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An Algorithm for Finding the Nucleolus of Assignment Games

TAMÁS SOLYMOSI and TIRUKKANNAMANGAI E. S. RAGHAVAN

Department of Mathematics, Statistics and Computer Science, University of Illinois at Chicago 851 South Morgan Street, Chicago, Ill. 60607-7045, USA

Abstract: Assignment games with side payments are models of certain two-sided markets. It is known that prices which competitively balance supply and demand correspond to elements in the core. The nucleolus, lying in the lexicographic center of the nonempty core, has the additional property that it satisfies each coalition as much as possible. The corresponding prices favor neither the sellers nor the buyers, hence provide some stability for the market.

An algorithm is presented that determines the nucleolus of an assignment game. It generates a finite number of payoff vectors, monotone increasing on one side, and decreasing on the other. The decomposition of the payoff space and the lattice-type structure of the feasible set are utilized in associating a directed graph. Finding the next payoff is translated into determining the lengths of longest paths to the nodes, if the graph is acyclic, or otherwise, detecting the cycle(s). In an (m, n)-person assignment game with $m = \min(m, n)$ the nucleolus is found in at most $1/2 \cdot m(m+3)$ steps, each one requiring at most $O(m \cdot n)$ elementary operations.

1 Introduction

The nucleolus was introduced by Schmeidler (1969) as a single-point solution concept for cooperative games. It is the unique payoff that lexicographically minimizes the vector of nonincreasingly-ordered excesses over the set of imputations. It is well known that the nucleolus is always in the core of a game, provided the latter is nonempty. The assignment games with side payments, introduced by Shapley and Shubik (1972), are games with always nonempty core. For these games the worth of any coalition is completely determined by knowing the worth of all two-person subcoalitions. The core has a lattice-type structure that implies the existence of two special vertices. The core can be characterized by a small set of linear inequalities and equations. In this paper we present an algorithm that exploits these properties of an assignment game to locate its nucleolus.

The practical methods to find the nucleolus of general cooperative games include the weighted-sum approach of Kohlberg (1972) leading to a single, but extremely large LP. In order to ensure that the highest excess gets the largest weight, the second highest excess gets the second largest weight, and so on, coefficients from a very wide range must appear in the constraints (causing some serious numerical difficulties even for 3 or 4 players), and all possible permutations of the coalitions must be present among the constraints (enlarging the size of the LP enormously). In Owen's (1974) improved version one has to solve a single minimization problem with $2^{n+1}+n$ variables and 4^n+1 constraints for an n-person game.

Maschler, Peleg and Shapley (1979) gave an alternative definition of the nucleolus. They describe a finite process that iteratively reduces the set of payoffs to a singleton, called the lexicographic center of the game. It is then shown that the unique final payoff is exactly the nucleolus. This constructive definition is more appropriate for a practical implementation. In fact, each iteration can be carried out by solving linear programs with $n+1$ variables and 2^n constraints including only -1, 0, or 1 coefficients. Recently Sankaran (1991) proposed such a formulation with at most $O(2^n)$ LPs to be solved for an n-person game.

Computing the nucleolus of a cooperative game can be cast as a special case of a multiobjective optimization problem, in which the conflict among the objective functions is resolved in a lexicographic way. In different general settings Maschler, Potters and Tijs (1992) investigate the properties of and provide axioms for such optimizations. They also present an extension of the iterative process described in Maschler et al. (1979) for their general setting.

For assignment games a method based on linear programming does not seem to be well-suited, since the combinatorial structure of the characteristic function cannot be effectively translated into a continuous problem formulation. While solving an assignment problem, the ordinary primal simplex method encounters difficulties caused by degeneracy. It is outperformed by specifically designed algorithms, such as Kuhn's (1955) well-known algorithm based on combinatorial arguments. In the same spirit we apply graph-related techniques instead of linear programming.

In the special case, when there are alternative optimal assignments such that the graph (whose nodes are all the mixed pairs appearing in some optimal assignment, with any two nodes connected if they have a player in common) contains a spanning tree, Granot and Granot (1992) characterize the location of the nucleolus. As a consequence of their investigation of the relationship between the kernel and the core, they show that under the above condition the nucleolus is the unique vector in which the individual payoffs are maximized in a lexicographic sense. Further, the payoff vectors in the core are determined by a single parameter, and the only coalitions which play a role in determining the nucleolus are the single-member coalitions. Thus, the nucleolus can be computed in linear time. However, without the spanning tree condition, as the authors themselves point out, the above characterization ceases to hold even for the well-known horse market example (see Shapley and Shubik (1972)) which has a line segment as its core. As for the general case, when the mixed pairs as well as the individual players play a role (due to the characteristic function they are sufficient), we are not aware of any specific method.

In a general assignment game the graph described in the preceeding paragraph is decomposed into several connected components. In fact, they are cliques, i.e. maximal complete subgraphs. For example, in case of a unique optimal assignment each one is an isolated node. Roughly speaking, the main problem is to find binding relationships between players related to different components, that would add new edges to the graph and reduce the number of components. Unfortunately, one cannot easily obtain an optimal assignment on a matrix from optimal assignments on some of its submatrices, so combining the components cannot be done by working merely on the components.

Geometrically, the problem translates to decomposing the payoff space into fewer and fewer orthogonal subspaces. All but one of these subspaces are in one-to-one correspondence with the above components. The remaining one is related to the missing edges, i.e. to the complement of the graph. That subspace contains the nucleolus, and it is shrinking until it becomes a line, so the above special case is obtained. The authors were able to exploit a similar geometric approach to locate the nucleolus for general cooperative games. That will be the topic of a subsequent paper.

