\in SL_2(\mathbf{Z})$. In other words, ψ is a meromorphic Jacobi form of weight k and index 0.

Proof. The function ψ is equal to

$$-c(0)\frac{B_k}{2k}E_k(\tau) + (2\pi i)^{-k}\sum_{r>0}c(r)\left(\wp^{(k-2)}((r,z),\tau) + \delta_k^2E_2(\tau)\pi^2/3\right)$$

so theorem 7.1 follows from the functional equations of the derivatives of the Weierstrass $\wp$ function and the Eisenstein series $E_k(\tau)$. (If $k=2$ then the assumption $\sum_{r\in K}c(r) = 0$ implies that the coefficient of $E_2(\tau)$ is 0.)

8. Generators for $O_M(\mathbf{Z})^+$

In this section we prove a technical result which says that the orthogonal group $O_M(\mathbf{Z})^+$ is generated by a Fourier and a Jacobi subgroup. We will use this result to construct automorphic forms for $O_M(\mathbf{Z})^+$ by showing that the form transforms correctly under both the Fourier and the Jacobi groups, and hence under the whole of $O_M(\mathbf{Z})^+$.

We let M be the lattice $II_{s+2,2}$ (with $8|s$). We let u be a primitive norm 0 vector of M and let U be a 2-dimensional primitive null sublattice containing u. We let F and J be the Fourier and Jacobi groups of u and U.

The main theorem of this section is

Theorem 8.1. *$O_M(\mathbf{Z})^+$ is generated by $F(\mathbf{Z})^+$ and $J(\mathbf{Z})^+$.*

We first prove two lemmas.

Lemma 8.2. *If $L = II_{s+1,1}$, $v \in L\otimes \mathbf{R}$ and $v \notin L$ then there is some $\lambda \in L$ with $1/2 \leqq (v-\lambda)^2 \leqq 3/2$.*

Proof. Choose a primitive norm 0 vector $u_1 \in L$ with (u_1, v) not an integer, which we can do because the norm 0 vectors of L generate L. Choose another norm 0 vector $u_2 \in L$ with $(u_1, u_2) = -1$. We can find an integer m so that $0 < |(v - mu_2, u_1)| \leqq 1/2$. But then $(v - mu_2 - nu_1)^2 = A + Bn$ for some fixed A and B with $0 < |B| \leqq 1$, so we can choose n so that $1/2 \leqq A + nB \leqq 3/2$. This proves lemma 8.2.

Lemma 8.3. *If u is a primitive norm 0 vector in M and G is the group generated by the reflection of norm 2 vectors r with $(r,u) = -1$ then G acts transitively on primitive norm 0 vectors of M.*

Proof. Suppose u_1 is a primitive norm 0 vector of M. If $(u_1, u) \geqq 0$ then it is easy to find an element $g \in G$ such that $(g(u_1), g) < 0$, so we can assume that $(u_1, u) < 0$. We will prove lemma 8.3 by induction on $-(u_1, u)$. If $-(u_1, u) = 1$ then the reflection of the root $u_1 - u$ maps u_1 to u. If $-(u_1, u) > 1$ then we choose coordinates $M = L \oplus II_{1,1}$ for M so that $u = (0,0,1)$. If $u_1 = (v, m, n)$ then v/m is not in L, otherwise m would divide v and hence n as $(v, v) = 2mn$, which is impossible as u is primitive and $m = -(u_1, u) > 1$. Therefore by lemma 8.2 we can choose a vector $\lambda \in L$ with $1/2 \leqq (\lambda - v/m)^2 \leqq 3/2$. Then a simple calculation shows that if g is the reflection of the vector $(\lambda, 1, \lambda^2/2 - 1)$ then $0 < m/4 \leqq -(g(u_1), u) \leqq 3m/4 < m = -(u_1, u)$, so by induction u_1 is conjugate to u under the group G. This proves lemma 8.3.

Now we prove theorem 8.1. We let G be the group generated by $F(\mathbf{Z})^+$ and $J(\mathbf{Z})^+$. We choose coordinates $M = K \oplus II_{1,1} \oplus II_{1,1}$ so that $u = (0,0,0,0,1)$ and U is the set of vectors of the form $(0,0,*,0,*)$. The reflection of $(0,1,-1,0,0)$ is in $F(\mathbf{Z})^+$, and is conjugate under $J(\mathbf{Z})^+$ to the reflection of $(0,0,0,1,-1)$ so that G contains the reflection of at least one root having inner product -1 with u. However the normal abelian subgroup of $F(\mathbf{Z})^+$ acts simply transitively on the set of such roots, so G contains the reflections of all roots r with

$(r,u) = -1$. By lemma 8.3 G therefore acts transitively on the primitive norm 0 vectors of M. As it contains the stabilizer $F(\mathbf{Z})^+$ of one such vector in $O_M(\mathbf{Z})^+$ it must therefore be the whole of $O_M(\mathbf{Z})^+$. This proves theorem 8.1.

9. The positive weight case

In this section we construct some examples of meromorphic automorphic forms whose poles are known explicitly, which can be used to construct holomorphic automorphic forms by multiplying them by other automorphic forms with zeros which cancel out the poles. The results of this section are not used elsewhere in this paper except in a few examples.

We recall that K is an even unimodular lattice of dimension s, and we let θ_K be the theta function of K of index 1, defined by

$$\theta_K(z,\tau) = \sum_{\lambda \in K} q^{(\lambda,\lambda)/2} \zeta^\lambda.$$

This is a holomorphic Jacobi form of index 1 and singular weight $s/2$.

Theorem 9.1. *The linear map taking $f(\tau)$ to $f(\tau)\theta(z,\tau)$ is an isomorphism from nearly holomorphic modular forms of weight k for $SL_2(\mathbf{Z})$ to nearly holomorphic Jacobi forms for $J(\mathbf{Z})^+$ of weight $k + s/2$ and index 1.*

We omit the proof of this as it is similar to the proof of theorem 3.5 in [E-Z], but simpler because the lattice L is even and unimodular so we only need one generator θ_K for the space of Jacobi forms of index 1 as a module over the ring of modular forms. This theorem is also proved in Gritsenko [G].

Lemma 9.2. *If $f(\tau)$ is a nearly holomorphic modular form for $SL_2(\mathbf{Z})$ of weight 2 then it is the derivative of a modular function, and in particular has zero constant term.*

Proof. The total number of zeros of f in a fundamental domain is 1/6 as f has weight 2 (where poles are counted as zeros of negative order), so f must have zeros of order at least 2 at conjugates of $(1+\sqrt{3}i)/2$ and a zero at i. Hence f is divisible by $j'(\tau)$. But $f/j'(\tau)$ is then a nearly holomorphic modular function and therefore a polynomial in $j(\tau)$, which easily implies that f is the derivative of a polynomial in $j(\tau)$. This proves lemma 9.2.

Theorem 9.3. *The map taking*

$$\phi(z,\tau) = \sum_{r,n} c(r,n) q^n \zeta^r$$

to $\Phi(z,\tau,\sigma) = \sum_{m \geq 0} p^m (\phi | V_m)(z,\tau)$ (where $p = e^{2\pi i \sigma}$, $q = e^{2\pi i \tau}$, and $\zeta = e^{2\pi i z}$) takes nearly holomorphic Jacobi forms ϕ of level 1 and weight $k > 0$ to meromorphic automorphic forms Φ of weight k for $O_M(\mathbf{Z})^+$ all of whose *singularities are poles of order k along rational quadratic divisors. The Hecke operator V_0 is defined by*

$$(\phi|V_0)(z,\tau) = -c(0,0)B_k/2k + \sum_{(r,n)>0}\sum_{d|(r,n)} d^{k-1}c(r,0)q^n\zeta^r$$

which is equal to $(-c(0,0)B_k/2k)E_k(\tau)$ if ϕ is holomorphic. The map from ϕ to Φ is an isomorphism from the space of holomorphic Jacobi forms ϕ of level 1 and weight k to the "Spezialschar" of all holomorphic automorphic forms $\Phi = \sum_{r,m,n} A(r,m,n)\zeta^r p^m q^n$ of weight k whose coefficients satisfy

$$A(r,m,n) = \sum_{d|(r,m,n)} d^{k-1}A(r/d,1,mn/d^2).$$

Proof. When ϕ is holomorphic this is a straightforward generalization of [E-Z, theorem 6.2] due to Gritsenko [G] and the proof in [E-Z] works with minor changes. (See also [M, paper I].) The first coefficient of the Fourier-Jacobi expansion of any automorphic form is a Jacobi form of level 1, and the coefficients of any form in the Spezialschar are obviously determined by those of the first Fourier-Jacobi coefficient, so the main thing to check is that Φ is an automorphic form. It transforms like an automorphic form under the Jacobi group because all its coefficients do. It is also invariant under the automorphism of L taking (λ,m,n) to (λ,n,m) because the formula for $A(\lambda,m,n)$ is symmetric in m and n. However this automorphism together with the Jacobi group generates the whole of $O_M(\mathbf{Z})^+$ by theorem 8.1, so $\sum_{m\geq 0} p^m(\phi|V_m)(z,\tau)$ is an automorphic form.

When ϕ has poles at the cusps the proof is similar except that we need the following extra arguments. Firstly, the series for Φ does not converge everywhere, so we need to use theorem 5.2 to show that the series for Φ can be analytically continued. Secondly, $\phi|V_0$ is no longer a modular form, so we need to use theorem 7.1 to show that $\Phi|V_0$ is a nearly holomorphic Jacobi form. If $k=2$ then the condition $\sum_{r\in K} c(r) = 0$ is satisfied by lemma 9.2, because it is the constant coefficient of the nearly holomorphic weight 2 form $\phi(0,\tau)$. This proves theorem 9.3.

Example 1. (Gritsenko [G].) If we let f be the constant form 1 of weight 0, we find a singular automorphic form for $O_M(\mathbf{Z})^+$ of weight $s/2$ whose Fourier coefficients $A(\lambda,m,n)$ are given by $A(\lambda,m,n) = \sigma_{s/2-1}(d)$ if $\lambda^2 = 2mn$ and the highest common factor of λ, m, and n is d (where $\sigma_{s/2-1}(0)$ is defined to be $-B_{s/2}/s$). In particular singular automorphic forms with nonzero constant terms exist for all the groups $O_{II_{8n+2,2}}(\mathbf{Z})^+$.

