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The Equilibria of a Multiple Objective Game

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Abstract: For a multiple objective game, we introduce its cooperative, non-cooperative, hybrid and quasi-hybrid solution concepts and prove their existence. JEL #: C70, C71, C72

1 Introduction

The conflicting interests or objectives are a perpetual object about which economics and other social sciences concern. There are essentially three classes of conflicting interests: the conflicts among several decision makers, the conflicts within an individual, and the conflicts in both cases. The first class of conflicts, the conflicting interests among several decision makers, is studied in the conventional game theory, where each player tries to maximize a scalar payoff. The second class of conflicts, the conflicting preferences within an individual, appears to be the subject of individual decision theory. Because single objective optimization problems are technical in that one could ask an expert or a computer to solve them, the individual decision making is conceptually non-trivial only in multiple objective mathematical programming (MOMP)[2]. While the third class of conflicts, the conflicting interests involved among several decision makers as well as within each individual, is the subject of multiple objective game (MOG)[3], which happens whenever players in a game have multiple objectives or a vector payoff to optimize.

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[2] MOMP is also called Multiple Criterion Decision Making (MCDM), or Multiple Attribute Problem or Vector Maximization.

[3] It is also called Games with Vector Payoffs or Multi-Payoff Game. We use the term of Multiple Objective Game (MOG) for its direct relation to the popular term multiple objective mathematical programming.

[4] It is well known that his paper is important in establishing the underlying foundation of Dynamic programming. Another important aspect of his paper – the MOG model was, unfortunately, not followed up by either many others or the author himself, perhaps because the state of art in both MOMP and game theory were not robust enough during the time.

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Though more than 35 years have passed since the first publication on MOG by Blackwell (1956)[4], only few subsequent papers appeared: Shapley (1959), Contini et al. (1966), Zeleny (1975), Hannan (1982), Charnes et al. (1987), Borm et al. (1988) and Charnes et al. (1990). These papers share two common features: they all consider the non-cooperative solutions; all of them except those of Charnes et al. deal with the Two-person games, and all these Two-person game studies except that of Contini et al. are restricted to Two-person matrix games.

Blackwell formalized the Two-person zero-sum vector matrix game, which is also the topic of Shapley and Zeleny. Blackwell concerns only the minimax property, Shapley shows the existence of strategic equilibria assuming each player will choose a weakly efficient or efficient solution given the choices of the rivals, while Zeleny uses the method of linear multiple objective mathematical programming to deal with the same problem. Hannan's paper is a short note indicating one error in the example of Zeleny's paper. Borm et al. consider the general Two-person matrix game (non-zero sum) and study its comparative statics (the continuity of solutions with respect to parameters in the game). Contini et al. begin with a general Two-person MOG but end up with a MOMP, the player simply maximizes his expected vector payoff because the other player is the Nature. While Charnes et al. (1987) study the more general n-person MOG where all players' choices are limited to a cross-constrained set, and both their results (1987, 1990) are also limited to the non-cooperative solutions.

Here we are interested in the general n-person MOG problem. We shall define the cooperative, non-cooperative, hybrid and quasi-hybrid solution concepts in the next section of the paper. As the readers shall see, the cooperative and non-cooperative solutions are two particular kinds of hybrid solutions. In Section 3, we shall provide the sufficient conditions for the existence of these solutions, and we shall conclude the paper with some remarks in Section 4.

2 Definitions of Equilibria

A multiple objective game (MOG) is defined as $\Gamma = \{N, X^i, u^i\}$, where $N = \{1,2,\ldots,n\}$ is the set of players. For each $i \in N$, X^i is player i's strategy set, which is assumed to be a non-empty subset in some finite-dimensional Euclidean space ($X^i \subset \mathbf{R}^{L(i)}$), $u^i : X = \Pi_{i=1}^n X^i \to \mathbf{R}^{m(i)}$ is player i's vector payoff, which is a real m_i-dimensional vector function. We shall adopt the following notations: For any two vectors a, b $\in \mathbf{R}^n$, $a \geq b \Leftrightarrow a_i \geq b_i$, all i ; $a > b \Leftrightarrow a \geq b$ and $a \neq b$; $a >> b \Leftrightarrow a_i > b_i$, all i. For any number in the forms of m_i, s_i etc., they shall be changed to $m(i)$ and $s(i)$ when they appear as subscript or subscript.