The organization of the paper is as follows. After recalling some basic definitions and known results in Section 2, we define the lexicographic center of an assignment game in Section 3. This is a specialized version of Maschler et al.'s (1979) definition, incorporating the simplifications which are possible for such games. For example, the set of coalitions can be restricted to individual players and certain two-player coalitions, and the lattice-type structure of the core is inherited by all subsequent payoff sets. In Section 4 we discuss the partition of the coalitions into so-called settled and unsettled blocks. This plays a fundamental role in associating a graph to the optimization problem to be solved in each iteration. The structure of the feasible payoff sets is investigated in Section 5. Following the arguments of Shapley and Shubik (1972) for the core, we establish the existence of the two special vertices for each set. Also here, the reformulation of the optimization problem into finding the lengths of longest paths to the nodes of a directed graph is introduced, and the equivalence of the two settings is shown. The algorithm is presented in Section 6. It generates a finite sequence of payoffs leading to the nucleolus, such that each one is the special vertex of the current feasible set, providing monotonicity for the subsequent payoffs. It is shown that for an (m, n)-person game with $m=\min(m, n)$, the algorithm determines the nucleolus in at most $1/2 \cdot m(m+3)$ steps, each one requiring at most $O(m \cdot n)$ elementary operations. We illustrate the process on a (4, 5)-person game in Section 7.

2 Preliminaries

Let us recall some basic definitions. A p-person cooperative game with side payments is given by the finite set of *players* $P=\{1, \ldots, p\}$ and by the *characteristic function* $V: 2^P \to \boldsymbol{R}$ satisfying $V(\emptyset)=0$ and $V(P) \geq \sum_{i \in P} V(\{i\})$. Given a game $(P; V)$, the *excess* of a *coalition* $S \subset P$ at a *payoff* vector $x \in \boldsymbol{R}^p$ is $e(S, x)=V(S)-x(S)$, where $x(S)=\sum_{i \in S} x_i$ if $S \neq \emptyset$, and $=0$ if $S=\emptyset$. The negative excess $f(S, x)=-e(S, x)$ will be called the *satisfaction* of S over x. It represents the contentment (or discontentment if negative) of the coalition as a whole with the given total payoff. An *imputation* is a vector $x=(x_1, \ldots, x_p)$ such that $x(P)=V(P)$ and $x_i \geq V(\{i\})$ for all $i \in P$. We denote the set of imputations by $I=I(P; V)$. The set $C=C(P; V)$ of imputations such that $x(S) \geq V(S)$ for all $S \subset P$ is called the *core*. The set of imputations is always a nonempty polytope, but the core can be empty for some games.

The *nucleolus*, one of the single-point solution concepts, was introduced by Schmeidler (1969). For each $x \in I(P;\ V)$ let $F(x)$ denote the 2^p-vector whose components are the numbers $f(S, x)$, $S \subset P$, arranged in a nondecreasing order, i.e., $F_j(x) \leq F_k(x)$ whenever $j<k$, and $F_j(x)=f(S, x)$ for some S such that every $S \subset P$ occurs exactly once as j runs through $1, \ldots, 2^p$. Let $>_L$ denote the usual lexicographic order of vectors (i.e. $y >_L z$ if there exists an index j_0 such that $y_j = z_j$ for all $j < j_0$ and $y_{j_0} > z_{j_0}$), and $\geq_L$ its weak form (i.e. $y \geq_L z$ if $y >_L z$ or $y=z$). The nucleolus of a game is the set of those imputations which lexicographically maximize the vector of ordered satisfactions over the set of imputations. Formally,

$$v = v(P;\ V) = \{x \in I(P;\ V) : F(x) \geq_L F(y) \quad \forall y \in I(P;\ V)\}. \tag{2.1}$$

Schmeidler showed that every game possesses a nonempty nucleolus that consists of a single point. Since

$$C = \{x \in I : f(S, x) \geq 0 \quad \forall S \subset P\}, \tag{2.2}$$

so $v \subset C$, provided the core is nonempty.

Assignment games with side payments were introduced by Shapley and Shubik (1972). These games are models of two-sided markets, where each player on one side can supply exactly one unit of some indivisible goods and exchange it for money with a player from the other side whose demand is also one unit. When a transaction between i and j takes place certain profit $a_{ij} \geq 0$ occurs. The worth of a coalition is given by an assignment of the players within the coalition which maximizes the total profit of the assigned pairs. Therefore the characteristic function is fully determined by the profits of the mixed pairs.

We call the two types of players as row and column players, and denote their sets by $\bar{M}$ and $\bar{N}$, respectively. Throughout the paper we assume without loss of generality that $m = |\bar{M}| \leq |\bar{N}| = n$. Since the worth of coalition $\bar{S} \cup \bar{T}$, $\bar{S} \subset \bar{M}$, $\bar{T} \subset \bar{N}$ depends only on the submatrix $(a_{ij})_{(\bar{S}, \bar{T})}$, we can identify the coalition with the index set $(\bar{S}, \bar{T}) = \{(i, j) : i \in \bar{S}, j \in \bar{T}\}$.

It will prove convenient to introduce a fictitious row and column player, and to agree that both are 'present' in any coalition. By labelling them with 0, the set of players is $P = (M, N) = (\bar{M} \cup \{0\}, \bar{N} \cup \{0\})$. Also, from now on (S, T) stands for the coalition of $S \subset M$, $0 \in S$ and $T \subset N$, $0 \in T$. With this notation we write $(0, 0)$ for the empty coalition, $(i, 0)$ for row player $i \in \bar{M}$, $(0, j)$ for column player $j \in \bar{N}$. Naturally, only fictitious profit can be made with a fictitious player. Thus we will assume $a_{00} = a_{i0} = a_{0j} = 0$ for all $i \in \bar{M}$, $j \in \bar{N}$ in the *augmented profit matrix* $A = (a_{ij})_{(M,N)}$. On the other hand, the payoff is fictitious too, so $u_0 = v_0 = 0$ in any payoff vector $(u,\ v) = (u_0,\ u_1,\ \ldots,\ u_m;\ v_0,\ v_1,\ \ldots,\ v_n)$. The set of payoffs is denoted by $UV \subset \boldsymbol{R}^{|M|} \times \boldsymbol{R}^{|N|}$. By an (S, T)-*matching* μ we mean a correspondence between S and T that induces a matching $\bar{\mu}$ on $(\bar{S}, \bar{T}) = (S \setminus \{0\}, T \setminus \{0\})$, and $(0, 0) \in \mu$, $(i, 0) \in \mu$ if $\{j \in \bar{T} : (i, j) \in \bar{\mu}\} = \emptyset$, $(0, j) \in \mu$ if $\{i \in \bar{S} : (i, j) \in \bar{\mu}\} = \emptyset$. That is to say, μ is a matching between 'real' players, and all unmatched players are assigned to the fictitious player of the other type.