When ϕ is not holomorphic the meromorphic automorphic form Φ will have poles along rational quadratic divisors. We can remove these poles by multiplying Φ by some of the functions produced in section 10, where we construct holomorphic automorphic forms with zeros along any given rational quadratic divisor.

Example 2. Let Φ be the meromorphic automorphic form for $O_{II_{26,2}}(\mathbf{Z})^+$ constructed from the nearly holomorphic modular form j' as in theorem 9.3. Then Φ has weight 14 and the singularities of Φ are poles of order 2 along all rational quadratic divisors of discriminant 2. In section 10 we will find a holomorphic automorphic form Φ_1 which has weight 12 and has a zero of order 1 along every rational quadratic divisor of discriminant 2. Hence $\Phi\Phi_1^2$ is a holomorphic automorphic form of weight 38 with the same zeros as Φ.

10. The zero weight case

This section is the heart of the paper where we put everything together to construct some automorphic forms on $O_M(\mathbf{Z})^+$ as infinite products. We let L be the even unimodular Lorentzian lattice $II_{s+1,1}$, and we let $M = L\oplus II_{1,1}$. We choose a negative norm vector in $L\otimes\mathbf{R}$ and write $r > 0$ to mean that r has positive inner product with this negative norm vector. We will prove

Theorem 10.1. *Suppose that $f(\tau) = \sum_n c(n)q^n$ is a nearly holomorphic modular form of weight $-s/2$ for $SL_2(\mathbf{Z})$ with integer coefficients, with $24|c(0)$ if $s = 0$. There is a unique vector $\rho \in L$ such that*

$$\Phi(v) = e^{-2\pi i(\rho,v)}\prod_{r>0}(1-e^{-2\pi i(r,v)})^{c(-(r,r)/2)}$$

is a meromorphic automorphic form of weight $c(0)/2$ for $O_M(\mathbf{Z})^+$. All the zeros and poles of Φ lie on rational quadratic divisors, and the multiplicity of the zero of Φ at the rational quadratic divisor of the primitive positive norm vector $r \in M$ (see section 5) is

$$\sum_{n>0} c(-n^2(r,r)/2).$$

In particular if this number is always nonnegative then Φ is holomorphic.

Proof. We write $L = K\oplus II_{1,1}$ where K is the lattice $E_8^{s/8}$. We let $\phi(\tau,z)$ be the nearly holomorphic Jacobi form $f(\tau)\theta_K(z,\tau)$. We define a vector system on $E_8^{s/8}$ to be the multiset of vectors $v \in K$ with multiplicities $c(v) = c(-(v,v)/2)$. This is a vector system by lemma 6.1, as $O_K(\mathbf{Z})$ acts irreducibly on K when K is $E_8^{s/8}$ so the set of vectors of any fixed norm is a vector system. We define the corresponding affine vector system V to be the multiset of vectors $(v,n) \in K\oplus\mathbf{Z}$ with multiplicities $c((v,n)) = c(-(v,v)/2)$. By theorem 6.5 the function $\psi(z,\tau)$ associated to V satisfies the following functional equations

$$\begin{aligned}
\psi(z,\tau+1) &= \psi(z,\tau)\\
\psi(z/\tau,-1/\tau) &= \tau^k e^{2\pi i m(z,z)/2\tau}\psi(z,\tau)\\
\psi(z+\mu,\tau) &= \psi(z,\tau)\\
\psi(z+\lambda\tau,\tau) &= q^{-m(\lambda,\lambda)/2}\zeta^{-m\lambda}\psi(z,\tau)
\end{aligned}$$

for any $\lambda, \mu \in K$. (This follows because the integer $k = s/2$ is divisible by 4 and d is divisible by 24, and $\rho \in K$ by the remark after lemma 6.4.) In particular ψ is a Jacobi form of weight k and index m for $J(\mathbf{Z})^+$, and therefore

$$p^m \psi(z, \tau)$$

transforms like an automorphic form of weight k under all elements of $J(\mathbf{Z})^+$.

On the other hand $\phi|V_\ell$ is a Jacobi form of weight 0 and index ℓ, so that

$$\exp(\sum_{\ell>0} p^\ell (\phi|V_\ell)(z, \tau))$$

transforms like an automorphic form of weight 0 for all elements in the Jacobi group $J(\mathbf{Z})^+$ whenever the product converges. If we multiply these two expressions together and use theorem 4.2 we find that

$$\Phi(z, \tau, \sigma) = p^m \psi(z, \tau) \exp(\sum_{\ell>0} p^\ell (\phi|V_\ell)(z, \tau))$$

transforms like an automorphic form of weight $k = c(0)/2$ for all elements in the Jacobi group $J(\mathbf{Z})^+$ whenever the product converges. By theorem 5.1 Φ can be analytically continued as a nonzero multivalued function to all vectors with imaginary parts in the positive cone, except for some singularities or zeros along rational quadratic divisors.

Next we check that Φ is invariant or antiinvariant under the Fourier group $F(\mathbf{Z})^+$. It is obviously invariant under the unipotent radical of $F(\mathbf{Z})^+$ (which is isomorphic to L), so we have to check invariance under the group $O_L(\mathbf{Z})^+$, which we will do by considering the Fourier expansion of Φ. We first check that it is invariant under the element g_1 taking $(z, \alpha, -\delta, \gamma, \beta)$ to $(z, -\delta, \alpha, \gamma, \beta)$. Under this transformation the factor $e^{-2\pi i(\rho, v)}$ of ψ is multiplied by

$$e^{2\pi i(\rho - g_1(\rho), v)},$$

and the factor $\prod(1 - e^{-2\pi i(r,v)})^{c(-(r,r)/2}$ of ψ is multiplied by a factor of $\prod_{r>0, g_1(r)<0} - e^{2\pi i c(-(r,r)/2)(r,v)}$. Hence to prove (anti)invariance under g_1 we have to show that

$$\rho - g_1(\rho) - \sum_{r>0, g_1(r)<0} c(-(r,r)/2)r = 0.$$

Before proceeding further with the proof of invariance of Φ under g_1 we need to calculate the Weyl vector ρ and check some of its properties, which we do in 10.2 to 10.7.

Lemma 10.2. *If $f(\tau), g(\tau)$ are nearly holomorphic modular functions for $SL_2(\mathbf{Z})$ (possibly transforming according to some nontrivial character of $SL_2(\mathbf{Z})$) then the constant term of the q expansion of $f(\tau)g'(\tau)$ vanishes.*

Proof. The $SL_2(\mathbf{Z})$ invariant differential form $f(\tau)g'(\tau)d\tau$ has only one pole on the compactification of the upper half plane modulo $SL_2(\mathbf{Z})$ (which is at the cusp $i\infty$) and therefore its residue there must vanish. But its residue is just the constant term of $f(\tau)g'(\tau)$. This proves lemma 10.2.

Lemma 10.3. *Suppose that $\theta(\tau)$ is a nearly holomorphic modular form of weight $s/2$ and $f(\tau)$ is a nearly holomorphic modular form of weight $-s/2$ (both of level 1). Then the constant term of the q expansion of*

$$s\theta(\tau)f(\tau)E_2(\tau)/24 - \theta'(\tau)f(\tau)$$

is zero.

Proof. This follows by applying lemma 10.2 to the modular functions $f(\tau)\eta(\tau)^s$ and $\theta(\tau)\eta(\tau)^{-s}$, since $\eta'(\tau)/\eta(\tau) = E_2(\tau)/24$. (Alternatively we can observe that $\theta'(\tau) - s\theta(\tau)E_2(\tau)/24$ is a nearly holomorphic modular form of weight $s/2+2$, so the expression in lemma 10.3 is a nearly holomorphic modular form of weight 2 with only one pole, whose residue must be 0.) This proves theorem 10.3.

Theorem 10.4. *The Weyl vector ρ is equal to*

$$\left(\sum_{(r,v)>0} c(-r^2/2)r/2, \quad m, \quad d/24\right)$$

where m is the constant coefficient of $\theta_K(\tau)f(\tau)E_2(\tau)/24$, and d is the constant coefficient of $\theta_K(\tau)f(\tau)$.

Proof. The Weyl vector is $(\rho_K, m, d/24)$ where ρ_K, m, and d are as in section 6. The formulas for ρ_K and d then follow immediately from the definitions in section 6. The integer m in section 6 is equal to the constant term of $\theta_K'(\tau)f(\tau)/s$, so we have to show that the constant term of

$$\theta_K'(\tau)f(\tau)/s - \theta_K(\tau)f(\tau)E_2(\tau)/24$$

vanishes. But this follows from lemma 10.3. This proves theorem 10.4.

Now we check that the Weyl vector ρ lies in L (and not just $L \otimes \mathbf{Q}$).

Lemma 10.5. *For any nonzero integer n the constant term of $\Delta(\tau)^n$ is divisible by 24.*

Proof. $\Delta'(\tau)/\Delta(\tau) = 1 - 24\sum_{m>0}\sigma_1(m)q^m$ is congruent to 1 mod 24, so

$$(\Delta(\tau)^n)' = n\Delta(\tau)^{n-1}\Delta'(\tau) \equiv n\Delta(\tau)^n \bmod 24n.$$

As the left hand side has zero constant coefficient, so does the right hand side mod $24n$, which proves lemma 10.5 as $n \neq 0$.

Lemma 10.6. *If f is a nearly holomorphic modular form of level 1 and negative weight then the constant term of f is divisible by 24.*

Proof. We can write f as an integral linear combination of functions of the form $\Delta^m E_4^n$ with $m < 0$, and lemma 10.6 then follows from lemma 10.5 and the fact that $E_4 \equiv 1 \bmod 24$.

