Invent. math. 120, 1 – 14 (1995)

Inventiones mathematicae

The number of critical points of a product of powers of linear functions

Peter Orlik, Hiroaki Terao

Department of Mathematics, University of Wisconsin, Madison, WI 53706, USA

Oblatum 16-V-1994 & 15-VIII-1994

1 Introduction

Let V be a complex affine space of dimension ℓ. Let $u_1, \ldots, u_\ell$ be affine coordinates of V. Let $1 \leqq i \leqq n$, $0 \leqq j \leqq \ell$ and let $\{a_{j,i}\}$ be complex numbers. Consider the linear functions

$$\alpha_i = a_{0,i} + a_{1,i}u_1 + \cdots + a_{\ell,i}u_\ell.$$

Let $H_i = \ker(\alpha_i)$ be the corresponding affine hyperplanes in V and let $\mathcal{A} = \{H_1, \ldots, H_n\}$ denote their arrangement. We assume that $\mathcal{A}$ is essential. This means that the lowest dimensional intersections of these hyperplanes are points. Let $N = N(\mathcal{A}) = \bigcup_{i=1}^n H_i$ be the divisor of $\mathcal{A}$ and let $M = M(\mathcal{A}) = V - N(\mathcal{A})$ be the complement of $\mathcal{A}$. Given a complex n-vector $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$, consider the multivalued holomorphic function defined on M by

$$\phi_\lambda(u) = \alpha_1^{\lambda_1} \cdots \alpha_n^{\lambda_n}.$$

Studying Bethe vectors in statistical mechanics, A. Varchenko [18] conjectured that for generic λ all critical points of ϕ_λ are nondegenerate and the number of critical points is equal to the absolute value of the Euler characteristic of $M, \chi(M)$. He proved the conjecture for complexified real arrangements in [18]. A similar result was known to K. Aomoto [1, Example 1] who stated without proof that for positive λ the number of critical points of ϕ_λ equals the number of bounded components of the complement of the real arrangement. See [2, Theorem 4.4.1.1] for details. In this paper we prove Varchenko's conjecture for all arrangements. More generally we give a formula for the number of critical points of the multivalued holomorphic function

$$\Phi_\lambda(u) = e^{f(u)}\phi_\lambda(u) = e^{f(u)}\alpha_1^{\lambda_1} \ldots \alpha_n^{\lambda_n}$$

on M where $f(u)$ is a polynomial of degree $r \geqq 0$. The formula for $r = 0$ is Varchenko's conjecture. The functions ϕ_λ and Φ_λ are integrands in the multivariable theory of hypergeometric functions and hypergeometric integrals They occur in work of Aomoto, Kita, Gelfand, Varchenko, and others [1], [5], [6], [8], [9], [14], [16], [18]. Consult forthcoming books by Aomoto and Kita [2] and by Varchenko [17] for further references.

Theorem 1.1 *Let $\mathcal{A}$ be an essential complex affine arrangement and let f be an $\mathcal{A}$-transverse polynomial of degree $r \geqq 0$.*

(i) *There exists a (Zariski-) closed algebraic proper subset Y of $\mathbb{C}^n$ such that for each $\lambda \in \mathbb{C}^n - Y$, Φ_λ has only finitely many critical points all of which are nondegenerate and the number of critical points of Φ_λ is independent of $\lambda \in \mathbb{C}^n - Y$. Denote this number by $\gamma(\mathcal{A})$.*

(ii) $\gamma(\mathcal{A}) = |\chi(\mathcal{A}, 1 - r)|$.

See Definition 2.3 for the notion of an $\mathcal{A}$-transverse polynomial and Definiton 2.1 for the characteristic polynomial $\chi(\mathcal{A}, t)$ of $\mathcal{A}$. It is known [11, Theorem 5.93] that $\chi(M) = \chi(\mathcal{A}, 1)$. Thus Theorem 1.1 specialized to $r = 0$ proves Varchenko's conjecture for all arrangements.

Let $C(\Phi_\lambda)$ be the set of critical points of Φ_λ in M. In Section 2 we use algebraic methods to study the set $C(\Phi_\lambda)$. It is easy to see that a point $v \in M$ is a critical point of Φ_λ if and only if it is a critical point of log Φ_λ so the 1-form

$$\omega_\lambda = d(\log \Phi_\lambda) = df + \sum_{i=1}^{n} \lambda_i \frac{d\alpha_i}{\alpha_i}$$

vanishes at v. Similar problems were studied in [12] and [15] and we use the terminology and methods of these papers. We interpret the set $C(\Phi_\lambda)$ in terms of a complex $\Omega^*_\lambda = (\Omega^*(\mathcal{A}), \omega_\lambda \wedge)$ of logarithmic forms with poles on $\mathcal{A}$ whose differential is $\omega_\lambda \wedge$. We show that the support in M of the top cohomology $H^\ell(\Omega^*_\lambda)$ is equal to the set $C(\Phi_\lambda)$. This is the set of points of M at which the localization of $H^\ell(\Omega^*_\lambda)$ is not zero. In Section 3 we show that there exists a closed algebraic proper subset Y_1 of $\mathbb{C}^n$ such that for each $\lambda \in \mathbb{C}^n - Y_1$, the cohomology groups $H^p(\Omega^*_\lambda)$ are finite dimensional and we calculate the Euler characteristic of the complex Ω^*_λ:

$$\chi(\Omega^*_\lambda) = \chi(\mathcal{A}, 1 - r). \tag{1}$$

In Section 4 we prove Theorem 1.1. First we find a closed algebraic proper subset Y_2 of $\mathbb{C}^n$ such that for each $\lambda \in \mathbb{C}^n - Y_2$, Φ_λ has no degenerate critical points on M. Then we find a closed algebraic proper subset Y containing $Y_1 \cup Y_2$. This proves the first part of Theorem 1.1. We call $\lambda \in \mathbb{C}^n - Y$ generic. For generic λ all critical points are nondegenerate and their number is the constant $\gamma(\mathcal{A})$:

$$|C(\Phi_\lambda)| = \gamma(\mathcal{A}). \tag{2}$$

It remains to find the value of $\gamma(\mathcal{A})$. We show the existence of generic λ for which

$$H^p(\Omega^*_\lambda) = 0, \quad 0 \leq p < \ell \text{ and } \dim_{\mathbb{C}} H^\ell(\Omega^*_\lambda) = |C(\Phi_\lambda)|. \tag{3}$$

Thus we get from (1), (2), and (3) that $\gamma(\mathcal{A}) = |\chi(\mathcal{A}, 1-r)|$. This proves the second part of Theorem 1.1.

