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The General Nucleolus and the Reduced Game Property

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Abstract: The nucleolus of a TU game is a solution concept whose main attraction is that it always resides in any nonempty ε-core. In this paper we generalize the nucleolus to an arbitrary pair (Π, F), where Π is a topological space and F is a finite set of real continuous functions whose domain is Π. For such pairs we also introduce the "least core" concept. We then characterize the nucleolus for *classes* of such pairs by means of a set of axioms, one of which requires that it resides in the least core. It turns out that different classes require different axiomatic characterizations.

One of the classes consists of TU-games in which several coalitions may be nonpermissible and, moreover, the space of imputations is required to be a certain "generalized" core. We call these games *truncated games*. For the class of truncated games, one of the axioms is a new kind of *reduced game property*, in which consistency is achieved even if some coalitions leave the game, being promised the nucleolus payoffs. Finally, we extend Kohlberg's characterization of the nucleolus to the class of truncated games.

1 Introduction

The *nucleolus* of a cooperative game with side payments (TU game) was introduced in [Schmeidler, 1969]. It quickly gained popularity and was applied in several areas[3]. Perhaps the most attractive property of the nucleolus is that it is a unique point in the *core* of the game, whenever the core is not empty. Thus, it may be a good candidate for situations in which it is desirable to have a rule which selects one outcome in the core of the game[4].

The nucleolus is also a unique point in every non-empty *ε-core* (Schmeidler [1969]). This, and other nice properties of the nucleolus induced Shubik ([1983], page 340) to say that "the nucleolus represents as nearly as any single imputation can the location of the core of the game", ... "its effective center", and, "if the core is empty, the nucleolus represents its 'latent' position".

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[3] We refer the reader to the survey [Maschler, 1992], where several properties of the nucleolus, as well as its applications are summarized and discussed.

[4] This desire is natural, for example, in problems of *cost allocation*. It is hard to imagine parties who are willing to share costs in a way in which they are asked to pay more than they would had they themselves bought the same benefits.

However, the nucleolus is not the only rule to choose a unique core point. One can think of others – the center of gravity of the core, for example. Why should one prefer the nucleolus? One reason might be the attractiveness of its definition as an outcome of a lexicographic minimization procedure. Other reasons are, perhaps, nice properties of the nucleolus which may be relevant to the particular application. But these criteria are hard to grasp. *What one needs is an axiomatic characterization of the nucleolus.* If one can characterize the nucleolus by means of *intuitively acceptable and simple axioms*, one can check for each application if these are relevant to the needs. If they are, then the logical choice must be the nucleolus; but – and this is equally important – if for some applications the axioms do not make sense, then the nucleolus should be rejected for them. Such a system of axioms was given for the *prenucleolus* – a related solution concept – in the intriguing paper [Sobolev, 1975][5]. Sobolev's axioms are intuitive indeed. The essential one among them is the requirement that the solution concept be *consistent* or, equivalently, satisfy a *reduced game property*. Heuristically, it requires that at the solution point every non-empty subset of the players, who look at their payments and at the same time examine their "own game" (the reduced game on them), will not want to move away, because they will find that their payments constitute the solution also for the reduced game[6].

The idea of lexicographically minimizing [maximizing] a vector of objective functions need not be applied only to TU games. Indeed, it was applied in several other conflict situations. Already in the forties, game theorists at RAND recommended a lexicographic maximization[7] for Player I and a lexicographic minimization for Player II, who participate in a zero-sum two-person game. The idea was to exploit an opponent's mistakes without sacrificing one's own safety levels[8]. This recommendation is described in [Brown, 1950]. Eventually it was published in [Dresher, 1961]. We shall call the set of recommended strategies *the nucleolus of the game* (for Player I/II). These nucleoli were encountered again more recently when van Damme proved that they constitute precisely the set of *proper equilibria* for the game (see van Damme [1983]).

Another application of a lexicographic minimization was given in [Justman, 1977]. There he tried to describe a process of negotiation between players, but actually his model could be applied in many other instances. Motivated by his results, Maschler and Peleg [1976] defined generalized nucleoli for a set-valued dynamic system which turned out to be the closed stable[9] sets of the system. In a somewhat different setup, Potters and Tijs [1992] introduced a *general nucleolus*[10], by means of which they were able to confront the nucleolus of TU games with the nucleolus of matrix games. They found interesting correspondences between the two concepts

[5] Following Sobolev, axiom systems were introduced in [Potters, 1991] and in [Snijders, 1991] to axiomatize the nucleolus itself.

[6] Consistency is a natural requirement. What is not so obvious is how to define the reduced game. Different definitions lead to different solution concepts (see, e.g., Hart and Mas-Colell [1989]).

[7] Precise definitions will be given in Section 5.

[8] We are indebted to Lloyd S. Shapley, who briefed us about the history of these discoveries.

[9] Stable in the sense of Lyapunov.

[10] This is the one defined in Section 2.

and were able to establish for matrix-nucleoli an analogue of Kohlberg's characterization of the TU-nucleolus in terms of balanced sets (see Kohlberg [1971]).

This paper is concerned with the general nucleolus of Potters and Tijs and further generalizations, as well as applications. Specifically, we consider *classes* of pairs (Π, F), where the Π's are topological spaces and the F's are finite vectors whose components are real and continuous functions defined on Π. The nucleolus for such a class is defined in Section 2. The class itself will be called *the domain of the nucleolus*. The main task of this paper is to characterize the nucleolus concept axiomatically. *It will turn out that the required axioms depend heavily on the domain*. Different domains require different sets of axioms!

This is an interesting phenomenon. The characterization of the nucleolus by means of a lexicographic minimization (Section 2), does not depend on the domain of the nucleolus: Lexicographic minimization characterizes the nucleolus of each individual pair (Π, F), regardless of the class to which it belongs. Characterization by axioms, as done in Sections 3–6, depends forcefully on the intended applications; namely, on the class of the pairs that we consider as the domain of the nucleolus. Thus, the axioms can be regarded as "social norms" that apply to various potential circumstances. Different sets of circumstances require different norms.

The paper is organized as follows: Section 2 provides the necessary notation and the definition of the general "least core" and the general "nucleolus". It also lists some basic properties from which, later on, axioms will be chosen. We prove that these are satisfied by the nucleolus whenever the domain is large enough to render them meaningful.

In Section 3 we provide a system of axioms that characterize the nucleolus for any sufficiently rich domain. The deepest among the axioms, in our opinion, is the requirement that the nucleolus is a subset of the least core. In Section 4 we specialize in classes in which the components of F are convex functions. All applications of the nucleolus, done so far, are included in this case. It turns out that in this case we can omit one of the previous axioms.

A special case of Section 4 is the case of the nucleolus of a matrix (zero-sum) game. Nevertheless, we cannot apply the axioms of Section 4 to this domain without some modifications. These we present in Section 5.

Section 6 studies the nucleolus of a class of cooperative games that contains as a subclass the ordinary TU-games. This class contains TU games in which certain coalitions are not permissible and, in addition, cooperation (to form the grand coalition) is prohibited unless some coalitions' excesses are smaller than *a priori* given numbers. We call such game *TU-games with permissible coalitions and permissible imputations*, or *truncated games*, for short. We provide axioms that characterize the nucleolus for games of this class, which are different from those given by Sobolev. One such set of axioms makes use of a "reduced game property". However, instead of "reducing" the game to a subset of players, as is done in Sobolev, we "reduce" the game to a smaller set of permissible coalitions. A confrontation between Sobolev's axioms and ours ends the section.

Section 7 generalizes Kohlberg's characterization of the nucleolus to the class of truncated games. This generalization throws light on the special role of the class $\mathscr{E}(x) = \{\{i\} : x_i = v(\{i\})\}$ in Kohlberg's characterization (see Kohlberg [1971] and also Sobolev [1975], Owen [1977] and Wallmeier [1983]).

2 The General Nucleolus

Our object of study is a class Ω of pairs (Π, F). In each pair, Π is a topological space and $F := \{F_j\}_{j\in M}$ is a finite set of real continuous functions on Π. This setup has many applications. We give here two examples which should be sufficient for motivation:

Example 2.1. Ω is derived from the class of all TU games $(N; v)$ on finite sets of players. For each game, Π is the set of *preimputations* of the game and $F = \{F_S\}_{S\subseteq N}$, where the various F_S's are the *excess functions*, $F_S(x) := v(S) - x(S)$.

Example 2.2. Ω is a class of potential "decision spaces". In each particular case a decision maker has to make a decision x which is a point in a "decision space" Π. Any such choice may affect a set of cities M. The effect can be measured in monetary terms. $F_j(x)$ is the damage caused to city j if the decision x is taken.

The central concepts in this paper are the *least core*[11] and the *nucleolus* defined by[12]

$$\mathscr{LC}(\Pi, F) := \{x\in\Pi \colon \bigvee_{j\in M} F_j(x) \le \bigvee_{j\in M} F_j(y) \text{ for all } y\in\Pi\} \tag{2.1}$$

and

$$\mathscr{N}(\Pi, F) := \{x\in\Pi \colon \theta\circ F(x) \lesssim_{\text{lex}} \theta\circ F(y) \text{ for all } y\in\Pi\}. \tag{2.2}$$

Here, $\theta \colon \mathfrak{R}^M \to \mathfrak{R}^m$ is the *coordinate* ordering map[13] and 'lex' is the lexicographic ordering[14] on $\mathfrak{R}^m$, $m = |M|$. For the special case $M = \emptyset$, we define:

$$\mathscr{LC}(\Pi, F) = \mathscr{N}(\Pi, F) = \Pi. \tag{2.3}$$

It should be noted that both the nucleolus and the least core may be empty sets. In view of possible applications in Section 6, we refrain from adding conditions that guarantee nonemptiness.