Corollary 10.7. *The Weyl vector ρ_W lies in L.*

Proof. Suppose that K is the lattice E_8^{3n}. We have to check that m and $d/24$ are integers and that $\rho_K = \sum_{(r,v)>0} c(-r^2/2)r/2$ lies in K. We know that $\rho_K \in K$ by the remark after lemma 6.4. The constant term d of $f(\tau)\theta_K(\tau)$ is divisible by 24 by lemma 10.6 and the fact that $\theta_K(\tau) = E_4(\tau)^{s/8} \equiv 1 \bmod 24$, so $d/24$ is integral. Also, $E_2 \equiv 1 \bmod 24$, so $m \equiv d/24 \bmod 1$ is an integer. This proves corollary 10.7.

We now continue with the proof that Φ is invariant under g_1, which was interrupted by the calculation of ρ. We know that $\rho = (\rho_K, m, d/24)$ by theorem 10.4, so that $g_1(\rho) = (\rho_K, d/24, m)$. The vector (κ, a, b) is positive if $a > 0$, or $a = 0, b > 0$, or $a = b = 0, \kappa > 0$, so that $(\kappa, a, b) > 0$ and $g_1(\kappa, a, b) = (\kappa, b, a) < 0$ if and only if $a > 0, b < 0$. Hence

$$\begin{aligned}\sum_{r>0, g_1(r)<0} c(-(r,r)/2)r &= \sum_\kappa \sum_{a>0} \sum_{b<0} c(ab - (\kappa,\kappa)/2)(\kappa, a, b)\\ &= \sum_\kappa \sum_{n>0} \sum_{a|n} c(-n - (\kappa,\kappa)/2)(0, a, -a)\\ &= \sum_\kappa \sum_{n>0} c(-n - (\kappa,\kappa)/2)(0, \sigma_1(n), -\sigma_1(n))\\ &= (0, x, -x)\end{aligned}$$

where x is the constant term of $\theta_K(\tau)f(\tau)\sum_{n>0}\sigma_1(n)q^n$. By theorem 10.4 $x = m - d/24$ (as $E_2(\tau) = 1 - 24\sum_{n>0}\sigma_1(n)q^n$). By comparing this with the expression for ρ we see that we have proved that

$$\rho - g_1(\rho) - \sum_{r>0, g_1(r)<0} c(-(r,r)/2)r = 0.$$

This completes the proof that Φ is (anti)invariant under g_1.

Next we see that it transforms like an automorphic form under the element g_2 taking $(z, \alpha, -\delta, \gamma, \beta)$ to $(z, \alpha, -\delta, \beta, \gamma)$, because this is conjugate to g_1 under an element of the Jacobi group. This element g_2 acts as $v \to 2v/(v,v)$ on $L \otimes \mathbf{C}$, and in particular commutes with of $O_L(\mathbf{Z})^+$. If g_3 is any element of $O_L(\mathbf{Z})^+$ then $\Phi(g_3(v)) = e^{2\pi i(\lambda,v)}\Phi(v)$ for some λ depending on g_3 because of the expression for Φ as an infinite product. We now see that

$$\begin{aligned}&((v,v)/2)^k e^{2\pi i(\lambda,v)}\Phi(v)\\ =&((g_3(v), g_3(v))/2)^k \Phi(g_3(v))\\ =&\Phi(g_2(g_3(v)))\\ =&\Phi(g_3(g_2(v)))\\ =&e^{2\pi i(\lambda, g_2(v))}\Phi(g_2(v))\\ =&e^{2\pi i(\lambda, 2v/(v,v))}((v,v)/2)^k\Phi(v)\end{aligned}$$

so that

$$e^{2\pi i(\lambda,v)} = e^{2\pi i(\lambda, 2v/(v,v))}$$

for all v, which implies that $\lambda = 0$ and hence that Φ is invariant or antiinvariant under $O_L(\mathbf{Z})^+$.

We have shown that Φ transforms correctly under both $F(\mathbf{Z})^+$ and $J(\mathbf{Z})^+$, so by theorem 8.1 Φ transforms like an automorphic form under the whole of $O_M(\mathbf{Z})^+$.

Next we have to find the singularities and zeros of Φ, which we know lie on rational quadratic divisors by theorem 5.1. If $v \in M$ is a primitive positive norm vector corresponding to a rational quadratic divisor, then there is some primitive norm zero vector u of U orthogonal to v, because U is 2-dimensional. As $J(\mathbf{Z})^+$ acts transitively on such vectors u we can assume that u is the standard choice $(0,0,0,0,1)$. This rational quadratic divisor is then just the linear divisor of points orthogonal to some positive norm vector of L. But such a divisor intersects the region where the infinite product for Φ converges, (except where one of the factors is zero or singular) so the only singularities or zeros along such a divisor must be where one of the factors in the infinite product for f has a zero or singularity. But if r is a primitive positive norm vector of L, then the order of the zero of Φ along the divisor of r is just

$$\sum_{n>0} c(-(nr,nr)/2),$$

coming from the factors

$$\prod_{n>0}(1-e^{-2\pi i(nr,v)})^{c(-(nr,nr)/2)}$$

in the infinite product for Φ. This completes the proof of theorem 10.1.

Example 1. If we take L to be $II_{1,1}$ and f to be $j(\tau)-744$ we recover the denominator formula for the monster Lie algebra (which can of course be proved easily without using theorem 10.1).

Example 2. Suppose we take L to be $II_{25,1}$ and f to be $1/\Delta(\tau)$. Then we find that Φ is an antiinvariant automorphic form of weight 12 for $O_{II_{26,2}}(\mathbf{Z})^+$ whose zeros are the rational quadratic divisors corresponding to vectors of norm 2. Any antiinvariant automorphic form must vanish at these zeros, and so must be divisible by Φ. By the Koecher boundedness principle the quotient is an invariant automorphic form. Hence multiplication by Φ is an isomorphism from invariant automorphic forms of weight k to antiinvariant automorphic forms of weight $k+12$. In particular any antiinvariant automorphic form of weight less that 24 must be a multiple of Φ because the only invariant forms of weight less than 12 are constant. (From theorem 9.3 we know that there is an invariant form of weight 12, so there is a nontrivial antiinvariant form of weight 24.) The form Φ is also the denominator function of the fake monster Lie algebra. As it has singular weight, all its nonzero Fourier coefficients correspond to vectors of norm 0. The multiplicities of norm 0 vectors are always easy to work out explicitly, so we find that

$$\Phi(v) = e^{-2\pi i(\rho,v)}\prod_{r>0}(1-e^{-2\pi i(r,v)})^{p_{24}(1-r^2/2)} = \sum_{w\in W}\sum_{n>0}\det(w)\tau(n)e^{-2\pi i n(w(\rho),v)},$$

which gives a new proof of the denominator formula of the fake monster Lie algebra. This is the only case when theorem 10.1 produces a holomorphic automorphic form of singular weight. When the weight is not singular, the Fourier coefficients are much harder to describe explicitly.

11. The negative weight case

In sections 9 and 10 we have shown how to construct holomorphic automorphic forms from Jacobi forms of positive or zero weight. In this section we show that Jacobi forms of negative weight do not seem to give new examples of automorphic forms, at least not in any obvious way.

Suppose that $f = \sum c(n)q^n$ is a nearly holomorphic modular form of weight $k < -s/2$. We can try to apply the construction of sections 9 or 10 to f to produce some function Φ which might be similar to an automorphic form. The first problem with this function is that it has polylogarithm singularities. We can turn these into poles by applying a high power of the Laplace operator, and then we get a meromorphic automorphic function. Unfortunately we will see in this section that this meromorphic automorphic is not new; it is the function associated to the modular form $(d/d\tau)^{1-k}f(\tau)$.

We would expect the Fourier coefficients $A(v)$ $(v \in L)$ of Φ to look something like

$$A(v) = \sum_{d|v} d^{k-1}c(-(v,v)/2d^2).$$

(We will not worry about what the coefficient of 0 is or whether this does somehow define an automorphic function, since the point of this section is that even if these problems can be solved we still do not seem to get new automorphic forms.) If we apply the $(1-k)$'th power of the Laplacian to this we get a function whose Fourier coefficients are

$$\sum_{d|v} d^{k-1}(-(v,v)/2)^{1-k}c(-(v,v)/2d^2).$$

On the other hand, if we apply the operator ${\frac{d}{d\tau}}^{1-k}$ to f we get a nearly holomorphic modular form of positive weight $2-k$ with Fourier coefficients $n^{1-k}c(n)$, because the $(1-k)$'th derivative of a meromorphic modular form of weight $k \leqq 0$ is a meromorphic modular form of weight $2-k$. The automorphic form associated to ${\frac{d}{d\tau}}^{1-k}f$ in theorem 9.3 has Fourier coefficients

$$\sum_{d|v} d^{(2-k)-1}(-(v,v)/2d^2)^{1-k}c(-(v,v)/2d^2).$$

These are equal to the Fourier coefficients above. So the good news is that if we apply the $(1-k)$'th power of the Laplacian to Φ we do seem to get a meromorphic automorphic form (assuming we can define Φ), but the bad news is that this is not a new automorphic form.

We can of course still apply powers of the Laplacian to Φ if f has weight $\geqq -s/2$, and this gives a few examples where some power of the Laplacian applied to some meromorphic automorphic form is a meromorphic automorphic form.

It may be possible to apply some of the Laplacian to Φ to produce a function with logarithmic singularities and then exponentiate this to get a function which can be written as an infinite product. One problem with this is that if we apply an arbitrary power of the Laplacian to something that transforms like an automorphic form, the result usually does not transform like an automorphic form.

12. Invariant modular products

In this section we will define Weyl vectors and Weyl chambers of modular products. These are sometimes the Weyl vectors and Weyl chambers of hyperbolic reflection groups, and even when they are not they still have many of the properties of hyperbolic reflection groups. Conversely, we can often find automorphic forms associated to hyperbolic reflection groups which have the same Weyl vectors and Weyl chambers. We will also give some applications to even unimodular lattices.

