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The Shapley Value in the Non Differentiable Case

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Abstract: The Shapley value is shown to exist even when there are essential non differentiabilities on the diagonal.

Introduction

In their book "values of Non Atomic Games", Aumann and Shapley [1] define the Shapley value for non atomic games, and prove existence and uniqueness of it for a number of important spaces of games like pNA and $bv'NA$. They also show that this value obeys the so-called diagonal formula, expressing the value of each infinitesimal player as his marginal contribution to the coalition of all players preceding him in a random ordering of the players: say if the worth $v(S)$ of coalition S is expressed as a function of finitely many non atomic probabilities $\mu_1 \ldots \mu_n$ by

$$v(S) = f(\mu_1(S), \ldots, \mu_n(S)) \quad f \in C^1,\ f(0) = 0$$

then the diagonal formula takes the form

$$[\phi(v)](S) = \Sigma\, \mu_i(S) \int_0^1 \frac{\partial f}{\partial x_i}(t, t, \ldots, t)dt$$

or in general, more symbolically

$$[\phi(v)](ds) = \int_0^1 [v(tI + ds) - v(tI)]dt.$$

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This interpretation in terms of a random order depends on the fact that, for a larger number of players, player ds will in a random order occur at some time t uniformly distributed on [0, 1], and that the set of players preceding him will be an almost perfect sample of size t of the whole population – so that its worth will be essentially $v(tI) = f(t\mu_1(I), \ldots, t\mu_n(I))$.

Those results have a large number of important applications – they do however depend on the differentiability of f along the diagonal.

The diagonal formula was later extended, in "Values and Derivatives" [4] to a much wider class of games, including spaces like $bv'NA$ where the function f cannot be called differentiable.

The extended formula applies say to majority games ($v(S) = I(\mu(S) \geqslant \alpha)\, 0 < \alpha < 1$); or even to majority games in several different houses ($v(S) = I(\mu_1(S) \geqslant \alpha_1, \mu_2(S) \geqslant \alpha_2, \ldots, \mu_n(S) \geqslant \alpha_n) - 0 < \alpha_i < 1$) provided all quota's α_i are different. ($I(\cdot)$ denotes the indicator function.)

But the case where the quotas would be the same – say all $\frac{1}{2}$ – would be excluded, at least when $n > 2$.

Similarly, in economic applications, economies with strong complementarities, like "n-handed glove markets" ($v(S) = \min_{i=1,\ldots,n} (\mu_i(S))$ would remain excluded – again at least when $n > 2$.

Moreover, no value operator at all was known to exist on any space of games that would include all n-handed glove markets – except (Y. Taumann [7]) when n is fixed and in addition all measures μ_i are mutually singular, i.e. the different types of gloves have disjoint sets of owners.

S. Hart's "measure-based values" [3] are an illuminating approach to this problem. They highlight the fact – which could already be seen in Aumann and Shapley's analysis [1] of the three-handed glove market – that in some sense different finite approximations to the game may yield quite different values, according as to one or another part of the player set – say the owners of one or another type of glove – approximates better the limiting game. For the approximations considered, the distribution of a random sample around the diagonal is essentially normal, with a covariance matrix that is quite sensitive to the relative degree of approximation in different parts of the player set.

Surprisingly, as we will show, in the limit the symmetry axiom – i.e. to ask that the solution depends only on the data of the game – is strong enough to force the distribution away from the normal distribution, and to impose an in some sense unique answer.

Here we extend the diagonal formula of Mertens [4] to include in addition all situations of this type.

We get in this way a value – of norm 1 – on a closed space that will include DIFF – and DIAG –, the closed algebra generated by $bv'NA$, and also all games generated by a finite number of algebraic and lattice operations from a finite number of measures, and all markets functions of finitely many measures. The space will also include the finite games and the "regular" games with countably many players.

Intuitively, the diagonal formula is extended by taking the derivative not on the diagonal, but at some small perturbation of it – say $tI + \epsilon\chi$ instead of tI – and by averaging the result for some probability distribution over perturbations. We prove further a weak form of uniqueness, in the sense that there is only one such probability distribution over perturbations that would yield a value.

In parallel, another extension is made to previous approaches, mainly in order to make the value invariant e.g. under all automorphisms of the lattice of coalitions mod. countable sets, instead of only all automorphisms of the player set. In particular, this allows us to deal with finitely additive measures just as well as countably additive ones.

The basic definitions are given in Section 1. Section 2 defines the probability distribution over perturbations and shows its uniqueness. An explicit formula for the value of games of the type discussed above (n-handed glove markets, majority in several different houses) is derived in Section 3.

Section 1

We follow basically the terminology of Aumann and Shapley [1]. $(I, \mathcal{C})$ denotes the player set, $\mathcal{C}$ being a σ-field of subsets of the set I. A game is a real valued function v on $\mathcal{C}$, with $v(\phi) = 0$. Its variation norm $\| v \|_{BV}$ is the supremum of the variation of v over all increasing chains $(C_1 \subseteq C_2 \subseteq \ldots \subseteq C_n)$ in $\mathcal{C}$. $(BV, \| \cdot \|_{BV})$ denotes the Banach algebra of all games of bounded variation.

FA is the subspace of BV consisting of additive set functions.

We are going to define a value – more precisely, a projection φ of norm 1 of some closed subspace Q of BV $(FA \subseteq Q)$ onto FA, such that $[\varphi(v)](I) = v(I)$ and such that φ is symmetric in the sense that for any automorphism θ of the Boolean algebra $\mathcal{C}$, if θ^t is defined on BV by $[\theta^t(v)](S) = v(\theta(S))$, then $\theta^t(Q) = Q$ and $\varphi \circ \theta^t = \theta^t \circ \varphi$.

In fact, φ will be constructed as the composition of different positive linear symmetric mappings of norm 1: $\varphi = \varphi_4 \circ \varphi_3 \circ \varphi_2 \circ \varphi_1$.

(1.1) φ_1 maps any game into the corresponding constant sum game, $[\varphi_1(v)](S) = \frac{1}{2}[v(S) - v(S^c) + v(I)]$: obviously φ_1 is a symmetric projection of norm 1 onto the space Q_1 of all constant sum games w $(w(S) + w(S^c) = w(I))$, such that $[\varphi_1(v)](I) = v(I)$.

(1.2) φ_2 is the extension operator:

$B(I, \mathcal{C})$ denotes the space of bounded measurable functions on $(I, \mathcal{C})$, $B_1^+(I, \mathcal{C})$ is the space of "ideal sets", i.e. $\{f | f \in B(I, \mathcal{C}), 0 \leqslant f \leqslant 1\}$.

For functions $\bar{v}$ on $B_1^+(I, \mathcal{C})$ (with $\bar{v}(0) = 0$), one defines as previously the variation norm $\| \bar{v} \|_{IBV}$ by considering all possible increasing chains in $B_1^+(I, \mathcal{C})$, and one defines $\bar{v}^+(\chi) = \sup_{0 \leqslant f_j \leqslant f_{j+1} \leqslant \chi} \sum_i (\bar{v}(f_{i+1}) - \bar{v}(f_i))^+$, and similarly for $\bar{v}^-$: obviously $\| \bar{v} \|_{IBV} = \bar{v}^+(I) + \bar{v}^-(I), \bar{v} = \bar{v}^+ - \bar{v}^-$.

Similar definitions are possible for the space F_ϵ of functions $\bar{v}$ ($\bar{v}(0) = 0$) defined only on the ϵ-neighborhood of the diagonal $V_\epsilon = \{f \in B_1^+(I, \mathcal{C}) : \sup(f) - \inf(f) \leqslant \epsilon\}$, and lead to $\| \bar{v} \|_{IBV, \epsilon}$ and $\bar{v}_\epsilon^+$, $\bar{v}_\epsilon^-$ by restricting all chains to remain in this neighborhood. By definition $\| \bar{v} \|_{IBV, 0} = \inf_{\epsilon > 0} \| \bar{v} \|_{IBV, \epsilon}$.

Following [5], we define $\mathcal{F}$ as the set of triplets (π, ν, ϵ), where π is a finite measurable partiiton of I, ν is a finite set of non atomic elements of FA and $\epsilon > 0$. $\mathcal{F}$ is ordered by $\alpha \leqslant \alpha'$ iff $\pi_\alpha \leqslant \pi_{\alpha'}$ ($\pi_{\alpha'}$ is a refiniement of π_α) and $\nu_\alpha \subseteq \nu_{\alpha'}$ and $\epsilon_\alpha \geqslant \epsilon_{\alpha'}$. $(\mathcal{F}, \leqslant)$ is filtering increasing.

C_n is the set of increasing sequences $0 \leqslant f_1 \leqslant f_2 \leqslant \ldots \leqslant f_n \leqslant 1$ of measurable functions, and E_n the set of increasing sequences $S_1 \subseteq S_2 \ldots \subseteq S_n$ in $\mathcal{C}$.

For any $\alpha \in \mathcal{F}$ and $f \in C_n$, we define $\mathcal{P}_{\alpha, f}$ as the set of all probabilities with finite support on E_n such that $\sum_i |E(I(S_i)) - f_i| < \epsilon$ uniformly on I, and such that $S \in \pi_\alpha$, $T \in \pi_\alpha, S \cap T = \phi$ imply

(α) $(S \cap S_i)_{i=1}^n$ is independent of $(T \cap S_i)_{i=1}^n$

(β) $\nu_\alpha(S \cap S_i) = \nu_\alpha(f_i \cdot I(S))$

and

(γ) $f_i \cdot I(S) = 0 \Rightarrow S \cap S_i = \phi$.

Intuitively $P \in \mathcal{P}_{\alpha, f}$ if P is the distribution of a random set (or sequence of sets) that is very similar to the ideal set f – "very" being measured by $\alpha \in \mathcal{F}$.

Obviously $\alpha \leqslant \alpha'$ implies $\mathcal{P}_{\alpha, f} \supseteq \mathcal{P}_{\alpha', f}$, and I showed in [5] that always $\mathcal{P}_{\alpha, f} \neq \phi$.

For any game v, and any $f \in B_1^+(I, \mathcal{C})$, let $\bar{v}(f) = \lim_{\alpha \in \mathcal{F}} \sup_{P \in \mathcal{P}_{\alpha, f}} \int v(S) dP(S)$, $\underline{v}(f) = \lim_{\alpha \in \mathcal{F}} \inf_{P \in \mathcal{P}_{\alpha, f}} \int v(S) dP(S)$. (The inclusion relations $\alpha \leqslant \alpha' \Rightarrow \mathcal{P}_{\alpha, f} \supseteq \mathcal{P}_{\alpha', f}$ imply that the limits exist, the corresponding sup's or inf's being monotonic in α.) Now v is in the domain D_ϵ of φ_2^ϵ iff $\forall \chi \in V_\epsilon$, $\bar{v}(\chi) = \underline{v}(\chi)$, and then $\varphi_2^\epsilon(v) \in F_\epsilon$ is defined by $[\varphi_2^\epsilon(v)](\chi) = \bar{v}(\chi) = \underline{v}(\chi)$.

Obviously D_ϵ is a closed (in the maximum pseudo-metric) vector subspace of the space of all games, and symmetric, and φ_2^ϵ is a symmetric linear operator from D_ϵ to F_ϵ.

Further φ_2^ϵ transforms non negative games into non negative elements of F_ϵ, and monotonic games into monotonic elements of F_ϵ, and is of norm 1 both in the maximum norm and in the variation norm – this follows from the fact that $\forall n, \forall f \in C_n$, $P_{\alpha,f} \neq \phi$.

Finally one has obviously $[\varphi_2^\epsilon(v)](1) = v(I)$, and $\epsilon_1 < \epsilon_2 \Rightarrow D_{\epsilon_1} \supseteq D_{\epsilon_2}$ and if $v \in D_{\epsilon_2}$ then $\varphi_2^{\epsilon_1}(v) = \varphi_2^{\epsilon_2}(v)$ on V_{ϵ_1}.

Observe that for games with finitely many players, φ_2 coincides with Owen's multilinear extension, and that for games in EXT (cfr "Values and Derivatives"), φ_2 coincides with the extension as defined in "Values and Derivatives".

Observe also that φ_2^ϵ obviously maps constant sum games $v \in D_\epsilon$ into constant sum games $w \in F_\epsilon$ [i.e.: $w(\chi) + w(1-\chi) = w(1)\ \forall \chi \in B_1^+(I, \mathcal{C})$], and that if $v \in D_\epsilon$, then $\varphi_1(v) \in D_\epsilon$ and $\varphi_2^\epsilon \varphi_1(v) = \varphi_1 \varphi_2^\epsilon(v)$, where φ_1 maps F_ϵ into F_ϵ by the formula $[\varphi_1(w)](\chi) = \frac{1}{2}[w(\chi) - w(1-\chi) + w(1)]$.

(1.3) φ_3 is essentially the derivative operator defined in "Values and Derivatives":

First, if $w \in F_\epsilon$, define $\bar{w}$ on $\{f \in B(I, \mathcal{C}) : \sup f - \inf f \leqslant \epsilon\}$ by $\bar{w}(f) = w[\max(0, \min(1, f))]$.

Obviously $w \to \bar{w}$ is symmetric, positive, linear, etc. ...; and if w is constant sum, $\bar{w}(\chi) + \bar{w}(1-\chi) = \bar{w}(1)$.

Let now $[\varphi_3(w)](\chi) = \lim\limits_{\tau \to 0} \int_0^1 \frac{\bar{w}(t+\tau\chi) - \bar{w}(t-\tau\chi)}{2\tau} dt$ – whenever $w \in \bigcup\limits_\epsilon F_\epsilon$ and the limit exists for all $\chi \in B(I, \mathcal{C})$ –.

Some remarks are in order.

First, if one deals only with games in BV, there would be no problem of existence of the integrals – otherwise, we make the explicit assumption that, for any χ, $\bar{w}(t + \tau\chi)$ is a.e. defined and integrable for all sufficiently small τ.

We also assume that, for all $\chi \in B(I, \mathcal{C})$, $\lim\limits_{\tau \overset{>}{\to} 0} \int_0^1 [w(\tau(u+\chi)^+) + w(\tau(u+\chi)^-)]du = 0$.

This is for instance satisfied, by the dominated convergence theorem, as soon as w is bounded and $\lim\limits_{\tau \overset{>}{\to} 0} w(\tau\chi) = 0\ \forall \chi \in B_1^+(I, \mathcal{C})$.

Obviously the mapping φ_3 is positive, linear, symmetric.

Let us show that

$$[\varphi_3(w)](\alpha + b\chi) = aw(1) + b[(\varphi_3(w))(\chi)] \quad \forall a, b \in R.$$

In particular we will have $[\varphi_3(w)](1) = w(1)$, so that $\varphi_3(w)$ is linear on every plane containing the constants.

It is obvious that $[\varphi_3(w)][b\chi] = b[\varphi_3(w))(\chi)]$ $\forall b$, $\forall \chi$. So we only have to show that $[\varphi_3(w)](1 + \chi) = w(1) + [\varphi_3(w)](\chi)$. Thus

$$\varphi_3(w)(1+\chi) = \lim_{\tau \overset{>}{\to} 0} \int_0^1 \frac{\bar{w}(t+\tau+\tau\chi) - \bar{w}(t-\tau-\tau\chi)}{2\tau}\,dt$$

$$= \lim_{\tau \overset{>}{\to} 0} \int_0^1 \frac{\bar{w}(t+\tau) - \bar{w}(t-\tau)}{2\tau}\,dt + \lim_{\tau \overset{>}{\to} 0} \int_0^1 \frac{\bar{w}(t+\tau\chi) - \bar{w}(t-\tau\chi)}{2\tau}\,dt$$

$$+ \lim_{\tau \overset{>}{\to} 0} \frac{1}{2\tau} \int_0^1 [\bar{w}(t+\tau+\tau\chi) - \bar{w}(t+\tau\chi) - \bar{w}(t+\tau) - \bar{w}(t-\tau-\tau\chi)$$

$$+ \bar{w}(t-\tau\chi) + \bar{w}(t-\tau)]dt.$$

The first integral equals

$$\frac{1}{2\tau}\left[\int_{\tau}^{1+\tau} \bar{w}(s)ds - \int_{-\tau}^{1-\tau} \bar{w}(s)ds\right] = \frac{1}{2\tau}\left[\int_{1-\tau}^{1+\tau} \bar{w}(s)ds - \int_{-\tau}^{\tau} \bar{w}(s)ds\right].$$

Since $\bar{w}(\chi) + \bar{w}(1-\chi) = \bar{w}(1)$, this equals

$$\bar{w}(1) - \frac{1}{2\tau}\left[2\int_{-\tau}^{\tau} \bar{w}(s)ds\right] = w(1) - \frac{1}{\tau}\int_0^{\tau} w(s)ds = w(1) - \int_0^1 w(\tau u)du,$$

and this last integral converges to zero by assumption. So the first integral converges to $w(1)$.

The second integral converges by definition to $[\varphi_3(w)](\chi)$, so there remains to show that the last integral converges to zero. It is equal to, writing $F_+(t)$ for $\bar{w}(t+\tau\chi) - \bar{w}(t)$, $F_-(t)$ for $\bar{w}(t-\tau\chi) - \bar{w}(t)$:

$$\frac{1}{2\tau}\left[\int_0^1 (F_+(t+\tau) - F_+(t))dt + \int_0^1 (F_-(t) - F_-(t-\tau))dt\right]$$

$$= \frac{1}{2\tau}\left[\int_1^{1+\tau} F_+(s)ds - \int_0^{\tau} F_+(s)ds + \int_{1-\tau}^{1} F_-(s)ds - \int_{-\tau}^{0} F_-(s)ds\right].$$

Now the relation $\bar{w}(\chi) + \bar{w}(1-\chi) = w(1)$ implies $F_+(t) = -F_-(1-t)$, so that the last integral equals

$$= \frac{1}{2\tau}\left[-\int_{-\tau}^{0} F_-(s)ds - \int_{0}^{\tau} F_+(s)ds - \int_{0}^{\tau} F_+(s)ds - \int_{-\tau}^{0} F_-(s)ds \right]$$

$$= \frac{-1}{\tau}\left[\int_{-\tau}^{0} F_-(s)ds + \int_{0}^{\tau} F_+(s)ds \right]$$

$$= \frac{-1}{\tau} \int_{0}^{\tau} [F_+(s) + F_-(-s)]ds$$

$$= -\int_{0}^{1} [F_+(\tau u) + F_-(-\tau u)]du$$

$$= -\int_{0}^{1} [\bar{w}(\tau(u+\chi)) - \bar{w}(\tau u) + \bar{w}(-\tau(u+\chi)) - \bar{w}(-\tau u)]du$$

$$= -\int_{0}^{1} [w[\tau(u+\chi)^+] + w[\tau(u+\chi)^-] - w(\tau u)]du \qquad (\text{for } \tau \leqslant [1 + \|\chi\|]^{-1})$$

and this last integral tends to zero by assumption.

Let us finally show that φ_3 is of norm 1. Let $\chi \leqslant \chi'$, and consider any increasing chain

$$\chi \leqslant \chi_1 \leqslant \chi_2 \leqslant \ldots \leqslant \chi_n \leqslant \chi'.$$

Denote by $V(v)[\chi, \chi']$ the supremum of the variation of v over all such finite chains. Let $\|\chi' - \chi\| = \delta$, then $V(v)[\chi, \chi'] \leqslant V(v)[\chi, \chi + \delta]$, and there is no loss in restricting the chains to satisfy $\chi_1 = \chi, \chi_n = \chi + \delta$.

If $v = \varphi_3(w)$, and we take $\tau > 0$ sufficiently small such that all $\bar{w}(t \pm \tau\chi_i)$ exist, then

$$\sum_i |v(\chi_{i+1}) - v(\chi_i)| = \lim \frac{1}{2\tau} \sum_i \left| \int_0^1 [\bar{w}(t + \tau\chi_{i+1}) - \bar{w}(t + \tau\chi_i) + \bar{w}(t - \tau\chi_i) \right.$$
$$\left. - \bar{w}(t - \tau\chi_{i+1})]dt \right|$$

$$\leqslant \lim \frac{1}{2\tau} \int_0^1 [\sum_i |\bar{w}(t + \tau\chi_{i+1}) - \bar{w}(t + \tau\chi_i)| + \sum_i |\bar{w}(t - \tau\chi_i) - \bar{w}(t - \tau\chi_{i+1})|]dt,$$

or, letting $|\bar{w}| = \overline{w_\epsilon^+} + \overline{w_\epsilon^-}$,

$$\leqslant \lim \frac{1}{2\tau} \int_0^1 [\,|\bar{w}|(t+\tau(\chi+\delta)) - |\bar{w}|(t+\tau\chi) + |\bar{w}|(t-\tau\chi) - |\bar{w}|(t-\tau(\chi+\delta))]dt$$

$$= \lim \frac{1}{2\tau}\left[\int_1^{1+\tau\delta} |\bar{w}|(t+\tau\chi)dt + \int_{1-\tau\delta}^{1} |\bar{w}|(t-\tau\chi)dt - \int_0^{\tau\delta} |\bar{w}|(t+\tau\chi)dt \right.$$
$$\left. - \int_{-\tau\delta}^{0} |\bar{w}|(t-\tau\chi)dt \right].$$

But, for all χ, we have $0 \leqslant |\bar{w}|(\chi) \leqslant |\bar{w}|(1) = w_\epsilon^+(1) + w_\epsilon^-(1) = \|w\|_{IBV,\epsilon}$. Thus

$$\sum_i |v(\chi_{i+1}) - v(\chi_i)| \leqslant \lim \frac{1}{2\tau}\left[\int_{1-\tau\delta}^{1+\tau\delta} \|w\|_{IBV,\epsilon} dt - \int_{-\tau\delta}^{\tau\delta} (0) dt\right]$$

$$= \delta\, \|w\|_{IBV,\epsilon},$$

and therefore, ϵ being arbitrary,

$$V(\varphi_3(w))[\chi, \chi'] \leqslant \|\chi' - \chi\| \cdot \|w\|_{IBV,0}.$$

In particular $\|\varphi_3(w)\|_{IBV} = V(\varphi_3(w))[0,1] \leqslant \|w\|_{IBV,0} \leqslant \|w\|_{IBV}$, which shows that φ_3 is of norm 1.

Since $v = \varphi_3(w)$ satisfies $v(a + b\chi) = a\,v(1) + b\,v(\chi)$, we have

$$\frac{1}{2\tau}(v(t+\tau\chi) - v(t-\tau\chi)) = v(\chi),$$

so that $\frac{1}{2\tau}[\bar{v}(t+\tau\chi) - \bar{v}(t-\tau\chi)] = v(\chi)$ for $\|\tau\chi\| \leqslant t \leqslant 1 - \|\tau\chi\|$. If now $\|v\|_{IBV} < \infty$, then by $v(a+\chi) = av(1) + v(\chi)$ we get $V(v)[a-\delta, a+\delta] = V(v)[0, 2\delta]$, which equals by homogeneity $2\delta\, V(v)[0,1] = 2\delta\,\|v\|$. Therefore, for all t we have

$$\left|\frac{1}{2\tau}[\bar{v}(t+\tau\chi) - \bar{v}(t-\tau\chi)]\right| \leqslant \|\chi\|\,\|v\|.$$

Also, $\|v\| < \infty$ implies that $\bar{v}(t + \tau\chi)$ is integrable (in t) – as a function of bounded variation-, so that, by Lebesgue's bounded convergence theorem

$$\int_0^1 \frac{1}{2\tau}[\bar{v}(t+\tau\chi) - \bar{v}(t-\tau\chi)]dt \to v(\chi).$$

Thus, to show that $v \in \text{Dom}(\varphi_3)$ and that $\varphi_3(v) = v$ there only remains to show that v is bounded and $\lim_{\tau \overset{>}{\to} 0} v(\tau\chi) = 0 \ \forall \ \chi \in B_1^+(I, C)$. This follows again immediately from the boundedness of the norm of v and from the homogeneity.