It is clear that MOG differs from the conventional game only in the payoff functions. Here each player has a vector payoff to optimize, while in the conventional games players all have scalar payoffs to optimize. So when $m_1 = \ldots = m_n = 1$, Γ becomes the conventional n-person game in normal form; when $n = 1$, $m_1 > 1$, Γ becomes the standard MOMP problem; and when $n = m_1 = 1$, Γ becomes the single objective mathematical programming problem. Thus MOG is much more general than either the conventional game theory or the MOMP model. Judging from the wide applications of the conventional game theory and MOMP, one can imagine the much wider applications of MOG that will emerge in the near future.

For future development, we shall review the concept of proper efficiency in MOMP (Geoffrion, 1968) and its existence (Lemma 1). A general multiple objective mathematical programming problem is defined as:

$$VM^5: \quad \underset{x \in Y}{\text{Max}}\, F(x) = \{f_1(x),\ldots,f_m(x)\}, \quad m > 1, \tag{1}$$

where Y is the feasible choice set, which is assumed to be a non-empty subset in some finite-dimensional Euclidean space, $f_i : Y \rightarrow \mathbf{R}$, $i = 1,\ldots, m$, are the objective functions to be maximized. Clearly, this is a degenerated MOG problem for the case of $n = 1$ and $m_1 = m > 1$.

For the above MOMP problem, its Weakly Efficient Solution set is defined as:

$$R^*_{we} = \left\{\bar{x} \in Y \mid \{x \in Y \mid F(x) >> F(\bar{x})\} = \emptyset\right\},$$

that is, $\bar{x}$ is a Weakly Efficient Solution if there is no $x \in Y$ such that $f_i(x) > f_i(\bar{x})$ for all i; its Pareto Efficient Solution or Efficient Solution set is defined as:

$$R^*_e = \left\{\bar{x} \in Y \mid \{x \in Y \mid F(x) > F(\bar{x})\} = \emptyset\right\},$$

that is, $\bar{x}$ is an Efficient Solution if there is no $x \in Y$ such that $f_i(x) \geq f_i(\bar{x})$ for all i and there is at least one j satisfying $f_j(x) > f_j(\bar{x})$; and its Properly Efficient or Properly Pareto Efficient Solution (Geoffrion, 1968) is defined as:

Definition 1: An efficient solution $\bar{x}$ ($\bar{x} \in R^*_e$) is a **Properly Efficient Solution**, if there is $L > 0$ such that for any i and $x \in Y$ satisfying $f_i(x) > f_i(\bar{x})$, there is always a $j \neq i$ such that $f_j(x) < f_j(\bar{x})$ and $\dfrac{f_i(x) - f_i(\bar{x})}{f_j(\bar{x}) - f_j(x)} \leq L$.

Let $\Delta^m_+ = \{\lambda \in \mathbf{R}^m \mid \lambda_i \geq 0, \Sigma^m_{i=1} \lambda_i = 1\}$ denote the $m-1$ simplex, and $\Delta^m_{++} = \{\lambda \in \mathbf{R}^m \mid \lambda_i > 0, \Sigma^m_{i=1} \lambda_i = 1\}$ be the interior of Δ^m_+. For each $\lambda \in \Delta^m_+$, let $f_\lambda(x) = \lambda' F(x) = \Sigma^m_{i=1} \{\lambda_i f_i(x)\}$, and $R^*(\lambda) = \{\bar{x} \in Y \mid \forall\, x \in Y, f_\lambda(\bar{x}) \geq f_\lambda(x)\} =$ Arg-Max $\{f_\lambda(x) \mid x \in Y\}$, where Arg-Max $\{f(x) \mid x \in Y\}$, denote the solution set of any optimization problem Max $\{f(x) \mid x \in Y\}$. Let R^*_{pe} denote the set of all properly efficient solutions, then Geoffrion's representation theorem (1968) can be given as:

Lemma 1: If the objective functions $f_i(x)$, $i = 1,2,\ldots, m$, are all continuous and concave in x, and Y is a convex and compact set, then

[5] *VM* stands for vector maximization.