We shall now state a few properties that a *solution concept*[15] Φ, defined on a class Ω, might have. We shall then show that the nucleolus possesses these properties. In subsequent sections, some of these properties will serve as axioms.

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[11] In Example 2.1 this is the smallest ε-*core* (Shapley and Shubik [1966]) that is not empty (Maschler, Peleg and Shapley [1979]).

[12] In this paper, $\bigvee$ and $\bigwedge$ are the max and min operators.

[13] I.e., $\theta\circ F(x)$ is an m-vector, $m = |M|$, with the same components as in $F(x)$, but ordered in a weakly decreasing order.

[14] An m-vector a is lexicographically smaller than an m-vector b if, in the first coordinate where they differ, the coordinate of a is smaller than the coordinate of b.

[15] A rule Φ that assigns to each element (Π, F) of Ω a subset $\Phi(\Pi, F)$ of Π.

($\mathbf{P}_0$) *(Restricted non-emptiness)* $\Phi(\Pi, F) \neq \emptyset$ *if* Π *is a nonempty compact set.*

($\mathbf{P}_1$) *(Non-discrimination)* $\Phi(\Pi, F) = \Pi$ *if* $M = \emptyset$. (In the absence of objective functions, every choice is equally good under Φ.)

($\mathbf{P}_2$) *(Redundancy)* $\Phi(\Pi, F) = \Phi(\Pi, F_{-j})$ *if* F_j *is constant on* Π. (If one of the functions makes no distinction between the points of Π, it has no influence on the outcomes under Φ.)

($\mathbf{P}_3$) *(Inclusion in the least core)* $\Phi(\Pi, F) \subseteq \mathscr{L}\mathscr{C}(\Pi, F)$. (If the largest value among the $F_j(x)$'s exceeds the largest value among the $F_j(y)$'s for some y, then x will not be chosen under Φ.)

($\mathbf{P}_4$) *(Restriction to the least core)* $\Phi(\Pi, F) = \Phi(\mathscr{L}\mathscr{C}(\Pi, F), F)$. (Points outside the least core do not affect the choices under Φ.)

($\mathbf{P}_5$) *(Invariance with respect to rearrangement) If* (Π, F) *and* $(\Pi, \tilde{F})$ *are elements of* Ω *and* $\{F_j(x): j \in M\} \doteq \{\tilde{F}_j(x): j \in M\}$ *for each* x *in* Π, *then* $\Phi(\Pi, F) = \Phi(\Pi, \tilde{F})$. *Here,* $\doteq$ *means equality of sets with counting multiplicities*[16]. (The solution concept considers only what and how often values of F occur and does not care which functions take them.)

($\mathbf{P}_6$) *(Invariance with respect to* max/min*)* $\Phi(\Pi, F) = \Phi(\Pi, F_i \vee F_j, F_i \wedge F_j, F_{-ij})$ *for every* $i, j \in M$, $i \neq j$. (The outcomes under Φ do not change if we replace an F_i and F_j by their maximum and their minimum and leave the other members of F unchanged.)

($\mathbf{P}_7$) *(Independence of irrelevant alternatives) If* Π' *is a subset of* Π, *with* $(\Pi', F) \in \Omega$, *and* $\emptyset \neq \Phi(\Pi, F) \subseteq \Pi'$, *then* $\Phi(\Pi, F) = \Phi(\Pi', F)$. (This is the well-known IIA property formulated for set-valued solution concepts.)

($\mathbf{P}_8$) *(Strong IIA property) If* Π' *is a subset of* Π, *with* $(\Pi', F) \in \Omega$ *and if* $\Phi(\Pi, F) \cap \Pi' \neq \emptyset$ *then* $\Phi(\Pi', F) = \Phi(\Pi, F) \cap \Pi'$. (This time one only requires that $\Phi(\Pi, F)$ intersects Π'.)

($\mathbf{P}_9$) *(Contravariance) If* $\Lambda: \Pi' \to \Pi$ *is a continuous map and* $\Lambda^{-1}\Phi(\Pi, F) \neq \emptyset$ *then* $\Phi(\Pi', F \circ \Lambda) = \Lambda^{-1}\Phi(\Pi, F)$. (This is an even stronger version of IIA.)

($\mathbf{P}_{10}$) *(Closedness)* $\Phi(\Pi, F)$ *is a closed set for all pairs* (Π, F) *in* Ω.

In the following we shall prove that the nucleolus $\mathscr{N}$ has the above properties whenever the appropriate statement is meaningful[17].

Theorem 2.3. The nucleolus $\mathscr{N}$*, defined on an arbitrary class* Ω, *satisfies* $\mathbf{P}_0$–$\mathbf{P}_{10}$, *whenever the appropriate statement is meaningful.*

Proof: $\mathbf{P}_1$ holds by definition.

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[16] I.e., every value occurs among the $F_j(x)$'s and among the $\tilde{F}_j(x)$'s the same number of times.

[17] For Example, property $\mathbf{P}_2$ would be meaningless if (Π, F_{-j}) had not been a member of Ω.

Property $\mathbf{P}_2$ follows from a basic property of lexicographic ordering: If[18] $u, v \in \Re^{m-1}$, $w \in \Re$, then $\theta(u, w) \lesssim_{\text{lex}} \theta(v, w)$ iff $\theta(u) \lesssim_{\text{lex}} \theta(v)$. Take $u = F_{-j}(x)$, $v = F_{-j}(y)$ and $w = F_j(x) = F_j(y)$.

Properties $\mathbf{P}_3$ and $\mathbf{P}_4$ are obtained as follows: if $\theta \circ F(x) \lesssim_{\text{lex}} \theta \circ F(y)$ then $\bigvee_{j \in M} F_j(x) \leq \bigvee_{j \in M} F_j(y)$. Thus, $x \in \mathcal{N}(\Pi, F)$ implies $x \in \mathscr{L}\mathscr{C}(\Pi, F)$.

If $x \in \mathcal{N}(\Pi, F)$ then $\theta \circ F(x) \lesssim_{\text{lex}} \theta \circ F(y)$ for all y in Π and, in particular, for all y in $\mathscr{L}\mathscr{C}(\Pi, F)$. But $x \in \mathscr{L}\mathscr{C}(\Pi, F)$, hence $x \in \mathcal{N}(\mathscr{L}\mathscr{C}(\Pi, F), F)$. Conversely, if $x \in \mathcal{N}(\mathscr{L}\mathscr{C}(\Pi, F), F)$, then $\theta \circ F(x) \lesssim_{\text{lex}} \theta \circ F(y)$ for all y in $\mathscr{L}\mathscr{C}(\Pi, F)$. For $y \in \Pi \backslash \mathscr{L}\mathscr{C}(\Pi, F)$ we also have $\theta \circ F(x) \prec_{\text{lex}} \theta \circ F(y)$ (look at the first coordinate). So, $x \in \mathcal{N}(\Pi, F)$.

For property $\mathbf{P}_5$: Whether a point x in Π is an element of the nucleolus or not is determined by the vector $\theta \circ F(x)$, $x \in \Pi$. Since $\theta \circ F(x)$ and $\theta \circ \tilde{F}(x)$ are the same vector for each point x in Π, we find $\mathbf{P}_5$.

Property $\mathbf{P}_6$ is a special case of $\mathbf{P}_5$. If we replace F_i and F_j by their maximum and minimum, then for each x the resulting $\tilde{F}(x)$ will be a rearrangement of $F(x)$.

For property $\mathbf{P}_9$, we take $y \in \mathcal{N}(\Pi', F \circ \Lambda)$. Then $\theta \circ (F \circ \Lambda)(y) \lesssim_{\text{lex}} \theta \circ (F \circ \Lambda)(y')$ for all $y' \in \Pi'$. For $x = \Lambda(y)$ we therefore have $\theta \circ F(x) \lesssim_{\text{lex}} \theta \circ F(x')$ for all x' in $\Lambda(\Pi')$. Because $\mathcal{N}(\Pi, F) \cap \Lambda(\Pi') \neq \emptyset$, it follows that $\Lambda(y) = x \in \mathcal{N}(\Pi, F)$. Conversely, if $\Lambda(y) \in \mathcal{N}(\Pi, F)$ for some element y in Π', then $\theta \circ F(\Lambda(y)) \lesssim_{\text{lex}} \theta \circ F(x')$ for all x' in Π and so, in particular, for $x' = \Lambda(y')$ with $y' \in \Pi'$. Then $y \in \mathcal{N}(\Pi', F \circ \Lambda)$.

$\mathbf{P}_7$ and $\mathbf{P}_8$ are special cases of $\mathbf{P}_9$. (Take for Λ the injection of Π' into Π.)