The characteristic function of an assignment game with augmented profit matrix $A=(a_{ij})_{(M,N)}$ is given by

$$V(S,\,T)=\mathrm{Max}\left\{\sum_{(i,j)\in\mu} a_{ij}\text{: for all } (S,\,T)\text{-matching } \mu\right\}. \tag{2.3}$$

Since $A\geq 0$ we can assume without loss of generality that μ is a *full* (S, T)-matching, i.e. $\bar{\mu}$ is *onto* for at least one of $\bar{S}$ or $\bar{T}$. The decomposable nature of the characteristic function is inherited by the satisfaction, providing significant reductions in finding the nucleolus. Namely, by introducing the notation $f_{ij}(u, v):=f((\{i\}, \{j\}), (u, v))$ we have the following

2.4. Lemma. Let $V(S, T)=\sum_{(i,j)\in\mu} a_{ij}$ for some (S, T)-matching μ. Then for any payoff $(u, v)\in UV$, $f((S, T), (u, v))=\sum_{(i,j)\in\mu} f_{ij}(u, v)$.

Proof. $f((S, T), (u, v))=(u, v)(S, T)-V(S, T)=\sum_{i\in S} u_i+\sum_{j\in T} v_j-\sum_{(i,j)\in\mu} a_{ij}$

$$=\sum_{(i,j)\in\bar{\mu}} (u_i+v_j-a_{ij})+\sum_{\substack{i\in\bar{S}\\(i,0)\in\mu}} (u_i+v_0-a_{i0})$$

$$+\sum_{\substack{j\in\bar{T}\\(0,j)\in\mu}} (u_0+v_j-a_{0j})+(u_0+v_0-a_{00})$$

$$=\sum_{(i,j)\in\mu} f_{ij}(u, v).$$ ●

The following alternative characterization of the core is immediate.

2.5. Corollary. For assignment game $A(M, N)$,

$$C=\{(u, v)\in UV{:}f((M, N), (u, v))=0, f_{ij}(u, v)\geq 0\ \forall (i, j)\in(M, N)\}.$$

Shapley and Shubik (1972) showed that for assignment games the core is never empty, and that the set in Corollary 2.5 is precisely the set of dual optimal solutions to the linear program that solves the combinatorial problem in (2.3) for $V(M, N)$. Their LP approach also provides a practical method to determine a point of the core. Further important properties, that are related to complementary slackness, are easy consequences of Lemma 2.4 and Corollary 2.5.

2.6. Corollary. Let σ be an optimal (M, N)-matching (i.e. $V(M, N)=\sum_{(i,j)\in\sigma} a_{ij}$). If $(i, j)\in\sigma$ then $f_{ij}(u, v)=0$ for all $(u, v)\in C$.

Proof. Let $(u, v)\in C$. Then $0=f((M, N), (u, v))=\sum_{(i,j)\in\sigma} f_{ij}(u, v)\geq 0$ implies $f_{ij}(u, v)=0$ for all $(i, j)\in\sigma$. ●

This means that the optimally matched players share the exact profit they can make, but an unmatched player receives nothing in any core allocation.

3 The Lexicographic Center of Assignment Games

An alternative definition of the nucleolus was given by Maschler, Peleg and Shapley (1979). It is an iterative process that constructs the set of payoffs which lexicographically maximize the vectors of ordered satisfactions. It is shown that this optimum set is a single point, called the lexicographic center of the game, that coincides with the nucleolus. Next we present a version of this definition specialized to assignment games, and certain results that are needed in the sequel.

3.1. Definition. Let σ be a fixed optimal (M, N)-matching for an assignment game $A(M, N)$. We construct a sequence $(\Delta^0, \Sigma^0), \ldots, (\Delta^{\rho+1}, \Sigma^{\rho+1})$ of partitions of (M, N) such that $\Sigma^0 \supset \ldots \supset \Sigma^{\rho+1}$, and a nested sequence $X^0 \supset \ldots \supset X^{\rho+1}$ of sets of payoff vectors, as follows:

Initially, let $\Delta^0 = \sigma$, $\Sigma^0 = (M, N) \setminus \sigma$, $X^0 = \{(u, v) \in UV : (u, v) \geq (0, 0), f_{ij}(u, v) = 0 \ \forall (i, j) \in \Delta^0, f_{ij}(u, v) \geq \alpha^0 \ \forall (i, j) \in \Sigma^0\}$, where $\alpha^0 = \operatorname*{Min}_{(i,j) \in \Sigma^0} f_{ij}(u^0, v^0)$ with $u_i^0 = a_{i\sigma(i)} \ \forall i \in M$, $v_j^0 = 0 \ \forall j \in N$.

For $r = 0, 1, \ldots, \rho$ define recursively

(1) $\alpha^{r+1} = \operatorname*{Max}_{(u,v) \in X^r} \operatorname*{Min}_{(i,j) \in \Sigma^r} f_{ij}(u, v)$

(2) $X^{r+1} = \left\{(u, v) \in X^r : g^r(u, v) := \operatorname*{Min}_{(i,j) \in \Sigma^r} f_{ij}(u, v) = \alpha^{r+1}\right\}$

(3) $\Sigma_{r+1} = \{(i, j) \in \Sigma^r : f_{ij}(u, v) = \text{constant on } X^{r+1}\}$

(4) $\Sigma^{r+1} = \Sigma^r \setminus \Sigma_{r+1}$, $\Delta^{r+1} = \Delta^r \cup \Sigma_{r+1}$,

where ρ is the last value of index r for which $\Sigma^r \neq \emptyset$.