2 Logarithmic forms

In this section we establish the fundamental connections between critical points of Φ_λ and a complex of logarithmic forms. For the rest of this paper we fix a complex affine essential ℓ-arrangement $\mathcal{A} = \{H_1, H_2, \dots, H_n\}$ where $H_i = \ker\alpha_i$ and let $Q = \prod_{i=1}^n \alpha_i$ be a defining polynomial for the arrangement A. Let $L = L(\mathcal{A})$ be the set of all nonempty intersections of hyperplanes in $\mathcal{A}$. We consider L as a partially ordered set by reverse inclusion so it has minimal element V. Let $\mu : L \times L \longrightarrow \mathbb{Z}$ be the Möbius function of L defined by $\mu(X,X) = 1$, the recursion $\sum_{X \leqq Z \leqq Y} \mu(X,Z) = 0$ for $X < Y$, and $\mu(X,Y) = 0$ otherwise.

Definition 2.1 *The characteristic polynomial of* $\mathcal{A}$ *is* $\chi(\mathcal{A},t) = \sum_{X\in L} \mu(V,X)\, t^{\dim X}$.

We construct a central $(\ell+1)$-arrangement from $\mathcal{A}$ called the cone over $\mathcal{A}$. Let cV be an affine $(\ell+1)$-space with coordinates $u_0, u_1, \dots, u_\ell$. Regard V as the affine hyperplane of cV defined by $u_0 = 1$. For $X \in L$, let X^c be the cone over X. We identify the $\mathbb{C}$-algebra of polynomial functions on V with the polynomial algebra $S = \mathbb{C}[u_1, \dots, u_\ell]$. Let $S^c = \mathbb{C}[u_0, u_1, \dots, u_\ell]$ be the coordinate ring of cV. For each $g \in S$ define the homogenization of g as

$$g^h = u_0^{\deg g} g(u_1/u_0, \dots, u_\ell/u_0)$$

where deg g is the degree of g. Clearly, $g^h \in S^c$ and it is homogeneous of degree deg g.

Definition 2.2 *Let* $\mathcal{A} = \{H_1, \dots, H_n\}$ *and* $Q = \prod_{i=1}^n \alpha_i$ *be a defining polynomial for* $\mathcal{A}$. *The* **cone** $c\mathcal{A}$ *of* $\mathcal{A}$ *is defined as the central arrangement in* cV *defined by the polynomial* $Q^c = u_0 Q^h = u_0 \prod_{i=1}^n \alpha_i^h$.

Definition 2.3 *Let* f *be a polynomial of degree* r. *If* $r > 0$, *then we say that* f *is* $\mathcal{A}$-**transverse** *provided for every* $X \in L(c\mathcal{A})$, *the restriction of* f^h *to* X *has no critical points outside the origin. If* $r = 0$, *then we agree that every nonzero constant is* $\mathcal{A}$*-transverse.*

For the rest of this paper we fix an $\mathcal{A}$-transverse polynomial f of degree $r \geqq 0$.

Proposition 2.4 *The following statements are equivalent for a point* $v \in M$:

(1) v *is a critical point of* Φ_λ, *so* $v \in C(\Phi_\lambda)$,

(2) v *is a critical point of* $\log \Phi_\lambda = f + \sum_{i=1}^n \lambda_i \log \alpha_i$,

(3) $\omega_\lambda(v) = df(v) + \sum_{i=1}^n \lambda_i \frac{d\alpha_i}{\alpha_i}(v) = 0$.

Proof. Since $\Phi_\lambda(v) \neq 0$, the assertions follow from the formula

$$\frac{\partial(\log \Phi_\lambda)}{\partial u_j} = \Phi_\lambda^{-1}\frac{\partial(\Phi_\lambda)}{\partial u_j}. \qquad \square$$

Proposition 2.5 *Let $v \in C(\Phi_\lambda)$ and let $H = H_\lambda$ be the Hessian matrix of* $\log \Phi_\lambda$ *with entries*

$$H_{i,j} = \frac{\partial^2 f}{\partial u_i \partial u_j} - \sum_{k=1}^{n} \lambda_k \frac{a_{k,i}a_{k,j}}{\alpha_k^2}.$$

The following statements are equivalent for a point $v \in M$:
(1) *v is a nondegenerate critical point of Φ_λ,*
(2) *v is a nondegenerate critical point of* $\log \Phi_\lambda$,
(3) $\det H_\lambda(v) \neq 0$.

Proof. Since $\partial\Phi_\lambda/\partial u_j(v) = 0$ for all j, the assertions follow from the formula

$$\frac{\partial^2(\log \Phi_\lambda)}{\partial u_i \partial u_j} = \Phi_\lambda^{-1}\frac{\partial^2 \Phi_\lambda}{\partial u_i \partial u_j} - \Phi_\lambda^{-2}\frac{\partial \Phi_\lambda}{\partial u_i}\frac{\partial \Phi_\lambda}{\partial u_j}. \qquad \square$$

A $\mathbb{C}$-linear map $\theta : S \to S$ is a derivation if $\theta(ab) = a\theta(b) + b\theta(a)$ for $a, b \in S$. Let $\mathrm{Der}(S)$ be the S-module of derivations of S. It is a free S-module with basis $\{D_i = \partial/\partial u_i\}$.The module of **$\mathcal{A}$-derivations** is an S-submodule of $\mathrm{Der}(S)$ defined by

$$D(\mathcal{A}) = \{\theta \in \mathrm{Der}(S) \mid \theta(Q) \in QS\}.$$

The Euler derivation is $\theta_E = \sum_{i=1}^{\ell} u_i D_i$. It lies in $D(\mathcal{A})$ if and only if $\mathcal{A}$ is a central arrangement so all hyperplanes contain the origin.

Let p be an integer. Let $\Omega^p[V]$ denote the S-module of all global regular (=polynomial) p-forms on V. Let $\Omega^p(V)$ denote the space of all global rational p-forms on V.

Definition 2.6 *The module $\Omega^p = \Omega^p(\mathcal{A})$ of* **logarithmic p-forms with poles along $\mathcal{A}$** *is defined as*

$$\Omega^p = \Omega^p(\mathcal{A}) = \{\omega \in \Omega^p(V) \mid Q\omega \in \Omega^p[V] \text{ and } Q(d\omega) \in \Omega^{p+1}[V]\}.$$

Let $\Omega^p = 0$ if $p < 0$ or $p > \ell$.