The proof of $\mathbf{P}_0$ and $\mathbf{P}_{10}$ will follow from the algorithmic scheme below:

Algorithm 2.4. In order to determine the nucleolus for a pair (Π, F) *in* Ω, *take the following steps:*

(Step 1) *Remove all functions from F which are constant on* Π.
(Step 2) *If* $M = \emptyset$ *then* $\mathcal{N}(\Pi, F) = \Pi$. *Go to Step 6.*
(Step 3) *If* $M \neq \emptyset$ *compute* $\mathscr{L}\mathscr{C}(\Pi, F)$. *If* $\mathscr{L}\mathscr{C}(\Pi, F) = \emptyset$ *then* $\mathcal{N}(\Pi, F) = \emptyset$. *Go to Step 6.*
(Step 4) *If* $m \geq 2$, *replace* $F = (F_1, \ldots, F_m)$ *by* $\tilde{F} = (\tilde{F}_1, \ldots, \tilde{F}_m)$, *where* $\tilde{F}_p = \bigvee_{j \leq p} F_j \wedge F_{p+1}$ *for* $p = 1, \ldots, m-1$ and $\tilde{F}_m = \bigvee_{j \leq m} F_j$.
(Step 5) *Replace* Π *with* $\mathscr{L}\mathscr{C}(\Pi, F)$. *Go to Step 1.*
(Step 6) *Stop. You have reached the general nucleolus.*

Proof: Step 1 is permissible by Property $\mathbf{P}_2$. Step 2 is correct by Property $\mathbf{P}_1$. Step 3 brings us to the nucleolus by property $\mathbf{P}_3$. Step 4 is obtained by applying property $\mathbf{P}_6$ several times. Step 5 is justified by property $\mathbf{P}_4$. At the end of Step 5 at least one of the functions, namely $\tilde{F}_m$, is constant and therefore it is removed when we return to Step 1. Thus, the algorithm ends after m runs at most. ■

(Continuation of the Proof of Theorem 2.3): To prove property $\mathbf{P}_0$, note that if Π is a nonempty compact set, then $\mathscr{L}\mathscr{C}(\Pi, F)$ is also not empty and compact because F

[18] We switch to vector notation here, instead of set notation, merely to shorten the exposition.

is continuous. So we never reach Step 6 via Step 3. Thus the nucleolus will not be found empty.

There remains to prove property $\mathbf{P}_{10}$. This follows from the fact that the algorithm stops at a certain least core, which is a closed set, because the F_j's are continuous functions (see (2.1) and (2.3)). ■

3 Axioms for the General Nucleolus

In this section we shall characterize the nucleolus for every class that is 'rich enough' in the sense that it satisfies the following properties:

For each (Π, F) in Ω,

(α) $(\mathscr{LC}(\Pi, F), F) \in \Omega$,

(β) $(\Pi, F_{-j}) \in \Omega$ whenever $j \in M$ and F_j is constant on Π,

(γ) For all pairs i, j in M, $i \neq j$, $(\Pi, \tilde{F})$ is also in Ω. Here, $\tilde{F} = (\tilde{F}_1, \ldots, \tilde{F}_m)$, where $\tilde{F}_k = F_k$ for $k \in M \backslash \{i, j\}$, $\tilde{F}_i = F_i \wedge F_j$, $\tilde{F}_j = F_i \vee F_j$.

Theorem 3.1. Let Ω be a class of pairs (Π, F) satisfying (α)–(γ) above. Let Φ be a solution concept satisfying: $\mathbf{P}_1$ *(non-discrimination),* $\mathbf{P}_2$ *(redundancy),* $\mathbf{P}_4$ *(restriction to the least core) and* $\mathbf{P}_5$ *(invariance with respect to rearrangement). Under these conditions, $\Phi(\Pi, F) = \mathscr{N}(\Pi, F)$.*

Discussion. To judge if these axioms make sense on intuitive grounds let us check them in the case of Example 2.2. Similar checks should be performed for other applications. For this example, a decision-maker should choose the nucleolus if he accepts the following norms.

($\mathbf{P}_1$) This axiom follows if we want a choice to be governed *solely* by the damages. In the absence of reported damages, every choice is equally good.

($\mathbf{P}_2$) If damage to a city does not depend on your actual choice, ignore the city. (Anyway, you cannot help the city.)

($\mathbf{P}_4$) This is perhaps the strongest norm. It says in this context that if the highest damage under a choice x can be reduced by another choice, not only x should not be taken[19], but it should not influence the actual decision[20]. In

[19] That alone would be requirement $\mathbf{P}_3$.

[20] This incorporates an IIA-type property; in fact, we could replace $\mathbf{P}_4$ by $\mathbf{P}_3$ and $\mathbf{P}_7$. We could then prove that if Φ satisfies $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$, $\mathbf{P}_5$ and $\mathbf{P}_7$ then Φ is the nucleolus for all pairs in which it is not empty. (Nonemptiness is needed: Indeed, let Ω be such that all the Π's of its pairs are connected. Then, $\Phi = \Pi$, whenever F is constant, and otherwise $\Phi = \emptyset$ satisfies all of these five axioms and, in general, it differs from the nucleolus.)

many social situations involving value judgement, lowering highest damages as much as possible is an acceptable desire. In other applications one has a different goal: To lower damage to as many cities as possible. In such cases one may, for example, prefer to keep, or even slightly worsen, a highest damage, if that greatly improves many (even some) cities. If one has such a goal, then one should look at so-called 'compromises' between conflicting desires. In such cases the nucleolus should be rejected, or modified.

($\mathbf{P}_5$) This axiom says again that the decision should be governed solely by the package of damages (sets with multiplicities) and not influenced, for example, by questions such as which city suffers what damage. Again, that might not be appropriate: Some cities may have many inhabitants, so that the damage (assumed additive) for each member is small. In other cases, it is the opposite case: Although financially the damage per person is small, the decision-maker may anger many people, thus risking re-election. In all such cases the nucleolus is a bad choice and should either be modified or abandoned.

Proof of Theorem 3.1: By induction on $|M|=m$. For $m=0$, $\mathbf{P}_1$ yields $\Phi(\Pi, F)=\Pi=\mathscr{N}(\Pi, F)$. Assume that the theorem is true for $|M|=m$, some m. Let $(\Pi, F)\in\Omega$ with $|M|=m+1$. Then, by (α) and $\mathbf{P}_4$, $\Phi(\Pi, F)=\Phi(\mathscr{L}\mathscr{C}(\Pi, F), F)$. The rearrangment property $\mathbf{P}_5$ implies $\mathbf{P}_6$, which is a special case. We now replace several times F_i and F_j by their maximum and minimum, as in Algorithm 2.4 and we find that

$$\Phi(\Pi, F)=\Phi(\mathscr{L}\mathscr{C}(\Pi, F), \tilde{F}), \tag{3.1}$$

where $\tilde{F}$ is the one defined in Step 4 of the algorithm. Now, $\tilde{F}_{m+1}$ is constant on the least core and can be omitted by $\mathbf{P}_2$. From the induction hypothesis we get:

$$\Phi(\Pi, F)=\mathscr{N}(\mathscr{L}\mathscr{C}(\Pi, F), \tilde{F}'), \tag{3.2}$$

where $\tilde{F}'=\tilde{F}_{-(m+1)}$. We follow the same procedure with the nucleolus and finally find:

$$\Phi(\Pi, F)=\mathscr{N}(\mathscr{L}\mathscr{C}(\Pi, F), \tilde{F}')=\mathscr{N}(\Pi, F). \tag{3.3} \blacksquare$$

Corollary 3.2. The nucleolus is characterized by $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_4$ *and* $\mathbf{P}_5$.

Proof: Theorem 2.3 and Theorem 3.1. ■

It is clear from the proof that we could replace the rearrangement axiom $\mathbf{P}_5$ by the much weaker one $\mathbf{P}_6$ (invariance with respect to max/min), which is only a very special case of $\mathbf{P}_5$. The reason for preferring $\mathbf{P}_5$ is simply because we do not have an intuitive scenario in which $\mathbf{P}_6$ makes sense.

Theorem 3.3. The axioms $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_4$, $\mathbf{P}_5$ *are logically independent; i.e., there are solutions which fail to satisfy exactly any one of them.*

Proof:

($\neg\mathbf{P}_1$) $\Psi_1(\Pi, F) := \emptyset$ for all pairs in Ω fails to satisfy $\mathbf{P}_1$ only (when Ω contains a pair (Π, F), where $\Pi \neq \emptyset$ and $M = \emptyset$).

($\neg\mathbf{P}_2$) $\Psi_2(\Pi, F) := \mathscr{L}\mathscr{C}(\Pi, F)$ satisfies all axioms except $\mathbf{P}_2$, as the following example shows. Let (Π, F) be an element of Ω, where $\Pi := [0, 1]$, $F := \{F_1, F_2\}$, $F_1(t) := t$ and $F_2(t) := 1/2$ for all $t \in [0, 1]$. Then $\mathscr{L}\mathscr{C}(\Pi, F) = [0, 1/2] \neq \{0\} = \mathscr{L}\mathscr{C}(\Pi, F_{-2})$.