Suppose that

$$\Phi(y) = e^{-2\pi i(\rho,y)} \prod_{x>0} (1 - e^{-2\pi i(x,y)})^{c(x)}$$

is a modular product for some Lorentzian lattice L which defines a holomorphic automorphic form. In particular Φ is invariant up to sign for some finite index subgroup G of $O_L(\mathbf{Z})^+$. The hyperplanes orthogonal to the positive norm vectors x with $c(x) \neq 0$ divide up the cone C into closed chambers that we will call the Weyl chambers of Φ. If W is a Weyl chamber of Φ, we define the Weyl vector ρ_W of W by

$$\Phi(y) = \pm e^{-2\pi i(\rho_W,y)} \prod_{(x,-W)>0} (1 - e^{-2\pi i(x,y)})^{c(x)}$$

(where $(x,-W) > 0$ means that $(x,w) > 0$ for any w in the interior of $-W$). We list some of the properties of Weyl vectors of Φ.

1. If $g \in G$ then $\rho_{g(W)} = g(\rho_W)$.
2. If W_1 and W_2 are Weyl chambers then

$$\rho_{W_1} = \rho_{W_2} + \sum_{(x,-W_1)<0,(x,-W_2)>0} c(x)x.$$

3. Any Weyl vector has coëfficient ± 1 in the Fourier expansion of Φ. In particular any Weyl vector lies in the closure of C and has norm at most 0, because this is true of any vector corresponding to a nonzero coefficient of a holomorphic automorphic form.
4. Any Weyl vector of maximal norm with Weyl chamber W has positive inner product with all the positive real roots of W. In particular ρ_W lies in the interior of W if ρ_W has negative norm, and on the boundary if it has zero norm. In either case, W is the only Weyl chamber containing

ρ_W. (If ρ_W does not have maximal norm I do not know whether or not it necessarily lies in W.)

5. If W_1 and W_2 are 2 adjacent chambers separated only by the hyperplane $x^\perp$ with $(x, W_1) > 0$, and x is a root of G, then $(\rho_{W_1}, x) = \sum_{r\in\mathbf{Q}, r>0} c(rx)(rx, rx)/2$. This follows by applying property 1 with g equal to reflection in $x^\perp$. In particular if $c(rx)$ is 1 when $r = 1$ and 0 otherwise then $(\rho_W, x) = -(x, x)/2$, so that ρ_W does indeed behave like a Weyl vector with respect to the simple root x.
6. Any two Weyl vectors differ by a vector of L.

Weyl chambers behave differently depending on whether the Weyl vector W has negative or zero norm. When its norm is negative, the Weyl chamber has only a finite number of sides, a finite automorphism group, and its image in hyperbolic space has finite volume. (The finite volume property follows because $W/\mathrm{Aut}(W)$ is a subset of a fundamental domain of G which has finite volume.) When the Weyl vector has zero norm, the Weyl chamber may have an infinite number of sides, an infinite automorphism group, and infinite volume. The automorphism group then has a free abelian subgroup of finite index, and the quotient of the Weyl chamber by this free abelian subgroup has finite volume. It is quite common for both of these cases to occur for the same function Φ. In any case we obtain a canonical decomposition of hyperbolic space into Weyl chambers each of which has finite volume modulo the action of a free abelian subgroup.

To avoid confusion we will also list a few properties that Weyl vectors and Weyl chambers do not always have. Weyl chambers are not always acted on transitively by some group. Weyl chambers may have quite different shapes, and their Weyl vectors may have different lengths. Reflection in the hyperplane separating two Weyl chambers is not always an automorphism of L.

We will apply these considerations about Weyl vectors to the Lorentzian lattice $II_{s+1,1}$. In the special case $s = 24$, our results immediately imply several well known results about Niemeier lattices (for example, the existence and uniqueness of the Leech lattice, Conway's result that the Leech lattice is the Dynkin diagram of the reflection group of $II_{25,1}$, and the fact that the number of roots of a Niemeier lattice is divisible by 24).

We let $\Phi(y)$ be the automorphic form

$$\Phi(y) = e^{-2\pi i(\rho_W, y)} \prod_{x>0} (1 - e^{-2\pi i(x,y)})^{c(x)}$$

where $c(x)$ is the coefficient of $q^{-(x,x)/2}$ in some nearly holomorphic modular form $f(\tau) = \sum_n c(n)q^n$ of level 1 and weight $-s$. We let K be the even s-dimensional lattice corresponding to u, and we identify $II_{s+1,1}$ with $K \oplus II_{1,1}$, so that vectors of $II_{s+1,1}$ can be written in the form (v, m, n) with $v \in K$, $m, n \in \mathbf{Z}$, and $(v, m, n)^2 = v^2 - 2mn$. We choose the vector u to be $(0, 0, 1)$. We choose a vector $v \in K$ not orthogonal to any vectors of K of small norm, and we let W be the Weyl chamber of $II_{s+1,1}$ containing u and $(v, 0, 0)$.

Theorem 12.1. *If* $24|s$ *and* $s > 0$ *then the constant term of* $\theta_K(\tau)/\Delta(\tau)^{s/24}$ *is divisible by 24.*

Proof. We take f to be $\Delta^{-s/24}$. By theorem 10.4 the constant term of $\theta_K(\tau)/\Delta(\tau)^{s/24}$ is equal to $d = 24(\rho, u_1)$ where ρ is the Weyl vector of the Weyl chamber containing u and $u_1 = (0, -1, 0)$. By corollary 10.7, $\rho \in L$ so the inner product (ρ, u) is an integer, and this proves theorem 12.1.

For example, when $s = 24$ and $f(\tau) = \Delta(\tau)^{-1} = q^{-1} + 24 + \ldots$, this theorem is just the well known fact that the number of norm 2 vectors of any Niemeier lattice K is divisible by 24.

The automorphic form $\Phi(y)$ is a cusp form if and only if there are no Weyl vectors of zero norm, which is true unless $f(\tau)$ has weight $-s/2$ and there is an extremal lattice of dimension s. (An extremal even unimodular lattice in dimension s is one with no nonzero vectors of norm at most $2[s/24]$.) According to [C-S, Chapter 7, section 7], there is exactly one extremal lattice in 24 dimensions (the Leech lattice), at least 2 in dimension 48, and none in dimensions larger than about 41000.

For Niemeier lattices it is well known that there are either no roots (the Leech lattice) or the roots span the vector space of the lattice (any other Niemeier lattice). For even unimodular lattices K of dimension divisible by 24 this has the following generalization:

Theorem 12.2. *An even unimodular lattice K of dimension s divisible by 24 is either extremal (no nonzero vectors of norm at most* $s/12$*), or the vectors of norm at most* $s/12$ *span the vector space* $K \otimes \mathbf{R}$.

Proof. Consider the Weyl chamber W containing a norm 0 vector u corresponding to K. If K is not extremal, then u is not a Weyl vector, so the intersection of the Weyl chamber with a small neighborhood of u has finite volume. This implies that the Weyl chamber has a cusp at u, which implies that the vectors of norm at most $s/12$ in K span K as a vector space.

We can also use the ideas of this section to give an amusing proof of the existence and uniqueness of the Leech lattice (i.e., a 24 dimensional even unimodular lattice with no norm 2 vectors). We do this by considering the automorphic function Φ for the group $O_L(\mathbf{Z})^+$ in theorem 10.1 for $L = II_{25,1}$ and $f(\tau) = 1/\Delta(\tau) = q^{-1} + 24 + 324q + \ldots$. This form has weight $24/2 = 12$ which is singular, so all its nonzero Fourier coefficients correspond to norm 0 vectors of L. In particular the Weyl vector ρ has norm zero, so by the remarks above it corresponds to an extremal lattice, i.e. a 24-dimensional unimodular lattice with no roots. This proves the existence of the Leech lattice. To prove uniqueness, we observe that all faces of any Weyl chamber are orthogonal to norm 2 vectors which are roots, so $O_L(\mathbf{Z})^+$ acts transitively on the Weyl chambers. Any norm 0 vector of an extremal lattice is the Weyl vector of some Weyl chamber, so there can be only one orbit of such norm zero vectors, so the Leech lattice is unique. This also proves Conway's result [C-S, chapter 27] that the Leech lattice is essentially the Dynkin diagram of $II_{25,1}$, which in turn easily implies that the Leech lattice has covering radius $\sqrt{2}$.

13. Heights of vectors

Suppose that we have fixed an automorphic form Φ for $O_M(\mathbf{Z})^+$ which is a modular product, which defines a system of Weyl chambers and Weyl vectors as in section 12. We define the **height** of a vector λ in the positive cone of $L = II_{s+1,1}$ to be the inner product $-(\rho, \lambda)$ where ρ is the Weyl vector of any Weyl chamber containing λ. The height is a continuous positive function on the positive cone which is linear in the interior of any Weyl chamber, and can be extended to norm 0 vectors of L. We have already found a formula for the height of v when v has norm 0 in theorem 10.4. In this section we will find formulas for the height when λ has norm -2 or $-2p$ for p a prime. We do this by looking at the restriction of f to multiples $\tau\lambda$ of λ. This is a modular form in τ for the group $\Gamma_0(p) = \{\binom{ab}{cd} \in SL_2(\mathbf{Z}) | c \equiv 0 \bmod p\}$ whose zeros in the upper half plane are known explicitly and whose zero at the cusps has an order related to the height of λ.

We let v be a vector of norm $-2N$, where for the moment N is any positive integer. We write $c(n)$ for the coefficients of $f(\tau) = \sum_n c(n)q^n$, where f is the nearly holomorphic modular form of weight $-s/2$ from which Φ is constructed as in section 10.