Clearly, $\omega_\lambda \in \Omega^1(\mathcal{A})$. It is easy to see that $\Omega^p(\mathcal{A})$ is a finitely generated S-module containing $\Omega^p[V]$. Also $\Omega^0(\mathcal{A}) = S$ and $\Omega^\ell(\mathcal{A}) = (1/Q)\Omega^\ell[V] \simeq S$. Recall the interior product form [11, 4.74]

$$\langle\ ,\ \rangle : D(\mathcal{A}) \times \Omega^p(\mathcal{A}) \longrightarrow \Omega^{p-1}(\mathcal{A})$$

which is an S-bilinear pairing. In particular, when $p = 1$, we have a pairing

$$\langle\ ,\ \rangle : D(\mathcal{A}) \times \Omega^1(\mathcal{A}) \longrightarrow S \qquad (6)$$

satisfying $\left\langle \sum_i f_i D_i, \sum_j g_j du_j \right\rangle = \sum_k f_k g_k$. For a complex manifold Z and a point $z \in Z$, let $T_z Z$ denote the holomorphic tangent space of Z at z. Define the evaluation map $\tau_z : D(\mathcal{A}) \to T_zM$ as follows. Given $\theta \in D(\mathcal{A})$, write $\theta = \sum u_i D_i$ and let $\tau_z(\theta) = \sum u_i(z)D_i$. Write $\theta_z = \tau_z(\theta)$ and $D(\mathcal{A})_z = \tau_z(D(\mathcal{A}))$. It follows from [11, 5.17] that if $v \in M$, then $D(\mathcal{A})_v = T_vM$. The pairing (6) induces the natural pairing

$$\langle\ ,\ \rangle_v : T_vM \times T_v^*M \longrightarrow \mathbb{C}$$

of the holomorphic tangent space and the holomorphic cotangent space of M at each point $v \in M$.

Proposition 2.7 *Define the ideal* $I_\lambda = \langle D(\mathcal{A}), \omega_\lambda \rangle = \{\langle \theta, \omega_\lambda \rangle \mid \theta \in D(\mathcal{A})\}$ *of* S. *Let* $V(I_\lambda)$ *denote the zero set of* I_λ. *Then* $C(\Phi_\lambda) = V(I_\lambda) \cap M$.

Proof. Let $v \in M$. Then $\omega_\lambda(v) = 0$ if and only if $\langle \xi, \omega_\lambda(v) \rangle = 0$ for all $\xi \in T_vM = D(\mathcal{A})_v$. Apply Proposition 2.4. □

Definition 2.8 *Let* $\lambda \in \mathbb{C}^n$. *Define maps* $\omega_\lambda \wedge : \Omega^p(\mathcal{A}) \to \Omega^{p+1}(\mathcal{A})$ *by sending* $\omega \in \Omega^p(\mathcal{A})$ *to* $\omega_\lambda \wedge \omega$. *We obtain the cochain complex*

$$\Omega_\lambda^* = \Omega_\lambda^*(\mathcal{A}) = (\Omega^*, \omega_\lambda \wedge) : \cdots \xrightarrow{\omega_\lambda \wedge} \Omega^p(\mathcal{A}) \xrightarrow{\omega_\lambda \wedge} \Omega^{p+1}(\mathcal{A}) \xrightarrow{\omega_\lambda \wedge} \cdots$$

Lemma 2.9 $H^\ell(\Omega_\lambda^*) \simeq S/I_\lambda$.

Proof. There is an isomorphism $\gamma : D(\mathcal{A}) \simeq \Omega^{\ell-1}(\mathcal{A})$ defined by

$$\gamma(\theta) = Q^{-1} \sum_{i=1}^{\ell} (-1)^{i-1} \theta(u_i) du_1 \wedge \cdots \wedge \widehat{du_i} \wedge \cdots \wedge du_\ell$$

for $\theta \in D(\mathcal{A})$. By abuse of notation, let γ also denote the isomorphism $\gamma : S \simeq \Omega^\ell(\mathcal{A})$ defined by $\gamma(g) = Q^{-1} g du_1 \wedge \ldots \wedge du_\ell$. Then the map $\delta : \theta \mapsto \langle \theta, \omega_\lambda \rangle$ makes the following diagram commutative:

$$\begin{array}{ccc} D(\mathcal{A}) & \xrightarrow{\gamma} & \Omega^{\ell-1}(\mathcal{A}) \\ \downarrow \delta & & \downarrow \omega_\lambda \wedge \\ S & \xrightarrow{\gamma} & \Omega^\ell(\mathcal{A}). \end{array}$$

□

Let N be an S-module. For $v \in V$, let N_v denote the localization of N at v. It is naturally an S_v-module. Define the support of N by $\mathrm{Supp} N = \{v \in V \mid N_v \neq 0\}$.

Proposition 2.10 (1) $\mathrm{Supp} H^\ell(\Omega_\lambda^*) \cap M = C(\Phi_\lambda)$.
(2) *If* $H^\ell(\Omega_\lambda^*)$ *is finite dimensional, then* $|C(\Phi_\lambda)| \leqq \dim H^\ell(\Omega_\lambda^*)$.

Proof. (1) is a direct consequence of Proposition 2.7 and Lemma 2.9 and (2) follows from (1). □

Proposition 2.11 *The ideal* I_λ *annihilates the* S-*modules* $H^p(\Omega_\lambda^*)$ *for all* p.

Proof. Let $\omega \in \Omega^p$ with $\omega_\lambda \wedge \omega = 0$. Let $\theta \in D(\mathcal{A})$. Then $\langle \theta, \omega \rangle \in \Omega^{p-1}$. We have

$$\omega_\lambda \wedge \langle \theta, \omega \rangle = -\langle \theta, \omega_\lambda \wedge \omega \rangle + \langle \theta, \omega_\lambda \rangle \omega = \langle \theta, \omega_\lambda \rangle \omega.$$

This shows that $\langle \theta, \omega_\lambda \rangle \omega$ is a coboundary. □

3 The Euler characteristic

In this section we find an algebraic set $Y_1 \subset \mathbb{C}^n$ so that for all $\lambda \in \mathbb{C}^n - Y_1$ the cohomology groups $H^p(\Omega^*_\lambda)$ are finite dimensional and the Euler characteristic $\chi(\Omega^*_\lambda)$ equals $\chi(\mathcal{A}, 1-r)$. (We may actually choose $Y_1 = \emptyset$ unless $r = 0$.)

Introduce a natural increasing filtration on $\Omega^p(\mathcal{A})$. Let $\beta \in \Omega^p[V]$. If each coefficient of β is a polynomial of degree at most $q - p$ then we say that the total degree of β is $\leqq q$ and write $\mathrm{tdeg}\beta \leqq q$. Let $\omega \in \Omega^p(\mathcal{A})$. It follows from the definition that ω can be written in the form $\omega = \beta/Q$ where $\beta \in \Omega^p[V]$. Let $n = \deg Q = |\mathcal{A}|$. We may formally consider the degree of $1/Q$ as $-n$ and say that the **total degree** $\mathrm{tdeg}\omega \leqq q$ if $\mathrm{tdeg}\beta \leqq q + n$. For example, if $\ell = 1$, $Q = u(u-1)$, and $\omega = du/u(u-1)$, then $\mathrm{tdeg}\omega \leqq q$ for $q \geqq -1$.