($\neg\mathbf{P}_4$) We shall provide two examples that satisfy all axioms except $\mathbf{P}_4$. One of them will satisfy $\mathbf{P}_7$ but not $\mathbf{P}_3$ and the other will satisfy $\mathbf{P}_3$ but not $\mathbf{P}_7$. This we do in order to show that each of the two requirements which are implicit in $\mathbf{P}_4$ (see last footnote) are independent of the remaining set of axioms.

($\neg\mathbf{P}_3$) Define $\Psi_3(\Pi, F) := \Pi$. It does not satisfy $\mathbf{P}_3$ if Ω contains a pair (Π, F) for which $\mathscr{L}\mathscr{C}(\Pi, F) \neq \Pi$. It satisfies the remaining axioms $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_5$, $\mathbf{P}_7$.

($\neg\mathbf{P}_7$) Take for Ω the class consisting of $([0, 1], F)$, where $F(t) = 1$ for $0 \leq t \leq 1/2$ and $F(t) = 2t$ for $1/2 \leq t \leq 1$, $([0, 1/2], F_{|[0, 1/2]})$ and $([0, 1/2], \emptyset)$. Let Ψ_7 be $\{1/4\}$ for the first pair, and $[0, 1/2]$ for the other two. Then, Ψ_7 satisfes $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ and $\mathbf{P}_5$, but not $\mathbf{P}_7$.

($\neg\mathbf{P}_5$) Note that $\mathbf{P}_6$ is a special case of $\mathbf{P}_5$. We shall provide here an example in which only $\mathbf{P}_6$ will not be satisfied. Let Π be the segment $[0, 1]$ and $F = \{F_1, F_2\}$ be defined by

$$F_1(t) = \begin{cases} 2t & \text{for } 0 \leq t \leq 1/2 \\ 1 & \text{for } 1/2 \leq t \leq 1 \end{cases} \qquad F_2(t) = \begin{cases} 1 & \text{for } 0 \leq t \leq 1/2 \\ 2 - 2t & \text{for } 1/2 \leq t \leq 1. \end{cases}$$

Let Ω be the smallest class containing (Π, F) and satisfying (α)–(γ) above. Then Ω consists of (Π, F), $(\Pi, \tilde{F}_1 := F_1 \wedge F_2, \tilde{F}_2 := F_1 \vee F_2)$, $(\Pi, \tilde{F}_1)(\{0, 1\}, \tilde{F}_1)$ and $(\{0, 1\}, \emptyset)$. Define $\Psi_6(\Pi, F) = \Pi$, $\Psi_6 = \{0, 1\}$ for all other elements of Ω. Ψ_6 satisfies $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_4$, but not $\mathbf{P}_6$ and therefore not $\mathbf{P}_5$. ■

4 The General Nucleolus in the Convex Case

We start this section by stating an unfortunate fact and a fortunate one. The unfortunate fact is that in the game theoretical applications of the nucleolus, the classes involved do *not* satisfy condition (γ) of the previous section. For instance, the maximum and the minimum of two excess functions in Example 2.1 are *not* themselves excess functions. The fortunate fact is that in these applications the F_j's are *convex* – even linear. We shall see in this section that in this case condition (γ) is *not needed.* Accordingly, we shall speak in this section about classes Ω_1 satisfying the following requirements:

For each (Π, F) in Ω_1,

(α) $(\mathscr{LC}(\Pi, F), F) \in \Omega_1$,
(β) $(\Pi, F_{-j}) \in \Omega_1$ whenever $j \in M$ and F_j is constant on Π,
(δ) Π is a *convex* and closed set in a topological vector space and each F_j is a real, continuous, convex function whose domain is Π.

Lemma 4.1. If $(\Pi, F) \in \Omega_1$ *and* $M \neq \emptyset$, *then there exists a function* F_{j_0}, $j_0 \in M$, *which is constant on* $\mathscr{LC}(\Pi, F)$.

Proof: We may assume that $\mathscr{LC}(\Pi, F) \neq \emptyset$. Let w be the minimum value of $\bigvee_{j \in M} F_j$ in Π. Then $F_j(x) \leq w$ for all $j \in M$ iff $x \in \mathscr{LC}(\Pi, F)$. If every function F_j has a point x_j in $\mathscr{LC}(\Pi, F)$ with $F_j(x_j) < w$ then, by convexity,

$$F_j\left(|M|^{-1} \sum_{k \in M} x_k\right) \leq |M|^{-1} \sum_{k \in M} F_j(x_k) < w$$

for all j in M. Then, $\bigvee_{j \in M} F_j$ has a value less than w at $|M|^{-1} \sum_{k \in M} x_k$ in Π, in contradiction to the definition of w. ■

We can now prove:

Theorem 4.2. Let Ω_1 *be a class of pairs* (Π, F) *satisfying* (α), (β), (δ) *above, then the nucleolus on* Ω_1 *is characterized by* $\mathbf{P}_1$ *(non-discrimination),* $\mathbf{P}_2$ *(redundancy) and* $\mathbf{P}_4$ *(restriction to the least core).*

Proof: The proof follows the same lines as the proof of Theorem 3.1 and Corollary 3.2, except that we use Lemma 4.1 to get a function constant on $\mathscr{LC}(\Pi, F)$. ■

Algorithm 2.4 can be adapted to an algorithmic scheme to compute the nucleolus in a class Ω_1. Step 4 is not needed to obtain a constant function and can be skipped. The relevant examples in Theorem 3.3 all satisfy (α), (β) and (δ) and therefore show that each of the axioms of Theorem 4.2 does not follow from the others.

For use later on, we state and prove two additional properties of the general nucleolus defined on a class Ω_1 satisfying (α), (β) and (δ). (They are not true in other classes.)

($\mathbf{P}_{11}$) *(Deletion of the smaller of two functions with constant difference) If* $(\Pi, F)\in\Omega_1$ *and* $F_i = F_j + \lambda$ *for some* i *and* j, $i \neq j$, *with* $\lambda \in \Re_+$, *then* $\Phi(\Pi, F) = \Phi(\Pi, F_{-j})$.

($\mathbf{P}_{12}$) *(Indifference)* F *is constant on* $\Phi(\Pi, F)$.

Theorem 4.3. The nucleolus of a class Ω_1 *satisfying* (α), (β) *and* (δ) *satisfies* $\mathbf{P}_{11}$ *and* $\mathbf{P}_{12}$.

Proof: Follow the Algorithm 2.4, skipping Step 4. The algorithm ends as soon as an empty set M is reached. As long as F_i is not deleted, F_j also is not deleted. When F_i becomes constant, F_j too becomes constant and both are deleted. The only place where the computation handles F_i and F_j differently is in Step 3, when the least core is computed. If $\lambda > 0$ then F_j plays no role and can be omitted. If $\lambda = 0$ F_j can also be omitted because F_i is sufficient for the computation. This proves $\mathbf{P}_{11}$. As to $\mathbf{P}_{12}$, note that during the process one obtains smaller and smaller least cores and therefore all the functions that were deleted are constant in the nucleolus. Eventually all functions are deleted[21]. ■

Remark. The following example shows that $\mathbf{P}_{11}$ and $\mathbf{P}_{12}$ do not hold if we consider classes of pairs satisfying (α), (β), (γ) (see previous section). Here, Ω contains the pair (Π, F), where $\Pi = [0, 1]$, $F = \{F_1, F_2, F_3\}$ with

$$F_1 = \begin{cases} 1 & \text{for } 0 \le t \le 1/2 \\ 2(1-t) & \text{for } 1/2 \le t \le 1, \end{cases} \qquad F_2 = \begin{cases} 2t & \text{for } 0 \le t \le 1/2 \\ 1 & \text{for } 1/2 \le t \le 1 \end{cases}$$

and $F_3 = F_2 - 1/2$. In addition, Ω contains (Π, F_{-3}) and the smallest set of pairs needed to be added to these pairs so as to satisfy (α), (β) and (γ) of the previous section. Then $\mathcal{N}(\Pi, F) = \{0\}$ whereas $\mathcal{N}(\Pi, F_{-3}) = \{0, 1\}$. Also, both F_1 and F_2 are not constant on $\{0, 1\} = \mathcal{N}(\Pi, F_1, F_2)$.

5 The Nucleolus of Matrix Games

Let A be an $m \times n$ matrix zero-sum game. We can associate with it a pair (Π, F), where $\Pi := \Delta_{II}(A)$ (the set of mixed strategies for Player II), and $F_i(q) = e_i A q$ for all q in $\Delta_{II}(A)$. Here, $M := \{1, \ldots, m\}$. Thus, $F_i(q)$ is the payment to Player I if he takes row i and II takes q. Our class Ω_2 will consist in this section of all such pairs, for all matrix games.

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[21] One can easily provide another proof by induction which is based directly on the axioms of Theorem 4.2.

As in [Potters and Tijs, 1992] (see also Dresher [1961] and Brown [1950], as well as Section 1) the *nucleolus of the game* for Player II was defined by

$$\mathcal{N}_{II}(A) := \mathcal{N}(\Delta_{II}(A), \{F_i\}_{i \in M}). \tag{5.1}$$

It is easy to see that $\mathcal{LC}(\Delta_{II}(A), F) = O_{II}(A)$, the set of optimal strategies of Player II in the matrix game A.