We define an isomorphism from $(L \oplus II_{1,1}) \otimes \mathbf{R}$ to $v^\perp \otimes \mathbf{R} \oplus sl_2(\mathbf{R})$ which takes (λ, m, n) to

$$\bar{\lambda} \oplus \begin{pmatrix} -(\lambda, v) & 2n \\ 2mN & (\lambda, v) \end{pmatrix}$$

where $\bar{\lambda}$ is the projection of λ into $v^\perp$ and $sl_2(\mathbf{R})$ is the set of real 2×2 matrices of trace 0. We let the group $SL_2(\mathbf{R})$ act on $sl_2(\mathbf{R})$ by conjugation, and on $v^\perp$ by the trivial action, which induces an action of $SL_2(\mathbf{R})$ on $(L \oplus II_{1,1}) \otimes \mathbf{R}$. This action is given by

$$\begin{pmatrix} ab \\ cd \end{pmatrix}(\lambda, m, n) = \Big(\lambda + \Big(bdm - b\frac{c}{N}(\lambda, v) - a\frac{c}{N}n\Big)\, v,$$
$$d^2m - \frac{c}{N}d(\lambda, v) - \frac{c^2}{N}n, \quad ab(\lambda, v) + a^2n - b^2mN\Big).$$

In particular, if $\binom{ab}{cd} \in \Gamma_0(N)$ (so that $N|c$) then this maps $L \oplus II_{1,1}$ into $L \oplus II_{1,1}$. This defines an action of $\Gamma_0(N)$ on $L \oplus II_{1,1}$, and hence an action on the Hermitian symmetric space H.

We embed the upper half plane into H by mapping τ to τv, which is represented by the point $(\tau v, 1, -\tau^2 N) \in M \otimes \mathbf{C}$. The action of $\Gamma_0(N)$ on H restricts to the usual action $\binom{ab}{cd}(\tau) = (a\tau + b)/(c\tau + d)$ on the upper half plane. In particular if we restrict an automorphic form of weight k on H to multiples of v we get a modular form of weight $2k$ for $\Gamma_0(N)$.

The restriction of Φ to multiples of v will often be identically 0. We define Φ_0 to be Φ divided by all the factors in the product defining Φ which are identically zero on multiples of v, so that

$$\Phi_0(y) = e^{-2\pi i(\rho, y)} \prod_{x \in L, x>0, (x,v) \neq 0} (1 - e^{-2\pi i(x,y)})^{c(x)}.$$

We define $H_{N,j}(-D)$ to be the number of complex numbers τ in a fundamental domain of $\Gamma_0(N)$ satisfying a nonzero quadratic equation of the form $a\tau^2 + b\tau + c = 0$ with $a,c \in \mathbf{Z}$, $N|a$, $b \equiv j \bmod 2N$, and $b^2 - 4ac = D < 0$ if $D < 0$, and define $H_{N,j}(0)$ to be $-|SL_2(\mathbf{Z})/\Gamma_0(N)|/12$ if $j \equiv 0 \bmod 2N$ and 0 otherwise. (Points on the boundary of the fundamental domain have to be counted with fractional multiplicity in the usual way.) We define the function $\mathbf{H}_{N,j}(\tau)$ by

$$\mathbf{H}_{N,j}(\tau) = \sum_n H_{N,j}(n)q^n.$$

We put $H_N(n) = \sum_{j \bmod 2N} H_{N,j}(n)$ and $\mathbf{H}_N(n) = \sum_{j \bmod 2N} \mathbf{H}_{N,j}(n)$. For example, the first few functions are

$$\mathbf{H}_1(\tau) = -1/12 + (1/3)q^3 + (1/2)q^4 + q^7 + q^8 + q^{11} + (4/3)q^{12} + \cdots$$
$$\mathbf{H}_2(\tau) = -1/4 + (1/2)q^4 + 2q^7 + q^8 + 2q^{12} + \cdots$$
$$\mathbf{H}_3(\tau) = -1/3 + (1/3)q^3 + 2q^8 + 2q^{11} + (4/3)q^{12} + \cdots$$

Theorem 13.1. *Suppose v is a vector of norm $-2N < 0$ and let $\Phi_0(v\tau)$ the function Φ_0 defined above, restricted to the multiples $v\tau$ of v for $\tau \in \mathbf{C}$, $\Im(\tau) > 0$. Then $\Phi_0(v\tau)$ is a modular form for $\Gamma_0(N)$ of weight k_0 equal to the constant term of $f(\tau)\theta_{v\perp}(\tau)$. If τ is a root of an equation $a\tau^2+b\tau+c = 0$ with $a/N, b, c \in \mathbf{Z}$, $(a/N,b,c) = 1$, $b^2 - 4ac = D < 0$, then the order of the zero of $\Phi_0(\tau v)$ is*

$$\sum_{d>0} \sum_{(\lambda,v)=db} c(d^2D/4N - \bar{\lambda}^2/2).$$

The sum of the orders of the zeros of $\Phi_0(\tau v)$ at the cusps of a fundamental domain of $\Gamma_0(N)$ is the constant term of

$$-\sum_{b \bmod 2N} f(\tau)\theta_{v\perp,b}\mathbf{H}_{N,b}(\tau/4N).$$

Under the Fricke involution $\tau \to -1/N\tau$ $\Phi_0(\tau v)$ transforms as

$$\Phi_0(-v/N\tau) = \pm(\sqrt{N}\tau)^{k_0}\Phi_0(\tau v).$$

Proof. Under the group $\Gamma_0(N)$, direct calculation shows that the function Φ transforms as

$$\Phi\left(\frac{-2N\bar{y} + (ay + bv, cy + dv)v}{(cy + dv, cy + dv)}\right) = \pm\left(\frac{(cy + dv, cy + dv)}{-2N}\right)^k \Phi(y).$$

From this we find that the function $\Phi_0(\tau v)$ transforms as

$$\Phi_0(v(a\tau + b)/(c\tau + d)) = \pm(c\tau + d)^{2k+2\sum_{x>0,(x,v)=0}c(-(x,x)/2)}\Phi_0(v\tau)$$

for $\binom{ab}{cd} \in \Gamma_0(N)$, so that $\Phi_0(\tau v)$ is a nonzero modular form for $\Gamma_0(N)$ of weight equal to

$$2k + 2\sum_{x>0,(x,v)=0} c(-(x,x)/2) = \sum_{(x,v)=0} c(-(x,x)/2)$$

which is the constant term of $f(\tau)\theta_{v^\perp}(\tau)$. This follows because $\Phi_0(\tau v)$ is essentially the first nonvanishing coefficient in the Taylor series expansion of Φ orthogonal to $\mathbf{C}v$ and is obtained by differentiating Φ $c(-(x,x)/2)$ times for each positive norm vector x orthogonal to v, and each differentiation contributes 2 to the weight of $\Phi_0(\tau v)$.

We have to calculate the zeros of $\Phi_0(\tau v)$. The zeros of $\Phi_0(y)$ are the divisors of positive norm vectors (λ, m, n) of M, each with multiplicity $c(-\lambda^2/2+mn)$. We have to remember only to count the zeros from one of the vectors (λ, m, n) and $-(\lambda, m, n)$, and also remember to count the zeros from positive multiples of (λ, m, n). This gives a contribution of $c(-\lambda^2/2+2mn)$ to the zero of $\Phi_0(\tau v)$ at τ whenever $((v\tau, 1, v^2\tau^2/2), (\lambda, m, n)) = 0$, or in other words when $mN\tau^2+(v,\lambda)\tau-n = 0$. In particular τ must be an imaginary quadratic irrational. Suppose that $a\tau^2+b\tau+c = 0$ with $a/N, b, c \in \mathbf{Z}$, $(a/N, b, c) = 1$, $b^2-4ac = D$. Then we must have $m = da$, $(v,\lambda) = db$, $-n = dc$ for some integer d. But then $d^2D/4N = (v,\lambda)^2/4N + mn = \bar{\lambda}^2/2 + (mn - \lambda^2/2)$ where $\bar{\lambda} = \lambda - v(\lambda, v)/(v, v)$ is the projection of λ into $v^\perp$. Hence the multiplicity of the zero of $\Phi_0(\tau v)$ at τ is

$$\sum_{d>0}\sum_{\lambda,(\lambda,v)=bd} c(d^2D/4N - \bar{\lambda}^2/2).$$

As $\Phi_0(\tau v)$ is a modular form for $\Gamma_0(N)$, the total number of zeros in a fundamental domain is equal to $|SL_2(\mathbf{Z})/\Gamma_0(N)|/12$ times its weight, which is the constant term of $-\mathbf{H}_{N,b}$ times the constant term of $f(\tau)\theta_{v^\perp}(\tau)$. Hence the number of zeros at all the cusps is this number minus the number of complex zeros of $\Phi_0(\tau v)$ in a fundamental domain. We have just worked out the multiplicity of a zero at any complex number τ, so we can work out the total number of complex zeros in a fundamental domain, and we find that the number of zeros at the cusps is as stated in 13.1.

Finally the transformation formula for $\Phi_0(\tau v)$ under the Fricke involution follows from the formula $\Phi(2y/(y,y) = \pm((y,y)/2)^k\Phi(y)$. This proves theorem 13.1.

Corollary 13.2. *Suppose that v is a vector of norm $-2N \leqq 0$.*

If $N = 0$ and v is primitive then the height of v is the constant term of

$$\theta_{v^\perp/v}(\tau)f(\tau)E_2(\tau)/24.$$

If $N = 1$ then the height of v is the constant term of

$$-\theta_{v^{\perp\prime}}(\tau)f(\tau)\mathbf{H}(\tau).$$

If N is prime then the height of v is the constant term of

$$-\frac{1}{2}\sum_{b \bmod 2N}\theta_{v^\perp,b}(\tau)f(\tau)\mathbf{H}_{N,b}(\tau).$$

Proof. The case $N = 0$ follows from theorem 10.4. If $N = 1$ then the height is just the order of the zero of Φ_0 at the cusp $i\infty$ and as $SL_2(\mathbf{Z})$ has only one cusp the corollary then follows from theorem 13.1. If N is prime then $\Gamma_0(N)$ has 2 cusps represented by 0 and $i\infty$. These two cusps are exchanged by the Fricke involution, so the $\Phi_0(\tau v)$ has zeros of the same order at both cusps. Therefore the order of the zero at $i\infty$ is half the sum of the orders at all cusps. This case of the corollary then follows from theorem 13.1.