Definition 3.1 *Total degree introduces an increasing filtration on* $\Omega^p(\mathcal{A})$ *for* $q \in \mathbb{Z}$ *by*

$$\Omega^p_{\leqq q} = \Omega^p(\mathcal{A})_{\leqq q} = \{\omega \in \Omega^p(\mathcal{A}) \mid \mathrm{tdeg}\omega \leqq q\}.$$

Define $\mathbb{C}$*-vector spaces for* $q \in \mathbb{Z}$ *by*

$$\mathrm{Gr}_q\Omega^p = \mathrm{Gr}_q\Omega^p(\mathcal{A}) = \Omega^p(\mathcal{A})_{\leqq q}/\Omega^p(\mathcal{A})_{\leqq q-1}.$$

Definition 3.2 *Suppose* $\lambda \in \mathbb{C}^n$. *Let* $q \in \mathbb{Z}$. *The cochain complex* Ω^*_λ *has a subcomplex* $\Omega^*_{\leqq q} = \Omega^*_{\leqq q}(\mathcal{A})$ *which is defined by*

$$\Omega^*_{\leqq q} = \Omega^*_{\leqq q}(\mathcal{A}) : \cdots \xrightarrow{\omega_\lambda \wedge} \Omega^p_{\leqq q+(p-\ell)r} \xrightarrow{\omega_\lambda \wedge} \Omega^{p+1}_{\leqq q+(p-\ell+1)r} \xrightarrow{\omega_\lambda \wedge} \cdots \xrightarrow{\omega_\lambda \wedge} \Omega^\ell_{\leqq q} \to 0.$$

This provides an increasing filtration of the cochain complex Ω^*. *For each* $q \in \mathbb{Z}$, *define the complex*

$$\mathrm{Gr}^*_q = \mathrm{Gr}_q\Omega^* = \mathrm{Gr}_q\Omega^*(\mathcal{A}) = \Omega^*_{\leqq q}/\Omega^*_{\leqq q-1}.$$

Denote by Gr^* *the direct sum of the complexes* $\mathrm{Gr}^*_q = \mathrm{Gr}_q\Omega^*$ *for all* q.

The following result was proved in [15, Theorem 7.1]:

Theorem 3.3 *Let* f *be an* $\mathcal{A}$*-transverse polynomial of degree* $r > 0$. *Then for every* $\lambda \in \mathbb{C}^n$

(1) *the cohomology groups* $H^p(\Omega^*_\lambda)$ *and* $H^p(\mathrm{Gr}^*)$ *are finite dimensional for all* p,

(2) $\chi(\Omega^*_\lambda) = \sum(-1)^p \dim H^p(\Omega^*_\lambda) = \sum(-1)^p \dim H^p(\mathrm{Gr}^*) = \chi(\mathcal{A}, 1-r)$. □

The argument in [15] uses the fact that when $r > 0$, the induced differential in the complex Gr^* is the highest degree homogeneous component of df and hence it is independent of λ. Note also that in this case the number of solutions of $\omega_\lambda = 0$ is finite for all λ. We need the analog of Theorem 3.3 for $r = 0$. In this case $df = 0$ and there may exist values of λ which give infinitely many solutions of $\omega_\lambda = 0$. Thus the analysis is somewhat more delicate. See Remark 3.11 and Example 4.7. In the rest of this section we assume that $r = 0$.

Let p be an integer. Define

$$K^p = \{\omega \in \Omega^p(c\mathcal{A}) \mid du_0 \wedge \omega = 0\}.$$

Then K^p is an S^c-module graded by total degree: $K^p = \oplus_{q\in\mathbb{Z}} K^p_q$. Let q be an integer. Define a $\mathbb{C}$-linear map $\sigma : K^{p+1}_q \to \Omega^p_{\leq q}$ by $\sigma(\omega) = (-1)^p \langle\theta_E, \omega\rangle \mid_{u_0=1}$. Here θ_E is the Euler derivation. By abuse of notation, we let σ also denote the induced $\mathbb{C}$-linear map $\sigma : K^{p+1}_q \to \mathrm{Gr}_q\Omega^p$. The next Proposition was proved in [15, Prop. 4.6]:

Proposition 3.4 *The following sequence is exact where the first map is multiplication by u_0:*

$$0 \to K^{p+1}_{q-1} \xrightarrow{u_0} K^{p+1}_q \xrightarrow{\sigma} \mathrm{Gr}_q\Omega^p \to 0. \qquad \square$$

Define

$$\omega^h_\lambda = \sum_{i=1}^n \lambda_i \frac{d\alpha^h_i}{\alpha^h_i} \in \Omega^1(c\mathcal{A}).$$

It is easy to see that the maps $\omega^h_\lambda \wedge : K^p_q \to K^{p+1}_q$ define complexes $K^*_q = (K^*_q, \omega^h_\lambda \wedge)$.

Proposition 3.5 *The homomorphisms σ in* Proposition 3.4 *define a cochain homomorphism of the complexes $K^*_q \to$ $Gr_q\Omega^*$ that decreases dimension by 1.*

Proof. It suffices to prove that σ commutes with the differentials. Let $\omega \in K^{p+1}$. Notice that $\omega|_{u_0=1} = 0$. Using this we have

$$\omega_\lambda \wedge \langle\theta_E, \omega\rangle \mid_{u_0=1} = \left(\omega^h_\lambda \wedge \langle\theta_E, \omega\rangle\right) \mid_{u_0=1}$$

$$= \left(-\left\langle\theta_E, \omega^h_\lambda \wedge \omega\right\rangle + \left\langle\theta_E, \omega^h_\lambda\right\rangle \omega\right) \mid_{u_0=1} = \left(-\left\langle\theta_E, \omega^h_\lambda \wedge \omega\right\rangle\right) \mid_{u_0=1}. \qquad \square$$

Denote by K^* the direct sum of the complexes K^*_q for all q. It follows from Propositions 3.4 and 3.5 that the sequence of complexes $0 \to K^* \xrightarrow{u_0} K^* \xrightarrow{\sigma} \mathrm{Gr}^* \to 0$ is exact.