The set $X^{\rho+1}$ is called the *lexicographic center* of $A(M, N)$.

Before we prove the correctness of the definition, and show that it gives the nucleolus, let us make the following

3.2. Remarks. Compared to the definition in Maschler et al. (1979), the main differences are:

(i) Excess is replaced by satisfaction, hence Min and Max are interchanged.
(ii) In (3) all constant functions are put aside, not just those at the *current guaranteed level* of satisfaction α^{r+1}.
(iii) The satisfactions of only the *relevant coalitions* (single players and mixed pairs) are considered, instead of all coalitions.

(iv) The initial feasible payoff set X^0 is not the whole imputation set, but a subset restricted by some constraints which are valid in the (nonempty) core, hence for the nucleolus.
(v) Changes (i) and (ii) can be made in general, but (iii) and (iv) are specific to the structure of assignment games.

Following the arguments of Maschler et al. (1979), first we show that the above iterative process is finite.

3.3. Lemma. The number ρ in Definition 3.1 is finite.
For $r=0, \ldots, \rho$,

(i) the α^{r+1} are well defined;
(ii) the X^{r+1} are nonempty, compact, and convex;
(iii) the $\Sigma'_{r+1}=\{(i, j)\in\Sigma^r : f_{ij}(u, v)=\alpha^{r+1}\ \forall (u, v)\in X^{r+1}\}$ are nonempty;
(iv) if $r\geq 1$ then $\alpha^{r+1}>\alpha^r$.

Proof. The functions $g^r = \operatorname{Min}_{(i,j)\in\Sigma^r} f_{ij}$ are well defined for all r such that $\Sigma^r\neq\emptyset$. As a minimum of finitely many continuous and affine functions, g^r is continuous and concave. It follows that claim (ii)$_{r-1}$ implies claim (i)$_r$, and (i)$_r$ implies (ii)$_r$ as long as $\Sigma^r\neq\emptyset$. Since X^0 satisfies (ii)$_{-1}$, both (i)$_r$ and (ii)$_r$ are proved by induction up to $r=\rho$.

We prove (iii)$_r$ for $0\leq r\leq\rho$ by contradiction. To this end, suppose that $\Sigma'_{r+1}=\emptyset$ for some $0\leq r\leq\rho$. It means that for each $(i, j)\in\Sigma^r\neq\emptyset$ there is a point $(u, v)_{ij}\in X^{r+1}$ such that $f_{ij}((u, v)_{ij})>\alpha^{r+1}$. Define $(\bar{u}, \bar{v})=(1/|\Sigma^r|)\sum_{(i,j)\in\Sigma^r}(u, v)_{ij}$. By convexity $(\bar{u}, \bar{v})$ is also in $X^{r+1}\subset X^r$. Since for every $(i, j)\in\Sigma^r$, $f_{ij}(u, v)\geq\alpha^{r+1}$ for any $(u, v)\in X^{r+1}$, and the inequality is strict for $(u, v)_{ij}$, so $f_{ij}(\bar{u}, \bar{v})>\alpha^{r+1}$. Hence $g^r(\bar{u}, \bar{v})>\alpha^{r+1}$, a contradiction to the definition of α^{r+1}. This also proves the finiteness of ρ, since at each iteration a nonempty set $\Sigma_{r+1}\supset\Sigma'_{r+1}$ is removed from Σ^r to obtain Σ^{r+1}.

If $1\leq r\leq\rho$ then $\Sigma^r\neq\emptyset$ and none of the functions f_{ij}, $(i, j)\in\Sigma^r$, are constant on X^r (this second assertion may be false for $r=0$). On the other hand, for each $(i, j)\in\Sigma^r$, $f_{ij}(u, v)\geq\alpha^r$ at any $(u, v)\in X^r$. By repeating the previous construction we can find a point $(\tilde{u}, \tilde{v})\in X^r$ such that $g^r(\tilde{u}, \tilde{v})>\alpha^r$. Therefore, $\alpha^{r+1}=\operatorname{Max}_{(u,v)\in X^r} g^r(u, v)>\alpha^r$. ●

To completely establish the correctness of Definition 3.1, next we show that the lexicographic center is independent of the choice of the optimal assignment.

3.4. Lemma. In Definition 3.1,

(i) $\alpha^0\leq 0$ implies $C\subset X^0$;
(ii) $\alpha^0\geq 0$ implies $X^0\subset C$;

(iii) $\alpha^1 \geq 0$ and $X^1 \subset C$;
(iv) if the optimal (M, N)-matching is not unique then $\alpha^1 = 0$, $X^1 = C$.

Proof. Claims (i) and (ii) are immediate from the definitions and Corollary 2.6.

As for claim (iii), in case $\alpha^0 \geq 0$ we have $\alpha^1 \geq \alpha^0 \geq 0$ and $X^1 \subset X^0 \subset C$. In case $\alpha^0 \leq 0$ let $(\bar{u}, \bar{v}) \in C (\neq \emptyset)$. By Corollary 2.5, $f_{ij}(\bar{u}, \bar{v}) \geq 0$ for all $(i, j) \in (M, N)$, thus $g^0(\bar{u}, \bar{v}) \geq 0$. Since $(\bar{u}, \bar{v}) \in X^0$ it follows that $\alpha^1 = \operatorname{Max}_{(u,v) \in X^0} g^0(u, v) \geq 0$, and that the set X^1 where g^0 attains its maximum on X^0 is in the core.