In this section we are going to axiomatize the nucleolus of A. We face two tasks: On the one hand, to formalize the axioms in game-theoretic terminology, and on the other, to overcome the fact that our class Ω_2 does not satisfy condition (α) of the previous section. There are two ways to circumvent the last difficulty. We can extend the theory of matrix games to zero-sum games defined on polytopes. We can then apply Theorem 4.2 to this class. We can, however, adopt a different approach: Consider Player II's set of optimal strategies. It is a polytope $\mathscr{P} := \operatorname{conv}\{q_1, q_2, \ldots, q_k\}$. We can therefore *identify* the game in which Player II's strategies are restricted to $\mathscr{P}$ with a classical matrix game B, with m rows and k columns, with $b_{ij} = \mathrm{e}_i A q_j$, $i \in M, j \in K := \{1, 2, \ldots, k\}$, and remain within the class of the classical games. Formally, that means that we replace axiom $\mathbf{P}_4$ by $\mathbf{P}_0$ $\mathbf{P}_3$ and $\mathbf{P}_9$ (contravariance). With these remarks we can now formulate:

Theorem 5.1. The nucleolus for Player II for the class of matrix games is characterized by the following axioms:

($\mathbf{P}_0$) $\Phi(A) \neq \emptyset$ *for all matrix games* A.
($\mathbf{P}_1$) $\Phi(A) = \Delta_{II}(A)$ *if the entries in each row are constant.*
($\mathbf{P}_2$) $\Phi(A) = \Phi(A_{-i})$ *if the i-th row has constant entries and*[22] $|M| \geq 2$.
($\mathbf{P}_3$) $\Phi(A) \subseteq O_{II}(A)$.
($\mathbf{P}_9$) *If* $\Lambda: \Delta \to \Delta_{II}(A)$ *is a linear map, where* Δ *is an arbitrary simplex, and if* $\Lambda^{-1}\Phi(A) \neq \emptyset$ *then* $\Phi(A \circ \Lambda) = \Lambda^{-1}\Phi(A)$. *Here,* $A \circ \Lambda =: B$ *is a matrix game with m rows and k columns,* $b_{ij} = \mathrm{e}_i A q_j$, $i \in \{1, \ldots, m\}$, $j \in \{1, 2, \ldots, k\}$, *where* q_j *is the image under* Λ *of the j-th extreme point of* Δ.

Proof: By induction on m. If $m = 1$ and the entries of the row of A are equal, then, by $\mathbf{P}_1$, $\Phi(A) = \Delta_{II}(A) = \mathcal{N}_{II}(A)$. If they are not equal then $O_{II}(A)$ is a proper face of $\Delta_{II}(A)$. By $\mathbf{P}_0$ and $\mathbf{P}_3$, $\Phi(A)$ is a nonempty subset of $O_{II}(A)$. Let Λ be the identity map of $O_{II}(A)$ into $\Delta_{II}(A)$ then, by $\mathbf{P}_9$, $\Lambda^{-1}\Phi(A) = \Phi(A \circ \Lambda)$. But $A \circ \Lambda$ is a 1-row matrix with equal entries. It follows that $\Phi(A) = \Lambda^{-1}\Phi(A) = O_{II}(A) = \mathcal{N}_{II}(A)$. Suppose now that $m > 1$. By $\mathbf{P}_3$, $\Phi(A) \subseteq O_{II}(A)$. The set $O_{II}(A)$ is a convex hull of finitely many extreme points and there is a simplex Δ and a linear map $\Lambda: \Delta \to \Delta_{II}(A)$ with image $O_{II}(A)$. Then, by $\mathbf{P}_9$, $\Phi(A \circ \Lambda) = \Lambda^{-1}\Phi(A)$. Now, at least one row i of $A \circ \Lambda$ has equal entries and can be omitted by $\mathbf{P}_2$; therefore, $\Lambda^{-1}\Phi(A) = \Phi((A \circ \Lambda)_{-i})$. The same equality holds for $\mathcal{N}_{II}$, so, by the induction hypothesis $\Lambda^{-1}\mathcal{N}_{II}(A) = \Lambda^{-1}\Phi(A)$, therefore, by $\mathbf{P}_9$, $\mathcal{N}_{II}(A) = \Phi(A)$. ■

[22] We do not wish to include matrix games with an empty matrix; therefore, we required $|M| \geq 2$ and added $\mathbf{P}_0$.

Remark. It can be proved that the axioms of Theorem 5.1 are logically independent. In fact, examples constructed on the basis of the examples of Section 3 will do. Here we show only that the solution $A \rightarrow O_{\Pi}(A)$ does not satisfy $\mathbf{P}_2$. Take, for example

$$A := \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix},$$

then $O_{\Pi}(A) = \Delta_{\Pi}(A)$, but $O_{\Pi}(A_{-1}) = \{e_2\}$.

6 The Nucleolus of a TU-Game with Permissible Coalitions and Permissible Imputations

Schmeidler's [1969] classical nucleolus is obtained from a TU-game $(N; v)$ if we take Π to be the set of imputations[23] (for a given coalition structure, usually $\{N\}$)[24], and $F = \{F_S\}_{S \subseteq N}$, where F_S is the excess function of the coalition S, namely, $F_S(x) = e(S, x) = v(S) - x(S)$. The class of these nucleoli does not satisfy (α) and (β) of the previous section, because the least core is usually not a set of imputations and because $\{F_S\}$ has to be the set of *all* excess functions of a given game. In order to be able to use the setup and results of the previous section we shall *extend the class of games:*

(i) by allowing games in which certain coalitions are *not permissible.*
(ii) by allowing games in which the set of imputations is *restricted a-priori* to a given polyhedral set[25].

The games restricted in this way will be called *truncated games*, or, more informatively, *TU-games with permissible coalitions and permissible imputations*. Formally, a truncated game will be a quadruplet $(N, \mathscr{S}, v, \Pi)$, where $N = \{1, 2, \ldots, n\}$ is the set of *players*, $\mathscr{S}$ is a subset of $2^N \setminus \{\emptyset, N\}$, called the *set of permissible coalitions*[26]. $v: \mathscr{S} \rightarrow \mathfrak{R}$ is the characteristic function and Π – *the set of permissible imputations* – is a set of the form:

$$\Pi = \{x \in \mathfrak{R}^N : x(N) = v(N),\ x(U) \geq a_U, \text{ for all } U \in \mathscr{U}\}. \tag{6.1}$$

[23] Preimputations, if we consider the prenucleolus.
[24] Note, however, that Schmeidler's original definition took Π to be an arbitrary closed set in $\mathfrak{R}^N$.
[25] One could deal with richer classes, but we wish to restrict the classes to a reasonable minimum for our purpose.
[26] Not allowing N in $\mathscr{S}$ is a technical convention. Actually, we *are* discussing the nucleolus for the grand coalition, so that N *will* form. The convention has been created because the excesses of N and $\emptyset$ are constant.

Here, $\mathscr{U}$ is a, possibly empty, collection of coalitions and the numbers a_U, $U \in \mathscr{U}$, are given numbers[27].

Remarks. Ordinary TU-games are, of course, truncated games. They are obtained by taking $\mathscr{S} = 2^N \setminus \{\emptyset, N\}$, $\mathscr{U} = \{\{i\}: i \in N \text{ and } a_{\{i\}} = v(\{i\})\}$.

Discussion. It is easy to interpret $\mathscr{S}$. As suggested by its name, we extend the scope of the cooperative games to situations in which the formation of some coalitions is out of the question: The members of such coalitions are not on speaking terms with each other, or an anti-trust law prevails, or communcation barriers exist, or, simply, people do not wish to be bothered; these are among the many good examples which may be found. It is more difficult to interpret Π. To do so, we have to go back to the idea of the core, remembering at the same time that we constantly discuss a cooperation of all the players towards forming the grand coalition. A requirement for being in the core means that the players will not cooperate unless each coalition gets at least its worth. Implicit in Π is a generalization of this concept. The players will not consent to cooperate in forming N unless it is guaranteed that *certain coalitions*[28] (members of $\mathscr{U}$) receive at least certain amounts (the amounts a_U). Such *a priori* restrictions may occur in real life. Actually[29], we can regard Π as a generalization of the (strong) ε-core. An outcome x belongs to the (strong) ε-core iff the excess of each coalition (other than $\emptyset$ and N) is not larger than ε. In Π we require that various coalitions have excesses not larger than certain numbers - not necessarily the same for all. For coalitions not in $\mathscr{U}$ we lay no *a priori* restrictions on their excesses. This way of looking at Π stresses again the fact that the nucleolus is tied to the core concept; namely, it s a rule which selects outcomes that reside in any non-empty "generalized" core.

Note that in ordinary TU games $v(N)$ and $v(\{i\})$, $i \in N$, have a double connotation: On the one hand they provide a monetary expression of the *worth* of each coalition, and in this capacity they resemble any other $v(S)$. On the other hand, they serve to form *a priori* reduction of the space of imputations to those which are both individually rational and efficient. Here the two roles are extended to other coalitions.