Proposition 3.6 *The following induced cohomology sequence is exact:*

$$\cdots \to H^{p+1}(K^*) \xrightarrow{u_0} H^{p+1}(K^*) \to H^p(\mathrm{Gr}^*) \to H^{p+2}(K^*) \xrightarrow{u_0} H^{p+2}(K^*) \to \cdots \square$$

Proposition 3.7 *Write $H^p = H^p(K^*)$. Define $D_0(c\mathcal{A}) = \{\theta \in D(c\mathcal{A}) \mid \theta(u_0) = 0\}$. Let $\theta \in D_0(c\mathcal{A})$. Then $\langle\theta, \omega^h_\lambda\rangle \in$* Ann$(H^p)$ *for every p.*

Proof. Let $\omega \in K^p$ and $\omega_\lambda^h \wedge \omega = 0$. Then we have

$$0 = \langle \theta, \omega_\lambda^h \wedge \omega \rangle = \langle \theta, \omega_\lambda^h \rangle \omega - \omega_\lambda^h \wedge \langle \theta, \omega \rangle.$$

Since $\theta \in D(c\mathcal{A})$, $\eta = \langle \theta, \omega \rangle \in \Omega^{p-1}(c\mathcal{A})$. To prove the result it suffices to check that $\eta \in K^{p-1}$. Thus we need that $du_0 \wedge \eta = 0$. We have

$$du_0 \wedge \langle \theta, \omega \rangle = \theta(u_0)\omega - \langle \theta, du_0 \wedge \omega \rangle = 0$$

since $\theta(u_0) = 0$ and $\omega \in K^p$. □

Let $H_0 = \ker(u_0)$. Then H_0 is an ℓ-dimensional hyperplane in cV. For any $X \in L$, define $\overline{X} = H_0 \cap X^c$. Thus $\overline{X}$ is a vector subspace of cV and $\dim \overline{X} = \dim X$. We can regard $\overline{X}$ as the parallel translate of X through the origin. For any $X \in L$, define the index set

$$I(X) = \{ i \mid \overline{X} \not\subseteq H_i^c, \quad 1 \leqq i \leqq n \}.$$

For example, $I(V) = \{1, \ldots, n\}$ and $I(H_1) = \{ i \mid H_i \text{ is not parallel to } H_1 \}$. Let $\pi_X = \prod_{i \in I(X)} \alpha_i^h$. For any nonzero vector $a \in cV$, let ∂_a be the derivation of S^c in the direction of a.

Corollary 3.8 *For any $X \in L$ and nonzero vector $a \in \overline{X}$, we have $\langle \pi_X \partial_a, \omega_\lambda^h \rangle \in$ $\mathrm{Ann} H^p$.*

Proof. By Proposition 3.7, it suffices to check that $\pi_X \partial_a \in D_0(c\mathcal{A})$. This is straightforward. □

Proposition 3.9 *Suppose $\mathcal{A}$ is nonempty and central. Then $C(\Phi_\lambda) = \emptyset$ unless $\sum_{i=1}^n \lambda_i = 0$.*

Proof. Since the Euler derivation $\theta_E \in D(\mathcal{A})$, we have

$$0 \neq \sum_{i=1}^{n} \lambda_i = \langle \theta_E, \omega_\lambda \rangle \in I_\lambda.$$

Thus $V(I_\lambda) = \emptyset$. Apply Proposition 2.7. □

Define $L^+ = \{ X \in L(\mathcal{A}) \mid \dim X > 0 \}$. For each $X \in L^+$, define a hyperplane F_X in $\mathbb{C}^n$ by

$$F_X = \left\{ \lambda \in \mathbb{C}^n \mid \sum_{i \in I(X)} \lambda_i = 0 \right\}.$$

Lemma 3.10 *Define $Y_1 = \bigcup_{X \in L^+} F_X$. Then Y_1 is a closed algebraic proper subset of $\mathbb{C}^n$. Suppose that $\lambda \in \mathbb{C}^n - Y_1$. Define the ideal*

$$I_\lambda^0 = \{ \langle \theta, \omega_\lambda^h \rangle \in S^c \mid \theta \in D_0(c\mathcal{A}) \}$$

of S^c. The radical of the ideal generated by u_0 and I_λ^0 contains the maximal ideal $S_+^c = (u_0, \ldots, u_\ell)$.

Proof. Denote the ideal of S^c generated by u_0 and I^0_λ by I. Let $V(I)$ be the set of common zeros of I. By the Nullstellensatz, it suffices to show that $V(I)$ is contained in $\{0\}$. Suppose $v \in V(I)$. Then $v \in H_0 = \ker(u_0)$. Let $X \cap H_i$, where the intersection is over $\{i \mid 1 \leqq i \leqq n, v \in H_i^c\}$. Then $\overline{X} = H_0 \cap X^c$ is the maximum (smallest as a set) element of $L(c\mathcal{A})$ which contains v. We also note $I(X) = \{i \mid \overline{X} \not\subseteq H_i^c\} = \{i \mid v \notin H_i^c\}$. We will show $\overline{X} = 0$. Define the arrangement $\mathcal{B} = \{\overline{X} \cap H_i^c \mid i \in I(X)\}$. Suppose $\overline{X} \neq 0$. Then $\mathcal{B}$ is a nonempty central arrangement in $\overline{X}$. Note that $v \in M(\mathcal{B})$. For simplicity write $\omega = \omega_\lambda^h = \sum_{i=1}^n \lambda_i (d\alpha_i^h/\alpha_i^h)$. By Corollary 3.8, $\langle \pi_X \partial_a, \omega \rangle \in I$ for every vector $a \in \overline{X}$. Define $\omega^X = \sum_{i \in I(X)} \lambda_i (d\alpha_i^h/\alpha_i^h)$. If $i \notin I(X)$, then $\overline{X} \subseteq H_i^c$ and thus $\langle \partial_a, d\alpha_i^h \rangle = 0$. Thus we have $0 = \langle \pi_X \partial_a, \omega \rangle_v = \langle \pi_X \partial_a, \omega^X \rangle_v$. Since $\pi_X(v) \neq 0$ and ω^X has no pole at v, the restriction $\overline{\omega}$ of the 1-form ω^X to $\overline{X}$ vanishes at $v \in M(\mathcal{B})$. On the other hand, by Proposition 3.9, since $\mathcal{B}$ is nonempty and central, $\overline{\omega}$ never vanishes on $M(\mathcal{B})$ unless $\sum_{i \in I(X)} \lambda_i = 0$. This is a contradiction. □

Remark 3.11 The set Y_1 has the following description in terms of the projectivized arrangement $\mathcal{P}$ of the cone $c\mathcal{A}$.