To show (iv), let $\tau \neq \sigma$ be an alternative optimal (M, N)-matching, and $(i_0, j_0) \in \tau \backslash \sigma \subset \Sigma^0$. By Corollary 2.6, $f_{i_0 j_0}(u, v) = 0$ for all $(u, v) \in C$. Hence α^1 cannot exceed 0 and X^1 cannot be a proper subset of the core. ●

3.5. Corollary. The lexicographic center $X^{\rho+1}$ is independent of the choice of σ.

Proof. In case of more than one optimal assignment, $\alpha^1 = 0$ and $X^1 = C$. Further, $\Sigma^1 \cap \tau = \emptyset$ for any optimal τ, so all the consequent parts of the definition (including $X^{\rho+1}$) are independent of the choice of σ. ●

Due to the removal of all the functions which are constant on the feasible set of the subsequent iteration, we have the following

3.6. Lemma. For any r, $1 \leq r \leq \rho$, $\dim(X^{r+1}) < \dim(X^r)$.

Proof. Let $(i, j) \in \Sigma_{r+1} \neq \emptyset$. The function $f_{ij}(u, v)$ is constant on X^{r+1} but not constant on X^r, because otherwise it would have been removed earlier $(r \geq 1)$. Suppose $\dim(X^{r+1}) = \dim(X^r)$. Since $X^{r+1} \subset X^r$ and f_{ij} is affine linear, so f_{ij} must be a constant on X^r as well, a contradiction. ●

3.7. Example. An assignment game with $\dim(X^1) = \dim(X^0)$.

Consider the game whose augmented profit matrix is

$$A = \begin{bmatrix} \boxed{0} & 0 & 0 & 0 \\ 0 & \boxed{4} & 1 & 1 \\ 0 & 2 & \boxed{0} & 0 \\ 0 & 2 & 0 & \boxed{0} \end{bmatrix}.$$

An optimal assignment is $\sigma = \{(i, i) : i = 0, \ldots, 3\}$. Then $\alpha^0 = -2$, and X^0 can be identified with $\{u_1 \in \boldsymbol{R} : 0 \leq u_1 \leq 4\}$ because once u_1 is fixed then $v_1 = 4 - u_1$ is given, the other variables all must be 0. So $\dim(X^0) = 1$. By Lemma 3.4 (iv), $X^1 = C$ because $\tau = \{(0, 0), (1, 1), (2, 3), (3, 2)\}$ is also an optimal assignment. For the core we get $u_1 \geq 1$, and $v_1 \geq 2$ implying $u_1 \leq 2$. It is easy to see that C can be identified with $\{u_1 \in \boldsymbol{R} : 1 \leq u_1 \leq 2\}$, thus $\dim(X^1) = 1$, too. ●

Next, we show that $\dim(X^{\rho+1}) = 0$.

3.8. Theorem. The lexicographic center of an assignment game consists of a single point.

Proof. It follows from Definition 3.1 that for every $(i, j) \in \Delta^r$ $(0 \leq r \leq \rho+1)$, f_{ij} is constant on X^r. Since $\Delta^{\rho+1} = (M, N)$ and $X^{\rho+1} \neq \emptyset$, so in particular every $f_{io}(u, v) = u_i$, $i \in M$, and $f_{oj}(u, v) = v_j$, $j \in N$, is constant on $X^{\rho+1}$. This is possible only if $X^{\rho+1}$ is a single point. ●

In order to prove that the lexicographic center coincides with the nucleolus we need the following

3.9. Lemma. For any r, $0 \leq r \leq \rho$, if $(u, v) \in X^{r+1}$ and $(y, z) \in X^r \setminus X^{r+1}$ then $F(u, v) >_L F(y, z)$.

Proof. First, we observe that if $(u, v) \in X^1 \subset C$ and $(y, z) \in I \setminus C$ then $F(u, v) >_L F(y, z)$, so we can assume without loss of generality that (y, z) is also in the core. Then for any coalition (S, T), $f((S, T), (u, v)) \geq 0$ and $f((S, T), (y, z)) \geq 0$.

Claim. If $f((S, T), (u, v)) < \alpha^{r+1}$ then $f((S, T), (y, z)) = f((S, T), (u, v))$. To prove this, let $f((S, T), (u, v)) < \alpha^{r+1}$, and for some (S, T)-matching μ, $V(S, T) = \sum_{(i,j) \in \mu} a_{ij}$ hold. By Lemma 2.4, $f((S, T), (u, v)) = \sum_{(i,j) \in \mu} f_{ij}(u, v)$. Further, for core elements the satisfactions are nonnegative, hence $f_{ij}(u, v) < \alpha^{r+1}$ for all $(i, j) \in \mu$. Since $(u, v) \in X^{r+1}$, none of $(i, j) \in \mu$ can be in Σ^r. Thus $\mu \subset \Delta^r$, and all the functions f_{ij}, $(i, j) \in \mu$, are constant on X^r. This implies $f_{ij}(u, v) = f_{ij}(y, z)$ for all $(i, j) \in \mu$, which in turn implies our claim $f((S, T), (u, v)) = f((S, T), (y, z))$.

On the other hand, since $(y, z) \in X^r \setminus X^{r+1}$, there must be an $(i, j) \in \Sigma^r$ such that $f_{ij}(y, z) < \alpha^{r+1}$. At the same time $f_{ij}(u, v) \geq \alpha^{r+1}$. This implies that more components of vector $F(y, z)$ are less than α^{r+1}, than of vector $F(u, v)$. Naturally, such components appear in the first positions in both vectors. Moreover, the claim shows that the entries of $F(u, v)$ which are smaller than α^{r+1} are repeated in $F(y, z)$. If k is the first index such that $F_k(u, v) \neq F_k(y, z)$ then $F_k(u, v) > F_k(y, z)$, hence $F(u, v) >_L F(y, z)$. ●

3.10. Theorem. The lexicographic center of an assignment game as given in Definition 3.1 coincides with the nucleolus of the game.