The fact that the height is always an integer can be used to find some congruences between the coefficients of theta functions of lattices. As an example we will work out these congruences for the theta functions of some unimodular lattices explicitly. The isomorphism classes of 25 dimensional unimodular lattices can be identified with the orbits of norm -4 vectors v in $II_{25,1}$, where the lattice $v^\perp$ is isomorphic to the lattice of vectors of even norm in the corresponding 25-dimensional unimodular lattice. We put $N = 2$ and $f(\tau) = 1/\Delta(\tau)$ and find that the constant term of

$$-\frac{1}{2}\sum_{b \bmod 4} \theta_{v^\perp,b}\mathbf{H}_{2,b}(\tau)f(\tau)$$

is the height of v and therefore is an integer. The first few coefficients of $\mathbf{H}_{2,b}$ are given by $\mathbf{H}_{2,0}(\tau/8) = -1/4+q+\cdots$, $\mathbf{H}_{2,1}(\tau/8) = \mathbf{H}_{2,3}(\tau/8) = q^{7/8}+\cdots$, and $\mathbf{H}_{2,2}(\tau/8) = (1/2)q^{1/2}+\cdots$. The coefficients of the theta functions are given by $\theta_{v^\perp,0}(\tau)+\theta_{v^\perp,2}(\tau) = \theta_K(\tau)$, and $\theta_{v^\perp,1}(\tau) = \theta_{v^\perp,3}(\tau) = aq^{1/8}+\cdots$, where $a = 1$ if K is the sum of a one dimensional lattice and a Niemeier lattice, and $a = 0$ otherwise. Putting everything together, we find that

$$8\text{height}(v) = 20 + r_2 - 2r_1 - 8a$$

where r_n is the number of vectors of norm n in K. In particular

$$r_2 \equiv 2r_1 + 4 \bmod 8.$$

A consequence of this is that any 25 dimensional even unimodular lattice has minimum norm at most 2. More generally we find that if K is a unimodular lattice of dimension $s+1 \equiv 1 \bmod 24$ with $s > 0$ then the constant term of

$$(\mathbf{H}_{2,0}(\tau/8) + \mathbf{H}_{2,2}(\tau/8))\theta_K(\tau)/\Delta(\tau)^{s/24}$$

is divisible by 2. As the Fricke involution acts on a fundamental domain of $\Gamma_0(2)$ fixed point freely except at the images of the points $\tau = i$ and $\tau = \sqrt{2}i$ the coefficients of the series $\mathbf{H}_{2,0}$ and $\mathbf{H}_{2,2}$ are usually even; more precisely $\mathbf{H}_{2,0}(\tau) \equiv -1/4 + \sum_{n>0} q^{8n^2} \bmod 2$ and $\mathbf{H}_{2,2}(\tau) \equiv (1/2)\sum_{n>0} q^{4n^2} \bmod 2$. This implies that the constant term of

$$(2\theta(\tau) + \theta(\tau/2))\theta_K(\tau)/\Delta(\tau)^{s/24}$$

is divisible by 8. We can get similar congruences for unimodular lattices whose dimension is not 1 mod 24 by adding on copies of the 1-dimensional unimodular lattice until the dimension is 1 mod 24. These congruences can probably also be deduced by constructing automorphic forms for the groups $O_{K\oplus I_{2,2}}(\mathbf{Z})^+$.

14. Product formulas for modular forms

In this section we will prove theorem 14.1 below. This immediately implies the product formula for the modular polynomial stated in the introduction, because the product $\prod_{[\sigma]}(j(\tau)-j(\sigma))$ obviously satisfies the conditions in theorem 14.1, and has zeros corresponding to a function $f_0(\tau)$ of the form $q^D + O(q)$.

Recall that $H(n)$ is the Hurwitz class number for the discriminant $-n$ if $n > 0$, and $H(0) = -1/12$. (So $\sum H(n)q^n = -1/12 + q^3/3 + q^4/2 + q^7 + q^8 + q^{11} + (4/3)q^{12} + \cdots$.)

Theorem 14.1. *Suppose that $f_0(\tau) = \sum c_0(n)q^n$ is a nearly holomorphic modular form of weight $1/2$ for $\Gamma_0(4)$ with integer coefficients whose coefficients $c_0(n)$ vanish unless n is 0 or 1 mod 4. We put*

$$\Psi(\tau) = q^{-h} \prod_{n>0} (1 - q^n)^{c_0(n^2)}$$

where h is the constant term of $f_0(\tau)\sum H(n)q^n$. Then $\Psi(\tau)$ is a meromorphic modular form for some character of $SL_2(Z)$, of integral weight, leading coefficient 1, whose coefficients are integers, and all of whose zeros and poles are either cusps or imaginary quadratic irrationals. This correspondence gives an isomorphism between the additive group of functions satisfying the conditions on f_0 and the multiplicative group of functions satisfying the conditions on Ψ. Under this isomorphism, the weight of the modular form Ψ is $c_0(0)$, and the multiplicity of the zero of Ψ at a quadratic irrational τ of discriminant $D < 0$ is $\sum_{d>0} c_0(Dd^2)$. (The discriminant of τ is $D = b^2 - 4ac$, where a, b, and c are integers with no common factor such that $a\tau^2 + b\tau + c = 0$.)

Example 1. Under this isomorphism $f_0(\tau) = 12\theta(\tau) = 12+24q+24q^4+24q^9+ \cdots$ corresponds to $\Psi(\tau) = \Delta(\tau) = q\prod_{n>0}(1-q^n)^{24}$ of weight $c_0(0) = 12$; this is the usual product formula for the Δ function.

Example 2. Put

$$\begin{aligned}
F(\tau) &= \sum_{n>0, n \text{ odd}} \sigma_1(n)q^n = q + 4q^3 + 6q^5 \cdots \\
\theta(\tau) &= \sum_{n\in Z} q^{n^2} = 1 + 2q + 2q^4 + \cdots \\
f_0(\tau) &= F(\tau)\theta(\tau)(\theta(\tau)^4 - 2F(\tau))(\theta(\tau)^4 - 16F(\tau))E_6(4\tau)/\Delta(4\tau) + 56\theta(\tau) \\
&= q^{-3} - 248q + 26752q^4 - \cdots \\
&= \sum_n c_0(n)q^n.
\end{aligned}$$

Then the corresponding infinite product $\Psi(\tau)$ has weight $c_0(0) = 0$ and has zeros of order $c_0(-3) = 1$ at the point of discriminant -3 (which are the conjugates of $(1+i\sqrt{3})/2)$ and nowhere else, so it must be $j(\tau)^{1/3}$. The Fourier series of $3f_0(\tau)$ is

$$3f_0(\tau) = 3q^{-3} - 744q + 80256q^4 - 257985q^5 + 5121792q^8 - 12288744q^9 + \cdots$$

so

$$\begin{aligned} j(\tau) &= q^{-1} + 744 + 196884q + 21493760q^2 + \cdots \\ &= q^{-1}\prod_{n>0}(1-q^n)^{3c_0(n^2)} \\ &= q^{-1}(1-q)^{-744}(1-q^2)^{80256}(1-q^3)^{-12288744}\cdots \end{aligned}$$

Example 3. The Eisenstein series E_4, E_6, E_8, E_{10}, and E_{14} all satisfy the condition on Ψ and so can be written as infinite products corresponding to some functions $f_0(\tau)$. An explicit formula for E_4 follows easily from the infinite product expansions of j and Δ because $j = E_4^3/\Delta$, and an explicit formula for E_6 is given in example 2 of section 15. The other cases follow from $E_8 = E_4^2$, $E_{10} = E_4E_6$, and $E_{14} = E_4^2E_6$. The remaining Eisenstein series cannot be written as modular products.

Example 4. There are exactly 13 integers n for which $j(\tau) - n$ satisfies the conditions on $\Psi(\tau)$ and hence can be written as a modular product; these are the values of $j(\tau)$ at values of imaginary quadratic τ for which $j(\tau)$ is integral which are well known to be $j((1+i\sqrt{3})/2) = 0$, $j(i) = 2^6.3^3$, $j((1+i\sqrt{7})/2) = -3^35^3$, $j(i\sqrt{2}) = 2^65^3$, $j((1+i\sqrt{11})/2) = -2^{15}$, $j(i\sqrt{3}) = 2^43^35^3$, $j(2i) = 2^33^311^3$, $j((1+i\sqrt{19})/2) = -2^{15}3^3$, $j((1+i\sqrt{27})/2) = -2^{15}3.5^3$, $j(i\sqrt{7}) = 3^3.5^3.17^3$, $j((1+i\sqrt{43})/2) = -2^{18}3^35^3$, $j((1+i\sqrt{67})/2) = -2^{15}3^35^311^3$, $j((1+i\sqrt{163})/2) = -2^{18}3^35^323^329^3$. There are therefore exactly 14 modular forms of weight 12 with integer coefficients for $SL_2(\mathbf{Z})$ which are modular products: the forms $\Delta(\tau)(j(\tau)-n)$ and the form $\Delta(\tau)$.

We will prove theorem 14.1 as follows. We find a spanning set for the set of modular forms f_0 in theorem 14.1 and check for each of them that the conclusion of theorem 14.1 is true using the automorphic forms on $II_{s+2,2}$ constructed in section 10. Then we check that all the modular forms Ψ satisfying the conclusion of theorem 14.1 can be written as a product of the modular products constructed from the forms f_0.