Projectivize V^c to get an ℓ-dimensional complex projective space $\mathbf{P}(V^c)$. For any vector subspace X of V^c, let $\mathbf{P}(X)$ denote its projectivization. Note that $\mathbf{P}(X^c)$ is the projective closure of X. The cone $c\mathcal{A}$ naturally determines a projective arrangement $\mathcal{P}$ by

$$\mathcal{P} = \{\mathbf{P}(H_0), \mathbf{P}(H_1^c), \dots, \mathbf{P}(H_n^c)\}.$$

For simplicity, write $P_0 = \mathbf{P}(H_0)$ and $P_i = \mathbf{P}(H_i^c)$ for $1 \leqq i \leqq n$. Choose the complex number $\lambda_0 = -\sum_{i=1}^n \lambda_i$ as the weight of P_0 so that the expression

$$\lambda_0 \frac{du_0}{u_0} + \sum_{i=1}^n \lambda_i \frac{d\alpha_i^h}{\alpha_i^h}$$

defines a global rational 1-form on $\mathbf{P}(V^c)$. Then for $X \in L^+$ we have

$$\sum_{i \in I(X)} \lambda_i = -\lambda_0 - \sum_{\substack{1 \leqq i \leqq n \\ \overline{X} \subseteq H_i^c}} \lambda_i = -\sum_{\mathbf{P}(\overline{X}) \subseteq P_i} \lambda_i.$$

Let $L(\mathcal{P})$ be set of all nonempty intersections of hyperplanes of $\mathcal{P}$. Note that $P_0 = \mathbf{P}(H_0) = \mathbf{P}(V^c) - V$ is "the hyperplane at infinity." Define

$$L(\mathcal{P})_\infty = \{Z \in L(\mathcal{P}) \mid Z \subseteq P_0\}.$$

It is the subset of $L(\mathcal{P})$ consisting of the elements lying at infinity. Then

$$L(\mathcal{P})_\infty = \{\mathbf{P}(\overline{X}) \mid X \in L^+\}.$$

Therefore

$$Y_1 = \bigcup_{X \in L^+} \left\{ \lambda \in \mathbb{C}^n \mid \sum_{\mathrm{P}(\overline{X}) \subseteq P_i} \lambda_i = 0 \right\} = \bigcup_{Z \in L(\mathcal{P})_\infty} \left\{ \lambda \in \mathbb{C}^n \mid \sum_{Z \subseteq P_i} \lambda_i = 0 \right\}.$$

Theorem 3.12 *Let $r = 0$. Suppose $\lambda \in \mathbb{C}^n - Y_1$. Then*

(1) *the cohomology groups $H^p(\Omega^*_\lambda)$ and $H^p(\mathrm{Gr}^*)$ are finite dimensional for all p,*

(2) $\chi(\Omega^*_\lambda) = \sum(-1)^p \dim H^p(\Omega^*_\lambda) = \sum(-1)^p \dim H^p(\mathrm{Gr}^*) = \chi(\mathcal{A}, 1)$.

Proof. (1) We only need to show that $H^p(\mathrm{Gr}^*)$ is finite dimensional. By Proposition 3.6, it suffices to prove that the map induced by multiplication by $u_0 : H^p(K^*) \to H^p(K^*)$ has finite dimensional kernel and cokernel. Recall that $H^p(K^*)$ is annihilated by $\langle \theta, \omega^h_\lambda \rangle$ for all $\theta \in D_0(c\mathcal{A})$ by Proposition 3.7. Thus both the kernel and the cokernel are annihilated by the ideal generated by u_0 and I^0_λ. Therefore (1) follows from Lemma 3.10.

(2) Let $E_r^{p,q}$ be the spectral sequence associated with the filtered complex $\{\Omega^*_{\leqq q}\}$. Then $E_1^{p,q} = H^{p+q}(\mathrm{Gr}^*_{-q}) = 0$ except for finitely many pairs (p,q) by (1). So we have

$$\begin{aligned}\sum_p(-1)^p \dim H^p(\mathrm{Gr}^*) &= \sum_{p,q}(-1)^{p+q} \dim E_1^{p,q} = \sum_{p,q}(-1)^{p+q} \dim E_2^{p,q} \\ &= \cdots \\ &= \sum_{p,q}(-1)^{p+q} \dim E_\infty^{p,q} = \sum_p(-1)^p \dim H^p(\Omega^*).\end{aligned}$$

Therefore it suffices to show the statement for Gr^*. Let

$$\mathrm{Poin}(\mathrm{Gr}^*; x, y) = \sum_{p,q}(\dim \mathrm{Gr}_q \Omega^p) x^q y^p$$

and set $y = -1$. We get

$$\begin{aligned}\mathrm{Poin}(\mathrm{Gr}^*; x, -1) &= \sum_{p,q}(\dim (\mathrm{Gr}_q \Omega^p) x^q (-1)^p \\ &= \sum_q x^q \sum_p (-1)^p \dim H^p(\mathrm{Gr}^*_q).\end{aligned}$$

Define $\Psi(\mathcal{A}; x, t) = \mathrm{Poin}\left(\mathrm{Gr}^*; x, \frac{t(1-x)-1}{x}\right)$. The formula $\Psi(\mathcal{A}; 1, t) = \chi(\mathcal{A}, t)$ was proved in [15, Theorem 5.3]. Thus we have

$$\begin{aligned}\chi(\mathcal{A}, 1) = \Psi(\mathcal{A}; 1, 1) &= \lim_{x \to 1} \mathrm{Poin}(\mathrm{Gr}^*; x, -1) \\ &= \sum_p (-1)^p \dim H^p(\mathrm{Gr}^*).\end{aligned}$$

4 The number of critical points

In this section we prove Theorem 1.1. First we find an algebraic set $Y \subset \mathbb{C}^n$ so that for each $\lambda \in \mathbb{C}^n - Y$, Φ_λ has only finitely many critical points, all of which are nondegenerate and the number of critical points of Φ_λ is independent of $\lambda \in \mathbb{C}^n - Y$. We call $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{C}^n - Y$ **generic**.

Proposition 4.1 *Define $Z = \{(\lambda, v) \in \mathbb{C}^n \times M \mid \omega_\lambda(v) = 0\}$. Then Z is an n-dimensional complex manifold*

Proof. Note that Z is the zero set of the ℓ equations

$$\frac{\partial f}{\partial u_j} + \sum_{i=1}^{n} \lambda_i \frac{\alpha_{j,i}}{\alpha_i(v)} = 0, \quad 1 \leqq j \leqq \ell.$$

The Jacobian at (λ, v) is an $(n + \ell) \times \ell$ matrix which may be written as $J = [A H]$. Here A is an $n \times \ell$ matrix with $A_{i,j} = \alpha_{j,i}/\alpha_i(v)$ and H is the Hessian matrix of Definition 2.5. Since $\mathcal{A}$ is essential, the matrix A has rank ℓ for all (λ, v). □

Proposition 4.2 *Define the projection $p : Z \to \mathbb{C}^n$ by $p(\lambda, v) = \lambda$. Then (λ, v) is a critical point of p if and only if v is a degenerate critical point of Φ_λ.*