$$R^*_{pe} = \bigcup_{\lambda \in \Lambda^m_{++}} \{R^*(\lambda)\}.$$

Now let us define the solutions for a MOG problem. Let $\mathcal{N}$ denote the set of all nonempty subsets of N, then each element of $\mathcal{N}$ represents a coalition of players. For each coalition $S \in \mathcal{N}$, let $|S|$ denote the number of elements in S, and $\mathbf{R}^S = \Pi_{i\in S}\mathbf{R}^{m(i)}$ denote the $\{\Sigma_{i\in S} m_i\}$-dimensional Euclidean space whose coordinates have as groups of subscripts the members in S. For any $x = \{x^1,...,x^n\} \in X$, $u = \{u^1,...,u^n\} \in \mathbf{R}^N = \Pi_{i\in N}\mathbf{R}^{m(i)}$, where $x^i = \{x^i_1, x^i_2,...,x^i_{L(i)}\} \in X^i$, $u^i = \{u^i_1, u^i_2,...,u^i_{m(i)}\} \in \mathbf{R}^{m(i)}$, let $x_S = \{x^i \mid i \in S\} \in X_S = \Pi_{i\in S} X^i$ be the strategies of coalition S; $x_{-S} = x_{N\setminus S} = \{x^i \mid i \notin S\} \in X_{-S} = \Pi_{i\notin S} X^i$ be the strategies of the players not in the coalition S (or in the complementary coalition $N\setminus S$); $u_S = \{u^i \mid i \in S\} \in \mathbf{R}^S$ and $u_{-S} = u_{N\setminus S} = \{u^i \mid i \notin S\} \in \mathbf{R}^{-S} = \Pi_{i\notin S}\mathbf{R}^{m(i)}$ be the projections of u on $\mathbf{R}^S$ and $\mathbf{R}^{-S}$ respectively; $\bar{u}_S : X_S \to \mathbf{R}^S$ be the worst vector payoffs to S and be defined as

$$\bar{u}_S(x_S) = \{\bar{u}_S(x_S)_i \mid i \in S\} \in \mathbf{R}^S$$

for each $x_S \in X_S$, where for each $i \in S$,

$$\bar{u}_S(x_S)_i = \{\inf_{x_{-S}\in X_{-S}} u^i_j(x_S, x_{-S}) \mid j = 1,2,...,m_i\} \in \mathbf{R}^{m(i)}$$

is player i's worst or guaranteed vector payoff, given coalition's choice x_S. For each coalition $S \in \mathcal{N}$, we shall write $(x_S, x_{-S}) = x$ for convenience. The cooperative and non-cooperative solutions are then defined as:

Definition 2: A joint strategy $\bar{x} = \{\bar{x}^1, \bar{x}^2,...,\bar{x}^n\} \in X$ is a **Nash equilibrium** of the MOG $\Gamma = \{N, x^i, u^i\}$, if for each $i \in N$, $\bar{x}^i$ is a properly efficient solution of the vector maximization problem

$$VM_i(x_{-i}): \tag{2}$$
$$\max_{x^i\in X^i}\{u^i(x^i,\bar{x}_{-i}) = [u^i_1(x^i,\bar{x}_{-i}), u^i_2(x^i,\bar{x}_{-i}),..., u^i_{m(i)}(x^i,\bar{x}_{-i})]\}.$$

That is, $\bar{x}$ is a Nash equilibrium if each player i chooses a properly efficient solution $\bar{x}^i$ as a best response to all others' strategies $\bar{x}_{-i}$.

Definition 3: A joint strategy $\bar{x} = \{\bar{x}^1, \bar{x}^2,...,\bar{x}^n\} \in X$ is a **core solution**[6] of the general MOG $\Gamma = \{N, X^i, u^i\}$, if for each coalition $S \in \mathcal{N}$,

$$\Omega_S(\bar{x}) = \{x_S \in X_S \mid \bar{u}_S(x_S) >> u(\bar{x})_S\} = \emptyset.$$

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6 As in the conventional game theory, the core so defined follows Aumman's idea of α-core (1961).

Where $\overline{u}_S(x_S)$ is the worst payoff of coalition S when x_S is chosen, and $u(\overline{x})_S$ is the projection of $u(\overline{x})$ onto $\mathbf{R}^S$. In other words, $\overline{x}$ is a core solution if no coalition S can, by choosing another strategy available to S, guarantee a higher payoff for each of its members indepedently of the actions of the outside players.

Similarly, a vector $y \in \mathbf{R}^N$ is a core vector (or in the core) of the MOG Γ if it is feasible and if for each coalition $S \in \mathcal{N}$,

$$\Phi_S(y) = \{x_S \in X_S \mid \overline{u}_S(x_S) >> y_S\} = \emptyset.$$

By feasible we mean that there exist $x \in X$ such that $u(x) = u_N(x) \geq y$. Thus a joint strategy x is a core solution if and only if $u(x)$ is in the core.