Thus if $v = \varphi_3(w)$, $\|v\| < \infty$ implies $v \in \text{Dom}(\varphi_3)$ and $\varphi_3(v) = v$. Therefore we get then $V(v)[\chi, \chi'] = V(\varphi_3(v))[\chi, \chi'] \leqslant \|\chi' - \chi\| \cdot \|v\|$ and this relation is anyway true if $\|v\| = +\infty$. To summarize, we have shown that

Proposition 1:

- φ_3 is positive, linear, symmetric.
- $[\varphi_3(w)](1) = w(1)$.
- $\|\varphi_3(w)\|_{IBV} \leqslant \|w\|_{IBV,0}$.
- $v \in \text{Range}(\varphi_3)$ implies
  - . v is linear on every plane containing the constants,
  - . $V(v)[\chi, \chi'] \leqslant \|\chi' - \chi\| \cdot \|v\| \quad \forall \ \chi, \chi' \in B(I, C)$,
  - . further, if $\|v\| \leqslant \infty$, then $v \in \text{Dom}\ \varphi_3$, and $\varphi_3(v) = v$.

For every $\epsilon > 0$, one gets a different domain for $\varphi_1 \circ \varphi_2 \circ \varphi_3$. However, the composition having norm 1, and the domains being increasing when $\epsilon \to 0$, the composition can be extended to the closure of the union of those domains. Let us call ψ this operator.

(In Section 4, we will show how to define directly an operator with closed domain extending ψ – the present approach seems however easier for getting the main idea through, being more closely related to the literature.)

(1.4) Let us now already prove part of our claims.

Let $Q = \{v \mid v \in \text{Dom}\ \psi,\ \psi(v) \in FA\}$: obviously Q is a closed symmetric space, and ψ is a value on Q.

It is obvious that Q contains DIFF, DIAG, and all games satisfying $v(S) = v(S^c)$ $\forall S \in \mathcal{C}$.

Let us show that Q also contains $bv'FA$ (and all "regular" games with countably many players).

$bv'FA$ is the closed space generated by all games of the form $v(S) = f(\mu(S))$, where $\mu \in FA_+^1$ and f is monotonic, continuous at zero and at 1, with $f(0) = 0, f(1) = 1$. It is sufficient to show that $v \in Q$ when v is a generator. After applying φ_1 to v, one may assume further that $f(x) + f(1-x) = 1$.

Let us apply φ_2. Let $\mu = \sum_{i \geqslant 0} \mu_i$ where $\forall\, i\ \mu_i \geqslant 0, i \neq j \Rightarrow \mu_i \neq \mu_j, \mu_i$ is two-valued for $i \geqslant 1$ and μ_0 is non atomic (i.e. $\sum_{i \geqslant 1} \mu_i(I)$ is maximal given the other conditions). Assume without loss of generality that $\mu_i(I)$ is monotonic in $i \geqslant 1$, and let $a_i = \mu_i(I)$, $\nu_i = a_i^{-1} \cdot \mu_i$ whenever $a_i \neq 0, n_a = \sup\ \{i \mid a_i \neq 0\}$.

Denote by π_n any partition of I such that

$$\forall\, i, j \in \{1, \ldots, n \wedge n_a\}\ (i \neq j)\ \exists\, A \in \pi_n : \mu_i(A) \neq 0, \mu_j(A) = 0.$$

Let $\alpha_n = (\pi_n, \mu_0, 2^{-n})$. Then for any $P \in \mathcal{P}_{\alpha_n, g}$ one has P-a.s. $\mu_0(S) = \mu_0(g)$, for $i \geqslant 1$ $\mu_i(S) \in \{0, \mu_i(I)\}$ has expectation $\mu_i(g)$ up to 2^{-n} and $\mu_1(S) \ldots \mu_n(S)$ are independent.

Let also $Y_g = \mu_0(g) + \sum_{i \geqslant 1} a_i X_i$, where the X_i's are independent random variables with value in $\{0, 1\}$, and with expectation $\nu_i(g)$.

It follows that, when n goes to infinity, the distribution of $\mu(S)$ $(= \sum_{i \geqslant 0} \mu_i(S))$ converges weakly to the distribution of Y_g. Further, when $n_a < \infty$, then $\mu(S)$ is concentrated on the (finite) set of atoms of Y_g. It follows that $E(v(S))$ converges to $\bar{v}(g) = Ef(Y_g)$ – except maybe when $n_a = +\infty$ and the distribution of Y_g has some atoms at discontinuities of f.

Recall that, for g arbitrary, one sets $\bar{v}(g) = \bar{v}(\max(0, \min(1, g)))$. Obviously $\bar{v}(t + g)$ is integrable – being monotonic. We now show that, even when $n_a = +\infty$, $\bar{v}(t+g)$ is the extension of v at $t + g$, except for at most two values of t. Indeed, the distribution of Y_g is obviously non atomic except when $\lim_{i \to \infty} \nu_i(g) \wedge (1 - \nu_i(g)) = 0$. Since we work only on some ϵ-neighborhood of the diagonal, we can assume $\sup(g) - \inf(g) < 1$, so that the only possible exceptions occur when $\lim_{i \to \infty} \nu_i[(t+g)^+] = 0$ and when $\lim_{i \to \infty} \nu_i[(t+g) \wedge 1] = 1$. It is sufficient to consider the first case, which is true for all t satisfying $0 \leqslant t \leqslant -\limsup_{i \to \infty} \nu_i(g) = t_0$. But if $0 \leqslant t < -\limsup_{i \to \infty} \nu_i(g) = t_0$, then $B = \{\omega \mid g(\omega) \leqslant -\frac{1}{2}(t + t_0)\}$ is some measurable set, and, since $\nu_i(B) = 0 \Rightarrow \nu_i(g) \geqslant -\frac{1}{2}(t + t_0)$, one has $\nu_i(B) = 1$ except at most finitely many times – otherwise one would have $t_0 = -\limsup \nu_i(g) \leqslant \frac{1}{2}(t + t_0)$, thus $t \geqslant t_0$ contrary to our assumption. Remark that on B one has $(g + t)^+ = 0$.

Thus, as soon as our partition π_α refines B, we will have that, with probability one, $B \cap S = \phi$ i.e. $S \subseteq B^c$; and that $\nu_i(B^c) > 0$ at most finitely many times – say $\nu_i(B^c) = 0 \ \forall\, i \geqslant n_0$. Therefore $\mu(S)$ will have the distribution of $\mu_0[(g + t)^+] + \sum_{i=1}^{n_0} a_i X_i$ where $|E(X_i) - \nu_i[(g + t)^+]| \leqslant 2^{-n}$. Since n_0 depends only on g and t, this implies $\mu(S)$ is a distribution on a fixed, finite set of atoms, that converges weakly to the distribution of $Y_{(g+t)^+}$ – and thus the probability of every atom converges: we still have that $\bar{v}(g + t)$ is the extension of v at $(g + t)^+$.

Thus the only possible troublesome value of t is $t = -\limsup \nu_i(g)$ (and dually $1 - t = \liminf \nu_i(g)$).

In particular, for any χ, $\bar{v}(t + \tau\chi)$ is a.e. defined and integrable for all sufficiently small τ. The second condition for $v \in \operatorname{Dom} \varphi_3$ was satisfied as soon as v is bounded and $\bar{v}(\tau\chi)$ converges to zero for all $\chi \in B_1^+(I, \mathcal{C})$; v being monotonic it is sufficient to show that $\lim_{\substack{> \\ \tau \to 0}} \bar{v}(\tau) = 0$; this follows from $\bar{v}(\tau) \leqslant (\operatorname{Cav} f)(\tau)$ because $\operatorname{Cav} f$ is continuous and vanishes at zero, f having this property.

Thus to show that $v \in Q$ there only remains to show that $\frac{1}{2\tau} \int_0^1 [\bar{v}(t + \tau\chi) - \bar{v}(t - \tau\chi)]dt$ converges to some element of FA. To facilitate this, we begin by the lemma.

Lemma: If

- $\bar{v}(t + \tau\chi)$ is, for every χ, a.e. defined and integrable for all sufficiently small τ
- $\int_0^1 (v[\tau(t + \chi)^+] + v[\tau(t + \chi)^-])dt$ converges to zero with τ (> 0) for all χ
- $\frac{1}{2\tau} \int_0^1 [\bar{v}(t + \tau\chi) - \bar{v}(t - \tau\chi)]dt$ converges for all $\chi \in B_1^+(I, \mathcal{C})$

then $v \in \operatorname{Dom} \varphi_3$ – i.e. the last expression converges for all $\chi \in B(I, \mathcal{C})$.

Proof: Our assumption immediately implies the convergence when $\chi \leqslant 0$, and it is sufficient to prove the convergence for any ψ satisfying $\psi \leqslant 1$. Write thus $\psi = 1 + \chi$, with $\chi \leqslant 0$; the computation we did when proving that $\varphi_3(w)$ is linear on every plane containing the constants proved also that $\varphi_3(w)(\psi)$ exists – and this finishes the proof. □

By virtue of this lemma, it is sufficient to show that $\phi(\tau, \chi) = \frac{1}{2\tau} \int_0^1 [\bar{v}(t + \tau\chi) - \bar{v}(t - \tau\chi)]dt$ converges to some element of FA for all $\chi \in B_1^+(I, \mathcal{C})$, where $\bar{v}(g) = Ef(Y_{\bar{g}})$, with $\bar{g} = 0 \vee (g \wedge 1)$, and $Y_g = a_0\nu_0(g) + \sum_{i=1}^{n_a} a_i X_i$, where the random variables X_i are independent, with values in $\{0, 1\}$ and with expectation $\nu_i(g)$.

Let now $f_L(x) = \sum_{y<x} (f(y+) - f(y))$ where $f(y+) = \lim_{\epsilon \overset{>}{\to} 0} f(y + \epsilon)$. Let also $f_R(x) = f(x) - f_L(x)$: then both f_L and f_R are increasing, f_L is left continuous and f_R is right continuous, so that

$$f(x) = \int_0^1 I(x \geqslant q)df_R(q) + \int_0^1 I(x > q)df_Lq).$$

Since $\phi(\tau, \chi)$ depends linearly on f, and since $0 \leqslant \phi(\tau, \chi) \leqslant \phi(\tau, 1) \leqslant 1$ using the monotonicity of v and the relation $0 \leqslant \chi \leqslant 1$, we can apply Fubini's theorem to get

$$\phi(\tau, \chi) = \int_0^1 \phi_{q,R}(\tau, \chi)df_R(q) + \int_0^1 \phi_{q,L}(\tau, \chi)df_L(q)$$

when $\phi_{q,R}$ and $\phi_{q,L}$ denote the function ϕ corresponding to the case where $f(x) = I(x \geqslant q)$ and $f(x) = I(x > q)$ respectively.

But $\phi_{q,R}(\tau, \chi)$ and $\phi_{q,L}(\tau, \chi)$ being uniformly bounded, the bounded convergence theorem implies that it is sufficient to prove that $\phi_{q,R}(\tau, \chi)$ and $\phi_{q,L}(\tau, \chi)$ converge to some element of FA.

Thus we have reduced the problem to the case where $f(x) = I(q <\cdot x)$, where $<\cdot$ stands for either $\leqslant$ or $<$.

Remark that for $i \geqslant 1$, $\nu_i(\bar{g}) = 0 \vee (\nu_i(g) \wedge 1)$, and that $\nu_i(t + \tau\chi) = t + \tau\nu_i(\chi)$. Let for short $p_i = \nu_i(\chi)$ (thus $0 \leqslant p_i \leqslant 1$), and let Z_i ($i = 1 \ldots n_a$) be independent random variables, uniformly distributed over $[0, 1]$. Then

$$\phi(\tau, \chi) = \frac{1}{2\tau} E \int_0^1 I\left[a_0\nu_0((t - \tau\chi)^+) + \sum_{i=1}^{n_a} a_i I(Z_i \leqslant t - \tau p_i) <: q \right.$$
$$\left. <\cdot\, a_0\nu_0((t + \tau\chi) \wedge 1) + \sum_{i=1}^{n_a} a_i I(Z_i \leqslant t + \tau p_i)\right| dt$$

where $<:$ stands for $<$ or $\leqslant$ when $<\cdot$ is $\leqslant$ or $<$ respectively. Let also $\bar{\phi}(\tau, \chi)$ be the same expression, with $\nu_0[(t - \tau\chi)^+]$ replaced by $t - \tau p_0$ and $\nu_0[(t + \tau\chi) \wedge 1]$ replaced by $t + \tau p_0$. Then obviously $\bar{\phi} \geqslant \phi$, and the integrands can differ only when $t \leqslant \tau$ or $t \geqslant 1 - \tau$, so that

$$\bar{\phi}(\tau,\chi)-\phi(\tau,\chi)\leqslant\frac{1}{2\tau}E\int_0^\tau I\left[q<a_0(t+\tau p_0)+\sum_{i=1}^{n_a}a_iI(Z_i\leqslant t+\tau p_i)\right]dt$$

+ a similar integral between $1-\tau$ and τ.

Obviously, the right hand member goes to zero with τ. Thus if we set

$$\psi(p)=\frac{1}{2}E\int_0^1 I[a_0(t-p_0)+\sum_{i\geqslant 1}a_iI(Z_i\leqslant t-p_i)<:q<a_0(t+p_0)$$
$$+\sum_{i\geqslant 1}a_iI(Z_i\geqslant t+p_i)]dt$$

we have to prove that ψ is differentiable at zero, i.e. that $\lim\limits_{\substack{>\\ \tau\to 0}}\frac{1}{\tau}\psi(\tau p)$ exists and is linear in p – i.e. a continuous linear functional on l_∞. We will even show that the limit is of the form $\Sigma\,\gamma_i p_i$, with $\gamma_i\geqslant 0, \Sigma\,\gamma_i=1$.

We will show that this is the case for all sequences p_i having only finitely many non zero terms.

The result will follow from this, because, for an arbitrary sequence p_i $(0\leqslant p_i\leqslant 1)$, one has then by monotonicity

$$\liminf_{\substack{>\\ \tau\to 0}}\frac{1}{\tau}\psi(\tau p)\geqslant\sum_{i=0}^{k}\gamma_i p_i,\quad\text{thus}$$

$$\liminf_{\substack{>\\ \tau\to 0}}\frac{1}{\tau}\psi(\tau p)\geqslant\sum_0^\infty\gamma_i p_i,\quad\text{and}$$

$$\liminf_{\substack{>\\ \tau\to 0}}\frac{1}{\tau}\psi(\tau(1-p))\geqslant 1-\sum_0^\infty\gamma_i p_i$$

and since $\frac{1}{\tau}[\psi(\tau p)+\psi(\tau(1-p))]$ converges to 1 (this is the computation we did when proving that any w in the range of ψ_3 is linear on every plane containing the constants), it follows that $\lim\limits_{\substack{>\\ \tau\to 0}}\frac{1}{\tau}\psi(\tau p)=\sum_0^\infty\gamma_i p_i$.

Thus we have to show that $\psi(p)$ is differentiable at zero as a function of the variables $p_0\ldots p_k$, the other p_i's being held fixed at zero.

Further if then $\gamma_i = \left(\frac{\partial \psi}{\partial p_i}\right)_{p=0}$, we have to show that $\sum_{i \geqslant 0} \gamma_i = 1$ – (because obviously $\gamma_i \geqslant 0$ by monotonicity of ψ).

Writing the expectation in the formula of $\psi(p)$ as the expectation of the conditional expectation given $Z_1 \ldots Z_k$ yields

$$\psi(p) = \frac{1}{2} \sum_{y \in \{0,1\}^k} \int_0^1 [H_t^y(p) - H_t^y(-p)]dt,$$

where

$$H_t^y(p) = \prod_{i=1}^k ([\tfrac{1}{2} + (2y_i - 1)(t - p_i - \tfrac{1}{2})]^+ \wedge 1) F_t^k\left(q + a_0 p_0 - \sum_{i=1}^k a_i y_i\right),$$

with

$$F_t^k(x) = P(a_0 t + \sum_{i > k} a_i I(Z_i \leqslant t) <: x).$$

Let

$$\bar{H}_t^y(p) = \sum_{i=1}^k [\tfrac{1}{2} + (2y_i - 1)(t - p_i - \tfrac{1}{2})] F_t^k\left(q + a_0 p_0 - \sum_{i=1}^k a_i y_i\right),$$

and

$$\bar{\psi}(p) = \frac{1}{2} \sum_{y \in \{0,1\}^k} \int_0^1 [\bar{H}_t^y(p) - \bar{H}_t^y(-p)]dt.$$

One shows, just as before for $\bar{\phi} - \phi$, that $\bar{\psi} - \psi$ is differentiable at zero with zero differential (the difference of the integrands is anyway small, and different from zero only a small part of the domain).

Thus to show the differentiability at zero, it is sufficient to show the differentiability at zero of the expression

$$\left(\prod_{i=1}^{k} p_i^{n_i}\right)\int_0^1 t^n F_t^k(x \pm a_0 p_0)dt$$

since $\bar{\psi}(p)$ is a linear combination of expressions of this type.

Since $\left(\prod_{i=1}^{k} p_i^{n_i}\right)$ is obviously differentiable, this amounts in turn to the differentiability at zero of $\int_0^1 t^n F_t^k(x \pm a_0 p_0)dt$.

If this differentiability is proved, then using $k = 1$,

$$\gamma_1 = \lim_{p_1 \to 0} \frac{1}{p_1}\bar{\psi}(p_1) = \lim_{p_1 \to 0} \frac{1}{2p_1}\int_0^1 2p_1[F_t^1(q) - F_t^1(q - a_1)]dt$$

$$= \int_0^1 [F_t^1(q) - F_t^1(q - a_1)]dt$$

$$= E\int_0^1 I[a_0 t + \sum_{i>1} a_i I(Z_i \leqslant t) <: q <\cdot a_0 t + a_1 + \sum_{i>1} a_i I(Z_i \leqslant t)]dt$$

or, since Z_1 is, like t, uniformly distributed on $[0, 1]$ and independent of the other Z_i's, we get

$$\gamma_1 = P[a_0 Z_1 + \sum_{i \geqslant 1} a_i I(Z_i < Z_1) <: q <\cdot a_0 Z_1 + \sum_{i \geqslant 1} a_i I(Z_i \leqslant Z_1)].$$

Let, for $k \geqslant 1, J_k(\omega)$ denote the random interval (of length a_k)

$$\{x \mid a_0 Z_k + \sum_{i \geqslant 1} a_i I(Z_i < Z_k) <: x <\cdot a_0 Z_k + \sum_{i \geqslant 1} a_i I(Z_i \leqslant Z_k)\}$$

(obviously the $J_k(\omega)$ are disjoint if we restrict ourselves to the set of ω's (with probability one) where $i \neq j \Rightarrow Z_i(\omega) \neq Z_j(\omega), 0 < Z_i(\omega) < 1$).

Let also $J_0(\omega) = [0, 1] \setminus \bigcup_{k \geqslant 1} J_k(\omega)$. In those notations

$$\gamma_i = \left(\frac{\partial \psi}{\partial p_i}\right)_{p=0} = P(q \in J_i(\omega)) \quad \text{for all } i \geqslant 1.$$

Similarly, using $k = 0$, we get, using $W(t) = a_0 t + \sum_{i\geq 1} a_i I(Z_i \leqslant t)$,

$$\gamma_0 = \lim_{p_0\to 0} \frac{1}{p_0} \bar{\psi}(p_0) = \lim_{p_0\to 0} \frac{1}{2p_0} \int_0^1 [F_t^0(q + a_0 p_0) - F_t^0(q - a_0 p_0)]dt$$

$$= \lim_{p_0\to 0} \frac{1}{2p_0} \int_0^1 P[q - a_0 p_0 <\cdot W(t) <: q + a_0 p_0]dt.$$

Thus, if $a_0 = 0$, then $\gamma_0 = 0$, and the differentiability condition to check is obvious, so there only remains to show that $\sum_{i\geq 1} \gamma_i = P(q \in \bigcup_i J_i(\omega)) = 1$. When there are only finitely many non zero a_i's, then $q \in \bigcup_i J_i(\omega) = \{x \mid 0 <: x <\cdot 1\}$ for any ω, while if there are countably many non zero a_i's, a recent result of Berbee [2] proves that $P(q \in \bigcup_i J_i(\omega)) = 1$ [even that $P(q \in \bigcup_i \mathring{J}_i(\omega)) = 1$].

There remains therefore to consider the case $a_0 > 0$. Since $1 - \sum_{i\geq 1} \gamma_i = P(q \in J_0(\omega))$, the property $\sum \gamma_i = 1$ amounts to

$$P(q \in J_0(\omega)) = a_0 \lim_{p_0\to 0} \frac{1}{2a_0 p_0} \int_0^1 P[q - a_0 p_0 <\cdot W(t) <: q + a_0 p_0]dt.$$

On the other hand, the differentiability at zero of $\int_0^1 t^n F_t^k(x + a_0 p_0)dt$, when $F_t^k(x) = P(a_0 t + \sum_{i>k} a_i I(Z_i \leqslant t) <: x)$ can be rewritten, by letting $a_i' = a_{k+i}$ for $i \geqslant 1$, $a_0' = a_0$, $\sigma = \sum_{i\geqslant 0} a_i'$ $(a_0 > 0 \Rightarrow \sigma > 0)$, $a_i'' = a_i'/\sigma$, $F_t(y) = F_t^k(\sigma y)(= P(a_0'' t + \sum_{i\geqslant 1} a_i'' I(Z_i \leqslant t) <: y)$, as the differentiability at zero of $\int_0^1 t^n F_t\left(\frac{x}{\sigma} + a_0'' p_0\right)dt$ – or, letting $z = x/\sigma$, and writing a_i for a_i'' – so F_t becomes F_t^0 – as the differentiability of $a_0 \int_0^1 t^n F_t^0(z)dt = \varphi_n(z)$ as a function of z.

To show also that $\sum \gamma_i = 1$, we have to show further that, when $n = 0$, the derivative is $P(z \in J_0(\omega))$. We have

$$\frac{1}{\delta}[\varphi_n(z + \delta) - \varphi_n(z)] = E\frac{a_0}{\delta} \int_0^1 t^n I[z <\cdot W(t) <: z + \delta]dt.$$

Let $T_z = \inf\{t \geqslant 0 \mid 1 \wedge z <\cdot W(t)\}$: if $z <\cdot W(t) <: z + \delta$, we have $T_z \leqslant t \leqslant T_z + \frac{\delta}{a_0}$, thus $T_z^n \leqslant t^n \leqslant T_z^n + \left[\left(1 + \frac{\delta}{a_0}\right)^n - 1\right]$, therefore, if $X = \frac{a_0}{\delta} T_z^n \int_0^1 I[z <\cdot W(t) <: z + \delta]dt$,

$$X \leqslant \frac{a_0}{\delta} \int_0^1 t^n I(z < W(t) < z + \delta)dt \leqslant X + \frac{a_0}{\delta} \left[\left(1 + \frac{\delta}{a_0}\right)^n - 1 \right] \left[\left(T_z + \frac{\delta}{a_0}\right) - T_z \right]$$

$$= X + \left(1 + \frac{\delta}{a_0}\right)^n - 1$$

Now $X = T_z^n \int_0^1 I_{J_0(\omega)}(z + \delta u)du$.