Proof. The tangent space $T_{(\lambda,v)}Z$ of Z at (λ, v) is naturally identified with the kernel of the matrix map $J = [A H] : \mathbb{C}^{n+\ell} \to \mathbb{C}^\ell$. Thus $(dp)_{(\lambda,v)} : T_{(\lambda,v)}Z \to T_\lambda\mathbb{C}^n$ is not surjective if and only if it is not injective if and only if det $H_\lambda(v) = 0$. □

Proposition 4.3 *There exists a closed algebraic proper subset $Y_2 \subset \mathbb{C}^n$ so that if $\lambda \in \mathbb{C}^n - Y_2$, then the critical points of the function Φ_λ in M are nondegenerate.*

Proof. Let $D \subset \mathbb{C}^n$ be the discriminant of the projection $p : Z \to \mathbb{C}^n$. By Sard's theorem, D is nowhere dense in $\mathbb{C}^n$. Since D is a constructible set, it is contained in a closed algebraic proper subset Y_2 of $\mathbb{C}^n$. The conclusion follows from Proposition 4.2. □

Theorem 4.4 *There exists a closed algebraic proper subset Y of $\mathbb{C}^n$ such that for each $\lambda \in \mathbb{C}^n - Y, \Phi_\lambda$ has only finitely many critical points, all of which are nondegenerate, and the number of critical points of Φ_λ is independent of $\lambda \in \mathbb{C}^n - Y$. Denote this number by $\gamma(\mathcal{A})$.*

Proof. Case 1. Assume that the map $p : Z \to \mathbb{C}^n$ is not dominant so the image of p is not dense in $\mathbb{C}^n$. Then the image $p(Z)$ is contained in a closed algebraic proper subset Y. Obviously, Y satisfies the condition. In this case $\gamma(\mathcal{A}) = 0$.

Case 2. Assume that the map $p : Z \to \mathbb{C}^n$ is dominant. Then there exists a closed algebraic proper subset which contains Y_1 and Y_2 such that for $U = \mathbb{C}^n - Y$ the map $p_{|p^{-1}(U)} : p^{-1}(U) \to U$ is a surjective covering map and the number of points in a fiber is constant. In this case $\gamma(\mathcal{A}) > 0$. □

This establishes the first part of Theorem 1.1. It remains to prove the equality $\gamma(\mathcal{A}) = |\chi(\mathcal{A}, 1-r)|$. The following lemma of de Rham type was proved by Saito [13].

Lemma 4.5 (de Rham-Saito) *Let A be a Noetherian ring. Let N be a free A-module of rank ℓ with basis $e_1, \dots, e_\ell$ so $N = \otimes_{i=1}^{\ell} Ae_i$, and let $\Lambda^p N$ be the p-th exterior power of N. Let $\omega \in N$. Write $\omega = \sum_{i=1}^{\ell} a_i e_i$. Let I be the ideal generated by the coefficients a_i for $1 \leqq i \leqq \ell$. Consider the cochain complex*

$$\dots \longrightarrow \Lambda^{p-1}N \longrightarrow \Lambda^p N \longrightarrow \Lambda^{p+1}N \longrightarrow \dots,$$

where the coboundary maps are given by $\phi \mapsto \omega \wedge \phi$. Let H^p denote the cohomology of this complex. Let $d = \operatorname{depth}_I A$ be the maximal length of an A-regular sequence in I. Then we have $H^p = 0$ for $0 \leqq p < d$. □

Define an algebraic proper subset Y_3 of $\mathbb{C}^n$ by

$$Y_3 = \bigcup_{X \in L(\mathcal{A})} \left\{ \sum_{x \subseteq H_i} \lambda_i = 0 \right\}.$$

Proposition 4.6 *If $\lambda \in \mathbb{C}^n - (Y \cup Y_3)$, then*
(1) *$H^p(\Omega_\lambda^*) = 0$ for $0 \leq p < \ell$, and*
(2) $\dim H^\ell(\Omega_\lambda^*) = |C(\Phi_\lambda)|$.

Proof. (1) It suffices to show that the localization $H^p(\Omega_\lambda^*)_v = 0$ for all $v \in V$. Choose an arbitrary $v \in V$ and fix it. By translating the coordinates we may assume that v is the origin.

Case 1. Suppose $v \notin M(\mathcal{A})$. We may assume that $\mathcal{A} = \{H_1, \dots, H_n\}$ with $v \in H_i$ for $i = 1, \dots, k$ and $v \notin H_i$ for $i = k+1, \dots, n$. Then $\alpha_1, \dots, \alpha_k$ are homogeneous of degree one. Define $\pi_v = \alpha_{k+1} \dots \alpha_n$. Then $\pi_v(v) \neq 0$. Suppose $\eta \in \Omega^p(\mathcal{A})_v$ with $\omega_\lambda \wedge \eta = 0 \in \Omega^{p+1}(\mathcal{A})_v$. Note that $\pi_v \theta_E \in D(\mathcal{A})$, where θ_E is the Euler derivation. Recall the ideal I_λ of S from Proposition 2.7. It annihilates the S-modules $H^p(\Omega_\lambda^*)_v$ for all p. Note that $\langle \pi_v \theta_E, \omega_u \rangle \in I_\lambda$. Write f as a sum of its homogeneous components, $f = \sum_{m=0}^{r} f_{(m)}$. We have

$$\langle \theta_E, \omega_\lambda \rangle = \sum_{m=0}^{r} m f_{(m)} + \sum_{i=1}^{k} \lambda_i + \sum_{i=k+1}^{n} \lambda_i \frac{\bar{\alpha}_i}{\alpha_i},$$

where $\bar{\alpha}_i$ is the degree one homogeneous part of α_i for $k+1 \leqq i \leqq n$. By assumption $\sum_{i=1}^{k} \lambda_i \neq 0$. The remaining terms lie in the maximal ideal of S_v. Thus $\langle \theta_E, \omega_\lambda \rangle$ is a unit in S_v. Since $\pi_v(V) \neq 0, \langle \pi_v \theta_E, \omega_\lambda \rangle \in I_\lambda$ is also a unit in S_v. This shows that $v \notin V(I_\lambda)$ and that $H^p(\Omega_\lambda^*)_v = 0$ for all p.

Case 2. Suppose $v \in M(\mathcal{A}) - C(\Phi_\lambda)$. Since $V(I_\lambda) \cap M(\mathcal{A}) = C(\Phi_\lambda)$ by Proposition 2.7, we have $v \notin V(I_\lambda)$. Since the ideal I_λ annihilates $H^p(\Omega_\lambda^*)_v$ by Proposition 2.11, we have $H^p(\Omega_\lambda^*)_v = 0$ for all p.