Proof. By Theorem 3.8, let $X^{\rho+1} = \{(\bar{u}, \bar{v})\}$. As in the proof of Lemma 3.9 we have $F(\bar{u}, \bar{v}) \geq_L F(y, z)$ for all $(y, z) \in I$, therefore $\nu = \{(\bar{u}, \bar{v})\}$. ●

4 Settled-Unsettled Partitions

In this section we investigate the structure of the partition (Δ^r, Σ^r) of (M, N) in a fixed iteration r, $0 \le r \le \rho$, of Definition 3.1. We call $X = X^r$ the current *feasible payoff set*, and $\Delta = \Delta^r$, $\Sigma = \Sigma^r$ the current set of *settled, unsettled coalitions*, respectively. Coalition $(i, j) \in \Delta$ is settled because its satisfaction f_{ij} is constant on X. Coalition $(i, j) \in \Sigma$ is unsettled because f_{ij} varies on X (except perhaps in some degenerate games, like in Example 3.7, but even there only for $X = X^0$). The fixed optimal (M, N)-matching σ of Definition 3.1 will often be used in the sequel, but it will turn out, as it did for the definition itself, that the particular choice of σ has relevance for the case $X = X^0$ only. Although in σ the fictitious row player might be matched to more than one column player (in addition to the fictitious column player, to the optimally unmatched 'real' players in case of $|M| < |N|$), it will prove convenient to write "$(i, \sigma(k)) \in \Delta$" instead of the precise "$(i, j) \in \Delta$ for some (or all) $(k, j) \in \sigma$" even for $k = 0 \in M$.

The most important feature of the partition is formulated in the following

4.1. Lemma. The settled-unsettled partition (Δ, Σ) satisfies

(i) $(i, \sigma(k)) \in \Delta$ implies $(k, \sigma(i)) \in \Delta$;
(ii) $(i, \sigma(k)) \in \Delta$ and $(k, \sigma(j)) \in \Delta$ implies $(i, \sigma(j)) \in \Delta$;

for all $i, j, k \in M$.

Proof. Let us recall that for $i \in \bar{M} = M \setminus \{0\}$ there is a unique partner $\sigma(i) \in \bar{N} = N \setminus \{0\}$, and for $i = 0 \in M$, $\sigma(0) = \{0\} \cup \bar{N} \setminus \sigma(\bar{M})$, so each column is a partner of a row. Remember also that throughout the paper $|M| \le |N|$.

To show (i), let $(i, \sigma(k)) \in \Delta$. It implies $f_{i\sigma(k)} \equiv$ constant on X. Since $f_{i\sigma(i)} \equiv 0$ and $f_{k\sigma(k)} \equiv 0$ on $X^0 \supset X$, by combining these three equations we get $f_{i\sigma(i)} + f_{k\sigma(k)} - f_{i\sigma(k)} \equiv$ constant on X. It follows that $u_k + v_{\sigma(i)} \equiv$ constant on X, hence $f_{k\sigma(i)} \equiv$ constant on X. Without loss of generality $i \ne k$, so $(i, \sigma(k)) \in \Delta = \Delta^r$ implies $r \ge 1$. Thus, $f_{k\sigma(i)} \equiv$ constant on $X = X^r$ implies $(k, \sigma(i)) \in \Delta$.

Claim (ii) is shown similarly. Let $(i, \sigma(k)) \in \Delta$ and $(k, \sigma(j)) \in \Delta$, implying that $u_i + v_{\sigma(k)}$ and $u_k + v_{\sigma(j)}$ are constant on X. By adding these two and subtracting $u_k + v_{\sigma(k)} = a_{k\sigma(k)}$ we get that $u_i + v_{\sigma(j)}$ is also constant on X. As before this means $(i, \sigma(j)) \in \Delta$. ●

Given a (Δ, Σ) partition, we define a relation on the rows by

$$i \sim k \text{ if and only if } (i, \sigma(k)) \in \Delta \tag{4.2}$$

for all $i, k \in M$, and we say that rows i and k are *tied*. By Lemma 4.1 we immediately have the following

4.3. Corollary. The relation $\sim$ is an equivalence on M.

Let us denote by $M_0, M_1, \ldots, M_d$ the equivalence classes of $\sim$, and by $N_0, N_1, \ldots, N_d$ the sets of the corresponding partners, i.e. $N_p=\sigma(M_p)$, $0\leq p\leq d$. We require $0\in M_0$, hence $0\in N_0$. Naturally, $N_0, N_1, \ldots, N_d$ is a partition of N, so it defines an equivalence on N. We say that two columns are tied if their partners are tied, and write $\sigma(i)\sim\sigma(k)$ if and only if $i\sim k$. The partitions of M and N induce a partition of (M, N) into blocks, namely $(M, N)=\bigcup_{0\leq p,q\leq d}(M_p, N_q)$. We also say that two mixed pairs are tied if the two row and the two column players are tied.

Now we are ready to state the main result. It asserts that (i) the settled coalitions are exactly the pairs of players such that one is tied to the optimally matched partner of the other, and (ii) the satisfactions of tied coalitions can vary only in an identical way.

4.4. Theorem. The settled-unsettled partition (Δ, Σ) satisfies

(i) $\Delta=\bigcup_{0\leq p\leq d}(M_p, N_p)$, hence $\Sigma=\bigcup_{0\leq p\neq q\leq d}(M_p, N_q)$;

(ii) if $(i, j), (k, \ell)\in(M_p, N_q)$ then $f_{ij}-f_{k\ell}$ is constant on X.

Proof. To show (i), first let $(i, j)\in(M_p, N_p)$ for some p, $0\leq p\leq d$. There exists $k\in M_p$ such that $j=\sigma(k)$. Since $i\sim k$ so $(i, j)=(i, \sigma(k))\in\Delta$. Hence $\bigcup_{0\leq p\leq d}(M_p, N_p)\subset\Delta$. Conversely, let $(i, j)\in\Delta$ and $j=\sigma(k)$, $k\in M_p$ for some p. Since $(i, j)=(i, \sigma(k))\in\Delta$, so $i\sim k$, implying $(i, j)\in(M_p, N_p)$. Hence $\Delta\subset\bigcup_{0\leq p\leq d}(M_p, N_p)$.