If $z \in \bigcup_i J_i(\omega)$, then $\lim_{x \to z} I_{\bigcup_i J_i(\omega)}(x) = 1$, except maybe if z is a boundary point of some $J_i(\omega)$ – but this event has probability zero, even conditionally to all $Z_j (j \neq i)$ (using $a_0 > 0$).

If $z \in J_0(\omega)$, then $\frac{1}{\delta} \int_z^{z+\delta} I_{J_0(\omega)}(x)dx = 1 - \sum_{i=1}^{\infty} \frac{1}{\delta} \int_z^{z+\delta} I_{J_i(\omega)}(x)dx$, and it is sufficient to show that the conditional expectation (given $z \in J_0(\omega)$ and given T_z) of the sum converges to zero. Now, if $z \in J_0(\omega), x > z$, then $I_{J_i(\omega)}(x) \leqslant I(T_{z \vee (x-a_i)} \leqslant Z_i \leqslant T_x) \leqslant I\left(T_{z \vee (x-a_i)} \leqslant Z_i \leqslant T_{z \vee (x-a_i)} + \frac{(x-z) \wedge a_i}{a_0}\right)$, and then

$$P(x \in J_i(\omega) \mid T_z, z \in J_0(\omega)) \leqslant \frac{1}{1-T_z} \left[\frac{(x-z) \wedge a_i}{a_0} \right]$$

Thus

$$P(x \in \bigcup_i J_i(\omega) \mid z \in J_0(\omega), T_z) \leqslant \frac{1}{a_0(1-T_z)} \sum_{i \geqslant 1} [(x-z) \wedge a_i].$$

Also, for $a_0 > 0$, $1 - T_z > 0$ with probability one if $z < 1$, and since $\Sigma a_i < \infty$, it follows that the right hand member goes to zero when $x \downarrow z < 1$. Thus the left hand member being bounded, we get, if $z < 1$ – and obviously also if $z \geqslant 1$ – $\lim_{x \overset{>}{\to} z} P(x \in \bigcup_i J_i(\omega) \mid z \notin \bigcup_i J_i(\omega)) = 0$; and therefore by symmetry $\lim_{x \to z} P[x \in \bigcup_i J_i(\omega) \mid z \notin \bigcup_i J_i(\omega)] = 0$ and thus $I_{\bigcup_i J_i(\omega)}(x)$ is continuous in probability. In particular $\sum_{i \geqslant 1} \gamma_i(x)$ is a continuous function of x, and also $T_z^n I_{J_0(\omega)}(x)$ converges in probability to $T_z^n I_{J_0(\omega)}(z) I\ (0 \leqslant x \leqslant 1)$ so that by the bounded convergence theorem $\frac{1}{\delta}[\varphi_n(z + \delta) - \varphi_n(z)]$ converges to $I(z + \delta \in [0, 1]) E[T_z^n I_{J_0(\omega)}(z)]$.

Since the equation $\varphi'(z) = P(z \in J_0(\omega))$ is needed only for $0 < z < 1$, we have proved our statement. (Remark that the differentiability condition of $\int_0^1 t^n F_t^k(x \pm a_0 p_0)dt$ at zero was only one-sided since $a_0 p_0 > 0$.)

Remark 1. A closer look at the above argument shows that in fact we proved more: if $\phi(\chi) = \frac{1}{2}\int_0^1 [\bar{v}(t+\chi) - \bar{v}(t-\chi)]dt$, then $\phi(\chi)$ is Fréchet-differentiable at zero. Indeed, the proof of the lemma shows that it is sufficient to consider $\chi \in B_1^+(I, C)$ – provided $\int_0^1 (v[\tau(u+\chi)^+] + v[\tau(u+\chi)^-])du$ converges to zero uniformly over the unit ball, which is obvious whenever v is norm continuous at zero. Similarly the bounded convergence theorem still permits to reduce oneself to the case where $f(x) = I(q \lessdot x)$. Also the approximation of ϕ by $\bar{\phi}$ and later of ψ by $\bar{\psi}$ are obviously uniform in $p \in [0, 1]^\infty$. Since, as we just mentioned, the convergence of $\frac{1}{\tau}[\psi(\tau p) + \psi(\tau(1-p))]$ to 1 is uniform in p for v norm continuous at zero, it will be sufficient to consider vectors p such that $p_i = 0\ \forall\, i > k$: indeed, the same conclusion will then hold when $p_i = 1\ \forall\, i > k$, so that if k is chosen such that $\sum_{i>k} \gamma_i < \epsilon$, then τ_0 such that, $\forall\, \tau : |\tau| \leqslant \tau_0$, $\forall\, p$ in one of these two classes $\left|\frac{1}{\tau}\psi(\tau p) - \Sigma\, \gamma_i p_i\right| \leqslant \epsilon$, the result will follow (from the monotonicity of $\bar{v}$) for arbitrary $p \in [0, 1]^\infty$ by sandwiching it between the two approximations $\underline{p}_i, \bar{p}_i$, where $\underline{p}_i = \bar{p}_i = p_i$ for $i \leqslant k$ and for $i > k$ $\underline{p}_i = 0, \bar{p}_i = 1$. As shown in the proof, the differentiability of $\bar{\psi}$ over p's having only k non zero coordinates amounts to the differentiability of a product of functions of 1 variable, which is true as soon as each factor in the product is differentiable, what we proved.

Also we did not need in fact the symmetry of ϕ. We thus obtain finally:

Proposition 2. Let

$$H(\chi) = \int_0^1 \bar{v}(t+\chi)dt.$$

Then H is Fréchet differentiable at zero, with as derivative the value of v:

$$\int_0^1 \sum_0^\infty \gamma_i(x) v_i(\cdot) df(x)$$

where in the integration a discontinuity to the right (left) of x is to be interpreted as the corresponding mass immediately to the right (left) of x.

The $\gamma_i(x)$ are defined in the following way. Assume the game v is of the form $v(S)= f(\mu(S))$, where $\mu = \sum_0^\infty a_i\nu_i$, $a_i \geqslant 0$, $\nu_i \geqslant 0$, $\nu_i(I) = 1$, $\sum_i a_i = 1$, ν_0 non atomic and $i \geqslant 1 \Rightarrow \nu_i$ two valued, $i \neq j \Rightarrow \nu_i \neq \nu_j$. Define random variables Z_i independent and uniformly distributed over $[0, 1]$, then expand each point Z_i to some half-open interval of length a_i, then shrink the remaining part of $[0,1]$ (of length 1) to length a_0 (proportionately). Denote by J_i the random interval thus obtained corresponding to Z_i. Then $\gamma_i(x) = P(x \in J_i)$ for $i \geqslant 1$, and $\gamma_0(x) = P(\bigcup_i \bar{J}_i$ has density 0 at $x)$[2]

$$\left[\text{i.e. } \gamma_0(x) = P\left(\limsup_{\delta \to 0} \frac{1}{2\delta} \int_{x-\delta}^{x+\delta} I_{\bigcup_i \bar{J}_i}(s)ds = 0\right)\right]$$

When there are infinitely many players, we also showed that the $\gamma_i(x)$ $(i = 0, 1, \ldots)$ are continuous on $[0, 1]$, with $\gamma_0(0) = \gamma_0(1) = 1$ if $a_0 > 0$ – in particular, if $a_0 > 0$, the series $\sum \gamma_i$ is uniformly convergent (to 1) on $[0, 1]$, and anyway $\sum_i \gamma_i(x) = 1$ $\forall x : 0 < x < 1$.

In particular, when there are infinitely many players, the value of $f(x)$ at a jump and the exact definition of the $\int \ldots df(x)$ play no role.

It is possible to draw still some sharper conclusions from the foregoing: let $\mu^n \in FA^1_+$ converge (in norm) to $\mu^0 \in FA^1_+$. Let ν_i $(i \geqslant 1)$ enumerate all atoms of all μ^n and let ν_0^n be the non atomic part of μ^n, with $\mu^n = a_0^n\nu_0^n + \sum_1^\infty a_i^n\nu_i$. One sees immediately that $a_0^n\nu_0^n$ is norm convergent to $a_0^0\nu_0^0$, and that $(a_i^n)_{i=0}^\infty$ is l_1-convergent to $(a_i^0)_{i=0}^\infty$. Assume now μ^0 has infinitely many players. Since this implies that when realizing the random

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[2] The equality $\gamma_0(x) = P(\bigcup_i \bar{J}_i$ has density 0 at $x)$ has to be proved only when $a_0 > 0$: we claim that a.s. on $x \notin \bigcup_i \bar{J}_i$, this set has density zero at x. It is indeed sufficient to prove this conditionally to the set of atoms and the fraction of a_0 coming after x (or before) – which reduces (by renormalization) the problem to the case $x = 0$. Let $X_t = \frac{1}{t}\sum_{i \geqslant 1}\left(\frac{a_i}{a_0} \wedge t\right) I(Z_i \leqslant t)$: X_t is an upper bound for the density of $\bigcup_i \bar{J}_i$ up to time t, so it is sufficient to show that $X_t \to 0$ a.s. But, if $\mathcal{F}_t$ denotes the σ-field generated by all variables $t \vee Z_i$, then, when reversing the usual order on the time interval $[0, 1]$, X_t becomes a positive supermartingale w.r.t. $\mathcal{F}_t$, whose expectation goes to zero as we have seen: thus X_t goes to zero a.s. We would like to stress that this equality cannot be dispensed with in a random order approach: indeed, if $f(x) = I(x > q)$, and if some player of the ocean pivots, since he is negligible, it is in fact the infinitesimal coalition ds that immediately follows him that pivots – so to impute this event to the credit of the ocean, one needs that this infinitesimal coalition consists essentially only of oceanic players – i.e. the ocean must have density 1 to the right of q. The same applies to the left of q if $f(x) = I(x \geqslant q)$.

ordering with the same set of random variables Z_i, we will have a.s. $J_i^n \to J_i^0$, and since $\gamma_i(q) = P(q \in \mathring{J}_i) = P(q \in \bar{J}_i)$, it will follow that $\forall\, i \geq 1$, $\gamma_i^n(q_n) \to \gamma_i^0(q_0)$ whenever $q_n \to q_0$ $(0 < q_0 < 1)$. But $\gamma_i^n(q) \leq \text{Prob}\,(q - a_i^n \leq W_n(Z_i-) \leq q) \leq a_i^n / a_0^n$, so that $\sum_{i \geq k} \gamma_i^n(q) \leq \frac{1}{a_0^n} \sum_{i \geq k} a_i^n$ if $a_0^n > 0$. Thus the l_1-convergence of a_i^n to a_i^0 implies that, if $a_0^0 > 0$, $\lim_{k \to \infty} \sup_n \sup_q \sum_{i \geq k} \gamma_i^n(q) = 0$ i.e. the convergence of the series $\sum_{i \geq 1} \gamma_i^n(q)$ is uniform in n and q. Since $\gamma_i^n(q_n) \to \gamma_i^0(q_0)$, it follows that $\sum_{i \geq 1} \gamma_i^n(q_n) \to \sum_{i \geq 1} \gamma_i^0(q_0)$, and the relation $\sum_{i \geq 0} \gamma_i = 1$ yields therefore $\gamma_0^n(q_n) \to \gamma_0^0(q)$. If $a_0^0 = 0$, then $\gamma_0^0 = 0$ and therefore $\liminf_n \gamma_0^n(q_n) \geq \gamma_0^0(q_0)$, so that the relations $\gamma_i^n \geq 0$, $\sum_{i=0}^{\infty} \gamma_i^n = 1$ $\forall\, n$ and $\liminf_{n \to \infty} \gamma_i^n \geq \gamma_i^0$ $\forall\, i$ imply again $\gamma_0^n(q_n) \to \gamma_0^0(q_0)$. Hence the l_1-convergence of $\gamma_i^n(q_n)$ to $\gamma_i^0(q_0)$.

Let $g^n :]0, 1[\to FA : g^n(q) = \gamma_0^n(q)\nu_0^n + \sum_{i=1}^{\infty} \gamma_i^n(q)\nu_i$. Since we have shown that ν_0^n is norm convergent to ν_0^0 (or $\gamma_0^0 \equiv 0$) and that the $\gamma_i^n(q_n)$ are l_1-convergent to $\gamma_i^0(q_0)$, it follows that the $g^n(q_n)$ converge in norm to $g^0(q_0)$.

Therefore, if f^n converges to f^0 at every point of continuity of f^0, and has uniformly bounded variation which is uniformly small in the neighborhood of 0 and 1, $\int_0^1 g^n(q)df^n(q)$ will converge to $\int_0^1 g^0(q)df^0(q)$: we have shown that:

Proposition 3: At every point where μ has infinitely many players, value $(f(\mu))$ – as a mapping from $bv'([0, 1]) \times FA_+^1$ to FA – is jointly continuous in f and μ, when FA is endowed with the norm topology and $bv'([0, 1])$ is endowed with the ("Arens-")topology of uniform convergence on uniformly bounded equicontinuous subsets of $C(]0, 1[)$.

Remark 2 (Regular games): Let v be a monotonic simple game with countably many players. Coalitions being points of $\{0, 1\}^{\infty}$, v is a $\{0, 1\}$ valued monotonic function on $\{0, 1\}^{\infty}$. Assume first v to be measurable for any product measure on $\{0, 1\}^{\infty}$ (in order for the extension to be defined – this assumption has to be made explicitly: indeed, using the continuum hypothesis, it is possible to construct such v's such that the lower integral would be zero for the product of any sequences p_i with lim sup $p_i < 1$ and the upper integral would be 1 whenever lim inf $p_i > 0$: there is little hope to be able to define a meaningful value for such things). We will also denote by v its extension to $[0, 1]^{\infty}$ defined by letting $v(p_1, p_2, p_3, \ldots)$ be the expectation of v under the corresponding product measure. We assume v to be continuous in the product topology in a uniform neighborhood of the diagonal – to exclude such obviously non regular games as: $v(S) = 1$ iff $\liminf_{n\to\infty}$ [proportion of players belonging to S among the first n players] $\geqslant \frac{1}{2}$. Such a game is called regular (or non-singular, or proper) (cfr Shapiro and Shapley [6], at least for weighted majority games) if $\Sigma\, \pi_i = 1$, where $\pi_i = P(i \text{ pivots})$.

Remark now that $\pi_i = \int_0^1 P(i \text{ pivots arriving at } t)dt = \int_0^1 [v(t, t, t \ldots, t, 1, t, t \ldots) - v(t, t, t \ldots, t, 0, t, t \ldots)]dt = \int_0^1 \left(\frac{\partial v}{\partial p_i}\right)(t, t, t, \ldots)dt$ this last formula because v is obviously multilinear in any finite number of p_i's). The same multilinearity yields therefore that, for any sequence (δ_i) with finitely many non zero terms,

$$\lim_{\tau\to 0} \int_0^1 \frac{v(t\cdot 1 + \tau\delta) - v(t\cdot 1)}{\tau} dt = \int_0^1 \lim_{\tau\to 0} \frac{v(t\cdot 1 + \tau\delta) - v(t\cdot 1)}{\tau} dt$$

$$= \int_0^1 \Sigma\, \delta_i \left(\frac{\partial v}{\partial p_i}\right)(t, t, \ldots)dt = \sum_i \delta_i \pi_i.$$

Therefore for any non negative sequence $\delta_i\ (0 \leqslant \delta_i \leqslant 1)$ if δ_i^n denotes the same sequence with all but the first n terms set to zero we get

$$\liminf_{\tau\to 0} \int_0^1 \frac{v(t\cdot 1 + \tau\delta) - v(t\cdot 1)}{\tau} dt \geqslant \lim_{n\to\infty} \lim_{\tau\to 0} \frac{v(t\cdot 1 + \tau\delta^n) - v(t\cdot 1)}{\tau} dt$$

$$= \lim_{n\to\infty} \sum_i \delta_i^n \pi_i = \sum_i \delta_i \pi_i$$

By an argument we already made before this implies, when applied also to the sequence $1 - \delta_i$, that if $\Sigma\, \pi_i = 1$, then

$$\lim_{\tau\to 0} \int_0^1 \frac{v(t\cdot 1 + \tau\delta) - v(t\cdot 1)}{\tau} dt = \Sigma\, \delta_i \pi_i, \quad \text{i.e. } v \in Q$$

$$\text{with } \psi(v) = (\pi_i)$$

Thus, if v is regular (i.e. $\Sigma\, \pi_i = 1$), then $v \in Q$. Conservely, if $v \in Q$, then $\psi(v)$ is the limit of a sequence of continuous functions on $[-1, 1]^\infty$, so is continuous at at least one point of this space, which implies $\psi(v) \in l_1$ (this argument is essentially similar to an argument already made in "Values and Derivatives"): $\psi(v)$ is some summable sequence ψ_i. Since by our above argument one has $\psi_i = \pi_i$, and since efficiency yields $\Sigma\,\psi_i = 1$ we get $\Sigma\,\pi_i = 1$: a monotonic, simple game with countably many players is in Q if and only if regular.

Remark 3 (Polynomial games): A polynomial on a vector space E can be viewed as a continuous multilinear functional $F(x_1, \ldots, x_n)$ on E^n, evaluated along the diagonal $x_1 = \ldots = x_n = x$.

Let us thus therefore define *POL* to be the set of all games $v \in BV$ for which there exists a continuous multilinear function $F_v : [B(I,\ \mathcal{C})]^n \to \underset{\sim}{R}$ such that $v(S) = F_v(I_S, I_S, \ldots, I_S)\ \forall\, S \in \mathcal{C}$.

F_v can always be chosen to be symmetric, and is then uniquely determined by the game v. Indeed consider any finite measurable partition π, and the corresponding partition π^n on I^n. Obviously F_v induces a unique measure μ on π^n. Now the functions $I_S \times I_S \times \ldots \times I_S$ – with S π-measurable – form a set of continuous functions on π^n, stable under multiplication. And they separate any two points which are not obtained from each other by permutation of coordinates. Thus the linear space spanned by these functions is the game of all symmetric functions on π^n. Thus the value of μ on any symmetric functions on π^n is determined by v. If F_v is symmetric, this determines the value $F_v(\chi_1, \chi_2, \ldots, \chi_n)$ where the χ_i are π-measurable step functions. Thus F_v is uniquely determined by v on all step functions, and hence by continuity on $[B(I, \mathcal{C})]^n$.

Since F_v induces a measure μ on π^n for all finite measurable partitions π, one sees that F_v – and therefore v – also determines a unique symmetric finitely additive set function μ_v on the product algebra $\mathcal{C} \times \mathcal{C} \times \ldots \times \mathcal{C}$.

Define *POLb* to be the set of these games $v \in POL$, such that μ_v is of bounded variation. Thus *POLb* can be described as the set of all games of the form $v(S) = \mu(S \times S \times \ldots \times S)$, for some (symmetric) $\mu \in FA((I,\ \mathcal{C})^n)$. (I have no example of some game in $POL \setminus POLb$ – or better: in $POL \setminus$ closure $(POLb)$.) Then we have

Claim 1: POL and *POLb* are subalgebras of *BV*.

Proof: Let $v, w \in POL$ (or *POLb*). Then $\lambda v \in POL$ (resp. *POLb*) by setting $F_{\lambda v} = \lambda F_v$ (or $\mu_{\lambda v} = \lambda \mu_v$). Assume F_v is n-linear and F_w k-linear, with $n \geqslant k$. Set $F_{v+w}(\chi_1 \ldots \chi_n) = F_v(\chi_1 \ldots \chi_n) + F_w(\chi_1 \chi_{k+1} \chi_{k+2} \ldots \chi_n, \chi_2, \chi_3, \ldots, \chi_k)$ and symmetrize: this shows $v + w \in POL$.

Set $F_{vw}(\chi_1, \ldots, \chi_n, \chi_{n+1}, \ldots, \chi_{n+k}) = F_v(\chi_1 \ldots\ _n) F_w(\chi_{n+1} \ldots \chi_{n+k})$: $vw \in POL$.

The above formulas show that μ_{v+w} is obtained by transporting μ_w to $\bar{\mu}_w$ on $(I,\ \mathcal{C})^n$ by identifying the first factor of $(I,\ \mathcal{C})^k$ to the diagonal in $(I,\ \mathcal{C})^n$ of the factors $1, k+1, k+2, \ldots, n$; and then adding to this μ_v: clearly this yields still some μ of bounded variation. Similarly μ_{vw} can be taken as the product of μ_v and μ_w, and is therefore of bounded variation.

Note that such games have a direct economic interpretation: assume for instance that individuals are patent holders, and that any technology requires (at most) n different patents: then, if $\mu(ds_1 ds_2, \ldots, ds_n)$ is the production set achievable by coalition $(ds_1 \cup ds_2 \cup \ldots \cup ds_n)$, the total production achievable by coalition S is $v(S) = \mu(S \times S \times \ldots \times S)$.

The continuity of F_v for $v \in POL$ implies immediately that, if $\bar{\mu}_v$ denotes the marginal distribution on $(I, \mathcal{C})$ generated by μ_v (this is, independent of the coordinate due to the symmetry of F_v), then $\bar{\mu}_v$ has bounded variation – i.e., $\bar{\mu}_v \in FA$.

It is easy to give examples of games in *POLb* that are not in *pNA* – not even in *pNA'* –, even when choosing for μ_v a countably additive, non atomic measure (with $\bar{\mu}_v \in NA$).

I don't know about the validity of the following proposition for the whole of *POL*, if is not contained in the closure of *POLb*.

Claim 2: POLb $\subseteq Q$.

Proof: We can without loss assume the measure μ on $(I, \mathcal{C})^n$ to be positive. Fix a step function χ in $B(I, \mathcal{C})$, $0 \leqslant \chi \leqslant 1$, and take any finite measurable partition $\pi = (A_1, \ldots, A_k)$ with respect to which χ is measurable. Denote by p_i the value of χ on A_i, and by $\tilde{S}$ the random coalition obtained by $A_i \subseteq \tilde{S}$ with probability p_i independently for all i. Then

$$Ev(\tilde{S}) = \sum_{j \in k^n} \mu(S_{j_1} \times S_{j_2} \times \ldots \times S_{j_n}) \text{ Prob } (S_{j_r} \subseteq \tilde{S} \ \forall\, r)$$

$$= \sum_{j \in k^n} \mu(S_{j_1} \times S_{j_2} \times \ldots \times S_{j_n}) \prod_{i=1}^{k} p_i^{k_{ij}}$$

when $k_{ij} = I\{i \in \{j_1, \ldots, j_n\}\}$.

One checks immediately that, when π is refined, $\prod_{i=1}^{k} p_i^{k_{ij}}$ decreases, thus $Ev(\tilde{S})$ also.

For every partition P of $\{1, \ldots, n\}$, let $X_P^\pi = U\{S_{j_1} \times \ldots \times S_{j_n} \mid S_{j_r} = S_{j_s} \equiv r$ and s are in the same element of $P\}$. Let μ_P^π denote the restriction of μ to X_P^π, and let for $p \in P$, s_p vary in the diagonal of all $i \in p$.