If the MOG in Definitions 1 and 2 are degenerated to a game with scalar payoffs, the two solutions so defined are exactly the same as their counterparts in the conventional game theory. In the non-cooperative equilibrium, each player i takes other's strategies as given and chooses a best response. Since the player solves a vector maximization problem (2), there are a variety of solutions that can be chosen as the best response: weakly efficient solutions, efficient solutions or properly efficient solutions. Both Shapley (1959) and Charnes et al. (1990) take the efficient or non-dominated solutions as the best response, while in this paper a properly efficient solution of (2) is chosen as the best response.

Though the Core and Nash equilibrium are widely applied, it has been shown that they are both particular hybrid solutions, which are associated with different partitions of players. Recall that a partition of players $\{1,2,\ldots,n\}$ is a collection of coalitions $\Delta = \{S_1, S_2, \ldots, S_k\}$ such that $\cup S_i = N$ and $S_i \cap S_j = \emptyset$ for $i \neq j$. For each partition $\Delta = \{S_1, S_2, \ldots, S_k\}$, let $k = k(\Delta) = |\Delta|$ denote the number of coalitions in Δ. Then a partition Δ will induce k parametric multiple objective games:

$$\Gamma_S(x_{-S}) = \{S, X^i, u^i(x_S, x_{-S})\} \tag{3}$$

for $S = S_1, S_2, \ldots, S_k$. For the fixed parameter x_{-S}, each $\Gamma_S(x_{-S})$ has $|S|$ players and is simply a new MOG. The concept of hybrid solutions can now be given as:

Definition 4: For each partition $\Delta = \{S_1, S_2, \ldots, S_k\}$ of players in the MOG $\Gamma = \{N, X^i, u^i\}$, a joint strategy $\overline{x} = \{\overline{x}_{S(1)}, \overline{x}_{S(2)}, \ldots, \overline{x}_{S(k)}\} \in X$ is the **hybrid solution** corresponding to Δ if for each coalition $S \in \Delta$, $\overline{x}_S$ is a core solution $\Gamma_S(\overline{x}_{-S})$.

Like the earlier cooperative and non-cooperative solutions, the hybrid solution so defined is also exactly the same as that in the conventinal game theory if the MOG is degenerated to a game with scalar payoffs. The following Lemma 2 is a existence theorem of hybrid solutions in a conventional game, whose proof can be found in Zhao (1990).

Lemma 2: Given a partition $\Delta = \{S_1, S_2, \ldots, S_k\}$ of players in the general n-person game in normal form $\Gamma = \{N, X^i, v^i\}$, the set of the corresponding hybrid solutions is nonempty if Γ satisfies: (1) for each player i, X^i is a closed bounded convex subset in $\mathbf{R}^{L(i)}$; (2) for each coalition $S \in \Delta$, $v^i(x)$, $i \in S$, are all continuous in $x = (x_S, x_{-S})$ and are quasiconcave in x_S. Where for each $i \in N$, $v^i : X = \Pi_{i=1}^n X^i \rightarrow \mathbf{R}$ is player i's scalar payoff.

As seen from the previous three definitions, the cooperative, non-cooperative and hybrid solutions have the following characterizations. First, the heart of the non-cooperative solution is the strategic behavior and the individual rationality. It requires that each player chooses a best response given all other players' strategies, and at the equilibrium no player has any incentive to deviate alone. A properly efficient solution of (2) is defined as the best response in this paper.

Next, the pith of the Core is the coalitional improving behavior and the group rationality. It assumes that players are free to from coalitions and each member in the coalition expects to be better off by joining. A core solution is reached if no coalition can guarantee a higher payoff for each of its members indepedently of the actions of the outside players, thus no coalition has any incentive to object or block such a joint arrangement.

Last, the essence of hybrid solution is the partition of players and the coexistence of competition and cooperation. It assumes competition across coalitions and cooperation within each coalition. A hybrid equilibrium is reached if each coalition in the given partition chooses a core solution of (3) as a best response to all other coalitions' strategies. Thus at the equilibrium no coalition in the partition has any incentive to deviate alone.

Note also that "strategic behavior" simply means "taking other's choices as given." In this aspect the hybrid solution generalizes the strategic behavior of an individual player to that of a group of players. In the Nash equilibrium, each player takes all other's strategies as given, because no cooperation with other players is allowed; while in the hybrid equilibrium, each coalition takes all other coalitions' strategies as given, because no cooperation with other coalitions is allowed.