For TU-truncated games we define the least core and the nucleolus by

$$\mathscr{LC}(N, \mathscr{S}, v, \Pi) := \mathscr{LC}(\Pi, \{e(S, \cdot)\}_{S \in \mathscr{S}}), \tag{6.2}$$

$$\mathscr{N}(N, \mathscr{S}, v, \Pi) := \mathscr{N}(\Pi, \{e(S, \cdot)\}_{S \in \mathscr{S}}). \tag{6.3}$$

Note that the classical nucleolus and the classical prenucleolus of a game are particular nucleoli of this type. Note, however, that if Π is not bounded and if some

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[27] Other classes can be defined; for example, permissible imputations for coalition-structures other than $\{N\}$.

[28] This can be the situation even if these coalitions are not permissible. A permissible coalition can threaten to form (in order to improve the payments to its members). In contrast, a coalition can simply refuse to cooperate to form N (for the same purpose) even if it is not allowed to form.

[29] We are indebted to R. J. Aumann for this remark.

coalitions are missing from $\mathscr{S}$, both the nucleolus and the least core may be empty[30].

Let Ω_3 be a class of truncated games satisfying (α) and (β) of Section 4[31]. Then, by Theorem 4.2, the nucleolus for this class is characterized by axioms P_1, P_2 and P_4, which make perfect sense within the framework of truncated games. The purpose of this section is to provide a *different axiomatic characterization*, using the concept of a *reduced game*. However, contrary to classical reduced games in which one discards sets of players, here we shall discard *sets of coalitions*.

Definition 6.1. Let $(N, \mathscr{S}, v, \Pi)$ be a truncated game and let $\mathscr{T}$ be a subset of $\mathscr{S}$. Let x be a point in Π. The *reduced game of* $(N, \mathscr{S}, v, \Pi)$ *on* $\mathscr{T}$ *at* x is the truncated game $(N, \mathscr{T}, v\,|\,\mathscr{T}, \Pi^x_{\mathscr{S}\to\mathscr{T}})$, where

$$\Pi^x_{\mathscr{S}\to\mathscr{T}} := \Pi \cap \{y \in \Re^N : y(S) = x(S) \ \text{ for all } \ S \in \mathscr{S} \setminus \mathscr{T}\}. \tag{6.4}$$

Here, $v\,|\,\mathscr{T}$ is the restriction of the domain of v to $\mathscr{T}$. Thus, in the reduced game, the set of permissible coalitions is restricted to $\mathscr{T}$, and Π is also reduced to those imputations that have the same excess at x for coalitions of S outside $\mathscr{T}$.

Comment. The original Davis-Maschler [1965] reduced game is nevertheless closely related to this one. Remember that a classical reduced game on a subset of players T, at a preimputation x, is given by

$$v^x_T(S) = \begin{cases} \max\{v(S \cup Q) - x(Q) : Q \subseteq N \setminus T\} & \text{if } \emptyset \neq S \subsetneq T \\ x(S) & \text{if } S \in \{T, \emptyset\}\,. \end{cases} \tag{6.5}$$

Compare this formula with the one obtained by reducing $(N, 2^N \setminus \{N, \emptyset\}, v, \Pi)$ at x on $\mathscr{T} := 2^N \setminus (\{N, \emptyset\} \cup \{\{i\} : i \in N \setminus T\})$. In this reduced game we freeze the payments to the members of $N \setminus T$ at $x^{N \setminus T}$; thus, in effect, only the players in T are playing. For them, the space of preimputation is the same as that of $(T; v^x_T)$ and every nonempty proper[32] subsets of T can ensure $v^x_T(S)$, and no more, by choosing proper partners Q from $N \setminus T$, paying them the fixed rate $x(Q)$. Now, formally, there are more options in the reduced game of this section: Several coalitions of the type $S \cup Q$ are permitted, for Q's being subsets of $N \setminus T$ and a fixed S, $S \subseteq T$. If $S = \emptyset$, the excesses of these coalitions in the reduced game are constant and they can be omitted for the purpose of computing the nucleolus (Axiom $\mathbf{P}_2$). If $S \neq \emptyset$ then the difference of the excesses in the reduced game,

$$\hat{e}(S \cup Q_1, y) - \hat{e}(S \cup Q_2, y) = [v(S \cup Q_1) - y(S) - x(Q_1)] - [v(S \cup Q_2) - y(S) - x(Q_2)]$$

[30] Precisely because of this feature we included the possibility of an empty nucleolus in the axiomatization of Sections 3 and 4.

[31] Clearly (δ) is also satisfied.

[32] Note the special role of T: even if $\max v(\{(T \cup Q) - x(Q) : Q \subseteq N \setminus T\})$ is greater than $x(T)$, this fact plays no role in determining the space of preimputations of both reduced games. Similarly, $\emptyset$ has no effect.

is constant in y, for every y in $\Pi^x_{\mathscr{S}\to\mathscr{T}}$; so by $\mathbf{P}_{11}$, only the Q's for which $v(S\cup Q)-x(Q)$ is maximal need to be considered for the purpose of looking for the nucleolus. Thus, for a society that believes in the nucleolus, the two reduced games are equivalent.

Definition 6.2. We say that a solution concept Φ, defined on a class of truncated games, *satisfies the reduced game property*, or *is consistent*, if it satisfies

($\mathbf{P}_{13}$) *(reduced game property)* $x\in\Phi(N, \mathscr{S}, v, \Pi)$ *implies* $x\in\Phi(N, \mathscr{T}, v\,|\,\mathscr{T}, \Pi^x_{\mathscr{S}\to\mathscr{T}})$, *whenever* $(N, \mathscr{T}, v\,|\,\mathscr{T}, \Pi^x_{\mathscr{S}\to\mathscr{T}})$ *belongs to the domain of* Φ.

Lemma 6.3. The nucleolus defined on an arbitrary class of truncated games satisfies the reduced game property.

Proof: The intersection of $\mathscr{N}(N, \mathscr{S}, v, \Pi)$ and $\Pi^x_{\mathscr{S}\to\mathscr{T}}$ for x in the nucleolus contains the point x. Therefore, by the strong IIA property $\mathbf{P}_8$, $\mathscr{N}(N, \mathscr{S}, v, \Pi^x_{\mathscr{S}\to\mathscr{T}})$ is equal to this intersection. On $\Pi^x_{\mathscr{S}\to\mathscr{T}}$, $e(S, y)=\text{constant}$ whenever $S\in\mathscr{S}\setminus\mathscr{T}$, so we can remove these coalitions without changing the nucleolus. Thus, we find that $x\in\mathscr{N}(N, \mathscr{T}, v\,|\,\mathscr{T}, \Pi^x_{\mathscr{S}\to\mathscr{T}})$. ■

Discussion. One interpretation of the reduced game property runs as follows: Suppose x in a solution Φ is being proposed, then someone may improve his payment by manipulation. He approaches members of a coalition S (or several coalitions) and tells them: "Please make your coalition unavailable (= nonpermissible). In return I shall give you $x(S)$ (or slightly more)". If this manipulation were beneficial, then Φ could be criticized for being *unstable*, or *inconsistent*. Satisfying the reduced game property means that this manipulation cannot benefit any player.

Although the reduced game property is essentially a special case of the strong IIA property, we can use it to axiomatize the nucleolus:

Theorem 6.4. Let Ω_N be the class of all[33] *truncated games on a fixed set of players N. Let Φ be a solution concept for this class that satisfies the following axioms:*

($\mathbf{P}_1$) *(non-discrimination)* $\Phi(N, \mathscr{S}, v, \Pi)=\Pi$ *if* $\mathscr{S}=\emptyset$,
($\mathbf{P}_2$) *(redundancy)* $\Phi(N, \mathscr{S}, v, \Pi)=\Phi(N, \mathscr{S}\setminus\{S\}, v, \Pi)$, *if* $e(S,\cdot)$ *is constant on* Π,
($\mathbf{P}_3$) $\Phi(N, \mathscr{S}, v, \Pi)\subseteq\mathscr{L}\mathscr{C}(N, \mathscr{S}, v, \Pi)$,
($\mathbf{P}_{13}$) Φ *satisfies the reduced game property.*

Under these conditions, $\Phi(N, \mathscr{S}, v, \Pi)\subseteq\mathscr{N}(N, \mathscr{S}, v, \Pi)$ *for every truncated game in* Ω_N.