In these notations, we get then

$$Ev(\tilde{S}) = \sum_P \int \Big(\prod_{p \in P} \chi(s_p) \Big) \mu_P^\pi(ds_1 \ldots ds_p \ldots ds_{|P|}).$$

Since $Ev(\tilde{S})$ decreases when π is refined, we may denote its limit by $\bar{v}(\chi)$. Denoting by μ_P a limit point of the μ_P^π when π is refined, we get

$$\bar{v}(\chi) = \sum_P \int (\prod_{p \in P} \chi(s_p)) d\mu_P \text{ for all step functions } \chi.$$

If one wants, this extends now by uniform continuity to all ideal set functions χ. Thus there only remains to show that any $\bar{v}$ of the form $\bar{v}(\chi) = \int_{I_n} \chi(s) \dots \chi(s_n) d\mu(s_1 \dots s_n)$ is in Dom ψ_3, with $\psi_3(\bar{v}) = \bar{\mu}$ the marginal of μ.

Obviously $\bar{v}$ is continuous at zero and at one, so let us compute $\psi_3(\bar{v})$ using the same formula for any $\chi \in B(I, C)$.

$$\begin{aligned}
\psi_3(\bar{v})(\chi) &= \int_0^1 \frac{\bar{v}(t + \tau\chi) - \bar{v}(t)}{\tau} dt \\
&= \frac{1}{\tau} \int_0^1 \int_{I^n} \{[t + \tau\chi(s_1)], \dots, [t + \tau\chi(s_n)] - t^n\} d\mu(s_1 \dots s_n) dt \\
&= \int_0^1 \int_{I^n} t^{n-1} [\chi(s_1) + \dots + \chi(s_n)] d\mu(s_1 \dots s_n) dt + 0(\tau) \\
&= \frac{1}{n} [\int \chi(s_1) d\bar{\mu}(s_1) + \int \chi(s_2) d\bar{\mu}(s_2) + \dots] = \bar{\mu}(\chi).
\end{aligned}$$

Section 2

In Section 1 we have shown how to reduce the problem of defining a value to the problem of defining a positive, symmetric linear operator (of norm 1) ψ to FA from a (closed, symmetric) space V of functions $v : B(I, C) \to R$ that satisfy $v(a + b\chi) = av(1) + bv(\chi)\ \forall a, b \in R\ \forall \chi \in B(I, C)$.

We have also seen that, for such functions v, one has

$$V(v)[\chi, \chi'] \leq \|\chi' - \chi\| \cdot \|v\| \qquad \forall \chi, \chi' \in B(I, C).$$

Therefore, if we let $D_\chi^\lambda(\tilde{\chi}) = [v(\tilde{\chi} + \lambda\chi) + v(\tilde{\chi} - \lambda\chi)]/2$, we get

$$V(D_\chi^\lambda)[\tilde{\chi}, \tilde{\chi}'] \leq \|\tilde{\chi}' - \tilde{\chi}\| \cdot \|v\|$$

and

$$D^{\lambda}_{a+b\chi}(c+d\tilde{\chi}) = cv(1) + dD^{|b\lambda/d|}_{\chi}(\tilde{\chi}) \quad \text{(in particular } D^{\lambda}_{\chi}(1) = v(1)\text{)}.$$

Let $D_\chi(\tilde{\chi})$ stand for $\lim_{\lambda\to\pm\infty} D^{\lambda}_{\chi}(\tilde{\chi})$ (if this limit does not necessarily exist, use any Banach limit; remark that $D^{\lambda}_{\chi}(\tilde{\chi})$ is necessarily an even function of λ). We get then

$$V(D_\chi)[\tilde{\chi}, \tilde{\chi}'] \leqslant \|\tilde{\chi}' - \tilde{\chi}\| \cdot \|v\|$$

and

$$D_{a+b\chi}(c+d\tilde{\chi}) = cv(1) + dD_\chi(\tilde{\chi}).$$

Thus, $\forall\ \chi$, $D_\chi(\cdot)$ is linear on every plane containing the constants, and satisfies $D_\chi(1) = v(1)$ and $\|D_\chi\| \leqslant \|v\|$.

In addition, the mapping $\chi \to D_\chi$ is constant on every plane containing the constants, and the mapping $v \to D_\chi$ is linear, positive and of norm 1.

$D_\chi(\tilde{\chi})$ is the (two-sided) derivative of v at χ in the direction of $\tilde{\chi}$:

$$\lim_{\tau\to 0} \frac{v(\chi + \tau\tilde{\chi}) - v(\chi - \tau\tilde{\chi})}{2\tau}.$$

We think of the typical situation where D_χ would already be in FA for "almost every" χ: for an average of the D_χ then to be a value, one only has to make sure to get the symmetry: the average should be computed with a (finitely additive) probability distribution on $B(I, \mathcal{C})$ that is invariant under all automorphisms of $(I, \mathcal{C})$.

The averaging should be well defined whenever v is a function of a vector measure, so for any vector measure $\mu = (\mu_1 \ldots \mu_n)$ and for any borel set B in R^n, $\mu^{-1}(B) = \{\chi \mid \mu(\chi) \in B\}$ should be measurable: this class of sets in $B(I, \mathcal{C})$ is the algebra of cylinder sets. Thus we look for a "cylinder probability" on $B(I, \mathcal{C})$, i.e. a finitely additive measure P on the cylinder sets, such that, for any vector measure $\mu = (\mu_1 \ldots \mu_n)$ the induced measure $P \circ \mu^{-1}$ is a (countably additive) probability on the borel sets of R^n.

Similarly, one may define the "conical sets" as being those cylinder sets C such that $\chi \in C \equiv a + b\chi \in C\ \forall\ a, b \in \underset{\sim}{R}$. One may then define a "conical probability" as a finite additive measure on the conical sets, such that any vector measure $\mu = (\mu_0, \ldots, \mu_n)$, with $\mu_i \in NA$, $\mu_i \in NA$, $\mu_i(I) = 0$, induces a countably additive distribution on n-dimensional projective space.

Recall that any cylinder probability on $B(I, C)$ is uniquely characterized by its Fourier transform, a function on the dual defined by

$$F(\mu) = E \exp i\langle \mu, \chi\rangle.$$

In the next theorem we use the classical concept of invariance (i.e. under all automorphisms of (I, C)); accordingly (I, C) is here required to be a standard borel space (i.e. isomorphic to $[0, 1]$ with the borel sets) and the duality used is that of $B(I, C)$ with the space NA non atomic, countably additive measures on (I, C).

Theorem 1:

A) The extreme points of the sets of invariant cylinder probabilities on $B(I, C)$ have Fourier transforms $F_{m,\sigma}(\mu) = \exp(im\mu(1) - \sigma \|\mu\|)$ where $m \in \underset{\sim}{R}, \sigma \geqslant 0$. More precisely, the formula $E \exp i\langle\mu, \chi\rangle = \int_{\underset{\sim}{R}\times\underset{\sim}{R}_+} F_{m,\sigma}(\mu) dP(m, \sigma)$ establishes a one to one correspondence between invariant cylinder measures [3] and (countably additive) measures P over $\underset{\sim}{R} \times \underset{\sim}{R}_+$. This correspondence is a positive, linear, convolution preserving isometry.

B) There exists only one invariant conical probability on $B(I, C)$, which is the restriction to the conical sets of any invariant cylinder measure of total mass 1 as described sub A).

Proof:

A) Consider first cylinder probabilities. Let μ_i denote a sequence of mutually singular non atomic probabilities. There exists a partition of (I, C) into a sequence of borel sets B_i, such that μ_i is carried by B_i – which has therefore the power of the continuum. Thus, for any permutation π of the integers, there exists an automorphism θ_π of (I, C) such that θ_π maps the sequence $(\mu_i)_{i=1}^\infty$ to the sequence $(\mu_{\pi(i)})_{i=1}^\infty$.

The sequence μ_i maps $B(I, C)$ to $\underset{\sim}{R}^\infty$, and the cylinder measure induces therefore a consistent system of probabilities on the borel sets of the $\prod_{i=1}^n (\underset{\sim}{R})$, and thus a (countably additive) probability Q on the borel sets of $\underset{\sim}{R}^\infty$. The invariance of the cylinder measure under θ_π implies then the invariance of this probability under any permutation π: the coordinates of $\underset{\sim}{R}^\infty$ are exchangeable under Q.

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3 i.e., for any P in the right hand member, there exists a unique cylinder measure with this Fourier transform, and this cylinder measure is invariant.

Thus, by the Finetti's theorem, if we denote by A the asymptotic σ-field on $\underset{\sim}{R}^\infty$, the random variables μ_i are i.i.d. conditionally to A, say with distribution F. The mapping from $\underset{\sim}{R}^\infty$ to the set $M(\underset{\sim}{R})$ of probabilities on $\underset{\sim}{R}$ that maps any sequence to it's distribution (if this exists – which has Q-probability one by the Glivenko-Cantelli and de Finetti theorems) is A-measurable, so Q induces a probability P on $M(\underset{\sim}{R})$, such that Q is the distribution of a sequence $F^{-1}(x_i)$ where F is selected according to P and the x_i are selected, independently of F and of each other, uniformly on $]0, 1[$. It follows in particular that any subsequence of the μ_i's would induce the same probability P on $M(\underset{\sim}{R})$.

Let now μ_i' be another such sequence; then there exists an uncountable borel set B in (I, C) which is negligible for all μ_i's and all μ_i''s: one can construct on B a third such sequence $\tilde{\mu}_i$. When the μ_i's and the $\tilde{\mu}_i$'s are arranged in sequence, they fulfill the requirements set out at the start of the proof, so the probability P on $M(R)$ induced by the two subsequences μ_i and $\tilde{\mu}_i$ is the same. The same would apply to the two sequences μ_i' and $\tilde{\mu}_i$, so it follows that P is independent of the particular sequence μ_i chosen, but depends only on the cylinder measure.

Since $\bar{\mu}_n = \frac{1}{n}\sum_{i=1}^{n} \mu_i$ is such that the sequence $(\bar{\mu}_n, \mu_{n+1}, \ldots)$ satisfies our requirements, and has the same asymptotic σ-field A as the original sequence, it follows that, for P-almost every F, $\mu_1 \ldots \mu_n$ are independently F-distributed and $\bar{\mu}_n = \frac{1}{n}\sum_{i\leq n} \mu_i$ is also F distributed. Thus P-almost every F is such that, for all n, the average of n independent F-distributed random variables is F-distributed, i.e. F is strictly stable of index 1. For univariate random variables, this is equivalent to say F is a Cauchy distribution.

If we parametrize the Cauchy distributions by their location and scale parameters m and σ, P becomes a probability distribution on $\underset{\sim}{R} \times \underset{\sim}{R}_+$ such that, for any sequence μ_i of mutually singular non atomic measures, the sequence $\mu_i(\chi)$ is distributed as the average under $P(dm, d\sigma)$ of the distribution of $(\|\mu_i^+\| \cdot X_i - \|\mu_i^-\| \cdot Y^i)_{i=1}^{\infty}$, where the X_i and Y_i are all independently distributed as $m + \sigma U$, where U is a standard Cauchy random variable.

Thus $\|\mu_i^+\| \cdot X_i - \|\mu_i^-\| \cdot Y_i$ is distributed like $m \cdot \mu_i(1) + \sigma \cdot \|\mu_i\| \cdot U_i$, where U_i is a standard Cauchy random variable.

In particular, $E[\exp(i\langle\mu,\chi\rangle) | m, \sigma] = \exp[-\sigma\|\mu\| + im\langle\mu, 1\rangle]$ $\forall$ $\mu \in NA$, and $E\exp(i\langle\mu,\chi\rangle) = \int_{R\times\underset{\sim}{R}_+} \exp[-\sigma\|\mu\| + im\langle\mu, 1\rangle]\,dP(m, \sigma)$.

It is clear from the above proof – or from the last formula and the uniqueness theorems for Fourier and Laplace transforms – that for any cylinder measure there can exist only one P such that the above formula holds.

Let us now show that for any such P there exists a (unique) cylinder measure with that Fourier transform.

Unicity follows immediately from the fact that distribution over finite dimensional spaces are uniquely characterized by their Fourier transform, and from the fact that all cylinder sets are finite dimensional. To show existence, recall that Bochner's theorem characterizes the characteristic functions on $\underset{\sim}{R}^n$ as the positive definite functions φ that are continuous at zero with $\varphi(0) = 1$. This immediately extends itself to Fourier transforms of cylinder probabilities (when continuity at zero is interpreted as continuity at zero of the restrictions to all finite dimensional subspaces of the dual).

Indeed every inequality for positive definiteness involves only finitely many points in the dual, so the condition is still necessary, and if it holds, we get by Bochner's theorem a consistent system of probability distributions on all finite dimensional quotient spaces of $B(I, \mathcal{C})$, i.e a cylinder probability.

Now our formula obviously has values 1 at zero, and is continuous there by the dominated convergence theorem. Thus we only have to show that it is positive definite. For this it is sufficient to show that, for every m, σ the function $\exp(-\sigma\|\mu\| + im\langle\mu, 1\rangle)$ is positive definite, the inequalities being linear.

To show this, it is sufficient to show that this function is the pointwise limit of a set of positive definite functions φ_α, since the inequalities each involve only a finite number of points in the dual and are weak inequalities.

For every borel partition $\alpha = (B_1^\alpha \ldots B_n^\alpha)$ of $(I, \mathcal{C})$, let $X_1(\omega) \ldots X_n(\omega)$ be independent Cauchy random variables with parameters m and σ, and let $f(\omega) \in B(I, \mathcal{C})$ have value $X_i(\omega)$ on B_i^α : $f(\omega)$ is a random variable with values in (a finite dimensional subspace of) $B(I, \mathcal{C})$, Thus by Bochner's theorem it's characteristic function φ_α will be positive definite. We have $\varphi_\alpha(\mu) = E \exp(i\langle\mu, \chi\rangle) = E \exp(i\langle\mu, f(\omega)\rangle) = E \exp i \sum_{j=1}^{n} \mu(B_j^\alpha)X_j(\omega)$.

NowNow $\sum_{j=1} \mu(B_j^\alpha)X_j(\omega)$ is Cauchy with parameters $m \sum \mu(B_j^\alpha)$ and $\sigma \sum |\mu(B_j^\alpha)|$, i.e. $m \cdot \langle\mu, 1\rangle$ and $\sigma\|\mu\|_\alpha$, where $\|\mu\|_\alpha$ is the norm of the restriction of μ to the (σ-field generated by the) partition α.

Thus $\varphi_\alpha(\mu) = \exp[-\sigma\|\mu\|_\alpha + im\langle\mu, 1\rangle]$ is positive definite, and obviously $\|\mu\|_\alpha \to \|\mu\|$ when α ranges over the increasing net of all partitions.

This proves that φ is positive definite, and thereby establishes the one to one character of this correspondence, when restricted to probabilities on both sides. (Obviously the cylinder probability has to be invariant, since its Fourier transform is so.)

It is now clear that, for any bounded measure P, there exists a corresponding invariant cylinder measure: let $P = \alpha P_1 - \beta P_2$, when P_1 and P_2 are two probabilities, $\alpha \geqslant 0$, $\beta \geqslant 0$, and use $\alpha Q_1 - \beta Q_2$ as invariant cylinder measure, where Q_i is the cylinder probability corresponding to P_i. Furthermore this cylinder measure is unique – if there were two of them, their difference would be a cylinder measure with zero Fourier transform, so the positive and negative parts of this difference would be two different positive cylinder measures with the same Fourier transfrom – and in particular with the same total mass (value of the Fourier transform at zero), so that by normalizing one would obtain two different cylinder probabilities with the same Fourier transform, in contradiction with what we have seen above.

We have just used the fact that the positive part λ^+ of a (bounded) cylinder measure λ is still a cylinder measure. Indeed, if $\mathcal{A}$ denotes the algebra of cylinder sets, λ^+ is defined by $\lambda^+(A) = \sup_{B \in \mathcal{A}} \lambda(A \cap B)\ \forall A \in \mathcal{A}$. One sees immediately that λ^+ is finitely additive, positive and bounded on $\mathcal{A}$, with $\lambda^+ \geqslant \lambda$. To show that λ^+ is still a cylinder measure, let $\mathcal{A}_\varphi = \{\varphi^{-1}(B) | B \text{ borel set in } R^n\}$ for φ ranging over all finite subsets $\{\varphi_1 \dots \varphi_n\}$ of NA. Then $\lambda^+ = \sup_\varphi \lambda_\varphi^+$, with $\lambda_\varphi^+(A) = \sup_{B \in \mathcal{A}_\varphi} \lambda(A \cap B)$. It is thus sufficient to show that, $\forall \varphi_0, \forall \varphi : \varphi \geqslant \varphi_0, \lambda_\varphi^+$ is countably additive on $\mathcal{A}_{\varphi_0}$ – (the supremum of a bounded, increasing net of countably additive measures is still such) or that, $\forall \varphi$, λ_φ^+ is countably additive on $\mathcal{A}_\varphi$: this is the Hahn decomposition theorem for countably additive measures.

Obviously, if further λ was invariant, λ^+ will also be: therefore, we can, in the same way as above for P, construct for any invariant cylinder measure λ a corresponding measure P on $\underset{\sim}{R} \times \underset{\sim}{R}_+$. Again, this P is unique, otherwise one could construct, as above, two different probabilities P_1 and P_2 with the same value of the integral $\int F_{m,\sigma}(\mu) dP(m, \sigma)$, contradicting our previous result for probabilities.

Thus the bijectivity of the correspondence is established. Its positivity was already established before, when dealing with probabilities, and it's linearity is now immediately obvious from the bijectivity – an integral is a linear function of the underlying measures. Being poisitive and linear, it is a isometry because it maps both ways probabilities to probabilities.

The assertion about extreme points is now immediate, so there only remains to establish the preservation of convolution.

Since a linear mapping from R^n to R^k maps the convolution of two measures to the convolutionof their images, it is clear that the convolution of two cylinder measures is a well defined cylinder measure, with the Fourier transform of the convolution being the product of the Fourier transforms of the individual measures. In particular, if the two measures were invariant, the convolution will still be. Similarly one checks immediately that the integral in the right hand side under the convolution of two measures P_1 and P_2 is the product of the corresponding integrals. This finishes the proof of A).

B) Choose as sub A) a sequence (μ_i) of mutually singular non atomic probabilities. Remark that $\mu_i - \mu_j \neq 0$ with probability one, otherwise it would not induce a distribution on (zero-dimensional) projective space.

Hence the $f_i(\chi) = \dfrac{\mu_i - \mu_1}{\mu_3 - \mu_2}$ $(i \geqslant 4)$ are a sequence of conically-measurable functions, and therefore they have, as in A), a countably additive distribution $P_{\mu_1, \mu_2, \dots}$ on $\underset{\sim}{R}^\infty$ (by Kolmogorov's theorem).

Also the f_i ($i \geqslant 4$) are exchangeable, hence we get – as in A) – a distribution P_{μ_1,μ_2,μ_3} on $\underset{\sim}{R} \times \underset{\sim}{R}_+$ such that any such sequence with the same (μ_1, μ_2, μ_3) is distributed like $m + \sigma U_i$, where U_i is a sequence of independent Cauchy (0, 1) variables, and (m, σ) is selected according to P_{μ_1,μ_2,μ_3}.

Let

$$g_i = \frac{f_i - f_4}{f_6 - f_5} = \frac{\mu_i - \mu_4}{\mu_6 - \mu_5} = \frac{m + \sigma U_i - (m + \sigma U_4)}{m + \sigma U_6 - (m + \sigma U_5)} = \frac{U_i - U_4}{U_6 - U_5} \; (i \geqslant 7).$$

They are distributed according to P_{μ_4,μ_5,μ_6}, and therefore for any 3 mutually singular non atomic probabilities (μ_4, μ_5, μ_6) P_{μ_4,μ_5,μ_6} induces for any admissible sequence μ_i the same distribution as if the μ_i's were independently distributed as Cauchy zero-one variables. In particular, $P_{\mu_1,\mu_2,\mu_3} = P$ does not depend on (μ_1, μ_2, μ_3).

Thus any invariant conical probability induces the same distribution as induced by the invariant cylinder probabilities of A) on any sequence (μ_i) of mutually singular non atomic probabilities $\left(\text{i.e., on their rations } \frac{\mu_i - \mu_j}{\mu_k - \mu_l}\right)$.

Now any conical set is determined by the ratios of such a finite sequence – where the μ_i's are not necessarily non singular – but can be taken as linearly independent.

Thus we need the distribution of $\left(\frac{\mu_i}{v}\right)_{i=1}^{n}$, where all measures are non atomic and of total mass zero, the μ_i's are linearly independent and v may in addition be chosen to be singular with respect to all μ_i's.

This distribution is fully determined (Fourier transform) by the distributions of all linear combinations $\sum_{i=1}^{n} t_i \frac{\mu_i}{v}$: we need only to know the distribution of $\frac{\mu}{v}$, where μ and v are two non zero mutually singular non atomic measures of total mass zero – and by normalisation we may assume $\|\mu\| = \|v\| = 2$. This is the known distribution of $\frac{\mu^+ - \mu^-}{v^+ - v^-}$, where (μ^+, μ^-, v^+, v^-) are 4 mutually singular non atomic probabilities. □

Denote by Q the closed, symmetric space generated by FA and all functions v satisfying $v(a + b\chi) = av(1) + bv(\chi)$, $\|v\| < \infty$ that are of the form $v(\chi) = f(\mu(\chi))$, where μ is a vector measure in NA.

Theorem 2: Let $v \in Q$, and let P be any invariant cylinder measure of total mass 1 on $B(I, C)$ which is non degenerate – i.e. the subspace of constant functions has probability zero, or: Prob $(\sigma = 0) = 0$ –. Then $D_\chi(\tilde{\chi})$ exists, for every $\tilde{\chi}$, for P almost every χ

(i.e. the difference $\sup_{\lambda \geq \lambda_0} D_\chi^\lambda(\tilde{\chi}) - \inf_{\lambda \geq \lambda_0} D_\chi^\lambda(\tilde{\chi})$ converges to zero in $L_1(dP(\chi))$ when $\lambda_0 \to \infty$) and is, as well as any $D_\chi^\lambda(\tilde{\chi})$, P-integrable in χ, and $\phi_v(\tilde{\chi}) = \int D_\chi(\tilde{\chi})dP(\chi) = \lim_{\lambda \to \infty} \int D_\chi^\lambda(\tilde{\chi})dP(\chi)$ is independent of the particular invariant P chosen.

Further $\phi_v \in FA$, so that the mapping $v \to \phi_v$ is positive, linear, symmetric, of norm 1, and satisfying $\phi_v(1) = v(1)$: $\phi : v \to \phi_v$ is a value on Q.

Proof: Since the mapping $v \to D_\chi^\lambda$ is positive, linear, of norm 1 and satisfies $D_\chi^\lambda(1) = v(1)$, the last sentence of the statement will follow from the other provided we prove the additivity of ϕ_v.

It also follows that it is sufficient to prove the statement on the generators of the space, since a uniform limit of P-integrable functions is P-integrable, with the integral being continuous along the sequence.

Finally, since D_χ^λ acts as the identity on FA, and since constant functions are P-integrable, it is sufficient to consider the generators of the form $v = f(\mu)$, with $\mu = (\mu_1 \ldots \mu_n)$ a vector measure in NA.

Also, since, by Theorem 1, P can be written as $\alpha P_1 - \beta P_2$, where the P_i are invariant cylinder probabilities and $\alpha - \beta = 1$, it is sufficient to consider the case where P is a cylinder probability.

There is no loss in assuming that μ has full dimensional range – otherwise one of the components of μ is a linear combination of the others, so v can be written only as a function of the other components.