To show (ii), first we prove that $i\sim k$, $i, k\in M$, implies $u_i-u_k\equiv$constant on X. If $i\sim k$ then $(i, \sigma(k))\in\Delta$. Thus $f_{i\sigma(k)}=u_i+v_{\sigma(k)}-a_{i\sigma(k)}\equiv$constant on X. By subtracting $f_{k\sigma(k)}=u_k+v_{\sigma(k)}-a_{k\sigma(k)}\equiv 0$ we get $u_i-u_k\equiv$constant on X. Similarly we can show that if columns j and ℓ are tied, then $v_j-v_\ell\equiv$constant on X. Now if (i, j), $(k, \ell)\in(M_p, N_q)$ then $(f_{ij}-f_{k\ell})(u, v)=(u_i-u_k)+(v_j-v_\ell)-(a_{ij}-a_{k\ell})$, a constant on X because $i\sim k$ and $j\sim\ell$. ●

4.5. Corollary. Let $\Delta=\Delta^r=(M_0, N_0)\cup\ldots\cup(M_d, N_d)$ and $r\geq 1$. Then $\dim(X^r)=d$.

Proof. By Theorem 4.4(ii), for each $p=1, \ldots, d$ there are linear functions of the form $(f_{i0}-f_{k0})(u, v)=u_i-u_k$, $i, k\in M_p$, which are constant on $X=X^r$. We can choose $|M_p|-1$ linearly independent ones for each $1\leq p\leq d$. Any two of these functions related to different equivalence classes are also linearly independent. In the settled class M_0 even the functions $u_i=u_i-u_0$ are constant on X^r. So there are $|M_0|+\sum_{p=1}^{d}(|M_p|-1)=|M|-d$ linearly independent linear functions which are constant on X^r. Hence $\dim(X^r)\leq d$, because once the values of u_i, $i\in M$, are fixed, the values of v_j, $j\in N$, are given by $v_j=a_{ij}-u_i$ for all $(i, j)\in\sigma$.

On the other hand, if $r \geq 1$ then the functions $f_{io}(u, v)=u_i$, $i \in \bigcup_{1 \leq p \leq d} M_p$ are not constant on X^r. Although the satisfactions of tied players must change by the same amount, but there is no binding relation between players from different classes. So we can find d linearly independent vectors in X^r, by changing the variables in only one class at a time, and keeping the others unchanged. (Again, the values of v_j, $j \in N$, are given by $v_j = a_{ij} - u_i$ for all $(i, j) \in \sigma$.) Hence $\dim(X^r) \geq d$. ●

4.6. Example. An assignment game with $\dim(X^0) < d(0) = |\bar{M}|$.

Consider the profit matrix $\bar{A} = [\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}]$. The unique optimal assignment is $\sigma(i)=i$, $i=1, 2$. It is easy to see that $u_2 = u_1$, $v_1 = 2 - u_1$, $v_2 = 2 - u_2$ must hold in X^0, so $X^0 = \{u_1 \in \boldsymbol{R} : 0 \leq u_1 \leq 2\}$. Therefore, $\dim(X^0) = 1 < 2 = |\bar{M}| = \min(|\bar{M}|, |\bar{N}|)$.

5 The Feasible Payoff Sets

In this section we investigate the structure of the feasible payoff sets X^r, $0 \leq r \leq \rho+1$, in Definition 3.1. It turns out that all these sets have the lattice-type property of the core, characterized by Shapley and Shubik (1972).

First let us give an explicit description of these sets.

5.1. Lemma. Let ν denote the nucleolus payoff, and let $c_{ij} = f_{ij}(\nu)$ for all $(i, j) \in (M, N)$. For $r = 1, \ldots, \rho+1$,

$$X^r = Y^r := \{(u, v) \in UV : f_{ij}(u, v) = c_{ij} \ \forall (i, j) \in \Delta^r, f_{ij}(u, v) \geq \alpha^r \ \forall (i, j) \in \Sigma^r\}.$$

Proof. For any r, $1 \leq r \leq \rho+1$, the $X^r \subset Y^r$ inclusion comes immediately from the definitions and from $\nu \in X^r$. We prove $X^r \supset Y^r$ by induction on r.

Let $r=1$ and $(u, v) \in Y^1$. The nonnegativity of (u, v) comes from $\alpha^1 \geq 0$ and $c_{ij} \geq 0$ for all $(i, j) \in \Delta^1$. In fact, $c_{ij} = 0$ for all $(i, j) \in \Delta^0 = \sigma$, and $\operatorname{Min}_{(i,j) \in \Sigma_1} c_{ij} = \alpha^1$ because $\nu \in X^1$. Since $\alpha^1 \geq \alpha^0$ and $\Sigma^0 = \Sigma^1 \cup \Sigma_1$, $(u, v) \in X^0$. It follows that $(u, v) \in X^1$ because $\operatorname{Min}_{(i,j) \in \Sigma^1} f_{ij}(u, v) \geq \alpha^1$ and $\operatorname{Min}_{(i,j) \in \Sigma_1} c_{ij} = \alpha^1$ imply $g^0(u, v) = \alpha^1$.

Let us now assume $X^r = Y^r$ for some r, $1 \leq r \leq \rho$. Let $(u, v) \in Y^{r+1} (\neq \emptyset)$. Since $\nu \in X^{r+1}$, $\operatorname{Min}_{(i,j) \in \Sigma_{r+1}} c_{ij} = \alpha^{r+1}$. This together with $\operatorname{Min}_{(i,j) \in \Sigma^{r+1}} f_{ij}(u, v) \geq \alpha^{r+1}$ imply $\operatorname{Min}_{(i,j) \in \Sigma^r} f_{ij}(u, v) = \alpha^{r+1}$. From $\alpha^{r+1} > \alpha^r$ and $\Delta^r \subset \Delta^{r+1}$ it follows that not only $(u, v) \in Y^r (= X^r)$ but also $(u, v) \in X^{r+1}$. ●

If we extend the above definition of Y^r also to the case $r=0$, then obviously $X^0 = Y^0 \cap \{(u, v) \geq (0, 0)\}$. Now for $0 \leq r \leq \rho$ and $\alpha^r \leq \alpha \leq \alpha^{r+1}$ we define

$$Y(r, \alpha) := \{(u, v) \in UV : f_{ij}(u, v) = c_{ij} \ \forall (i, j) \in \Delta^r, f_{ij}(u, v) \geq \alpha \ \forall (i, j) \in \Sigma^r\}. \quad (5.2)$$

Evidently, as α increases $Y(r, \alpha)$ shrinks. In fact α^{r+1} is the maximum value of α for which $Y(r, \alpha)$ is nonempty.