Proof: The proof is by induction on the number m of permissible coalitions. If $|\mathscr{S}|=0$, then Φ is the nucleolus by $\mathbf{P}_1$. Assume the claim is true for truncated

[33] Actually, the theorem is true for every class of truncated games that, with each (N, S, v, Π), contains all the other truncated games mentioned in the theorem and in the proof.

games with fewer than m permissible coalitions and let $(N, \mathscr{S}, v, \Pi)$ be a truncated game with m permissible coalitions. We can assume that none of them has a constant excess on Π and that $\Phi(N, \mathscr{S}, v, \Pi) \neq \emptyset$. By $\mathbf{P}_3$,

$$\Phi(N, \mathscr{S}, v, \Pi) \subseteq \mathscr{L}\mathscr{C}(N, \mathscr{S}, v, \Pi) \subseteq \Pi^x_{\mathscr{S} \to \mathscr{T}} \tag{6.6}$$

where $\mathscr{T} := \{S \in \mathscr{S}: e(S, \cdot)$ is not constant on $\mathscr{L}\mathscr{C}(N, \mathscr{S}, v, \Pi)\}$ and x is *any* point in $\Phi(N, \mathscr{S}, v, \Pi)$. By Lemma 4.1, $(N, \mathscr{T}, v \mid \mathscr{T}, \Pi^x_{\mathscr{S} \to \mathscr{T}})$ has fewer than m permissible coalitions; therefore, by the reduced game property and the induction hypothesis,

$$\Phi(N, S, v, \Pi) \subseteq \Phi(N, \mathscr{T}, v \mid \mathscr{T}, \Pi^x_{\mathscr{S} \to \mathscr{T}}) \subseteq \mathscr{N}(N, \mathscr{T}, v \mid \mathscr{T}, \Pi^x_{\mathscr{S} \to \mathscr{T}}). \tag{6.7}$$

If $x \in \mathscr{N}(N, \mathscr{T}, v \mid \mathscr{T}, \Pi^x_{\mathscr{S} \to \mathscr{T}})$, then $\theta \circ \{e(S, x)\}_{S \in \mathscr{T}} \lesssim_{\text{lex}} \theta \circ \{e(S, y)\}_{S \in \mathscr{T}}$ for all y in $\Pi^x_{\mathscr{S} \to \mathscr{T}}$. Therefore, $\theta \circ \{e(S, x)\}_{S \in \mathscr{S}} \lesssim_{\text{lex}} \theta \circ \{e(S, y)\}_{S \in \mathscr{S}}$ for all y in $\Pi^x_{\mathscr{S} \to \mathscr{T}}$. But $\Pi^x_{\mathscr{S} \to \mathscr{T}} \supseteq \mathscr{L}\mathscr{C}(N, \mathscr{S}, v, \Pi)$ (see 6.6), therefore, $x \in \mathscr{N}(N, \mathscr{S}, v, \Pi)$. ■

In view of Theorem 2.3 and Lemma 6.3 we can now paraphrase Theorem 6.4 by

Corollary 6.5. The nucleolus of the class Ω_N is the largest solution concept satisfying $\mathbf{P}_1$, $\mathbf{P}_2$, $\mathbf{P}_3$ *and* $\mathbf{P}_{13}$.

Comparison between our axioms and the axioms of Sobolev [1975]. Sobolev's class of games is richer in the sense that it requires that the domain of the nucleolus consists of *all* TU n-person games[34], whereas we can stay with the class of all truncated games on a *fixed* set of players N. Our class is richer in the sense that it contains *truncated games* – not only ordinary TU games. Sobolev's axioms characterize only the prenucleolus. Ours characterize simultaneously the nucleolus, the prenucleolus and many other nucleoli.

However, Sobolev's result is deeper, and his proof is quite ingenious, whereas our proofs are much simpler. This is due to the fact that we require the solution to be in the least core – an axiom that is not needed in Sobolev's theory. We feel that this is an important axiom because it points out what the nucleolus is all about: A desire to minimize maximum excess as a primary goal. This makes it a core-motivated solution; namely, a solution concept that "points" to the location of the core, and if it is empty – to its latent image (Shubik [1983], see also Section 1).

'Being a subset of the least core' is equivalent to 'being a subset of every nonempty (strong) ε-core'. The importance of nonempty ε-cores to economics is known. For example, Wooders [1991] shows that under proper conditions, replications of economies (even with empty cores) lead to nonemptyness of the strong ε-cores, for

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[34] For example, if the domain is the class of all TU games on 4 players, or less, take as a solution any non-nucleolus kernel point for some 4-person games, together with the nucleoli points for all other games. This "solution" satisfies all Sobolev's axioms, but it is not the nucleolus (B. Peleg, oral communication).

all sufficiently large replications and to convergence of the ε-cores to the Walrasian payoffs of the limit market.

7 Characterization of the Nucleolus of a Truncated Game by Balanced Collections

Let x be an imputation in a TU-game $(N; v)$. Define

$$\mathscr{B}_t(x) := \{S : e(S, x) \geq t\}, \tag{7.1}$$

$$\mathscr{C}(x) := \{\{i\} : x_i = v(\{i\})\}. \tag{7.2}$$

Kohlberg [1971] proved that a necessary and sufficient condition that x is the *nucleolus* of the game is that whenever $\mathscr{B}_t(x) \neq \emptyset$, $\mathscr{B}_t(x) \cup \mathscr{C}(x)$ is weakly balanced[35] with positive coefficients for the coalitions of $\mathscr{B}_t(x)$. Sobolev [1975] showed that a similar condition for the *prenucleolus* holds, if one replaces $\mathscr{C}(x)$ by the empty set. Related theorems for other nucleoli appear in Owen [1977], Wallmeier [1983], Potters and Tijs [1990]. Note the special role of the single person coalitions. These are the coalitions responsible for the determination of the space of imputations. This suggests that a similar characterization holds for the nucleolus of truncated games.

Definition 7.1. Let $\mathscr{B}$ and $\mathscr{C}$ be two collections of coalitions. We shall say that $\mathscr{B}$ *is balanced with the help of* $\mathscr{C}$, if positive λ_S's exist for each S in $\mathscr{B}$ and non-negative μ_U's exist for every U in $\mathscr{C}$ such that

$$\sum_{S \in \mathscr{B}} \lambda_S \mathbf{e}_S + \sum_{U \in \mathscr{C}} \mu_U \mathbf{e}_U = \mathbf{e}_N. \tag{7.3}$$

Here $\mathbf{e}_T$ is an n-tuple consisting of zeroes and ones, whose i-th coordinate, $i \in N$, is 1 iff $i \in T$ (i.e., e_T is *the characteristic vector of* T).

For an imputation x in a truncated game $(N, \mathscr{S}, v, \Pi)$ we now define

$$\mathscr{B}_t(x) := \{S \in \mathscr{S} : e(S, x) \geq t\}, \tag{7.4}$$

$$\mathscr{C}(x) := \{U \in \mathscr{U} : x(U) = a_U\}, \tag{7.5}$$

(see (6.1)). We can prove:

Theorem 7.2. An imputation x in Π is a nucleolus point of a truncated TU-game $(N, \mathscr{S}, v, \Pi)$ if and only if $\mathscr{B}_t(x)$ is balanced with the help of $\mathscr{C}(x)$, whenever $\mathscr{B}_t(x) \neq \emptyset$.

[35] I.e., the balancing coefficients are allowed to be zeros.

To provide a proof we need some notation, definitions, and two lemmas.

Definition 7.3. The *strict least core* of a truncated game, denoted $\mathscr{L}\mathscr{C}(N, \mathscr{S}, v, \Pi)^{\#}$, is given by

$$\mathscr{L}\mathscr{C}(N, \mathscr{S}, v, \Pi)^{\#} := \{x \in \mathscr{L}\mathscr{C}(N, \mathscr{S}, v, \Pi) : \mathscr{B}(x) = \mathscr{B}\}, \tag{7.6}$$

where

$$\mathscr{B}(x) := \{S \in \mathscr{S} : e(S, x) = \bigvee_{T \in \mathscr{S}} e(T, x)\}, \tag{7.7}$$

$$\mathscr{B} := \{S \in \mathscr{S} : e(S, x) = \bigvee_{T \in \mathscr{S}} e(T, x) \text{ for all } x \in \mathscr{L}\mathscr{C}(N, \mathscr{S}, v, \Pi)\}. \tag{7.8}$$

Thus, $\mathscr{B}(x)$ is the set of maximum excess permissible coalitions at x, whereas B consists of those permissible coalitions that achieve maximum excess at all points of the least core. The strict least core contains the relative interior[36] of the least core, but may contain also some boundary points.

Lemma 7.4. Let $(N, \mathscr{S}, v, \Pi)$ be a truncated TU-game, $\mathscr{S} \neq \emptyset$. A necessary and sufficient condition for an imputation x in Π to belong to the strict least core is that $\mathscr{B}(x)$ is balanced with the help of $\mathscr{C}(x)$ (see (7.7), (7.5), and Definition 7.1).

Proof: Consider the following pair of dual *LP* problems:

(P) $x \in \mathfrak{R}^N$, $t \in \mathfrak{R}$		(D)
$-x(N) = -v(N)$ $x(U) \geq a_U$ for all $U \in \mathscr{U}$ $x(S) + t \geq v(S)$ for all $S \in \mathscr{S}$	$z \in \mathfrak{R}$ $w_U \geq 0$ $y_S \geq 0$	$\sum_{U \in \mathscr{U}, i \in U} w_U + \sum_{S \in \mathscr{S}, i \in S} y_S = z$ for all $i \in N$ $\sum_{S \in \mathscr{S}} y_S = 1$
min t		max $\sum_{U \in \mathscr{U}} w_U a_U + \sum_{S \in \mathscr{S}} y_S v(S) - z v(N)$.

Let $\hat{t}$ be the minimum value of (P). A point $(x, \hat{t})$ is in the solution space of the primary program (P), iff x belongs to the least core of the truncated game. Let $(\hat{x}, \hat{t})$ be a point of the *relative interior*[37] of the solution space of (P), then,

$$\mathscr{B}(\hat{x}) = \{S \in \mathscr{S} : x(S) = v(S) - \hat{t} \text{ for all solutions } (x, \hat{t})\} = \mathscr{B}, \tag{7.9}$$

and

$$\mathscr{C}(\hat{x}) = \{U \in \mathscr{U} : x(U) = a_U \text{ for all solutions } (x, \hat{t})\}. \tag{7.10}$$

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[36] Namely, the interior of the core relative to the least manifold that contains the least core. This convention will be used in this section.
[37] ibid.