Denote by B_μ the image under μ of the unit ball of $B(I, C)$ – i.e. $B_\mu = 2(\text{Range of } \mu) - \mu(1)$. B_μ being compact, convex, symmetric around zero, and full dimensional, it is a neighborhood of zero (by the absorption theorem say). The relation $v(a + b\chi) = av(1) + bv(\chi)$ implies now $f(a \cdot e + bx) = af(e) + bf(x)$, $\forall\, x \in R^n$, $\forall\, a, b \in R$, where $e = \mu(1) \in R^n$.

Finally, the relation $V(v)[\chi, \chi'] \leq \|\chi' - \chi\| \cdot \|v\|$ implies that, if $x = \mu(\tilde{\chi})$, and $y - x \in \delta B_\mu$, then $\exists\, \bar{\chi}$ with $\|\bar{\chi} - \tilde{\chi}\| \leq \delta$ and $y = \mu(\bar{\chi})$ so that

$$|f(y) - f(x)| = |v(\bar{\chi}) - v(\tilde{\chi})| \leq |v(\bar{\chi}) - v(\tilde{\chi} - \delta \cdot 1)| + |v(\tilde{\chi}) - v(\tilde{\chi} - \delta \cdot 1)|$$

$$\leq V(v)[\tilde{\chi} - \delta \cdot 1, \tilde{\chi} + \delta \cdot 1] + V(v)[\tilde{\chi} - \delta \cdot 1, \tilde{\chi}] \leq 3\delta \|v\|.$$

B_μ being a neighborhood of zero, $\exists\, \epsilon \geq 0 : \|x\| \leq \epsilon \Rightarrow x \in B_\mu$, and thus we have shown that $\|y - x\| \leq \epsilon\delta \Rightarrow |f(y) - f(x)| \leq 3\delta \|v\|$ for all δ, y and x: thus $|f(y) - f(x)| \leq \frac{3\|v\|}{\epsilon} \cdot \|y - x\|$: f is Lipschitz.

Conversely if f is Lipschitz it follows immediately that $\|v\| < \infty$, so our assumptions reduce simply to $v = f(\mu)$, where μ is a vector in NA with full n-dimensional range and where f is Lipschitz satisfying $f(a \cdot e + bx) = af(e) + bf(x)$ where $e = \mu(1)$, $\forall x \in R^n$, $\forall a, b \in \underset{\sim}{R}$.

We have $D_\chi^\lambda(\tilde{\chi}) = [f(x + \lambda y) + f(x - \lambda y)]/2$, where $x = \mu(\tilde{\chi}), y = \mu(\chi)$ or $D_\chi^{\tau^{-1}}(\tilde{\chi}) = [f(y + \tau x) - f(y - \tau x)]/(2\tau)$.

Remark that, f being Lipschitz, the limit (for $\tau \to 0$) will, for each x, exist λ-almost everywhere in y, λ being Lebesgue measure. This follows from Lebesgue's a.e. differentiability theorem. Indeed, if x is zero, there is nothing to prove, otherwise x can be taken as the first basis vector in R^n: for any $z_2 \ldots z_n, f(z, z_2, \ldots, z_n)$ is a Lipschitz function of z, so the first partial derivative exists for almost every z, by Lebesgue's theorem. Since f is Lipschitz on R^n, the set of points where the first partial derivative does not exist is a borel set, and therefore this set of points has Lebesgue measure zero by Fubini's theorem.

The probability induced by P on R^n has characteristic function $\varphi_\mu(t) = E \exp i\langle t, x\rangle = E \exp i\langle t, \mu(\chi)\rangle = E \exp i(\langle t, \mu\rangle(\chi)) = \int_{\underset{\sim}{R} \times \underset{\sim}{R}_+} \exp [-\sigma \|\langle t, \mu\rangle\| + im(\langle t, \mu\rangle(1))] d\tilde{P}(m, \sigma)$ for some probability $\tilde{P}$.

Now $\|\langle t, \mu\rangle\| = \sup_{\|\chi\| \leq 1} \langle t, \mu(\chi)\rangle = \sup_{x \in B_\mu} \langle t, x\rangle = N_\mu(t)$ where N_μ is the norm on the dual generated by the ball B_μ.

And $\langle t, \mu\rangle(1) = \langle t, e\rangle$. Thus

$$\varphi_\mu(t) = \int_{\underset{\sim}{R} \times \underset{\sim}{R}_+} \exp [-\sigma N_\mu(t) + im\langle t, e\rangle] d\tilde{P}(m, \sigma).$$

Now obviously, for any given m and σ (>0), $\exp [-\sigma N_\mu(t) + im\langle t, e\rangle]$ is Lebesgue integrable in t, so the corresponding probability distribution has, by the Fourier inversion theorem, a density with respect to Lebesgue measure. Since the conditional distribution on R^n given m and σ is absolutely continuous, the unconditional distribution is also certainly so.

Thus we may conclude that, for any invariant P, and for any $\tilde{\chi}$, the limit $D_\chi(\tilde{\chi})$ will exist for P almost every χ.

Thus, for any P, and any x, $[f(y + \tau x) - f(y - \tau x)]/(2\tau)$ is uniformly bounded (f being Lipschitz) and converges P a.e. to it's limit: by the dominated convergence theorem, the limit is P-integrable and the limit of the integrals is the integral of the limit function $\varphi_x(y)$.

Now $f(a \cdot e + by) = af(e) + bf(y)$ yields $\varphi_x(a \cdot e + by) = \lim_{\lambda \to \infty} [f(x + \lambda(a \cdot e + b \cdot y)) + f(x - \lambda(a \cdot e + b \cdot y))]/2 = \lim_{\lambda \to \infty} [f(x + \lambda by) + f(x - \lambda by)]/2 = \varphi_x(y)$ if $b \neq 0$ (and $= f(x)$ if $b = 0$).

Let Z denote a random variable having characteristic function $\exp -N_\mu(t)$. Then $m \cdot e + \sigma Z$ where (m, σ) is selected, independently of Z, according to $\tilde{P}(m, \sigma)$, has the correct characteristic function $\int_{\underset{\sim}{R} \times \underset{\sim}{R}_+} \exp [-\sigma N_\mu(t) + im\langle t, e\rangle] d\tilde{P}(m, \sigma)$. Thus

$\int \varphi_x(y)dP(y) = E[\varphi_x(m \cdot e + \sigma Z)] = E\varphi_x(Z)$ since $\tilde{P}(\sigma = 0) = 0$: the integral of the limit – which is the limit of the integrals does not depend on the choice of $\tilde{P}$, i.e. on the choice of a particular invariant cylinder probability.

There only remains to establish the additivity, i.e. that $E\varphi_x(Z)$ is a linear function of x.

Let $f_\epsilon(x) = f(x) \exp(-\epsilon \|x\|^2)$ $(\epsilon > 0)$ (here $\|\cdot\|$ is the euclidean norm).

We want to show that the f_ϵ are uniformly Lipschitz (i.e. with a Lipschitz constant independent of ϵ).

Since f is Lipschitz, each of them is obviously locally Lipschitz, so by the above mentioned theorem of Lebesgue, it will be sufficient to show that the directional derivatives of the f_ϵ are uniformly bounded whenever they exist.

By choosing appropriate axes, we can assume our directional derivative is in the direction of the x_1 axis.

We have

$$\frac{\partial f_\epsilon}{\partial x_1} = \left(\frac{\partial f}{\partial x_1}\right) \exp[-\epsilon \|x\|^2] - 2\epsilon x_1 f \exp[-\epsilon \|x\|^2].$$

If K is the Lipschitz constant of f, then $\left|\frac{\partial f}{\partial x_1}\right| \leqslant K$ and $|f| \leqslant K\|x\|$ – bounding also $|x_1|$ by $\|x\|$, we get

$$\left|\frac{\partial f_\epsilon}{\partial x_1}\right| \leqslant K \exp -\epsilon \|x\|^2 + 2K(\epsilon \|x\|^2) \exp -(\epsilon \|x\|^2) \leqslant 2K$$

$$\text{since } e^{-z} + 2ze^{-z} \leqslant 2.$$

Thus the f_ϵ have uniformly the Lipschitz constant $2K$. Further the formula shows that, whenever the directional derivative of f exists, the corresponding directional derivatives of the f_ϵ will also exist and converge to that of f when $\epsilon \to 0$.

Thus $\varphi_x(y) = \lim_{\epsilon \to 0} \lim_{\tau \to 0} [f_\epsilon(y + \tau x) - f_\epsilon(y - \tau x)]/(2\tau)$, all functions involved being $\leqslant 2K\|x\|$ in absolute value. Thus, by the dominated convergence theorem,

$$E\varphi_x(Z) = \lim_{\epsilon \to 0} \lim_{\tau \to 0} \frac{1}{2\tau} \int [f_\epsilon(z + \tau x) - f_\epsilon(z - \tau x)] g_\mu(z) dz$$

where g_μ is the density of Z (which we have already shown to exist).

But since f_ϵ is a bounded function, it is integrable, so

$$\int [f_\epsilon(z+\tau x)-f_\epsilon(z-\tau x)]g_\mu(z)dz = \int f_\epsilon(z)g_\mu(z-\tau x)dz - \int f_\epsilon(z)g_\mu(z+\tau x)dz$$

$$= \int f_\epsilon(z)[g_\mu(z-\tau x)-g_\mu(z+\tau x)]dz.$$

Now g_μ has characteristic function $\exp -N_\mu(t)$, and $\|t\|\exp(-N_\mu(t))$ is integrable for Lebesgue measure. Therefore, by the Riemann-Lebesgue theorem, g_μ is continuously differentiable with its gradient going to zero at infinity. In particular the $[g_\mu(z-\tau x) - g_\mu(z+\tau x)]/2\tau$ are uniformly bounded and converge pointwise to $\langle -(\nabla g_\mu)(z), x\rangle$, where $(\nabla g_\mu)(z)$ is the gradient of g_μ at z.

Since $f_\epsilon(z)$ is integrable, it follows (dominated convergence) that $\lim_{\tau\to 0}\frac{1}{2\tau}\int f_\epsilon(z)[g_\mu(z-\tau x)-g_\mu(z+\tau x)]dz = \int f_\epsilon(z)\langle -\nabla g_\mu(z), x\rangle dz$ and thus

$$E\varphi_x(z) = \lim_{\epsilon\to 0}\int e^{-\epsilon\|z\|^2} f(z)\langle -(\nabla g_\mu)(z), x\rangle dz,$$

$$= -\langle x, \lim_{\epsilon\to 0}\int e^{-\epsilon\|z\|^2} f(z)\nabla g_\mu(z)dz\rangle$$

which is linear in x (the limit being some form of Cauchy principal value of $\int f(z)\nabla g_\mu(z)dz$).

This finishes the proof. □

Remarks:

1) One can use the same formula ($\exp - \|\mu\|$) to define the Fourier transform on the whole of FA, thereby defining a cylinder probability on $B(I, C)$ when cylinder sets are defined as inverse images of borel sets by any vector $(\mu_1 \ldots \mu_n)$ in FA. This cylinder probability would obviously be even more invariant-treating all elements of FA symmetrically.

Theorem 2 remains then valid when μ in the definition of Q is allowed to be any vector in FA – provided one interprets "invariant P" as "P having the Fourier transform prescribed by Theorem 1".

It might be that this formula could be justified by some type of uniqueness argument on the space of non atomic elements of FA – using maybe a weaker concept of automorphism. But certainly for the atomic part no such argument could be hoped for.

However, as our analysis of regular games with countably many players at the end of Section 1 may indicate, it could be that in general the "atomic part of the game" is already essentially linearized by the first derivative operation, so that the end result would anyway be canonical. This certainly deservers further study.

2) Define for any vector measure $\mu, N_\mu(t)$ as $\sup_{x \in B_\mu} \langle t, x\rangle = \|\langle t, \mu\rangle\| = \int |\langle t, x\rangle \| d\nu(x)$, where ν is the distribution – under a common dominating measure μ_0 – of the Radon Nikodym derivatives $f = (f_1 \ldots f_n)$ of $\mu = (\mu_1 \ldots \mu_n)$ w.r.t. μ_0. For any norm n, one could replace f by $f' = f/n(f)$, and $d\mu_0$ by $d\mu_0' = n(f)d\mu_0$ to normalize ν on the n – unit sphere – say, for a canonical choice, ν could be carried by the boundary of B_μ.

Our proof then shows that, in this case, $\exp -N_\mu(t)$ is positive definite. Conversely however, if the support function $N_\mu(t) = \sup_{x \in B_\mu} \langle t, x\rangle$ of some compact, convex, symmetric set B_μ is such that $\exp -N_\mu(t)$ is positive definite, then, since N_μ is positively homogeneous of degree one, $\exp(-N_\mu(t))$ is the characteristic function of a strictly stable distribution of index one, and has therefore as classical Levy representation $\exp - \int |\langle t, x\rangle| d\nu(x)$, where ν is the normalization of the Levy measure of the process on some sphere – say on the boundary of B_μ : there exists a positive measure ν on the boundary of B_μ such that $N_\mu(t) = \int |\langle t, x\rangle| d\nu(x)$. If now we define μ by $d\mu_i = x_i d\nu$, where x_i is the i-th coordinate mapping, we get immediately $N_\mu(t) = \|\langle t, \mu\rangle\|$: B_μ is indeed the ball corresponding to the vector measure μ.

This interpretation in terms of the Levy measure allows therefore to view the random perturbation around the diagonal as the sum of a very large number of independent contributions – those of the players preceding the given player in a random order – the direction of each being according to the distribution of Radon Nikodym derivatives of the given measure. This type of interpretation will be pursued much further in a subsequent paper.

3) A large number of definitions of "spaces on which there is a value" are possible in view of what preceeds – depending among others on the exact order in which the various limiting operations and averaging operations are to be done, on how much "a.e." is put into the definitions, etc.

We prefer to leave this matter to the taste of the reader, as long as no theorems are available that would show clearly which option is to be prefered. For a foretaste, the reader may want to look at Section 4.

Section 3

In many applications of the above results, whether to majority games with several houses or to n handed glove markets for instance, the function v of Section 2 will be of the form $f(\mu_1 \ldots \mu_n)$, where μ is a vector measure and f is piecewise linear. Elementary transformations reduce this to the case where μ is a full-dimensional vector of probability measures. Say f_i $(i = 1 \dots k)$ are the different linear functions appearing as pieces of f $(i \neq j \Rightarrow f_i \neq f_j)$. Then the set $\{x \mid f_i(x) = f_j(x)\}$ being of lower dimension, has zero probability under the invariant measure of last paragraph (since this is absolutely continuous with respect to Lebesgue measure), so that we can neglect ties among the f_i's. Then, for any order $\prec$ on the indexes $1 \dots k$, the set $\{x \mid \forall i, j, i \prec j .\equiv. f_i(x) < f_j(x)\} = C(\prec)$ is an open convex cone – thus connected – where, by continuity, f is constantly equal to one of the f_i's say $f_{i(\prec)}$.

Thus, by the results of Section 2, the value of this game takes the form $\sum_{\prec} P[C(\prec)] f_{i(\prec)}(\mu)$.

So, to compute the value of such games, we have to compute the probability that $f_{i_1}(\mu(\chi)) < f_{i_2}(\mu(\chi)) < \dots < f_{i_k}(\mu(\chi))$ – or, letting φ_j stand for the measure $f_{i_j}(\mu)$, the probability that $\varphi_1 < \varphi_2 < \dots < \varphi_k$, when φ is some vector measure. Remark also that the property $f(t \cdot 1 + \alpha \cdot x) = tf(1) + \alpha f(x)$ implies that, for all i needed to represent f (i.e. $f = f_i$ on some open set), one has not only f_i linear and not merely affine, so the φ_j are indeed measures, but also $f_i(1) = f(1)$, so they also have the same total mass.

Thus, letting $\nu_i = \varphi_{i+1} - \varphi_i$, we have a vector measure with total mass zero, and we have to compute the probability that $\nu(\chi)$ falls in the positive orthant.

If the ν_i's are not linearly independent, those inequalities determine a convex polyhedral cone in the space generated by ν. This cone can be written as a finite union of convex simplicial cones (neglecting boundaries that have probability zero), and for each convex simplical cone one can take its extreme rays as new coordinate axes, thus reverting to the case where the ν_i's are linearly independent.

This is the probability we are going to compute in this section: $\nu = (\nu_1 \dots \nu_n)$ is a full dimensional vector measure with total mass zero, and we want $P(\nu(\chi) \in R^n_+)$.

Obviously, this probability does not depend on the particular invariant measure chosen, so we will use $m = 0, \sigma = 1$.

Let us first recall that for any norm N on R^n, any point $x \in R^n$ can be written in polar coordinates $r = N(x)$ and $s = x/r$, and that Lebesgue measure $dx_1 \dots dx_n = r^{n-1} dr d\sigma(s)$, by definition of the surface measure $d\sigma$ on the unit N-sphere. One gets the following "change of variables" formula: if τ is any other such surface measure (i.e. originating from some other norm), then for any positive measurable function f on the unit N-sphere,

$$\int f(s) d\sigma(s) = \int f\left[\frac{\alpha}{N(\alpha)}\right] \frac{d\tau(\alpha)}{N^n(\alpha)}.$$

From now on we denote shortly by N the support function N_ν of B_ν.

We observed in Section 2 that the characteristic function $\exp -N(t)$ is integrable, so the Fourier inversion formula holds. Thus

$$P = P(\nu(\chi) \in R_+^n) = \frac{1}{(2\pi)^n} \int_{R_n^+} dx_1 \dots dx_n \int [\exp -N(y)][\exp -i\langle y, x\rangle] \, dy_1 \dots dy_n$$

or, going to polar coordinates

$$P = (2\pi)^{-n} \int_{R_n^+} (dx_1 \dots dx_n) \int [\exp -r(1 + i\langle s, x\rangle)] r^{n-1} dr d\sigma(s)$$

$$= \frac{(n-1)!}{(2\pi)^n} \int_{x_i \geqslant 0} dx_1 \dots dx_n \int \frac{d\sigma(s)}{[1 + i\langle s, x\rangle]^n}.$$

The inner integral being a density, it is positive, so we get from the montone convergence theorem

$$P = \frac{(n-1)!}{(2\pi)^n} \lim_{M \to \infty} \int_{0 \leqslant x_i \leqslant M} dx_1 \dots dx_n \int \frac{d\sigma(s)}{[1 + i\langle s, x\rangle]^n}.$$

Now $\dfrac{1}{[1 + i\langle s, x\rangle]^n}$ is bounded ($|\ | \leqslant 1$) and thus integrable on the product of any cube in x and the unit N-sphere. Using thus Fubini's theorem, we get

$$P = \frac{(n-1)!}{(2\pi)^n} \lim_{M \to \infty} \int d\sigma(s) \int_{0 \leqslant x_i \leqslant M} \frac{dx_1 \dots dx_n}{[1 + i\langle s, x\rangle]^n}.$$

Let $\phi_n(c, s) = i^n (n-1)! \left(\prod_1^n s_i \right) \int_{0 \leqslant x_i \leqslant M} \dfrac{dx_1 \dots dx_n}{[c + i\langle s, x\rangle]^n}$ $(Re(c) = 1)$ (ϕ_n depends only on the first n coordinates of the sequence s_i). An elementary integration over x_n yields

$$\phi_n(c, s) = \sum_{\delta_n \in \{0,1\}} (-1)^{\delta_n} \phi_{n-1}[c + i\delta_n s_n M, s]$$

and this formula still holds for $n = 1$ if one sets $\phi_0(c) = -\ln c$.

One gets now immediately by induction that

$$\phi_n(c, s) = -\sum_{\delta \in \{0,1\}^n} (-1)^{\sum_j \delta_j} \, ln\,(c + iM \sum_j \delta_j s_j),$$

and thus

$$P = \frac{-1}{(2\pi i)^n} \lim_{M\to\infty} \int [\sum_{\delta\in\{0,1\}^n} (-1)^{\Sigma\delta_j} \, ln\,(1 + iM \Sigma\, \delta_j s_j)] \frac{d\sigma(s)}{\prod_j s_j}.$$

Since $d\sigma(s)$ is symmetric around zero, we can replace each

$$\frac{-1}{i^n} \frac{ln\,(1 + iM \Sigma\, \delta_j s_j)}{\Pi s_j}$$

by the average of its value at s and at $(-s)$, i.e. by its real part. We get thus

$$P = \frac{1}{(2\pi)^n} \lim_{M\to\infty} \int [\sum_{\delta\in\{0,1\}^n} (-1)^{\Sigma\delta_j} F_n(M \Sigma\, \delta_j s_j)] \frac{d\sigma(s)}{\prod_j s_j}$$

where

$$F_k(x) = -\tfrac{1}{2}\, ln\,(1 + x^2), \quad -\tan^{-1}(x), \tfrac{1}{2}\, ln\,(1 + x^2), \quad \tan^{-1} x$$

according as to $k = 0, 1, 2$ or $3 \bmod 4$.

Here $\tan^{-1} x$ denotes the inverse of the tangens function, with values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

Using now the change of variables formula, we can rewrite this as

$$P = \frac{1}{(2\pi)^n} \lim_{M\to\infty} \int \left[\sum_{\delta\in\{0,1\}^n} (-1)^{\Sigma\delta_j} F_n\left(\frac{M}{N(s)} \Sigma\, \delta_j s_j\right)\right] \frac{d\tau(s)}{\Pi s_j}$$

where τ denotes the surface measure corresponding to an arbitrary norm $\| \|$. Henceforth we will use $\|x\| = \Sigma\, |x_i|$. For this norm, the unit sphere has 2^n faces, each with τ-area equal to $1/(n-1)!$.

Letting $\Delta_n = \{s \mid s \geqslant 0, \Sigma s_i = 1\}$ we get, folding all faces back on Δ_n,

$$P = (2\pi)^{-n} \int_{\Delta_n} \left[\sum_{\substack{\delta \in \{0,1\}^n \\ \epsilon \in \{-1,1\}^n}} (-1)^{\Sigma \delta_j} (\prod_j \epsilon_j) F_n \left(\frac{M}{N(\epsilon \cdot s)} \sum_j \delta_j \epsilon_j s_j \right) \right] \frac{d\tau(s)}{\Pi s_j}.$$

The next part of the computation is for n even.

Let

$$\phi_\delta(M, s) = \frac{1}{2} \sum_{\epsilon \in \{-1,1\}^n} (\Pi \epsilon_j) \ln \left(1 + \left[\frac{M}{N(\epsilon \cdot s)} \sum_j \delta_j \epsilon_j s_j \right]^2 \right).$$

Claim 1:

$$\sup_M \left| \frac{\phi_\delta(M, s)}{\prod_j s_j} \right| \quad \text{is locally integrable on} \quad \{s \in \Delta_n : \Sigma \delta_j s_j > 0\}$$

(i.e. any point – and thus also any compact subset – of this set has a neighborhood on which the function is integrable).