Case 3. Suppose $v \in C(\Phi_\lambda)$. The 1-form ω_λ vanishes at v. Note that α_i is a unit in S_v for $1 \leqq i \leqq n$. Write

$$\omega_\lambda = \sum_{i=1}^{\ell} g_i du_i \in \Omega^1(\mathcal{A})_v,$$

with $g_i \in S_v$ for $1 \leqq i \leqq l$. Since $g_i(v) = 0$ for $1 \leqq i \leqq l$, we can define a holomorphic map germ

$$G = (g_1, \ldots, g_\ell) : (\mathbb{C}^\ell, 0) \longrightarrow (\mathbb{C}^\ell, 0).$$

The Jacobian matrix of the map germ G is equal to the Hessian matrix $H = H_\lambda$ from Proposition 2.5. Since $\lambda \in \mathbb{C}^n - Y$, v is a nondegenerate critical point of Φ_λ. Thus $\det G$ does not vanish at v. This implies that the map germ G is locally biholomorphic so v is an isolated zero of G with multiplicity one. Therefore $g_1, \ldots, g_\ell$ form a regular sequence. We can apply Lemma 4.5 to $A = S_v$, $N = \Omega^1[V]_v$, $\omega = \omega_\lambda, e_i = du_i$, and $d = \ell$ to prove (1).

(2) In the proof of (1) we showed that $V(I_\lambda) = C(\Phi_\lambda)$. Let $v \in C(\Phi_\lambda)$. We use the notation of *Case 3* above. Since v is a zero of $G = (g_1, \ldots, g_\ell)$ with multiplicity one, we have

$$\dim S_v/(g_1, \ldots, g_\ell)S_v = 1.$$

Note that $D(\mathcal{A})_v = S_v(\partial/\partial u_1) + \cdots + S_v(\partial/\partial u_\ell)$ because $v \in M$. Thus $(I_\lambda)_v = (g_1, \ldots, g_\ell)S_v$. Therefore

$$\dim H^\ell(\Omega^*_\lambda) = \sum_{v \in C(\Phi_\lambda)} \dim H^\ell(\Omega^*_\lambda)_v$$

$$= \sum_{v \in C(\Phi_\lambda)} \dim S_v/(I_\lambda)_v = \sum_{v \in C(\Phi_\lambda)} \dim S_v/(g_1, \ldots g_\ell)S_v = |C(\Phi_\lambda)|\,. \ \square$$

To complete the proof of the second part of Theorem 1.1, let $\lambda \in \mathbb{C}^n - (Y \cup Y_3)$ so $\gamma(\mathcal{A}) = |C(\Phi_\lambda)|$. Apply Proposition 4.6, Theorem 3.3, and Theorem 3.12 to get

$$\gamma(\mathcal{A}) = |C(\Phi_\lambda)| = \dim H^\ell(\Omega^*_\lambda) = |\chi(\Omega^*_\lambda)| = |\chi(\mathcal{A}, 1 - r)|.$$

This completes the proof of Theorem 1.1.

Example 4.7 *Let* $\ell = 1, n = 3$, *and* $r = 0$. *Consider* $\Phi_\lambda = (u-1)^{\lambda_1} u^{\lambda_2} (u+1)^{\lambda_3}$.

Here $M = \mathbb{C} - \{-1, 0, 1\}$. Y_1 is defined by $\lambda_1 + \lambda_2 + \lambda_3 = 0$. Y_1 contains the points where $|C(\Phi_\lambda)| = \infty$. Y_2 is defined by $(\lambda_3 - \lambda_1)^2 + 4\lambda_2(\lambda_1 + \lambda_2 + \lambda_3) = 0$. Y_2 contains the discriminant. At these points $|C(\Phi_\lambda)| = 1$. We may choose Y as the union of Y_1, Y_2 and the three coordinate planes. All $\lambda \in \mathbb{C}^3 - Y$ are generic and Φ_λ has $|\chi(M)| = 2$ nondegenerate critical points in M.

Acknowledgements. We thank J. Damon, M. Kita, A. Varchenko and S. Yuzvinsky for valuable suggestions. J. Damon has indicated that there may be an alternate proof of this theorem using results in [3] and [4].

References

1. Aomoto, K.: On vanishing of cohomology attached to certain many valued meromorphic functions. J. Math. Soc. Japan **27**, 248–255 (1975)
2. Aomoto, K., Kita, M.: Hypergeometric functions (in Japanese). Tokyo: Springer 1994
3. Damon, J.: Higher multiplicities and almost free divisors and complete intersections (preprint)
4. Damon, J., Mond, D.: $\mathcal{A}$-codimension and the vanishing topology of discriminants, Invent. math. **106**, 217–242 (1991)
5. Esnault, H., Schechtman, V., Viehweg, E.: Cohomology of local systems of the complement of hyperplanes, Invent. math. **109**, 557–561 (1992), Erratum, ibid. **112**, 447 (1993)
6. Gelfand, I.M.: General theory of hypergeometric functions. Soviet Math. Dokl. **33**, 573–577 (1986)
7. Goresky, M., MacPherson, R.: Stratified Morse Theory. Berlin Heidelberg New York: Springer, 1988
8. Kita, M.: On hypergeometric functions in several variables I. New integral representations of Euler type. Japan J. Math. **18**, 25–74 (1992)
9. Kita, M.: On hypergeometric functions in several variables II. The Wronskian of the hypergeometric functions. J. Math. Soc. Japan **45**, 645–669 (1993)
10. Orlik, P., Solomon, L.: Combinatorics and topology of complements of hyperplanes. Invent. math. **56**, 167–189 (1980)
11. Orlik, P., Terao, H.: Arrangements of Hyperplanes. Grundlehren der Math. Wiss. **300**, Berlin Heidelberg New York: Springer , 1992
12. Orlik, P., Terao, H.: Arrangements and Milnor fibers. Math. Ann. (to appear)
13. Saito, K.: On a generalization of de Rham lemma. Ann. Inst. Fourier **28**, 165–170 (1976)
14. Schechtman, V.V., Varchenko. A.N.: Arrangements of hyperplanes and Lie algebra homology. Invent. math. **106**, 139–194 (1991)
15. Terao, H., Yuzvinsky, S.: Logarithmic forms on affine arrangements Nagoya Math. J. (to appear)
16. Varchenko, A.N.: The Euler beta-function, the Vandermonde determinant, Legendre's equation, and critical values of linear functions on a configuration of hyperplanes. I. Math. Ussr Izvestiya **35**, 543–571 (1990) II. Math. Ussr Izvestiya **36**, 155–167 (1991)
17. Varchenko, A.N.: The functions $\prod_{i<j}(t_i - t_j)^{\alpha_{ij}/\kappa}$ and the representation theory of Lie algebras and quantum groups. World Scientific Publishers (to appear)
18. Varchenko, A.N.: Critical points of the product of powers of linear functions and families of bases of singular vectors (preprint)