Then, by the strong complementary slackness theorem, we know that there is an optimal solution $(\hat{z}, \{\hat{y}_S\}_{S\in\mathscr{S}}, \{\hat{w}_U\}_{U\in\mathscr{U}})$ of the dual problem (D), with $\hat{y}_S>0$ iff $S\in\mathscr{B}(\hat{x})$ (and $\hat{w}_u>0$ iff $U\in\mathscr{C}(\hat{x})$). Since at least one $\hat{y}_S$ is positive, $\hat{z}$ must be positive and hence,

$$\lambda_S:=\hat{y}_S/\hat{z} \quad \text{for } S\in\mathscr{S} \quad \text{and} \quad \mu_U:=\hat{w}_U/\hat{z} \quad \text{for } U\in\mathscr{U} \tag{7.11}$$

provide a solution of (7.3) for $\mathscr{B}$ and $\mathscr{C}(\hat{x})$. If $x\in\mathscr{LC}(N,\mathscr{S},v,\Pi)^{\#}$ then $\mathscr{B}(x)=\mathscr{B}$ and $\mathscr{C}(x)\supseteq\mathscr{C}(\hat{x})$. Hence, $\mathscr{B}(x)$ is balanced with the help of $\mathscr{C}(x)$.

Conversely, let x be a point in Π for which $\mathscr{B}(x)$ is balanced with the help of $\mathscr{C}(x)$. Let $t:=\bigvee_{S\in\mathscr{S}} e(S,x)$. With this notation, (x,t) is feasible for the primary problem (P). There exist positive λ_S, $S\in\mathscr{B}(x)$ and non-negative μ_U, $U\in\mathscr{C}(x)$ such that (7.3) is satisfied for $\mathscr{B}=\mathscr{B}(x)$ and $\mathscr{C}=\mathscr{C}(x)$. Define: $z:=(\sum_{S\in\mathscr{B}(x)}\lambda_S)^{-1}$, $y_S:=\lambda_S z$ for $S\in\mathscr{B}(x)$, $y_S:=0$ for $S\in\mathscr{S}\setminus\mathscr{B}(x)$, $w_U:=\mu_U z$ for $U\in\mathscr{C}(x)$, $w_U:=0$ for $U\in\mathscr{U}\setminus\mathscr{C}(x)$. Then, $(z, \{y_S\}_{S\in\mathscr{S}}, \{w_U\}_{U\in\mathscr{U}})$, is a solution of the dual problem (D) which satisfies the complementary slackness conditions with (x,t). Thus, both solutions are optimal for their respective LP programs. In particular, x is a least core point. Further, if $S\in\mathscr{B}(x)$, then $e(S,y)=t$ for *every* point y in $\mathscr{LC}(N,\mathscr{S},v,\Pi)$, since, otherwise, (y,t) and $(z, \{y_S\}_{S\in\mathscr{S}}, \{w_U\}_{U\in\mathscr{U}})$ fail to satisfy the complementary slackness condition. Thus, $\mathscr{B}(x)=\mathscr{B}$ and $x\in\mathscr{LC}(N,\mathscr{S},v,\Pi)^{\#}$. ∎

Let $\Gamma=(N,\mathscr{S},v,\Pi)$ be a truncated game and let x be a fixed point in Π. Arrange the excesses of the permissible coalitions in a strictly decreasing order: $e_1(x)>e_2(x)>\cdots>e_k(x)$, $(k\le|\mathscr{S}|)$ and denote by $\mathscr{B}_i(x)$ the set of coalitions having the i-th excess:

$$\mathscr{B}_i(x)=\{S\in\mathscr{S}: e(S,x)=e_i(x)\}, \quad i=1,2,\ldots,k. \tag{7.12}$$

We shall now define new games, $\Gamma_i^x:=(N,\mathscr{S},v_i^x,\Pi)$ by

$$v_i^x(S)=\begin{cases} v(S)-e_j(x)+e_i(x) & \text{if } S\in\mathscr{B}_j(x), \quad j<i, \\ v(S) & \text{otherwise}. \end{cases} \tag{7.13}$$

Note that

$$\mathscr{B}(x\,|\,\Gamma_i^x)=\bigcup_{1\le j\le i}\mathscr{B}_j(x), \tag{7.14}$$

i.e., Γ_i^x was obtained from Γ by decreasing the worths of all coalitions in $\bigcup_{1\le j\le i}\mathscr{B}_j(x)$ so that their excesses at x will all be $e_i(x)$. (Here, $|\Gamma_i^x$ means "referred to Γ_i^x".)

Lemma 7.5. $x\in\Pi$ is a nucleolus point of Γ iff $x\in\mathscr{LC}(\Gamma_i^x)^{\#}$ for all i, $i=1, 2,\ldots,k$.

Proof: Let $x\in\Pi\setminus\mathscr{LC}(\Gamma_i^x)$ for some i. Then, there is a point y in Π with

$$\bigvee_{S\in\mathscr{S}} e(S,y\,|\,\Gamma_i^x) < \bigvee_{S\in\mathscr{S}} e(S,x\,|\,\Gamma_i^x) = e_i(x)\,. \tag{7.15}$$

It follows from (7.13) and (7.15) that if $S\in\mathscr{B}_j(x)$, $j<i$, we have

$$e(S,y)=v(S)-y(S)<e_i(x)+e_j(x)-e_i(x)=e_j(x)\le e_1(x)\,,$$

and for

$$S\in\mathscr{B}_j(x),\ j\ge i,\ e(S,y)=e(S,y\,|\,\Gamma_i^x)<e_i(x)\le e_1(x)\,.$$

This shows that x is *not* a nucleolus point, or, in other words, any nucleolus point x must belong to $\mathscr{L}\mathscr{C}(\Gamma_i^x)$ for every i.

Let x be a point in $\mathscr{L}\mathscr{C}(\Gamma_i^x)\backslash\mathscr{L}\mathscr{C}(\Gamma_i^x)^{\#}$ for some i. We may assume that i is the first index with this property. Let y be a point of $\mathscr{L}\mathscr{C}(\Gamma_i^x)^{\#}$. Then, $e(S,y)=e(S,x)=e_i(x)$ for all coalitions S in $\mathscr{B}(\Gamma_i^x)$. Here, $\mathscr{B}(\Gamma_i^x):=\{S\in\mathscr{S}: e(S,y) =\bigvee_{T\in\mathscr{S}} e(T,y)$, for all $y\in\mathscr{L}\mathscr{C}(N,\mathscr{S},v_i^x,\Pi)\}$. Further, $e(S,y)<e_i(x)$ for all S not in $\mathscr{B}(\Gamma_i^x)$, and there is at least one such coalition S^*, not in $\mathscr{B}(\Gamma_i^x)$, satisfying $e(S^*,x)=e_i(x)$. This shows that $x\notin\mathscr{N}(\Gamma)$. We have shown that any nucleolus point x belongs to $\mathscr{L}\mathscr{C}(\Gamma_i^x)^{\#}$, for every i.

Conversely, let x be an element of $\mathscr{L}\mathscr{C}(\Gamma_i^x)^{\#}$ for every i, $i=1, 2, \ldots, k$, and let y be any other point of Π. We shall prove that

$$\theta\circ\{e(S,x)\}_{S\in\mathscr{S}} \lesssim_{\text{lex}} \theta\circ\{e(S,y)\}_{S\in\mathscr{S}}\,. \tag{7.16}$$

Indeed, if not, then there exists a smallest index i having the property that $\mathscr{B}_i(x)$ contains a permissible coalition S^* with $e(S^*,y)\ne e(S^*,x)$. If there is a permissible coalition S, not in $\mathscr{B}_1(x)\cup\mathscr{B}_2(x)\cup\cdots\cup\mathscr{B}_{i-1}(x)$ with $e(S,y)>e_i(x)$ then (7.16) holds. Therefore, we may assume that $e(S,y)\le e(S,x)=e_i(x)$ for all S in $\mathscr{B}_i(x)$. In this case, $e(S^*,y)<e(S^*,x)=e_i(x)$ and $x\notin\mathscr{L}\mathscr{C}(\Gamma_i^x)^{\#}$. This contradiction shows that (7.16) holds after all; namely, $x\in\mathscr{N}(\Gamma)$. ■

Proof of Theorem 7.2: If $\mathscr{S}=\emptyset$ then $\mathscr{N}(\Gamma)=\Pi$ and the requirement of balancedness is vacuously satisfied. Suppose $\mathscr{S}\ne\emptyset$, then x is a nucleolus point iff $x\in\mathscr{L}\mathscr{C}(\Gamma_i^x)^{\#}$ for all indices i (Lemma 7.5). This, by Lemma 7.4, is equivalent with $\mathscr{B}(x\,|\,\Gamma_i^x)=\mathscr{B}_1(x)\cup\cdots\cup\mathscr{B}_i(x)$ being balanced with the help of $\mathscr{C}(x)$ for all i, $i=1, 2, \ldots, k$. ■

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