Proof: Fix one such point s_0, and consider first a neighborhood $\tilde{V}_{s_0}$ of s_0 where all strict inequalities among the functions $\{0; s_1 \ldots s_n; (\sum_j \delta_j \epsilon_j s_j)_{\epsilon \in \{-1,1\}^n}\}$ that hold at s_0 are preserved. Let $\eta > 0$ be strictly smaller than the absolute value at s_0 of any of those functions that does not vanish at s_0, and assume further that, on $\tilde{V}_{s_0}$, all functions that vanish at s_0 remain $< \eta$ in absolute value, while all the other functions remain $> \eta$ in absolute value. Finally write $\tilde{V}_{s_0}$ as the finite union of sets V_{s_0} (and a null set) where on each set V_{s_0} the ordering of all those functions is constant (and strict). It is sufficient to prove integrability on V_{s_0}. Assume in particular without loss of generality that, on V_{s_0}, we have $0 < s_1 < s_2 < \ldots < s_k < \eta < s_{k+1} < \ldots < s_n (0 \leqslant k < n)$. By assumption $\exists\, j > k : \delta_j = 1$. We have

$$\phi_\delta = \frac{1}{2} \sum_{(\epsilon_2 \ldots \epsilon_n) \in \{-1,1\}} (\Pi \epsilon_j) \ln \frac{1 + \left[\dfrac{M[\delta_1 s_1 + \sum_{j>1} \delta_j \epsilon_j s_j]}{N(s_1, \epsilon_2 s_2, \ldots, \epsilon_n s_n)} \right]^2}{1 + \left[\dfrac{M[-\delta_1 s_1 + \sum_{j>1} \delta_j \epsilon_j s_j]}{N(-s_1, \epsilon_2 s_2, \ldots, \epsilon_n s_n)} \right]^2}$$

and we will bound individually the absolute value of every logarithm in this sum. Let $f_i(s) = -(-1)^i \delta_1 s_1 + \sum_{j>1} \delta_j \epsilon_j s_j$, $n_i(s) = N(-(-1)^i s_1, \ \epsilon_2 s_2, \ldots, \epsilon_n s_n)$ $(i = 1, 2)$. Now $\left| \ln \frac{1 + (Mf_1/n_1)^2}{1 + (Mf_2/n_2)^2} \right|$ increases monotonically with M to it's limit, so that

$$\sup_M \frac{1}{\Pi s_j} \left| \frac{1}{2} \ln \frac{1 + [Mf_1/n_1]^2}{1 + [Mf_2/n_2]^2} \right| = \left| \frac{\ln |f_1/f_2|}{\Pi s_j} - \frac{\ln (n_1/n_2)}{\Pi s_j} \right|.$$

Thus we only have to show that $\frac{\ln |f_1/f_2|}{\Pi s_j}$ and $\frac{\ln (n_1/n_2)}{\Pi s_j}$ are integrable on V_s.

For the second term, remark that, N being a norm, and any two norms on R^n being equivalent, n_1 and n_2 are bounded away from 0 and from ∞, and $|n_1 - n_2| \leqslant N(2s_1, 0, 0, \ldots) \leqslant Ks_1$. So $|\ln n_1/n_2| \leqslant K's_1$ – thus we only have to show the integrability of $s_1/\Pi s_j$ on $\{s \in \Delta_n : \forall\, i,\, s_i \geqslant s_1\}$.

The measure τ on Δ_n has a bounded density w.r.t. $ds_1 \ldots ds_{n-1}$, so it is sufficient to prove that $\int_0^1 ds_1 \prod_{i=2}^{n-2} \int_{s_1}^1 \frac{ds_i}{s_i} < \infty$, i.e. $\int_0^1 |\ln s_1|^{n-2} ds_1 < \infty$, which is well known.

The term $\frac{\ln |f_1/f_2|}{\pi s_j}$ appears only if $\delta_1 = 1$ – so assume this. Let $\varphi(s) = \sum_{j>1} \delta_j \epsilon_j s_j$: since $f_1 = \varphi + s_1$, $f_2 = \varphi - s_1$, we can repeat with f_1 and f_2 the same argument as with n_1 and n_2 if φ does not vanish at s_0. So assume furthermore $\varphi(s_0) = \sum_{j>k} \delta_j \epsilon_j s_j^0 = 0$. Since by assumption $\sum_{j>k} \delta_j s_j^0 > 0$, it follows that there exist two indexes $> k$, say $n-1$ and n (renumbering coordinates), such that $\delta_{n-1} = \delta_n = 1$, $\epsilon_{n-1} = -1$, $\epsilon_n = 1$.

Do now the change of coordinates $(s_1, \ldots, s_n) \to (\frac{1}{2} f_1, \frac{1}{2} f_2, s_2, \ldots, s_{n-2})$ using the formulas for f_1 and f_2 and the equation $\Sigma s_j = 1$. Since under our assumptions the change of coordinates has nonzero determinant, it will be sufficient to prove integrability of $\frac{|\ln |f_1/f_2||}{(f_1 - f_2) \prod_2^{n-2} s_i}$ on $1 \geqslant s_i > f_1 - f_2 > 0$, $|f_i| \leqslant \frac{1}{2}$.

Integrating the s_i's, this becomes

$$\int \left| \frac{\ln |f_1/f_2|}{f_1 - f_2} \right| |\ln |f_1 - f_2||^{n-3} df_1 df_2$$

on $-\frac{1}{2} \leqslant f_2 \leqslant f_1 \leqslant \frac{1}{2}$ or equivalently on $|f_i| \leqslant \frac{1}{2}$.

Letting $x_i = |f_i|$, we get, bounding the integrand,

$$\int \frac{\ln (x_1/x_2)}{x_1 - x_2} | \ln |x_1 - x_2| |^{n-3} dx_1 dx_2 \quad \text{on } 0 \leqslant x_i \leqslant \tfrac{1}{2},$$

or, by symmetry around $x_1 = x_2$,

$$\int_{0 \leqslant x_1 \leqslant x_2 \leqslant \frac{1}{2}} \frac{\ln (x_2/x_1)}{x_2 - x_1} [-\ln (x_2 - x_1)]^k dx_1 dx_2 < \infty \quad (k \geqslant 0)$$

– or, using polar coordinates and increasing slightly the area of integration,

$$\int_{\substack{0 \leqslant r \leqslant 1 \\ 0 \leqslant \theta \leqslant \pi/4}} \frac{-\ln \tan \theta}{\cos \theta - \sin \theta} [-\ln r - \ln (\cos \theta - \sin \theta)]^k dr d\theta$$

since $\int_0^1 [-\ln r - A]^k dr$ is a polynomial in A, we have reduced the problem to showing the finiteness of

$$\int_{0 \leqslant \theta \leqslant \pi/4} \frac{\ln \cos \theta - \ln \sin \theta}{\cos \theta - \sin \theta} [-\ln (\cos \theta - \sin \theta)]^l d\theta \quad (l \geqslant 0).$$

It is sufficient to show local integrability at 0 and $\frac{\pi}{4}$; the ratio being bounded at $\frac{\pi}{4}$, it amounts at this point to the well known integrability of $|\ln x|^n$ near zero; and at $\theta = 0$, the argument is just as easy, and reduces to the integrability of $|\ln x|$ near zero.

This proves the claim. □

Using now Lebesgue's dominated convergence theorem, we get for any $\eta > 0$

$$\lim_{M \to \infty} \int_{\Delta_n \cap \{s | \Sigma \delta_i s_i \geqslant \eta\}} \frac{\phi_\delta(M, s)}{\Pi s_i} d\tau(s) = \int_{\Delta_n \cap \{s | \Sigma \delta_i s_i \geqslant \eta\}} \frac{\lim_{M \to \infty} \phi_\delta(M, s)}{\Pi s_i} d\tau(s)$$

and

$$\lim_{M\to\infty} \phi_\delta(M,s) = \sum_{\epsilon\in\{-1,1\}^n} (\prod_j \epsilon_j) \ln |\Sigma\, \delta_j \epsilon_j s_j| - \sum_{\epsilon\in\{-1,1\}^n} (\prod_j \epsilon_j) \ln N(\epsilon\cdot s)$$

if $\exists\, j : \delta_j = 0$, the first sum is zero, so

$$\lim_{M\to\infty} \phi_\delta(M,s) = I(\delta = (1,1,\ldots,1)) \sum_{\epsilon\in\{-1,1\}^n} (\prod_{jm} \epsilon_j) \ln |\Sigma\, \epsilon_j s_j|$$

$$- \sum_{\epsilon\in\{-1,1\}^n} (\Pi\, \epsilon_j) \ln N(\epsilon\cdot s).$$

We have also seen in the above proof that both $\frac{1}{\Pi s_j} \sum_\epsilon (\Pi\, \epsilon_j) \ln |\Sigma\, \epsilon_j s_j|$ and $\frac{1}{\Pi s_j} \sum_\epsilon (\Pi\, \epsilon_j) \ln N(\epsilon\cdot s)$ are integrable over Δ_n (for $\delta = (1,1,\ldots,1)$, $\Sigma\, \delta_i s_i = 1 > 0$ everywhere); so for $\delta \neq 0$,

$$\lim_{M\to\infty} \int_{\Delta_n} \frac{\phi_\delta(M,s)}{\Pi(s_j)}\, d\tau(s) = I(\delta = (1,\ldots,1)) \int_{\Delta_n} \frac{\sum_\epsilon (\Pi\, \epsilon_j) \ln |\Sigma\, \epsilon_j s_j|}{\Pi s_j}\, d\tau(s)$$

$$- \int_{\Delta_n} \frac{\sum_\epsilon (\Pi\, \epsilon_j) \ln N(\epsilon\cdot s)}{\Pi s_j}\, d\tau(s) + \lim_{\eta\to 0} \lim_{M\to\infty} \int_{\Delta_n \cap \{s \mid \Sigma \delta_i s_i < \eta\}} \frac{\phi_\delta(M,s)}{\Pi s_j}\, d\tau(s)$$

Therefore, summing over all nonzero δ's,

$$P(\nu_i \geqslant 0\ \forall\, i) = (-1)^{n/2} (2\pi)^{-n} \Bigg[- \int_{\Delta_n} \frac{\sum_\epsilon (\Pi\, \epsilon_j) \ln |\Sigma\, \epsilon_j s_j|}{\Pi s_j}\, d\tau(s)$$

$$- \int_{\Delta_n} \frac{\sum_\epsilon (\Pi\, \epsilon_j) \ln N(\epsilon\cdot s)}{\Pi s_j}\, d\tau(s)$$

$$- \sum_{\delta\in\{0,1\}^n} (-1)^{\Sigma \delta_j} \lim_{\eta\to 0} \lim_{M\to\infty} \int_{\Delta_n \cap \{s \mid \Sigma \delta_i s_i < \eta\}} \frac{\phi_\delta(M,s)}{\Pi s_j}\, d\tau(s) \Bigg]$$

Let us now compute the last limit.

Assume without loss of generality $\delta_j = 1$ iff $j \leqslant k$: we want to compute

$$\lim_{\eta \to 0} \lim_{M \to \infty} \int_{\substack{s \geqslant 0 \\ \sum_{i \leqslant n} s_i = 1 \\ \sum_{i \leqslant k} s_i \leqslant \eta}} \frac{\sum_{\epsilon} (\Pi\, \epsilon_j) \ln [1 + (M(\sum_{j \leqslant k} \epsilon_j s_j)/N(\epsilon \cdot s))^2]}{\Pi\, s_j} d\tau(s)$$

Represent $s \in \Delta_n$ as $\alpha x + (1 - \alpha)y$, with $x \in \Delta_k, y \in \Delta_{n-k}, \alpha \in [0, 1]$. Denote by $\bar{\tau}_n$ the uniform distribution on Δ_n: we have $\tau = \bar{\tau}_n/(n-1)!$ as noted earlier.

One checks easily that, under $\bar{\tau}_n$, α, x and y are independent, x and y being uniform and α having the beta-density

$$\frac{(n-1)!}{(k-1)!\,(n-k-1)!} \alpha^{k-1}(1-\alpha)^{n-k-1}.$$

Thus we get

$$\lim_{\eta \to 0} \lim_{M \to \infty} \int_{\Delta_{n-k}} \frac{d\bar{\tau}_{n-k}(y)}{(n-k-1)!} \int_{\Delta_k} \frac{d\bar{\tau}_k(x)}{(k-1)!}$$

$$\int_0^\eta \frac{\sum_{\epsilon} (\Pi\, \epsilon_j) \ln [1 + (M\alpha(\Sigma\, \epsilon_j x_j)/N(\alpha(\epsilon \cdot x) + (1-\alpha)(\epsilon \cdot y)))^2]}{\alpha^k (1-\alpha)^{n-k}\, \Pi\, x_j\, \Pi\, y_j}$$

$$\alpha^{k-1}(1-\alpha)^{n-k-1} d\alpha$$

$$= \lim_{\eta \to 0} \lim_{M \to \infty} \int_{\Delta_{n-k}} \frac{d\tau_{n-k}(y)}{\Pi\, y_j} \int_{\Delta_k} \frac{d\tau_k(x)}{\Pi\, x_j}$$

$$\int_0^\eta \sum_{\epsilon} (\Pi\, \epsilon_j) \ln [1 + (M\alpha(\Sigma\, \epsilon_j x_j)/N(\alpha(\epsilon \cdot x) + (1-\alpha)(\epsilon \cdot y)))^2]\, d\alpha/\alpha(1-\alpha).$$

We first want to show that:

Claim 2: The limit (when $\eta \to 0$) is not affected if we replace $N(\alpha(\epsilon \cdot x) + (1-\alpha)(\epsilon \cdot y))$ by $N(\epsilon \cdot y)$, and $d\alpha/\alpha(1-\alpha)$ by $d\alpha/\alpha$.

A) First replacement

Indeed, as before we get from the equivalence of norms of R^n that

$$\left| \ln \frac{N(\alpha(\epsilon \cdot x) + (1-\alpha)(\epsilon \cdot y))}{N(\epsilon \cdot y)} \right| \leqslant K\alpha.$$

Since $\sup_{A \geqslant 0} \left| \ln \frac{1+An_1}{1+An_2} \right| = \left| \ln \frac{n_1}{n_2} \right|$, we get that, after the first replacement, the error in the sum $\sum_{\epsilon}$ is bounded by $K \cdot \alpha$ – for some $K > 0$.

For the same reason, pairing the terms where, for $j > k$, ϵ_j is +1 and −1, one finds that, both before and after replacement, the sum $\sum_{\epsilon}$ is bounded in absolute value by $K \cdot y_j$ and for $j < k$, one finds the bound

$$Kx_j + K' \left| \ln \left| \frac{\sum_{i \neq j} \epsilon_i x_i + x_j}{\sum_{i \neq j} \epsilon_i x_i - x_j} \right| \right|.$$

Thus, to show that we commit a negligible error in this first replacement when $\eta \to 0$, we have to show that

$$\int_{\Delta_k} \frac{d\tau_k(x)}{\Pi x_j} \int_0^{e^{-m}} \frac{d\alpha}{\alpha} \int_{\Delta_m} \operatorname{Min} \left[\alpha, \left(x_j + \left| \ln \left| \frac{\langle \epsilon^{j+}, x \rangle}{\langle \epsilon^{j-}, x \rangle} \right| \right| \right)_{j=1}^{k}, (y_j)_{j=1}^{m} \right] \frac{d\tau_m(y)}{\Pi y_j} < \infty$$

where, for any $\epsilon \in \{-1, 1\}^k$, $(\epsilon^{j+})_i = \epsilon_i$ for $i \neq j$, $= +1$ for $i = j$ and $(\epsilon^{j-})_i = \epsilon_i$ for $i \neq j$, $= -1$ for $i = j$.

Let us first bound the inner integral

$$\int_{\Delta_m} \operatorname{Min} (\beta, (y_j)_{j=1}^m) \frac{d\tau(y)}{\Pi y_j} \leqslant m \int \operatorname{Min} (\beta \operatorname{Max}_j y_j, (y_j)_{i=1}^m) \frac{d\tau(y)}{\Pi y_j}$$

$$\leqslant m \int_{0 \leqslant y_j \leqslant 1} \operatorname{Min} (\beta \operatorname{Max}_j y_j, (y_j)_{j=1}^m) \prod_{j=1}^m \frac{dy_i}{y_j}$$

$$= m^2 (m-1) \int_{0 \leqslant y_1 \leqslant y_j \leqslant y_m \leqslant 1} \operatorname{Min} [\beta y_m, y_1] \prod_{j=1}^m \frac{dy_j}{y_j}$$

$$= m^2(m-1) \int_{0 \leqslant y_1 \leqslant y_m \leqslant 1} [\ln (y_m/y_1)]^{m-2} \operatorname{Min} (y_m^{-1}, \beta y_1^{-1}) dy_1 dy_m$$

$$= m^2(m-1) \int_0^1 dy \int_1^{y^{-1}} [\ln u]^{m-2} \operatorname{Min} [\beta, u^{-1}] du.$$

Now the inner integral is

$$\text{for } y \leqslant \beta \text{ equal to } \beta \int_1^{\beta^{-1}} [\ln u]^{m-2} du + \int_{\beta^{-1}}^{y^{-1}} [\ln u]^{m-2} \frac{du}{u}$$

and

$$\text{for } y \geqslant \beta \quad \text{to} \quad \beta \int_1^{y^{-1}} [\ln u]^{m-2} du.$$

Therefore our upper bound equals

$$m^2(m-1) \left[\beta^2 \int_1^{\beta^{-1}} [\ln u]^{m-2} du + \int I(0 \leqslant y \leqslant u^{-1} \leqslant \beta) [\ln u]^{m-2} dy \frac{du}{u} \right.$$
$$\left. + \beta \int I(\beta \leqslant y \leqslant u^{-1} \leqslant 1) [\ln u]^{m-2} du dy \right]$$

$$= m^2(m-1) \left[\beta^2 \int_1^{\beta^{-1}} [\ln u]^{m-2} du + \int_0^{\beta} [\ln v^{-1}]^{m-2} dv + \beta \int_1^{\beta^{-1}} u^{-1} [\ln u]^{m-2} du \right.$$
$$\left. - \beta^2 \int_1^{\beta^{-1}} [\ln u]^{m-2} du \right]$$

$$= m^2(m-1) \left[\int_0^{\beta} (-\ln v)^{m-2} dv + \frac{\beta}{m-1} (-\ln \beta)^{m-1} \right]$$

$$= m^2 \int_0^{\beta} (-\ln v)^{m-1} dv$$

by integration by parts. The same integration by parts gives by induction that this last integral equals

$$(m-1)! \beta \sum_{i=0}^{m-1} \frac{(-\ln \beta)^i}{i!}.$$

Therefore, since we are only interested in values of $\beta \leqslant e^{-m} < 1$, we get

$$\int_{\Delta_m} \operatorname{Min}(\beta, (y_j)_{j=1}^m) \frac{d\tau(y)}{\Pi y_j} \leqslant K_m \beta |\ln \beta|^{m-1}.$$

Let $F_m(\beta) = \beta |\ln \beta|^m$: we thus have to prove that

$$\int_{\Delta_k} \frac{d\tau(x)}{\Pi x_j} \int_0^{e^{-m}} F_{m-1}\left(\operatorname{Min}\left[\alpha, \left(x_j + \left|\ln \left|\frac{\langle e^{j+}, x\rangle}{\langle e^{j-}, x\rangle}\right|\right|\right)_{j=1}^k\right]\right) \frac{d\alpha}{\alpha} < \infty.$$

To evaluate the inner integral, let for short $y_j = x_j + \left|\ln \left|\dfrac{\langle e^{j+}, x\rangle}{\langle e^{j-}, x\rangle}\right|\right|$ – then the inner integral is bounded by

$$\operatorname{Min}_j \int_0^{e^{-m}} F_{m-1}(\min(\alpha, y_j)) \frac{d\alpha}{\alpha}$$

F_{m-1} being increasing in $[0, e^{-m}]$. Call this last integral $\phi(y_j)$; we get letting $\rho = \min(y, e^{-m})$

$$\phi(y) = \int_0^\rho |\ln x|^{m-1} dx - F_{m-1}(\rho)(\ln \rho + m)$$

$$\leqslant K_m F_{m-1}(\rho) - (\ln \rho + m) F_{m-1}(\rho)$$

$$\leqslant K'_m F_m(\rho) \quad \text{(using our previous bound for the integral).}$$

Thus it will be sufficient to show that, letting $\psi(y) = F_m(\min(y, e^{-m}))$

$$\int_{\Delta_k \cap \{x_1 \leqslant x_i\}} \psi(y_1) \frac{d\tau(x)}{\Pi x_i} < \infty.$$

Since $u \geqslant 0$, $v \geqslant 0$ implies $\psi(u + v) \leqslant \psi(u) + \psi(v)$, it will be sufficient to show separately that

$$\int_{x_1 \leqslant x_i} |\ln x_1|^m \frac{d\tau(x)}{\prod_{i>1} x_i} < \infty$$

and that

$$\int_{x_1 \leqslant x_i} \psi\left(\left|\ln\left|\frac{\langle \epsilon^{1+} \cdot x\rangle}{\langle \epsilon^{1-} \cdot x\rangle}\right|\right|\right)\frac{d\tau(x)}{\Pi\, x_i} < \infty.$$

The first integral is bounded by – letting $\|x\|_\infty = \text{Max}\ |x_i|$, and τ' being Lebesgue measure on $\{x \mid \|x\|_\infty = 1\}$ –

$$\int_{x_1 \leqslant x_i}\left[\ln\frac{k\cdot\|x\|_\infty}{x_1}\right]^m \frac{d\tau(x)}{\prod_{i>1} x_i} \leqslant (k-1)\int_{x_1 \leqslant x_i \leqslant x_2}\left[\ln\frac{kx_2}{x_1}\right]^m \frac{d\tau'(x)}{\prod_{i>1} x_i}$$

$$= C_k \int_{0 \leqslant x_1 \leqslant x_i \leqslant x_2 = 1}\left[\ln\frac{kx_2}{x_1}\right]^m dx_1 \prod_{i>2}\frac{dx_i}{x_i}$$

$$= C_k \int_0^1 \left(\ln\frac{k}{x_1}\right)^m \left(\ln\frac{1}{x_1}\right)^{k-2} dx_1 < \infty.$$

For the second integral, we will prove local integrability, i.e. that, for any $x \in \Delta_k \cap \{x_1 \leqslant x_i\}$ there is a neighborhood of x in this set where the function is integrable.

If $x_1 > 0$, then $x_i > 0\ \forall\ i$ so that the integrand is locally bounded. Otherwise, one has $\langle \epsilon, x\rangle = \langle \epsilon^{1+}, x\rangle = \langle \epsilon^{1-}, x\rangle$: if $\langle \epsilon, x\rangle \neq 0$, then locally $\left|\ln\left|\frac{\langle \epsilon^{1+}, x\rangle}{\langle \epsilon^{1-}, x\rangle}\right|\right| \leqslant Kx_1$, so $\psi \leqslant K'\psi(x_1)$, and we have just shown this bound to be integrable.

Thus there just remains to consider the case where $\langle \epsilon^{1+}, x\rangle = \langle \epsilon^{1-}, x\rangle = 0$.

Since $x_1 = 0$, $\Sigma x_i = 1$, there exists an index $j \neq 1$ with $x_j \geqslant 1/k$, and since further $\epsilon \cdot x = 0$, there exists another index $j' \neq 1$ with $x_{j'} \geqslant k^{-2}$, and $\epsilon_j \epsilon_j' = -1$. Assume without loss of generality that $j' = k-1, j = k$, and make the change of variables

$$(x_1 \ldots x_k) \to (f_1, f_2, x_2 \ldots x_{k-2})$$

using the equations

$$\langle \epsilon^{1+}, x\rangle = f_1, \quad \langle \epsilon^{1-}, x\rangle = f_2, \quad \Sigma x_i = 1.$$

The integrability on $\frac{\psi}{\Pi x_i}\, d\tau(x)$ is equivalent to that of $\frac{\psi}{\prod_{i<k-1} x_i}$, which is equivalent to that of

$$\frac{\psi(|\ln|f_1/f_2||)}{(f_1-f_2)}df_1df_2\prod_{i=2}^{k-2}\left(\frac{dx_i}{x_i}\right)$$

over $\{|f_i|\leqslant\frac{1}{2}, 0\leqslant f_1-f_2\leqslant x_i\leqslant 1\}$ – or, integrating over the x_i's:

$$\frac{|\ln(f_1-f_2)|^l\psi(|\ln|f_1/f_2||)}{f_1-f_2}df_1df_2.$$

The integrand is only increased if we replace f_1-f_2 by $||f_1|-|f_2||$ – so we can assume $0\leqslant f_i\leqslant 1$, inserting absolute value of differences.