Now given coalition $S \in \mathcal{N}$ and given the complementary strategies x_{-S}. The coalition S can, in stead of playing the cooperative MOG as in (3), solve the following parametric vector maximization problem:

$$VM_S(x_{-S}): \quad \operatorname*{Max}_{x_S \in X_S} \; u_S(x_S, x_{-S}), \tag{4}$$

where the complementary strategies x_{-S} are the fixed parameters, and there are $\Sigma_{i \in S} m_i$ objectives to be maximized.

Thus under the assumption that each coalition takes the complementary strategies as given, there are two different classes of solutions available to the coalition: one comes from the earlier multiple objective game (3), and the other comes from the vector maximization problem (4). If players in the coalition S play the cooperative MOG, a core solution will be chosen; and if the coalition solves the VM problem (4), a properly efficient solution will be chosen by S. This leads to the following concept of S-efficiency[7]:

[7] It appears that Dubey (1986) first defines the S-efficiency in a conventional n-person game, where he originally calls it "T-efficient" for each coalition $T \in \mathcal{N}$. Here we use "S" because the conventional symbol for a coalition is S rather than T. It should also be noted that the word "S-efficient" has appeared earlier in the context of multiple objective mathematical programming (Zhao, 1983). This is defined as follows. Consider the VM problem as given in (1). For each subset S of $\{1,2,\ldots,m\}$, $\bar{x}$ is called S-efficient if $\bar{x}$ is an efficient solution of Max $\{F_S(x) \mid x \in Y\}$, where $F_S(x) = \{f_i(x) \mid i \in S\}$.

Definition 5: For any coalition $S \in \mathcal{N}$, a joint strategy $\bar{x} = \{\bar{x}^1, \bar{x}^2, ..., \bar{x}^n\} = (\bar{x}_S, \bar{x}_{-S}) \in X$ is an **S-properly efficient solution** of Γ if $\bar{x}_S$ is a properly efficient solution of the vector maximization problem $VM_S(\bar{x}_{-S})$.

Similarly, $\bar{x} = (\bar{x}_S, \bar{x}_{-S})$ is an *S*-efficient (*S*-weakly efficient) solution of Γ if $\bar{x}_S$ is an efficient (weakly efficient) solution of $VM_S(\bar{x}_{-S})$.

Now assume that each coalition *S* solves the parametric MOMP problem (4). A partition $\Delta = \{S_1, S_2, ..., S_k\}$ of players will then be associated with $|\Delta| = k$ parametric MOMP problems:

$$VM_S(x_{-S}): \max_{x_S \in X_S} u_S(x_S, x_{-S}), \text{ for } S = S_1, S_2, ..., S_k. \tag{5}$$

Based on these behavioral assumptions, we can define the quasi-hybrid solution as:

Definition 6: For each partition $\Delta = \{S_1, S_2, ..., S_k\}$ of players in the MOG $\Gamma = \{N, X^i, u^i\}$, a joint strategy $\bar{x} = \{\bar{x}^1, \bar{x}^2, ..., \bar{x}^n\} = \{\bar{x}_{S(1)}, \bar{x}_{S(2)}, ..., \bar{x}_{S(k)}\}$ is the **Quasi-hybrid solution** corresponding to Δ if for each coalition $S \in \Delta$, $\bar{x}_S$ is a properly efficient solution of vector maximization problem $VM_S(\bar{x}_{-S})$.

Note that the behavioral assumption of solving the parametric MOMP problem (4) is neither cooperative nor non-cooperative nor hybrid in the standard sense, that is why we call it quasi-hybrid solution.

With the Definitions 4 and 6, we are now ready to turn to the existence theorems in next section.

3 Existence Theorems

In the following Theorems 1 and 2, we shall adopt the convention that a vector function is continuous and concave if and only if all its components are continuous and concave.

Theorem 1: Given a partition $\Delta = \{S_1, S_2, ..., S_k\}$ of players in the MOG $\Gamma = \{N, X^i, u^i\}$. The corresponding hybrid solution set is non-empty If Γ satisfies: (1) for each player i, X^i is a closed bounded convex subset in $\mathbf{R}^{L(i)}$; (2) for each coalition $S \in \Delta$, $u^j(x)$, $j \in S$, are all continuous in $x = (x_S, x_{-S})$ and are quasiconcave in x_S.

That is, if all the strategy sets are closed bounded convex, the payoffs of each coalition are quasiconcave in the coalition's own strategies and are continuous in all strategies, then there exists at least one joint strategies $\bar{x} = \{\bar{x}^1, \bar{x}^2, ..., \bar{x}^n\} = \{\bar{x}_{S(1)}, \bar{x}_{S(2)}, ..., \bar{x}_{S(k)}\}$ such that for each coalition $S \in \Delta$, $\bar{x}_S$ is a core solution of $\Gamma_S(\bar{x}_{-S})$.