Lemma 14.2. *Every sequence of integers $c_0(n)$ for $n \leqq 0$, $n \equiv 0, 1 \bmod 4$ which are almost all zero is the set of coefficients of nonpositive degree for a unique modular form f_0 satisfying the conditions of theorem 14.1.*

Proof. This is similar to the proof of theorem 5.4 of [E-Z] with a few sign changes. If $f_0(\tau) = \sum_n c_0(n)q^n$ is a modular form satisfying the conditions of theorem 14.1 with $c_0(n) = 0$ for $n \leqq 0$ then we define $h_0(\tau) = \sum_n c_0(4n)q^n$ and $h_1(\tau) = \sum_n c_0(4n+1)q^{n+1/4}$. If we use the fact that $f_0(\tau) = h_0(4\tau) + h_1(4\tau)$

satisfies the relation $f_0(\sigma/(4\sigma+1)) = \sqrt{(4\sigma+1)}f_0(4\sigma)$ and put $\tau = 4\sigma+1$ we find that h_0 and h_1 satisfy the relation

$$h_0(-1/\tau) - ih_1(-1/\tau) = \sqrt{\tau}(h_0(\tau) - ih_1(\tau)).$$

If we let τ be imaginary and take real and imaginary parts of this we find that h_0 and h_1 satisfy the relations

$$\begin{aligned} h_0(\tau+1) &= h_0(\tau) \\ h_1(\tau+1) &= ih_1(\tau) \\ h_0(-1/\tau) &= (1/2 - i/2)\sqrt{\tau}(h_0(\tau) + h_1(\tau)) \\ h_1(-1/\tau) &= (1/2 - i/2)\sqrt{\tau}(-h_0(\tau) + h_1(\tau)). \end{aligned}$$

This implies that h_0 and h_1 modular forms of weight 1/2 which have zeros of order at least 1/4 at all cusps, so h_0 and h_1 are both zero because h_0/η and h_1/η are holomorphic modular functions vanishing at all cusps. This proves that the form f_0 in lemma 14.2 is unique if it exists.

To prove the existence of f_0 we must exhibit such a form f_0 whose Laurent series starts off $q^n = \cdots$ for every $n \leqq 0$ with $n \equiv 0, 1 \bmod 4$. It is sufficient to do this for $n = 0$ or -3, because we can then get all values of n by multiplying by powers of $j(4\tau)$. For $n = 0$ we can use the function $\sum_{n\in\mathbf{Z}} q^{n^2}$, and for $n = -3$ we can use the function $F(\tau)\theta(\tau)(\theta(\tau)^4 - 2F(\tau))(\theta(\tau)^4 - 16F(\tau))E_6(4\tau)/\Delta(4\tau) + 56\theta(\tau)$ where $F(\tau)$ is defined in example 2 above. This proves lemma 14.2.

Lemma 14.3. *There is a norm -2 vector $v \in II_{25,1}$ such that $\theta_{v^{\perp\prime}}(\tau) = 1 + 2q^{1/4} + O(q)$.*

Proof. We take v to be the image of a norm -2 vector of $II_{1,1}$ in $II_{25,1} = II_{1,1} \oplus$ (Leech lattice).

Lemma 14.4. *There is a norm -2 vector $v \in II_{25,1}$ such that $\theta_{v^{\perp\prime}}(\tau) = 1 + 6q + O(q^{5/4}) = 1 + O(q)$.*

Proof. If we take ρ to be a primitive norm 0 vector of $II_{25,1}$ corresponding to the Leech lattice, then ρ is a Weyl vector for the reflection group of the Leech lattice and the simple roots are the norm 2 vectors r with $(r, v) = -1$ and they correspond to vectors of the Leech lattice. If we take r_1 and r_2 to be simple roots having inner product -1 (corresponding to two vectors of the Leech lattice at distance $\sqrt{6}$) then the vector $v = \rho + r_1 + r_2$ is a norm -2 vector of the Leech lattice. It is easy to check that it is in the Weyl chamber of the reflection group of $II_{25,1}$, and that the simple roots of $v^\perp$ are just r_1 and r_2 so that $v^\perp$ has root system a_2 and therefore has 6 norm 2 vectors. It is also easy to check that there are no norm 0 vectors of $II_{25,1}$ having inner product 1 with v, which implies that there are no vectors of norm 1/2 in $v^{\perp\prime}$. This proves lemma 14.4.

Lemma 14.5. *The* **Z**-*module of functions f_0 satisfying the conditions of theorem 14.1 is spanned by functions of the form $\theta_{v^{\perp\prime}}(4\tau)f(4\tau)$, where v is a* *norm -2 vector of a lattice $II_{s+1,1}$ and $f = \sum_n c(n)q^n$ is a nearly holomorphic modular form of level 1 and weight $-s/2$.*

Proof. By using lemmas 14.1 and 14.4 we can find functions which are linear combinations of the functions mentioned in lemma 14.5, and whose Fourier series start off $q^{-4n} + \ldots$ or $2q^{-4n-3} + \ldots$ for any nonnegative integer n. By lemma 14.2 these functions span the module of functions f_0 satisfying the conditions of theorem 14.1.

We can now prove theorem 14.1. Suppose that v is a norm -2 vector of $II_{25,1}$ and f is a nearly holomorphic modular form of weight -12, and

$$\Phi_0(y) = e^{-2\pi i(\rho,y)} \prod_{x\in L, x>0, (x,v)\neq 0} (1 - e^{-2\pi i(x,y)})^{c(x)}$$

is the function defined in section 13. By theorem 13.1 $\Psi(\tau) = \Phi_0(\tau v)$ is a modular form for $SL_2(\mathbf{Z})$ of weight equal to $2k + 2\sum_{x>0,(x,v)=0} c(-(x,x)/2) = \sum_{(x,v)=0} c(-(x,x)/2)$ which is the constant term of $f(\tau)\theta_{v'}(\tau)$. By theorem 13.1 again the zeros of Ψ are as stated in theorem 14.1.

We can find the infinite product decomposition of $\Phi_0(\tau v)$ from that of $\Phi_0(y)$ by restriction, and we see that

$$\Phi_0(\tau v) = q^h \prod_{n>0} (1 - q^n)^{c_0(n^2)}$$

where h is the height of v and $c_0(n)$ is the coefficient of q^n in $f(4\tau)\theta_{v^{\perp\prime}}(4\tau) = f_0(\tau)$. By theorem 13.1 the height h is given by the formula stated in theorem 14.1.

Finally we have to check that the map from functions f_0 to functions Ψ in theorem 14.1 is an isomorphism. If the image of f_0 is 1, then Ψ has no zeros and weight 0, so the coefficients $c_0(n)$ are 0 if $n \leqq 0$, so f_0 is 0 by lemma 14.2, which proves that the map is injective. If Ψ is any function satisfying the conditions of theorem 14.1, then again by lemma 14.2 we can find a function f_0 such that the corresponding infinite product has the same complex zeros and poles as Ψ. By taking the quotient we can assume that Ψ has no zeros or poles except at cusps. But this implies that Ψ must be a positive or negative power of $\eta(\tau)^2$ (as Ψ has integral weight), and we obtain these functions Ψ by taking f_0 to be an integral multiple of $\theta(\tau) = 1 + 2q + 2q^4 + \cdots$. This proves theorem 14.1.

Remark In the case when we take v to be a norm -2 vector in the lattice $II_{1,1}$ we can work out the Weyl vector explicitly. By identifying its height with the integer h in theorem 14.1 we recover the classical Hurwitz formula

$$\sum_{t\in\mathbf{Z}} H(4m - t^2) = \sum_{d|m} \max(d, m/d)$$

for positive integers m.

15. Generalized Kac-Moody algebras

Many automorphic forms for $O_M(\mathbf{Z})^+$ that are modular products, especially those of singular weight, are the denominator functions of generalized Kac-Moody algebras. We give a few examples of this.

Example 1. If $f(\tau)$ is the Hauptmodul of an element of the monster simple group then $f(\sigma) - f(\tau)$ is an automorphic function on $O_{2,2}(\mathbf{R})$ with respect to some discrete subgroup, and can be written as an infinite product whose exponents can be described explicitly. See [B] for details. More generally, if A and B are elements of $SL_2(\mathbf{Z})$ then $f(A\sigma) - f(B\tau)$ can often be written explicitly as an infinite product, and these expressions are often the denominator formulas for generalized Kac-Moody algebras.

Example 2. The product formula

$$\begin{aligned} E_6(\tau) &= 1 - 504\sum_{n>0}\sigma_5(n)q^n \\ &= 1 - 504q - 16632q^2 - 122976q^3 - \cdots \\ &= \prod_{n>0}(1-q^n)^{a(n^2)} \\ &= (1-q)^{504}(1-q^2)^{143388}(1-q^3)^{51180024}\cdots \end{aligned}$$

where

$$\begin{aligned} \sum_n a(n^2)q^n &= q^{-4} + 6 + 504q + 143388q^4 + 565760q^5 \\ &\quad + 18473000q^8 + 51180024q^9 + O(q^{12}) \\ &= (j(4\tau) - 876)\theta(\tau) \\ &\quad - 2F(\tau)\theta(\tau)(\theta(\tau)^4 - 2F(\tau))(\theta(\tau)^4 - 16F(\tau))E_6(4\tau)/\Delta(4\tau) \end{aligned}$$

(see section 16) is the denominator formula for a generalized Kac-Moody algebra of rank 1 whose simple roots are all multiples of some root α of norm -2, the simple roots are $n\alpha$ ($\alpha > 0$) with multiplicity $504\sigma_3(\alpha)$, and the multiplicity of the roots $n\alpha$ is $a(n^2)$. The positive subalgebra of this generalized Kac-Moody algebra is a free Lie algebra, so we can also state this result by saying that the free graded Lie algebra with $504\sigma_5(n)$ generators of each positive degree n has a degree n piece of dimension $a(n^2)$. There are similar examples corresponding to the infinite products for the Eisenstein series E_{10} and E_{14}.

Example 3. In [B] there is an example of an infinite product formula for every element of $2^{24}.O_\Lambda(\mathbf{Z})$ (where Λ is the Leech lattice) given by taking the trace of this element on the cohomology of the fake monster Lie algebra. This is probably always an automorphic form of singular weight for some group $O_M(\mathbf{Z})^+$, although I have not checked this for all cases. This automorphic form is often the denominator function for some generalized Kac-Moody algebra or superalgebra. This gives several examples of automorphic forms of singular weight on groups $O_M(\mathbf{Z})^+$ with level greater than 1.

Example 4. There seems to be a superalgebra of rank 10 associated with the E_8 lattice in the same way that the fake monster Lie algebra is associated with the Leech lattice. In fact, there seem to be 2 closely related superalgebras, one with zero Weyl vector and one with Weyl vector equal to the Weyl vector of the reflection group of $O_{I_{9,1}}(\mathbf{Z})$ generated by the reflections of norm 1 vectors. I have a construction for these superalgebras but have not yet checked all the details. The denominator formula for one of these superalgebras is proved in [B]. The superalgebra is probably acted on by a group $2^8.W(E_8)$ where $W(E_8)$ is the Weyl group if the E_8 lattice. There are presumably twisted versions of the denominator formula for this superalgebra associated to conjugacy classes in $2^8.W(E_8)$. These twisted denominator functions are probably automorphic forms of singular weights, and this can probably be proved case by case using the methods of this paper.