Further by symmetry it is sufficient to consider the case $f_1\geqslant f_2$:

$$\int_{0\leqslant f_2\leqslant f_1\leqslant 1}\frac{[-\ln(f_1-f_2)]^l F_m[\ln(f_1/f_2)\wedge(1+\delta_m))]}{f_1-f_2}df_1df_2.$$

Let $f_1=y, f_2/f_1=1-x$: our integral becomes

$$\int_0^1\frac{F_m[\ln(1-x)^{-1}\wedge(1+\delta_m))]}{x}dx\int_0^1[-\ln(xy)]^l dy;$$

the inner integral is $\frac{1}{x}\int_0^x[-\ln z]^l dz$, which by a previous computation is equal to a polynomial in $[-\ln x]$: everything amounts to show that

$$\frac{[-\ln x]^{l'}F_m[\ln((1-x)^{-1}\wedge(1+\delta_m))]}{x}dx$$

is integrable on $[0, 1]$. The integrand is bounded except at $x=0$, where it is bounded by $\frac{[-\ln x]^{l'}F_m(2x)}{x}$, i.e. a polynomial in $(-\ln x)$ – this we know to be integrable.

Thus, we finished proving that the first replacement can be made.

B) Second replacement

Once this first replacement is done, the sum in the integrand is, by our previous argument, bounded by

$$K \sum_{\epsilon} \operatorname{Min}\left\{ (y_j)_{j=1}^{n-k}, \left(\left| \ln \left| \frac{\langle \epsilon^{j+}, x\rangle}{\langle \epsilon^{j-}, x\rangle} \right| \right| \right)_{j=1}^{k} \right\}.$$

So, to show that the second replacement can be done, we have to show that this function is integrable for

$$\frac{d\tau(y)}{\Pi y_i} \cdot \frac{d\tau(x)}{\Pi x_j} \cdot \frac{d\alpha}{1-\alpha} \qquad \left(\text{since } \frac{1}{\alpha(1-\alpha)} = \frac{1}{\alpha} + \frac{1}{1-\alpha} \right)$$

over say $\alpha < \frac{1}{2}$ – thus that

$$\int \frac{d\tau(x)}{\Pi x_j} \int_0^{\frac{1}{2}} d\alpha \int \frac{d\tau(y)}{\Pi y_j} \operatorname{Min}\left\{ (y_j)_{j=1}^{n-k}, \left(\left| \ln \left| \frac{\langle \epsilon^{j+}, x\rangle}{\langle \epsilon^{j-}, x\rangle} \right| \right| \right)_{j=1}^{k} \right\} < \infty.$$

For the first integral we can use our previous computation, and the integral over α disappears, so we are left to prove that

$$\int_{x_1 \leqslant x_j} \frac{F_m\left(\left| \ln \left| \frac{\epsilon^{1+} \cdot x}{\epsilon^{1-} \cdot x} \right| \right| \wedge e^{-m} \right)}{\prod_j x_j} d\tau(x) < \infty$$

and this we have shown previously also. So Claim 2 is proved. □

Thus we have to compute

$$\lim_{\eta \to 0} \lim_{M \to \infty} \frac{1}{2} \int \frac{d\tau_{n-k}(y)}{\Pi y_j} \int \frac{d\tau_k(x)}{\Pi x_j} \int_0^{\eta}$$

$$\sum_{\epsilon^x, \epsilon^y} (\Pi \epsilon_j^x)(\Pi \epsilon_j^y) \ln [1 + [M\alpha \langle \epsilon^x, x\rangle / N(0 \cdot x, \epsilon^y \cdot y)]^2] \frac{d\alpha}{\alpha}.$$

Letting $M\alpha = u$, this becomes

$$\lim_{\eta\to 0}\lim_{M\to\infty}\frac{1}{2}\int\frac{d\tau_{n-k}(y)}{\Pi\, y_j}\int\frac{d\tau_k(x)}{\Pi\, x_j}\int_0^{M\eta}$$

$$\sum_{\epsilon^x,\epsilon^y}(\Pi\,\epsilon_j^x)(\Pi\,\epsilon_j^y)\ln\left[1+[u\langle\epsilon^x,x\rangle/N(0\cdot x,\epsilon^y\cdot y)]^2\right]\frac{du}{u}.$$

Now, the limit when M goes to infinity becomes independent of η, so we get, using for short ϵ for ϵ^x and η for ϵ^y:

$$\lim_{M\to\infty}\frac{1}{2}\int\frac{d\tau_{n-k}(y)}{\Pi\, y_j}\int\frac{d\tau_k(x)}{\Pi\, x_j}\int_0^{M}$$

$$\sum_{\epsilon,\eta}(\Pi\,\epsilon_j)(\Pi\,\eta_j)\ln\left[1+[u\langle\epsilon,x\rangle/N(0\cdot x,\eta\cdot y)]^2\right]\frac{du}{u}.$$

Denote the inner integral by $\phi_M(x, y)$. We have

$$\phi_M(x,y)=\sum_{\epsilon,\eta}(\Pi\,\epsilon_j)(\Pi\,\eta_j)\int_0^{M|\langle\epsilon,x\rangle|/N(0\cdot x,\eta\cdot y)}\ln(1+v^2)\frac{dv}{v}$$

$$=\sum_{\epsilon,\eta}(\Pi\,\epsilon_j)(\Pi\,\eta_j)\int_0^{|\langle\epsilon,x\rangle|/N(0\cdot x,\eta\cdot y)}\ln(1+M^2w^2)\,\frac{dw}{w}$$

$$=\int_0^\infty\Big[\sum_{\epsilon,\eta}(\Pi\,\epsilon_j)(\Pi\,\eta_j)I[w\leqslant|\langle\epsilon,x\rangle|/N(0\cdot x,\eta\cdot y)]\Big]\ln(1+M^2w^2)\,\frac{dw}{w}.$$

If instead of $\ln(1+M^2w^2)$ in this integral we had a constant, say $\ln M^2$, the integral would still exist – because the integrand vanishes in a neighborhood of the origin –, and would have a value equal to $\ln M^2$ times

$$\sum_{\epsilon,\eta}(\Pi\,\epsilon_j)(\Pi\,\eta_j)\ln|\langle\epsilon,x\rangle| - \sum_{\epsilon,\eta}(\Pi\,\epsilon_j)(\Pi\,\eta_j)\ln N(0\cdot x,\eta\cdot y)=0$$

– the first term being zero because of $\sum_\eta(\Pi\,\eta_j)$ and the second because $\sum_\epsilon(\Pi\,\epsilon_j)$.

So we still have

$$\phi_M(x, y) = \int_0^\infty \Big[\sum_{\epsilon,\eta} (\Pi\, \epsilon_j)(\Pi\, \eta_j) I[w \leqslant |\langle \epsilon, x\rangle| / N(0 \cdot x, \eta \cdot y)]\Big] \ln (w^2 + M^{-2}) \frac{dw}{w}.$$

Now the integrand is uniformly bounded, and vanishes outside some closed interval disjoint from zero, so that by Lebesgue's dominated convergence theorem

$$\lim_{M\to\infty} \phi_M(x, y) = \int_0^\infty \Big[\sum_{\epsilon,\eta} \Big] (\ln w^2) \frac{dw}{w} = \int_{w=0}^\infty \Big[\sum_{\epsilon,\eta} \Big] 2 \ln w d \ln w$$

$$= \int_{w=0}^\infty \Big[\sum_{\epsilon,\eta} \Big] d \ln^2 w = \sum_{\epsilon,\eta} (\Pi\, \epsilon_i)(\Pi\, \eta_j) \ln^2 \frac{|\langle \epsilon, x\rangle|}{N(0 \cdot x, \eta \cdot y)}$$

$$= \sum_{\epsilon,\eta} (\Pi\, \epsilon_j)(\Pi\, \eta_j) \ln^2 |\langle \epsilon, x\rangle| + \sum_{\epsilon,\eta} (\Pi\, \epsilon_j)(\Pi\, \eta_j) \ln^2 N(0 \cdot x, \eta \cdot y)$$

$$-2 \sum_{\epsilon,\eta} (\Pi\, \epsilon_j)(\Pi\, \eta_j) \ln |\langle \epsilon, x\rangle| \ln N(0 \cdot x, \eta \cdot y)$$

$$= -2 \Big[\sum_{\epsilon} (\Pi\, \epsilon_j) \ln |\langle \epsilon, x\rangle|\Big] \Big[\sum_{\eta} (\Pi\, \eta_j) \ln N(0 \cdot x, \eta \cdot y)\Big]$$

(the first of the three sums is zero because $\sum_{\eta} (\Pi\, \eta_j)$ and the second because $\sum_{\epsilon} (\Pi\, \epsilon_j)$).

Now we have to show that when we apply the first two integrations to $\phi_M - \lim_M \phi_M$, we get something going to zero, i.e. that

Claim 3:

$$\lim_{M\to\infty} \int \frac{d\tau_{n-k}(y)}{\Pi\, y_j} \int \frac{d\tau_k(x)}{\Pi\, x_j} \int_0^\infty \Big[\sum_{\epsilon,\eta} (\Pi\, \epsilon_j)(\Pi\, \eta_j) I[w \leqslant |\epsilon \cdot x| / N(0 \cdot x, \eta \cdot y)]\Big]$$

$$\ln (1 + (Mw)^{-2}) \frac{dw}{w} = 0.$$

By Lebesgue's dominated convergence theorem, since $\ln (1 + (Mw)^{-2})$ decreases pointwise to zero, it will be sufficient to show that

$$\int \frac{d\tau_{n-k}(y)}{\Pi y_j} \int \frac{d\tau_k(x)}{\Pi x_j} \int_0^\infty \left| \sum_{\epsilon,\eta} (\Pi \epsilon_j)(\Pi \eta_j) I[w \leqslant |\langle \epsilon, x\rangle| / N(0 \cdot x, \eta \cdot y)] \right|$$

$$\ln(1 + w^{-2}) \frac{dw}{w} < \infty$$

or, replacing w by $z^{-1/2}$, that

$$\int \frac{d\tau_{n-k}(y)}{\Pi y_i} \int \frac{d\tau_k(x)}{\Pi x_j} \int_0^\infty \left| \sum_{\epsilon,\eta} (\Pi \epsilon_j)(\Pi \eta_j) I[z \langle \epsilon, x\rangle^2 \geqslant N^2(0 \cdot x, \eta \cdot y)] \right|$$

$$\ln(1 + z) \frac{dz}{z} < \infty.$$

If z is close enough to zero (smaller than $\min_y N^2(0 \cdot x, y)$) then the sum $\sum_{\epsilon,\eta}$ is identically zero (i.e., for all x and y), so we can replace dz/z by $\frac{dz}{1+z}$, and get

$$\int \frac{d\tau_{n-k}(y)}{\Pi y_j} \int \frac{d\tau_k(x)}{\Pi x_j} \int_0^\infty \left| \sum_{\epsilon,\eta} (\Pi \epsilon_j)(\Pi \eta_j) I[z \langle \epsilon, x\rangle^2 \geqslant N^2(0 \cdot x, \eta \cdot y)] \right|$$

$$d \ln^2(1 + z) < \infty?$$

Let us now try to bound the sum $\sum_{\epsilon,\eta}$.

Pairing the terms where η_1 has opposite signs we get

$$\sum_{\epsilon,\eta} = \sum_{\epsilon,\eta} (\Pi \epsilon_j)(\Pi \eta_j) I(n_+ \leqslant z \langle \epsilon, x\rangle^2 < n_-)$$

using $n_\pm$ for $N^2(0 \cdot x, \pm \eta_1 y_1, \eta_2 y_2, \eta_3 y_3, \ldots)$.

Pairing now the terms where ϵ_1 has opposite signs, we get, letting $u = \sum_{j>1} \epsilon_j x_j$, $u_+ = |u + \epsilon_1 x_1|^2, u_- = |u - \epsilon_1 x_1|^2$:

$$\sum_{\epsilon,\eta} = \sum_{\epsilon,\eta} (\Pi \epsilon_j)(\Pi \eta_j) \left\{ \left[I\left(\frac{n_-}{n_+} \leqslant \frac{u_+}{u_-}\right) + I\left(\frac{n_-}{n_+} \leqslant \frac{u_-}{u_+}\right) \right] I\left(\frac{n_+}{u_+} \leqslant z < \frac{n_-}{u_+}\right) \right.$$

$$\left. + I\left(\frac{u_+}{u_-} < \frac{n_-}{n_+}\right) I\left(\frac{n_+}{u_+} \leqslant z < \frac{n_+}{u_-}\right) + I\left(\frac{u_-}{u_+} < \frac{n_-}{n_+}\right) I\left(\frac{n_-}{u_-} \leqslant z < \frac{n_-}{u_+}\right) \right\}.$$

Thus we get by integrating

$$\int \Big| \sum_{\epsilon,\eta} \Big| d \ln^2 (1+z) \leqslant \sum_{\epsilon,\eta} \left\{ \left[I\left(\frac{n_-}{n_+} \leqslant \frac{u_+}{u_-}\right) + I\left(\frac{n_-}{n_+} \leqslant \frac{u_-}{u_+}\right) \right] \Delta_1 + I\left(\frac{u_+}{u_-} < \frac{n_-}{n_+}\right) \Delta_2 + I\left(\frac{u_-}{u_+} < \frac{n_-}{n_+}\right) \Delta_3 \right\}$$

where

$$\Delta_1 = I(n_- > n_+)\left(\ln^2\left(1 + \frac{n_-}{u_+}\right) - \ln^2\left(1 + \frac{n_+}{u_+}\right)\right) \leqslant I(n_- > n_+)\left(\ln \frac{n_- + u_+}{n_+ + u_+}\right)$$

$$(\ln(1+n_-) - \ln u_+) \cdot 2 \leqslant KI(n_- > n_+)\left(\ln \frac{n_-}{n_+}\right)(1 - \ln u_+)$$

$$\Delta_2 = I(u_+ > u_-)\left(\ln^2\left(1 + \frac{n_+}{u_-}\right) - \ln^2\left(1 + \frac{n_+}{u_+}\right)\right) \leqslant I(u_+ > u_-)\left(\ln \frac{1/n_+ + 1/u_-}{1/n_+ + 1/u_+}\right)$$

$$(\ln(1+n_+) - \ln u_-) \cdot 2 \leqslant KI(u_+ > u_-)\left(\ln \frac{u_+}{u_-}\right)(1 - \ln u_-)$$

and similarly

$$\Delta_3 \leqslant KI(u_+ < u_-)\left(\ln \frac{u_-}{u_+}\right)(1 - \ln u_+).$$

Remark that Δ_2 is used only when $1 < \frac{u_+}{u_-} < \frac{n_-}{n_+} \leqslant K_1$, so that, by modifying the constant K, one can replace the factor $(1 - \ln u_-)$ by $(1 - \ln u_+)$.

Obviously this formula remains valid – readjusting K – when reinterpreting $n_\pm$ (resp $u_\pm$) as $\sqrt{\text{previous } n_\pm \text{ (resp } u_\pm)}$.

Remarks also that in all three cases, one uses the smaller of the factors $\left|\ln \frac{u_+}{u_-}\right|$, $\left|\ln \frac{n_-}{n_+}\right|$. Thus we get simply

$$\int \Big| \sum_{\epsilon,\eta} \Big| d \ln^2 (1+z) \leqslant K \sum_{\epsilon,\eta} I(n_- > n_+)\left[\min\left(\ln \frac{n_-}{n_+}, \left|\ln \frac{u_+}{u_-}\right|\right)\right](1 - \ln u_+)$$

As already remarked before, the equivalence of norms in R^n implies that $\ln \frac{n_-}{n_+} \leqslant K' y_1$ ($\leqslant K'$). In particular, if we assume for instance $u_+ > u_-$, we can replace $\left|\ln \frac{u_+}{u_-}\right|$ by $\ln\left[\min\left(\frac{u_+}{u_-}, e^{K'}\right)\right] \leqslant K'' \frac{u_+ - u_-}{u_+}$. Now $u_+ > u_-$ is equivalent to $\epsilon_1 u > 0$, so $u_+ - u_- = \epsilon_1 u + x_1 - |\epsilon_1 u - x_1| = 2\min(\epsilon_1 u, x_1) = 2\min(|u|, x_1)$, and $u_+ = |u| + x_1$. So we can replace $\left|\ln \frac{u_+}{u_-}\right|$ by $\frac{\min(|u|, x_1)}{|u| + x_1}$ – if $u_+ > u_-$, and thus also in the dual case. Thus we get

$$\int \Big|\sum_{\epsilon,\eta}\Big| \, d \ln^2 (1+z) \leqslant C \sum_{\epsilon} \left(y_1 \wedge \frac{x_1 \wedge |u|}{|u| + x_1} \right) (1 - \ln |\langle \epsilon, x \rangle|).$$

But y_1 could have been any y_j, and in particular their minimum $\bigwedge_j y_j$. Using now our previous formula

$$\int_{\Delta_m} \operatorname{Min}(\beta, (y_j)_{j=1}^m) \frac{d\tau(y)}{\prod y_j} \leqslant K_m \beta |\ln \beta|^{m-1}$$

we get

$$\int_{\Delta_m} \frac{d\tau(y)}{\prod y_j} \int \Big|\sum_{\epsilon,\eta}\Big| \, d \ln^2 (1+z) \leqslant C' \sum_{\epsilon} (1 - \ln |\langle \epsilon, x \rangle|) y \, |\ln y|^{m-1}$$

where

$$y = \frac{x_1 \wedge |u|}{x_1 + |u|}.$$

We have to show that this is integrable $\frac{d\tau(x)}{\prod x_j}$, at all points of $\Delta_k \cap \{x_1 \leqslant x_i \ \forall\, i\}$ – if another coordinate was minimal, let this play the role of x_1.

There is no problem if $x_1 > 0$ – because $\ln |\langle \epsilon, x \rangle|$ is integrable for Lebesgue measure, and $y \, |\ln y|^{m-1}$ is bounded.

Fix now an ϵ. If $x_1 = 0$, and $|u| > 0$, then the factor $(1 - \ln |\langle \epsilon, x \rangle|)$ is locally bounded, and y is locally of the order of x_1, so that we have to show the integrability of $x_1 |\ln x_1|^r \frac{d\tau(x)}{\prod x_i}$ on $\{x_1 \leqslant x_i\}$ which we have already done before.

There remains thus only the case where $x_1 = u = 0$. In that case, as we argued already before, there exists two different coordinates j and j', different from 1, such that $x_j > 0$, $x_{j'} > 0$, $\epsilon_j \epsilon_{j'} = -1$. We can assume without loss of generality that $j = k$, $j' = k - 1$, and can change coordinates

$$x_1 \ldots x_k \to u, \quad x_1 \ldots x_{k-2}$$

using the equations $\sum_{i=2}^{k} \epsilon_i x_i = u$, $\sum_{i=1}^{k} x_i = 1$.

We therefore have in effect to prove that – assuming without loss of generality that $\epsilon_1 = 1$:

$$\int_{\substack{|u| \leq 1 \\ 0 \leq x_1 \leq x_i \leq 1}} (1 - \ln |u + x_1|) \frac{x_1 \wedge |u|}{x_1 + |u|} \ln^r \left(\frac{x_1 + |u|}{x_1 \wedge |u|} \right) du \prod_{i=1}^{k-2} \frac{dx_i}{x_i} < \infty$$

or, integrating over x_i for $i > 1$:

$$\int_{\substack{|u| \leq 1 \\ 0 \leq x \leq 1}} (1 - \ln |u + x|) \frac{x \wedge |u|}{x + |u|} \ln^r \left(\frac{x + |u|}{x \wedge |u|} \right) |\ln x|^{k-3} du \frac{dx}{x} < \infty$$

or

$$\int_0^1 \int_0^1 [1 - \ln |x - u|] \frac{x \wedge u}{x + u} \ln^r \left(\frac{x + u}{x \wedge u} \right) |\ln x|^s du \frac{dx}{x} < \infty$$

Replacing $|\ln x|^s$ by $|\ln (x \wedge u)|^s$, and $\frac{dx}{x}$ by $\frac{dx}{x \wedge u}$, one sees it is sufficient to consider $x \leq u$:

$$\int I(0 \leq x \leq u \leq 1)[1 - \ln (u - x)] \frac{1}{x + u} \ln^r \left(1 + \frac{u}{x} \right) |\ln x|^s du dx < \infty ?$$

Since $\ln^r \left(1 + \frac{u}{x} \right)$ can be written as a polynomial in $\ln (x + u)$ and $\ln u$, and since $|\ln (x + u)| \leq |\ln (u - x)|$, the whole thing amounts to prove that

$$\int_{0 \leq x \leq u \leq 1} |\ln (u - x)|^r |\ln x|^s \frac{du dx}{u + x} < \infty \quad (\text{whatever be } r \geq 0, s \geq 0)$$

or, letting $z = \frac{x}{u}$

$$\int_0^1 \frac{dz}{1+z} \int_0^1 (-\ln u - \ln(1-z))^r (-\ln u - \ln z)^s du < \infty$$

the integrand in the second integral is a polynomial in $\ln u$, whose coefficients are polynomials in $\ln z$ and $\ln(1-z)$.

Since any power of $\ln u$ is integrable, the first integral yields a polynomial in $\ln z$ and $\ln(1-z)$; since $1/1+z$ is bounded, the outer integral boils down to

$$\left(\int_0^{1/2} + \int_{1/2}^1\right) |\ln z|^r \, |\ln(1-z)|^s dz,$$

which is finite for the same reason.