Proof of Theorem 1: Our proof consists of three steps. Given the partition $\Delta = \{S_1, S_2, ..., S_k\}$, in Step 1 we first construct a game Γ^* with scalar payoffs. In Step 2 we show that Γ^* satisfies the sufficient conditions in Lemma 2, and thus Γ^* has at least one hybrid solution for the corresponding partition. In Step 3, we shall derive a hybrid solution for the original MOG Γ from the hybrid solution of Γ^* found in Step 2.

Step 1: The definition of $\Gamma = \{N, X^i, u^i\}$ tells us that for each $i \in N$, player i has m_i objectives and L_i choice variables. For each player i, let us introduce m_i new image players $\{a_1^i, a_2^i, ..., a_{m(i)}^i\}$ whose scalar payoffs are the components of u^i respectively; that is, for $j = 1,2,...m_i$, the scalar payoff to image player a_j^i is u_j^i, where $u^i = \{u_1^i, u_2^i, ..., u_{m(i)}^i\} \in \mathbf{R}^{m(i)}$ is the vector payoff of the original player i in Γ.

Next assign each image player a_j^i some choice variables $y(i,j)$ from the original choice variables $x^i = \{x_1^i, x_2^i, ..., x_{L(i)}^i\} \in X^i$. Three cases should be distinguished. Case (1): $m_i < L_i$. In this case player i has more choice variables than objectives. For $j = 1,..., m_i -1$, let image player a_j^i controls x_j^i; for $j = m_i$, let $a_{m(i)}^i$ controls $\{x_{m(i)}^i, ..., x_{L(i)}^i\}$; that is, for $j = 1,..., m_i -1$, $y(i,j) = x_j^i$; for $j = m_i$, $y_{(i,m_i)} = (x_{m(i)}^i, ..., x_{L(i)}^i)$. Case (2): $m_i = L_i$. In this case player i has the same number of objectives as that of choice variables. For $j = 1,..., m_i$, let $y(i,j) = x_j^i$. Case (3): $m_i > L_i$. In this case player i has more objectives than choice variables. For $j = 1,..., L_i$, let $y(i,j) = x_j^i$; for $j = L_i+1,..., m_i$, let $y(i,j) = t(i,j) \in [0,1]$. Here the image players a_j^i ($j = L_i+1,..., m_i$) are dummy players in the sense that their choice variables $t(i,j)$ ($j = L_i+1,..., m_i$) have no effect on any player's (including themselves) payoff. For those $y(i,j)$'s that are components of $x^i \in X^i$, their ranges are taken as the projections of X^i on the corresponding subspaces respectively, these projections are obviously closed bounded convex sets. Thus for all $y(i,j)$ ($i = 1,..., n; j=1,..., m_i$), their ranges are closed bounded convex sets in some finite-dimensional Euclidean space.

When we have finished above assignments for all $i \in N$, we get a game Γ^* that has $\bar{n} = \Sigma_{i=1}^n m_i$ players, and each player has a scalar payoff u_j^i and some choice variables $y(i,j)$ ($i = 1,..., n$; $j = 1,..., m_i$).

Step 2: The partition $\Delta = \{S_1, S_2, ..., S_k\}$ of players in the original MOG Γ apparently defines a partition $\Delta^* = \{S_1^*, S_2^*, ..., S_k^*\}$ of players in the new game Γ^*, where $S_i^* = \{a_r^j \mid j \in S_i, r = 1,..., m_j\}$ is a coalition of $|S_i^*| = \Sigma_{j \in S(i)} m_j$ players in Γ^*. Consequently, the parametric MOG (3) played by each coalition S_i ($i = 1,2,..., k$) is changed to a parametric game with scalar payoffs played by $|S_i^*| = \Sigma_{j \in S(i)} m_j$ new players. For each $S \in \Delta = \{S_1, S_2, ..., S_k\}$, since $u^i(x)$, $i \in S$, are all continuous in $x = (x_S, x_{-S})$ and are quasiconcave in x_S, it is obvious that for each coalition of the image players $S^* \in \Delta^* = \{S_1^*, S_2^*, ..., S_k^*\}$, its payoffs are all continuous in $y = \{y(j,r) \mid j \in N, r = 1,..., m_j\} = \{y_{S^*}, y_{-S^*}\}$ and are quasiconcave in y_{S^*} ($y_{S^*} = \{y(j,r) \mid j \in S, r = 1,..., m_j\}$). By our construction in step 1, the domain of each

$y(i,j)$ is a closed bounded convex set. Thus by Lemma 2, Γ^* has a hybrid solution $\bar{y} = \{\bar{y}_{S^*(1)}, \bar{y}_{S^*(2)},\ldots,\bar{y}_{S^*(k)}\}$ corresponding to the partition Δ^*.