Example 5. The product formula

$$\sum_{m,n\in\mathbf{Z}}(-1)^{m+n}p^{m^2}q^{n^2}r^{mn} = \prod_{a+b+c>0}\left(\frac{1-p^aq^cr^b}{1+p^aq^cr^b}\right)^{f(ac-b^2)}$$

where $f(n)$ is defined by $\sum f(n)q^n = 1/(\sum_n(-1)^nq^{n^2}) = 1+2q+4q^2+8q^3+14q^4+\dots$. is the denominator formula for a generalized Kac-Moody superalgebra of rank 3. This superalgebra is graded by $\mathbf{Z}^3$, and the subspace of degree (a,b,c) had dimension 3 if $(a,b,c)=(0,0,0)$, and $f(ac-b^2)|f(ac-b^2)$ otherwise. (The symbol $m|n$ for the dimension of a superspace means that it is the sum of an ordinary part of dimension m and a super part of dimension n.) This product formula can be proved using the ideas of this paper, except that we need to use Jacobi forms of level 2 rather than level 1. The left hand side is essentially Siegel's theta function of genus 2, and so is an automorphic form for $Sp_2(\mathbf{Z})$. As $Sp_2(\mathbf{R})$ is locally isomorphic to $O_{3,2}(\mathbf{R})$, it is also an automorphic form for the group $O_M(\mathbf{Z})^+$ where M is the even lattice of determinant 2, dimension 5 and signature 1.

16. Hyperbolic reflection groups

There is often an automorphic form with a modular product expansion associated with the hyperbolic reflection group of a Lorentzian lattice, especially when the reflection group of the lattice has finite index in the automorphism group. We will give several examples of this.

Example 1. We let L be the 10-dimensional even Lorentzian lattice $II_{9,1}$, whose reflection group has Dynkin diagram e_{10}. We let f be the automorphic form of weight 252 corresponding to the weight -4 modular form $E_4(\tau)^2/\Delta(\tau) = q^{-1}+504+\cdots$. All the real vectors of the corresponding vector system are norm 2 roots and have multiplicity 1, so they are exactly the roots of the reflection group of L. The Weyl chambers of f are all conjugate and any Weyl vector has norm -1240, and these are the same as the Weyl chambers and Weyl vectors of the reflection group of L. Unfortunately the norm 0 vectors all have multiplicity 504 which is much larger than the multiplicity 8 of the norm 0 vectors of the Kac-Moody algebra e_{10}, so this seems to give no useful information about the root multiplicities of this Kac-Moody algebra. The function f has a simple zero at all rational quadratic divisors of norm 2 and no other zeros, so it divides any antiinvariant automorphic form for the group $O_{II_{10,2}}(\mathbf{Z})^+$, and therefore gives an isomorphism from invariant automorphic forms of weight k to antiinvariant forms of weight $k+252$. In particular any antiinvariant form of weight less than 256 is a multiple of f.

Example 2. We let L be the 18-dimensional even Lorentzian lattice. We let f be the form associated to the weight -8 form $E_4(\tau)/\Delta(\tau) = q^{-1} + 256 + \cdots$. As in the previous example, we find that the Weyl chambers and Weyl vectors (norm -620) of f are the same as those of the reflection group of L, and multiplication by f is an isomorphism from invariant forms of weight k to antiinvariant ones of weight $k+128$.

Example 3. We let L be the even sublattice of index 2 in $I_{21,1}$, which is the orthogonal complement of a d_4 lattice in $II_{25,1}$. The reflection group of L has finite index in the full automorphism group and is generated by the reflections of vectors of norms 2 and 4. If we take an automorphic form Φ for $O_{II_{26,2}}(\mathbf{Z})^+$ and divided it by the factors vanishing on L, we get a function Φ_0 which restricts to an automorphic form for $O_{L\oplus II_{1,1}}(\mathbf{Z})^+$. If we take Φ to be the automorphic form associated to $1/\Delta(\tau)$ then the positive norm vectors of L of nonzero multiplicity are the norm 2 vectors with multiplicity 1, and half the norm 4 roots with multiplicity 8. This gives an automorphic form whose positive norm vectors of nonzero multiplicity are multiples of the roots of L. This example cannot correspond to any generalized Kac-Moody algebra because all the positive norm roots of a generalized Kac-Moody algebra have multiplicity 1.

Example 4. More generally if we take any subdiagram of the Dynkin diagram Λ of $II_{25,1}$ whose components are all of the form d_4, d_n for $n \geqq 6$, e_6, e_7, or e_8 and we let L be the orthogonal complement in $II_{25,1}$ of the lattice of this subdiagram, then we get an example similar to the previous example of an automorphic form whose positive norm vectors of nonzero multiplicity are closely related to the roots of some reflection group of finite index in the full automorphism group of L. Examples 1,2, and 3 correspond to the subdiagrams e_8^2, e_8, and d_4.

17. Open problems

1. Can the methods for constructing automorphic forms as infinite products be used for semisimple groups other than $O_{s+2,2}(\mathbf{R})$?
2. Extend the methods of this paper to level greater than 1. Can any nearly holomorphic Jacobi form of nonnegative weight be used to construct a

meromorphic automorphic form? In particular, can the explicit calculations used in section 10 to prove the existence of a Weyl vector be replaced by a more general argument? Can any automorphic form (with rational integral Fourier coefficients) all of whose zeros are rational quadratic divisors be written as a modular product?

3. Are there a finite or infinite number of automorphic forms of singular weight that can be written as modular products? Are there any such forms on $O_{s+2,2}(\mathbf{R})$ for $s > 24$? The forms of singular weight which are modular products are particularly interesting because they often correspond to generalized Kac-Moody algebras. If a form has singular weight then its Weyl vectors must be of norm 0, and the lattices corresponding to them have no vectors of small norm.
4. Find some interesting cases of the generalized Macdonald identities of chapter 6 such that the sum of theta functions times modular forms can be written down explicitly.
5. What are the eigenvalues of the Hecke operators acting on the weight 12 form for $O_{II_{26,2}}(\mathbf{Z})^+$ (which is an eigenform of the Hecke operators)? Does it correspond to some Galois representation?
6. Can the automorphic forms that are modular products be understood in terms of representation theory or Langlands philosophy? (I do not even know how to understand the product formula for $\Delta(\tau)$ in terms of representation theory.) One problem with this is that the automorphic forms which are modular products are often not eigenforms of Hecke operators; for example there are 14 modular forms of level 1 and weight 12 which are modular products, only one of which (Δ) is an eigenform.
7. Many automorphic forms that are modular products can be interpreted as the denominator formulas of generalized Kac-Moody algebras. Is it possible to construct these generalized Kac-Moody algebras explicitly (other than by generators and relations)? For example, the fake monster Lie algebra is the Lie algebra of chiral strings on a 26-dimensional torus.
8. Given an automorphic form for $O_M(\mathbf{Z})^+$ with a modular product we obtain a decomposition of the cone C into Weyl chambers, each of which either has finite volume, or has finite volume modulo the action of a free abelian group. In the only cases I know of where this decomposition has been worked out explicitly (mostly reflection groups or the examples associated with $II_{25,1}$ in chapter 14) the automorphism group acts transitively on the Weyl chambers, but this is certainly not usually the case. What happens in general? I would guess that there are usually a large number of Weyl chambers, each one of which has a fairly simple structure and not too many sides, but I have no real evidence for this. The structure of any given Weyl chamber can probably be worked out using some sort of extension of Vinberg's algorithm for the case of reflection groups.
9. Is there any connection between the 3 product formulas for $j(\tau) - j(\sigma)$? (The three formulas are the Gross-Zagier formula when both of σ, τ are imaginary quadratic integers, the formula of this paper when one of them

is, and the product formula for the monster Lie algebra when neither of them are.)

10. Extend theorem 14.1 to higher levels. Can any modular form (of any level) with integral coefficients, such that all its zeros are cusps or imaginary quadratic irrationals, be written as a modular product? What happens if the coefficients are not required to be rational? Is there a proof of theorem 14.1 which only uses modular forms on SL_2 and not automorphic forms on larger groups? In theorem 14.1 the coefficients of negative powers of q in f_0 are related to the orders of the zeros of Φ_0, and the coefficients of q^{n^2} appear in the infinite product expansion of Φ_0. Is there any interpretation of the coefficients of q^n for positive values of n that are not squares? Does the correspondence in theorem 14.1 commute with some action of a Hecke algebra? (The Hecke operators would have to act multiplicatively rather than additively on meromorphic modular forms.) The Shimura-Kohnen isomorphism [Ko] is an isomorphism from the space of weight $k/2$ forms of level 4 whose Fourier coefficients $c(n)$ vanish unless $n \equiv 0, (-1)^{(k-1)/2} \bmod 4$, to the space of modular forms of level 1 and weight $k-1$ (k odd). In this isomorphism the coefficients $c(n^2)$ are closely related to the eigenvalues of the Hecke operators. Is there any connection with the isomorphism of theorem 14.1, which has the same condition on the coefficients $c(n)$ and for which the coefficients $c(n^2)$ are also particularly important? (Notice that the Shimura-Kohnen isomorphism is additive, while the one in theorem 14.1 is multiplicative.)
11. The extension of theorem 14.1 to higher levels appears to give many modular functions whose zeros are the imaginary quadratic irrationals that are used to define Heegner divisors of modular curves. R. Taylor suggested that this could be used to produce linear relations between the Heegner points on the Jacobian.
12. Suppose L is a Lorentzian lattice whose reflection group has finite index in its automorphism group. Is there always an automorphic form with a modular product expansion such that the positive vectors of nonzero multiplicity are more or less the same as the roots of L (possibly multiplied by nonzero rational numbers)?

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