This finishes the proof of Claim 3. □

It follows that

$$\lim_{M\to\infty} \int \frac{d\tau_{n-k}(y)}{\Pi y_j} \int \frac{d\tau_k(x)}{\Pi x_j} \phi_M(x,y) = \int \frac{d\tau_{n-k}(y)}{\Pi y_j} \int \frac{d\tau_k(x)}{\Pi x_j} (\lim_{M\to\infty} \phi_M(x,y))$$

$$= -2 \left[\int_{\Delta_k} \frac{\sum_{\epsilon} (\Pi \epsilon_j) \ln |\langle \epsilon, x \rangle|}{\Pi x_j} d\tau_k(x) \right] \left[\int_{\Delta_{n-k}} \frac{\sum_{\eta} (\Pi \eta_j) \ln N(0 \cdot x, \eta \cdot y)}{\Pi y_j} d\tau_{n-k}(y) \right]$$

and therefore that

$$\lim_{\eta\to 0} \lim_{M\to\infty} \int_{s\in\Delta_n, \sum_{i\leqslant k} s_i \leqslant \eta} \frac{1}{2} \frac{\sum_{\epsilon} (\Pi \epsilon_j) \ln [1 + (M(\sum_{j\leqslant k} \epsilon_j s_j)/N(\epsilon \cdot s))^2]}{\Pi s_j} d\tau_n(s)$$

$$= -A_k \int_{\Delta_{n-k}} \frac{\sum_{\eta} (\Pi \eta_j) \ln N(0 \cdot x, \eta \cdot y)}{\Pi y_j} d\tau_{n-k}(y)$$

where

$$A_k = \int_{\Delta_k} \frac{\sum_{\epsilon} (\Pi \epsilon_j) \ln |\langle \epsilon, x \rangle|}{\Pi x_j} d\tau_k(x).$$

If we use also $A_0 = -1$, we get therefore

$$P(\nu_i \geqslant 0 \ \forall \ i) = (-1)^{n/2}(2\pi)^{-n} \left[-A_n + A_0 \int_{\Delta_n} \frac{\sum_{\eta} (\Pi \eta_j) \ln N(\eta \cdot y)}{\Pi y_i} d\tau_n(y) \right.$$

$$\left. + \sum_{\substack{\partial \in \{0,1\} \\ \partial \neq (0,\ldots,0),(1,\ldots,1)}} A_{\Sigma\delta_j} \int_{\substack{y \in \Delta_n \\ \langle \delta, y \rangle = 0}} \frac{\sum_{\eta \in \{-1,1\}^n, \eta_i \delta_i \geqslant 0} (\Pi \eta_j) \ln N(\eta \cdot y)}{\prod_{j:\delta_j=0} y_j} d\tau_{n-\Sigma\delta_j}(y) \right]$$

(remarking that $A_k = 0$ if k is odd). Thus:

$$P(\nu_i \geqslant 0 \ \forall \ i) = (-1)^{n/2}(2\pi)^{-n} \sum_{\delta \in \{0,1\}^n} A_{\Sigma\delta_j} \int_{\substack{\| y \| = 1 \\ \forall j : \delta_j y_j = 0}} \frac{\ln N(y)}{\prod_{j:\delta_j=0} y_j} d\tau_{n-\Sigma\delta_j}(y)$$

where

- $\tau_{n-\Sigma\delta_j}$ is Lebesgue measure on the corresponding set,
- the integrals are Cauchy principal values
- for $\delta = (1, \ldots, 1)$, the integral over the empty set is set to -1
- $A_0 = -1, \quad A_k = \int_{\Delta_k} \frac{\sum_{\epsilon \in \{-1,1\}^k} (\Pi \epsilon_j) \ln |\langle \epsilon, x \rangle|}{\Pi x_j} d\tau_k(x).$

If each ν_i has norm 2, total mass zero, and all ν_i are mutually singular, $P(\nu_i \geqslant 0 \ \forall \ i) = 2^{-n}$ obviously. Also $R_\nu = [-1, 1]^n$, so that $N(y) = \sup_{x \in [-1,1]^n} \langle x, y \rangle = \Sigma |y_i| = \|y\|$.

Since we integrate on the unit ball, it follows that in this case our equation yields

$$2^{-n} = (-1)^{n/2}(2\pi)^{-n}[-A_n]$$

thus

$$A_n = -(-1)^{n/2}\pi^n$$

thus

$$\begin{aligned}(-1)^{n/2}(2\pi)^{-n}A_k &= -2^{-n}[(-1)^{n/2}\pi^{-n}(-1)^{-k/2}\pi^k] \\ &= -2^{-n}[(-1)^{-(n-k)/2}\pi^{-(n-k)}] \\ &= \frac{2^{-n}}{A_{n-k}}.\end{aligned}$$

Thus:

$$P(\nu_i \geqslant 0 \ \forall\ i) = 2^{-n}\left[1 - \sum_{\substack{\delta\in\{0,1\}^n \\ \delta\neq 0 \\ \Sigma\delta_j \text{ even}}} \frac{(-1)^{\Sigma\delta_j/2}}{\pi^{\Sigma\delta_j}} \int_{\substack{\|y\|=1 \\ \forall j:(1-\delta_j)y_j=0}} \frac{\ln N(y)}{\prod_{j:\delta_j=1} y_j} d\tau_{\Sigma\delta_j}(y)\right]$$

This formula is valid for the case n even, say $n = 2k$. But $P(\nu_1 \ldots \nu_{2k-1} \geqslant 0) = P(\nu_1 \ldots \nu_{2k} \geqslant 0) + P(\nu_1, \nu_2, \ldots, \nu_{2k-1}, -\nu_{2k} \geqslant 0)$ can be computed from this formula, and yields then the same formula with $n = 2k-1$: thus the formula is valid for all $n \geqslant 1$.

Thus, for all $\#I \geqslant 1$:

$$P(\nu_i \geqslant 0 \ \forall\ i \in I) = 2^{-(\#I)}\left[1 - \sum_{\substack{\phi\neq J\subseteq I \\ [\#J \text{ even}]}} \frac{1}{(\pi i)^{\#J}} \int_{\substack{y\in R^J \\ \|y\|=1}} \frac{\ln N_J(y)}{\prod_{j\in J} y_j} d\tau(y)\right]$$

where

$$N_J(y) = \sup_{\|x\|\leqslant 1} \sum_{j\in J} y_j \nu_j(x).$$

Remark that, by the symmetry of the norm ($N_j(y) = N_j(-y)$), the restriction to $\#J$ even is not necessary: the integral will be zero for $\#J$ odd.

The integrals have to be understood as Cauchy principal values, in the following sense: define a set $C \subseteq R^J$ to be symmetric iff $y \in C .\equiv. (|y_j|)_{j \in J} \in C$; say that C consists only of non zero elements iff $y \in C \Rightarrow y_j \neq 0 \ \forall j \in J$. Then the integral is to be understood as the limit of the integrals over an arbitrary sequence of closed symmetric sets C_i consisting only of non zero elements, and such that the measure of the complement of C_i goes to zero.

The norm $\|y\|$ used to derive the formula was the l_1-norm $\Sigma |y_j|$, but the formula of change of variables for surface measures yields now that it remains valid for any norm $\|\cdot\|$ on R^J such that $\|(y_1 \dots y_k)\| = \|(|y_1|, |y_2|, \dots, |y_k|)\|$.

The same formula permits to rewrite our expression using the surface measure σ_J on the unit sphere of the norm N_J:

$$P(v_i(\chi) \geqslant 0 \ \forall \, i \in I) = 2^{-\#I} \left[1 + \sum_{\substack{\phi \neq J \subseteq I \\ [\#J \text{ even}]}} \frac{1}{(\pi i)^{\#J}} \int \frac{\ln \|y\|}{\prod_{j \in J} y_j} \, d\sigma_J(y) \right].$$

Remark: Obviously the formula we got is not very transparent – this may be due to the fact that it has to reflect the peculiar geometry of the positive orthant. It would therefore be interesting to have also an expression for the density over directions – i.e. on projective space.

Section 4: To Mess Everything up: Some Extension Possibilities

1 Extension of the Cylinder Measure

Given a cylinder measure μ on a locally convex space E with dual E', one can use Kolmogorov's existence theorem for a projective limit of measures as done in the proof of Theorem 1, and a Hamel basis of E', to obtain an equivalent characterization of μ as a countably additive measure on the Baire σ-field of the weak completion $\tilde{E}$ of E, using also a recent result of Edgar[4].

Using this, one can then best define the corresponding integral in the following way: let α vary in the increasing net of all finite subsets of E'. For any α, and any

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[4] "Measurability in a Banach space" Indiana University Mathematics Journal, Vol. 26, n° 4, pp. 663–677 (1977).

$x \in \bar{E}\backslash E$, let $V_\alpha(x) = \{y \in E | \varphi(y) = \varphi(x)\ \forall \varphi \in \alpha\}$. For any function f on E, define its extension $\bar{f}$ to $\bar{E}$ by $\bar{f}(x) = \lim_\alpha \sup_{y \in V_\alpha(x)} f(y)$ at all $x \in \bar{E}\backslash E$. Finally define the upper integral $\bar{\mu}(f)$ as the upper integral of $\bar{f}$ for the countably additive measure μ on the baire σ-field of $\bar{E}$.

Given $\bar{\mu}$, one can use finitely additive integration theory in the standard way (cfr for instance Dunford and Schwartz, Linear Operators, Part I). More precisely, one has:

1) $\bar{\mu}(1) = 1, \bar{\mu}(-1) = -1, \bar{\mu}$ is monotonic;

2) $\alpha > 0$ implies $\bar{\mu}(\alpha f) = \alpha\bar{\mu}(f)$;

3) $\bar{\mu}(f+g) \leqslant \bar{\mu}(f \vee g) + \bar{\mu}(f \wedge g) \leqslant \bar{\mu}(f) + \bar{\mu}(g)$ whenever $\bar{\mu}(f) < +\infty$, $\bar{\mu}(g) < +\infty$ (the first inequality is subadditivity, the second follows from the corresponding formula for upper integrals, and from $\bar{f} \vee \bar{g} = \overline{f \vee g}$, $\bar{f} \wedge \bar{g} = \overline{f \wedge g}$);

4) $\bar{\mu}(f \vee (-n)) \to \bar{\mu}(f)\ \forall f, \bar{\mu}(f \wedge 0) > -\infty \Rightarrow \bar{\mu}(f) = \lim_{n\to\infty} \bar{\mu}(f \wedge n)$.

Those properties immediately imply that $L = \{f | \bar{\mu}(f) + \bar{\mu}(-f) \leqslant 0\}$ is a vector lattice containing the constants, and that $\bar{\mu}$ is a positive linear functional on L. Hence $\mathcal{A} = \{A | I_A \in L\}$ is a boolean algebra and $\bar{\mu}$ a finitely additive probability on $\mathcal{A}$ (also denoted μ).

Hence $f \in L$ and $s < t$ imply $\mu_*\{f > s\} \geqslant \mu^*\{f \geqslant t\}$ (reduce to $s = 0 \leqslant f \leqslant t = 1$, then $\int f d\mu$ is in between – we use $\mu_*(A) = \sup\{\bar{\mu}(f) | f \in L, f \leqslant I_A\}$, and $\mu^*(A) = \inf\{\bar{\mu}(f) | f \in L, f \geqslant I_A\}$). Therefore, if $f \in L$, then for all but countably many t's, $\mu^*\{f \geqslant t\} = \mu_*\{f > t\}$: $\{f \geqslant t\}$ and $\{f > t\}$ are in $\mathcal{A}$. Hence any bounded $f \in L$ can be approximated uniformly by $\mathcal{A}$-measurable step functions, and thus

$$L \subseteq \mathcal{L}_1(\mathcal{A}, \mu), \quad \text{with} \quad \bar{\mu}(f) = \int f d\mu \quad \text{for } f \in L.$$

Conversely properties (1) to (4) imply also that

$$\forall f, \forall (f_n)_{n \in \mathbb{N}}, [\bar{\mu}(f_n) > -\infty, \bar{\mu}(I\{f_n < f - \epsilon\}) \underset{n\to\infty}{\to} 0\ \forall \epsilon > 0,$$

$$\limsup_{n\to\infty} \sup_{k \geqslant n} \bar{\mu}[(f_k - f_n)^+] = 0] \Rightarrow \lim \bar{\mu}(f_n) \geqslant \bar{\mu}(f)$$

and hence that, if f_n is a Cauchy sequence in L converging in $\bar{\mu}$-measure to f, then $f \in L$ is the norm limit of f_n. In particular, choosing $f \in \mathcal{L}_1(\mathcal{A}, \mu)$ and f_n step functions, one obtains $L = \mathcal{L}_1(\mathcal{A}, \mu)$.

One concludes now easily that $\bar{\mu}$ is at least as good as the finitely additive integral: $\forall f,\ \bar{\mu}(f) \leqslant \int^* f d\mu$. Obviously, $L = L_1(A, \mu)$ contains both the cylindrically integrable functions and the bounded continuous functions on $\bar{E}$.

Of course, one could still get conceivably more integrable functions by refining $\bar{\mu}$ – for instance if one could prove τ-smoothness of μ on $\bar{E}$, one could use its regular extension to the borel sets of $\bar{E}$ for defining $\bar{\mu}$; or one could try to get a lower $\bar{f}$, for instance by restricting the $y \in V_\alpha(x)$ to be of essentially minimal norm.

2 Using More Smooth Cylinder Measures

By Theorem 1, the invariant cylinder measures corresponding to different pairs (m, σ) are mutually singular. Thus the integral of a function – and even its integrability – may depend in a highly irregular way on the pair (m, σ). To smooth this out, one could choose m and σ by some probability distribution $P(m, \sigma)$. Since the correspondence preserves convolution, and because of the idea that in some sense the sum of two independent random elements of $B(I, C)$ is a fortiori random, one should certainly take P absolutely continuous with respect to Lebesgue measure, and in some sense invariant under convolution. Since

$$\int D^\lambda_\chi(\tilde{\chi})dQ(\chi) = \int D^\lambda_{m+\sigma\chi}(\tilde{\chi})dP(m, \sigma)dQ_0(\chi)$$

$$\int D^1_{(m\lambda+\sigma\lambda\chi)}(\tilde{\chi})dP(m, \sigma)dQ_0(\tilde{\chi})$$

$$\int D^1_{m+\sigma\chi}(\tilde{\chi})dP^\lambda(m, \sigma)dQ_0(\chi)$$

we see that for defining the value, we consider integrals of a fixed function with respect to the distribution P^λ of $(\lambda m, \lambda\sigma)$ (where (m, σ) is P-distributed), and let the scale factor λ go to ∞. Asking that this family P^λ be invariant under convolution is asking that P be stable. This leads to choose m and σ independently, m with the symmetric stable distribution of index α, and σ with the one side stable distribution of index α (thus $\alpha < 1$). The lim sup of the (upper-) integrals when the scale factor λ goes to ∞ is then clearly decreasing when $\alpha \downarrow 0$, since for $\beta < \alpha$ the stable distribution with index β can be viewed as a mixture of stable distributions with index α (choosing their scale factors according to the stable one-sided distribution with index β/α).

One is thus led to a formulation of the following type:

– for any bounded measurable function f on R_+, let

$$p(f) = \lim_{\alpha \to 0} \limsup_{\lambda \to \infty} \int_0^\infty f(\lambda x)dP^\alpha(x).$$

where P^{α} is a one sided stable distribution with index α (its scale factor does not matter).

- Let $\bar{\mu}$ denote a suitable extension (cfr n° 1) of the invariant cylinder measure where m and σ are chosen independently with stable distributions of index α ($\alpha < 1$) – symmetric for m and one sided for σ.

- If f is a function of several variables, let $p_x(f)$ denote p of the function of a real variable x obtained by holding all variables but x fixed in f. Similarly $\bar{\mu}_{\chi}(\varphi)$ will indicate that all variables but χ are held fixed in φ.

- Let $\bar{\varphi}_v(\ddot{\chi}) \mp p_{\lambda}(\bar{\mu}_{\chi}(D_{\chi}^{\lambda}(\ddot{\chi})))$, and $\underline{\varphi}_v(\ddot{\chi}) \mp -p_{\lambda}(\bar{\mu}_{\chi}(-D_{\chi}^{\lambda}(\ddot{\chi})))$: then v has a value φ_v if $\bar{\varphi}_v = \underline{\varphi}_v$ and is additive.

3 Interverting More Limits and Integrals?

As a general rule, one gets functionals with a large domain by averaging before going to limits rather than after – in our context, this was already illustrated in "Values and Derivatives". Now the v appearing in the above formula for $\bar{\varphi}_v$ is not the given game, but obtained from it by the operator ψ of section 1, which itself involves both averaging and limit operations.

Let us first show how ψ could be replaced by an operator – say $\bar{\bar{\psi}}$ – where the averaging occurs before the limit operations, in order to remove as much as possible the basic restriction that we can talk only of games that have an extension in some sense.

Remark first, that it is sufficient to compute $w(\chi) = \psi(v)(\chi)$ for step functions χ – either because $V(w)[\chi, \chi'] \leqslant \|\chi' - \chi\| \cdot \|w\|$ implies (for $\|w\| < \infty$) that w can anyway be uniquely extended to $B(I, C)$, or using the fact that the cylinder measures on $B(I, C)$ are also cylinder measures or the space of step functions $\in B(I, C)$.

We return to the basic idea underlying the proof in [5] of both theorem B and its application to the extension of games – that was used in Section 1.

Let χ denote a step function, and let π denote a finite measurable partition such that χ is constant on every element of π.

Given any vector ν of non atomic elements of FA, let, for any $A \in \pi$, O_t^A denote an increasing family of measurable subsets of A with $\nu(O_t^A) = t\nu(A)$ ($\forall\, t : 0 \leqslant t \leqslant 1$) (and with $O_0^A = \phi$, $O_1^A = A$).

For any $n > 0$, for any permutation σ of $\{1, \ldots, n\}$, and for $\epsilon \in \{-1, 1\}^n$, define $X_O^i = O_{i/n}^A \setminus O_{(i-1)/n}^A$, and, denoting (σ, ϵ) by ω, let $O_t^{A,\omega}$ be defined by $(\bigcup_{i \leqslant nt} X_O^{\sigma(i)})$

$\cup B_{\sigma([nt]+1)}$ where $[x]$ denotes the integer part of x, and $B_k = O^A_{((k-1)/n)+t_n} \setminus O^A_{(k-1)/n}$ if $\epsilon_k = 1$, $B_k = O^A_{k/n} \setminus O^A_{(k/n)-t_n}$ if $\epsilon_k = -1$ – and where $t_n = t - \frac{[nt]}{n}$.

Then, for every ω and t we still have $\nu(O_t^{A,\omega}) = t\nu(A)$, and if ω is chosen at random, we have, for all $x \in A$, $|P(x \in O_t^{A,\omega}) - t| \leqslant \frac{1}{2n}$. (The ϵ is not strictly necessary, it is just introduced to preserve the symmetry with the opposite order.)

Let now, for any π, and any collection $\mathcal{O}^{\pi,\nu}$ of such increasing families $(O_t^A)_{\substack{A \in \pi, \\ 0 \leqslant t \leqslant 1}}$ and any n, Ω_n denote the finite probability space where independently for each $A \in \pi$, some $\omega = \omega_A$ is chosen at random.

Also. for any π-measurable ideal set function χ, and any $\omega \in \Omega_n$, let $\chi_\omega = \bigcup_{A \in \pi} O^{A,\omega_A}_{\chi(A)}$: then $\chi^1 \leqslant \chi^2 \Rightarrow \chi^1_\omega \leqslant \chi^2_\omega$, $\nu(\chi_\omega) = \nu(\chi)\ \forall\ \chi$, and $\|E(\chi_\omega) - \chi\| \leqslant 1/(2n)$.

Let also, for a general π-measurable function χ,

$$\chi_\omega = [\max(0, \min(1, \chi))]_\omega .$$

Define now

$$\psi^{\tau,v}_{\pi,\mathcal{O}^{\pi,\nu},n}(\chi) = \frac{1}{2\tau} \int_0^1 (E_{\Omega_{(n!)}}(v[(t+\tau\chi)_\omega] - v[(t-\tau\chi)_\omega]))dt$$

(where v still the constant sum game denotes corresponding to the originally given game).

For any given π, and any vector ν, denote by $F_{\nu,\pi}$ the set of all possible families $\mathcal{O}^{\pi,\nu}$. For any given π, the $F_{\nu,\pi}$ form a filter $\mathcal{F}_\pi$, when ν ranges over the increasing filtering set of finite subsets of the nonatomic elements of FA.

Similarly the partitions π can be ordered by refinement. Then $\lim_{\tau \to 0} \lim_{\pi} \lim_{\mathcal{F}_\pi} \lim_{n \to \infty} \psi^{\tau,v}_{\pi,\mathcal{O}^{\pi,\nu},n}(\chi) = \psi^v(\chi)$ should be the analog of our ψ from Section 1 but with all limits done after any averaging.

More formally, define a filter $\mathcal{F}$ on 4-tuples $(\tau, \pi, \mathcal{O}^{\pi,\nu}, n)$ [more formally on $(\underset{\sim}{R} \times (\sum_\pi F_{\phi,\pi}) \times \underset{\sim}{N})$] by $F \in \mathcal{F}$ if $\exists\ \epsilon : \forall\ \tau : 0 < |\tau| < \epsilon\ \exists\ \pi_0 : \forall\ \pi > \pi_0,\ \exists\ \nu(=(\nu_1, \ldots \nu_k)) : \forall\ \mathcal{O}^{\pi,\nu} \in F_{\nu,\pi}\ \exists\ n_0 : \forall\ n \geqslant n_0\ (\tau, \pi, \mathcal{O}^{\pi,\nu}, n) \in F$.

Then, we define ψ by $v \in \mathrm{Dom}(\psi) \equiv [\lim_{\mathcal{F}} \psi^{\tau,v}_{\pi,\mathcal{O}^{\pi,\nu},n}(\chi)$ exists for any step function $\chi] \Rightarrow [\psi(v)](\chi) = \lim_{\mathcal{F}} \psi^{\tau,v}_{\pi,\mathcal{O}^{\pi,\nu},n}(\chi)\ \forall\ \chi$ step function.

Obviously Dom (ψ) is a closed (using $\|\psi\| = 1$), symmetric space, and ψ is a positive linear symmetric operator on Dom (ψ). Further $\|\psi\| = 1$ – this follows from completely similar computations as those in Section 1, and is the main point where the specific structure of the $0^{\pi,\nu}$ $(\chi \leqslant \chi' \Rightarrow \chi_\omega \leqslant \chi'_\omega \ \forall\ \omega)$ is used. Similarly one gets, under mild continuity assumptions on v at ϕ and I, that $[\psi(v)](I) = v(I)$.

under mild continuity assumptions on v at ϕ and $I\left(\lim_{F} \int_0^1 E_{\Omega_{(n!)}}[v(\tau(u+\chi)^+) + v(\tau(u+\chi)^-)]du = 0\ \forall \chi\right)$, that $[\psi(v)](I) = v(I)$ and $\psi(v)$ is linear on every plane containing the constants.

One could thus use this ψ followed by the operation described in the previous section. However it is now tempting – and possible – to put all averagings before any limit operation.

But to do this, one may want to consider an alternative to integrating with respect to an appropriate extension of the (finitely additive) cylinder measure – in order to sidestep the difficulties of finitely additive integration theory (in what concerns the integrability of functions, and in what concerns changing the order of integration and the permutation of limits and integrals – although the old paper[5] helps a good way for those last two questions).

The cylinder measure Q can be obtained – as shown in the proof of Theorem 1 – in the following way: first select m and σ at random according to P, next, for any partition π, select independently on each partition element the (constant) value of χ on that partition element as a Cauchy (m, σ) random variable. This gives an approximation Q_π to Q, that converges weakly to Q on $\bar{E}$ when π is refined. Q_π is a (countably additive) probability carried by the finite dimensional subspace of $B(I, C)$ of all π-measurable step functions.

Now the operator D and the averaging for Q_π can without problem be pushed before the $\lim_{F}$, together with all other averagings there is no integrability problem at least if v is of bounded variation. On the other hand the limit over all refinements of π is best retained after the $\lim_{F}$ has been done (and before the lim over λ (p_λ)).

This was just to point out that the formulation adopted in this paper is by no means unique or optimal – and that in particular one could to some extent dispense altogether with the assumption that the game has an extension. It was adopted chiefly for expository reasons.

Certainly a lot remains to be done – i.e., convincing theorems – to get a good formulation.

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[5] J. F. Mertens: "Intégration des mesures non dénombrablement additives: une généralisation du lemme de Fatou et du théorème de convergence de Lebesgue". Annales de la Société Scientifique de Bruxelles, t. 84, 88, 231–239 (1970).

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