Step 3: Given the partition $\Delta = \{S_1,S_2,\ldots,S_k\}$ and $\Delta^* = \{S_1^*,S_2^*,\ldots,S_k^*\}$, let the hybrid solution $\bar{y}$ obtained above be written as $\bar{y} = \{\bar{y}_{S^*(1)}, \bar{y}_{S^*(2)},\ldots,\bar{y}_{S^*(k)}\} = \{\bar{x}_{S^*(1)}, \bar{x}_{S^*(2)},\ldots,\bar{x}_{S^*(k)}, \bar{t}\} = \{\bar{x},\bar{t}\}$, where $\bar{x} = \{\bar{x}_S,\bar{x}_{-S}\}$ (for each $S \in \Delta$) is the projection of $\bar{y}$ on $X = \Pi_{i\in N}X^i$, that is, $\bar{x}$ is obtained by dropping off those components of $\bar{y}$ that are $t(i,j)$'s in Step 1. Then we claim that for each coalition $S \in \{S_1,S_2,\ldots,S_k\}$, $u_S = \{u^i(\bar{y}) \mid i \in S\} = \{u_r^j(\bar{y}) \mid j \in S, r = 1,\ldots,m_j\} = \{u^i(\bar{x}) \mid i \in S\}$ is also a core payoff vector of $\Gamma_S(\bar{x}_{-S})$ (defined in (3)). Since u_S is the core vector payoff of coalition $S^* \in \Delta^*$ in Γ^*, and any sub-coalition of S corresponds to a class of sub-coalitions of $S^* = \{a_r^j \mid j \in S, r = 1,\ldots,m_j\}$, the claim is obvious. Thus $\bar{x}_S$ is a core solution of $\Gamma_S(\bar{x}_{-S})$ for each coalition $S \in \Delta$. This proves our theorem.

Q.E.D.

Notice that the arbitrariness in assigning choice variables to the image players in Γ^* suggests that for any partition of players, the set of the corresponding hybrid solutions in the MOG Γ is not only nonempty but also not unique, because for each particular way of assignment, we can get at least one core paoff vector corresponding to it.

Our proof also suggests an algorithm for solving a hybrid solution in general and for solving a core solution in particular to the MOG problem, which can be implemented with three stages. In stage 1 we construct a new game with scalar payoffs, as done in the above proof; in stage 2 we construct a map satisfying the conditions of the Kakutani's fixed point theorem (as done in Zhao (1990)); then in stage 3 we can approximate the solution (any fixed point of the map) by employing the available algorithms of finding a fixed point.

Theorem 2: Given a partition $\Delta = \{S_1,S_2,\ldots,S_k\}$ of players in the MOG $\Gamma = \{N, X^i, u^i\}$. The set of the corresponding quasi-hybrid solutions is non-empty If Γ satisfies: (1) for each player i, X^i is a closed bounded convex subset in $\mathbf{R}^{L(i)}$; (2) for each coalition $S \in \Delta$, $u^j(x)$, $j \in S$, are all continuous in $x = (x_S,x_{-S})$ and are quasiconcave in x_S.

That is, if all the strategy sets are closed bounded convex, the payoffs of each coalition are quasiconcave in the coalition's own strategies and are continuous in all strategies, then there exists at least one joint strategies $\bar{x} = \{\bar{x}^1,\bar{x}^2,\ldots,\bar{x}^n\} = \{\bar{x}_{S(1)},\bar{x}_{S(2)},\ldots,\bar{x}_{S(k)}\}$ such that for each coalition $S \in \Delta$, $\bar{x}_S \in X_S$ is an S-properly-efficient solution of $VM_S(\bar{x}_{-S})$.

Note that Theorem 2 generalizes the earlier existence theorems by Nash (1950), Shapley (1959) and Charnes et al. (1990). First, consider the finest partition where each coalition consists of a single player. In this particular case Theorem 2 says each player i chooses a best response to all other's strategies $\bar{x}_{-i}$ in the sense that he or she chooses a properly efficient solution of $VM_i(\bar{x}_{-i})$ as in (5). Since proper efficiency implies efficiency, the existence result in Theorem 2 is stronger than that of Shapley. Obviously, when $m_1 = \ldots = m_n = 1$, our solution is exactly the original

Nash equilibrium. Next consider the general partition where each coalition has at least two players. Under the assumption that each coalition behaves strategically or competitively, our theorem says that each coalition S in the given partition chooses a properly efficient solution of $VM_S(\bar{x}_{-S})$ as a best response to the complementary strategies $\bar{x}_{-S}$. While in Carnes et al. (1990), no strategic behavior across coalitions is considered, and our result is apparently more general and stronger.

Proof of Theorem 2: For each coalition $S(S = S_1, S_2, \ldots, S_k)$ in the given partition Δ, define a map $\delta_S : X_{-S} \rightarrow {}_2X_S$ by

$$\delta_S(x_{-S}) = \text{Arg-} \underset{x_S \in X_S}{\text{Max}} \ \Sigma_{j \in S} \Sigma_{r=1}^{m(j)} \{u_r^j(x_S, x_{-S})\} \tag{6}$$

for each $x_{-S} \in X_{-S}$. Since $u_r^j(x_S, x_{-S}), j \in S, r = 1, \ldots, m_j$, are all continuous in $x = (x_S, x_{-S})$ and quasi-concave in x_S, $X_S = \Pi_{j \in S} X^j$ is closed bounded convex set, it is clear that δ_S, whose image is the set of optimal solutions to the parametric programming problem (6), has a nonempty closed bounded convex value and has a closed graph. Thus for $X = \Pi_{i \in N} X^i = \Pi_{S \in \Delta} X_S = \Pi_{i=1}^k X_{S(i)}$, the map δ: $X \rightarrow {}_2X$ defined by

$$\delta(x) = \Pi_{S \in \Delta}\, \delta_S(x_{-S}) = \Pi_{i=1}^k\, \delta_{S(i)}(x_{-S(i)}) \tag{7}$$

for each $x = \{x^1, \ldots, x^n\} = \{x_{S(1)}, x_{S(2)}, \ldots, x_{S(k)}\} \in X$, satisfies the conditions of Kakutani's fixed point theorem. Thus δ has a fixed point $\bar{x} = \{\bar{x}_{S(1)}, \bar{x}_{S(2)}, \ldots, \bar{x}_{S(k)}\} \in \delta(\bar{x})$. Then, by Lemma 1, for each coalition $S \in \Delta$, $\bar{x}_S$ is an S-properly-efficient solution of $VM_S(\bar{x}_{-S})$.

Q.E.D.

Although the conditions in Theorem 1 and Theorem 2 are exactly the same, we have two different classes of solutions. This is because we have assumed two different behavioral assumptions. To see the distinction of the two solutions, consider a special case where each player has a scalar payoff and Δ is the coarsest partition consisting of the grand coalition alone. In Theorem 2 players are assumed to solve the vector maximization problem given in (5), and they shall achieve the proper efficiency; while in Theorem 1 players are assumed to play the cooperative game (3), they shall choose a core solution and achieve only the weak efficiency. Thus the hybrid and quasi-hybrid solutions for a given partition have different levels of efficiency or welfare.

4 Conclusions

We have discussed the conventional game theory studies the conflicts among several individuals, multiple objective mathematical programming (MOMP) studies the conflicts within an individual, and multiple objective game (MOG) studies the general conflicts in both cases.

For a multiple objective game, we have defined and proved the existence of two classes of solutions: the hybrid solution and the quasi-hybrid solution. Both solutions are associated with a partition of players, and both assume the strategic behavior of each coalition. A coalition in the quasi-hybrid solution solves a parametric vector maximization problem (5), while players of each coalition in the hybrid solution play a cooperative multiple objective game (3). Similarly as in the conventional game theory, the cooperative and non-cooperative solutions are two particular kinds of hybrid solutions. Nash equilibrium is exactly the quasi-hybrid solution corresponding to the finest partition, while the core is exactly the hybrid solution corresponding to the coarsest partition.

It is clear that our Theorems 1 and 2 generalize the existence theorems of the Core, Nash equilibrium and hybrid solution in conventional game theory. If the MOG in Theorems 1 and 2 are degenerated to a game with scalar payoffs, then Theorem 1 for the coarsest partition becomes the Scarf core existence theorem (1971), Theorem 1 for the general partition becomes the existence of hybrid solution as Lemma 2, and both Theorems 1 and 2 for the finest partition become the existence of Nash equilibrium (1950).

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