Next we show that the lattice-type property of the core holds for any of the sets $Y(r, \alpha)$.

5.3. Lemma. For any $0 \le r \le \rho$ and $\alpha^r \le \alpha \le \alpha^{r+1}$, if (u^1, v^1), $(u^2, v^2) \in Y(r, \alpha)$ then $(u^{1\vee 2}, v^{1\wedge 2}) \in Y(r, \alpha)$, where $u_i^{1\vee 2} = \max(u_i^1, u_i^2)$ $\forall i \in M$ and $v_j^{1\wedge 2} = \min(v_j^1, v_j^2)$ $\forall j \in N$.

Proof. Let (u^1, v^1), $(u^2, v^2) \in Y(r, \alpha)$. Obviously $(u^{1\vee 2}, v^{1\wedge 2}) \in UV$. For $(i, j) \in \Sigma^r$ we have $u_i^{1\vee 2} + v_j^1 \ge u_i^1 + v_j^1 \ge a_{ij} + \alpha$ and $u_i^{1\vee 2} + v_j^2 \ge u_i^2 + v_j^2 \ge a_{ij} + \alpha$, implying $u_i^{1\vee 2} + v_j^{1\wedge 2} \ge a_{ij} + \alpha$. For $(i, j) \in \Delta^r$ we have $u_i^1 + v_j^1 = a_{ij} + c_{ij} = u_i^2 + v_j^2$, so if $u_i^{1\vee 2} = u_i^1$ (or u_i^2) then $v_j^{1\wedge 2} = v_j^1$ (or v_j^2). In either case $u_i^{1\vee 2} + v_j^{1\wedge 2} = a_{ij} + c_{ij}$. ●

Since the lemma also holds for the set $\{(u, v) \ge (0, 0)\}$, the set $X(0, \alpha) := Y(0, \alpha) \cap \{(u, v) \ge (0, 0)\}$ also possesses the lattice-type property. In light of Lemma 5.1, we define $X(r, \alpha) := Y(r, \alpha)$ for $1 \le r \le \rho$ and $\alpha^r \le \alpha \le \alpha^{r+1}$.

It is now easy to show the existence of the two special vertices.

5.4. Theorem. Fix any $0 \le r \le \rho$ and $\alpha^r \le \alpha \le \alpha^{r+1}$. Let $\bar{u}_i = \bar{u}_i(r, \alpha)$ denote the maximum payoff for $i \in M$, and $\underline{v}_j = \underline{v}_j(r, \alpha)$ the minimum payoff for $j \in N$, over all payoff vectors in $X = X(r, \alpha)$. Then $(\bar{u}, \underline{v}) \in X$. Also, for the analogously defined $\underline{u}_i, i \in M, \bar{v}_j, j \in N, (\underline{u}, \bar{v}) \in X$.

Proof. Since X^0 is compact and $X \subset X^0$, all the $\bar{u}_i$, $\underline{v}_j$ are well defined. In fact, every $\bar{u}_i$, $i \in M$, and $\underline{v}_j$, $j \in N$, is attained in an extreme point of the polytope X. Applying Lemma 5.3 repeatedly to these finite number of payoff vectors, we get the theorem. ●

We refer to $(\bar{u}, \underline{v})$ as the *u-best* (*v-worst*) *corner* of X. Clearly, if $\alpha < \alpha'$ then $\bar{u}_i(r, \alpha) \ge \bar{u}_i(r, \alpha')$ for all $i \in M$, and $\underline{v}_j(r, \alpha) \le \underline{v}_j(r, \alpha')$ for all $j \in N$.

Next we introduce the tools to be used to find these special vertices of the subsequent feasible sets.

5.5. Definition. Fix any $0 \le r \le \rho$ and $\alpha^r \le \alpha \le \alpha^{r+1}$. Let $\Sigma = \Sigma^r$. Let $(\bar{u}, \underline{v})$ denote the u-best corner of $X = X(r, \alpha)$. We call the unsettled coalitions in $\Sigma_= = \Sigma_=(r, \alpha) = \{(i, j) \in \Sigma : f_{ij}(\bar{u}, \underline{v}) = \alpha\}$ *active*, the ones in $\Sigma_> = \Sigma_>(r, \alpha) = \{(i, j) \in \Sigma : f_{ij}(\bar{u}, \underline{v}) > \alpha\}$ *passive*.

Recall that by Theorem 4.4 both the rows and columns are partitioned into $d+1$ equivalence classes for some $d \ge 0$. Further, these two partitions induce a partition of (M, N) into $(d+1)^2$ blocks of tied coalitions.

5.6. Definition. Fix any $0 \le r \le \rho$ and $\alpha^r \le \alpha \le \alpha^{r+1}$. Let $d = d(r)$. Define the directed graph $G = G(r, \alpha) = (D, E)$ on the set of nodes $D = \{0, \ldots, d\}$ by the set of arcs $E = \{(p, q) : 0 \le p \ne q \le d, (M_p, N_q) \cap \Sigma_=(r, \alpha) \ne \emptyset\}$. The graph G is *proper* if it is acyclic and contains no incoming arc to node 0, and *improper* otherwise.

A fundamental result for the algorithm is the following

5.7. Theorem. For any $0 \le r \le \rho$ and $\alpha^r \le \alpha \le \alpha^{r+1}$, $G = G(r, \alpha)$ is proper if and only if $\alpha < \alpha^{r+1}$.

Proof. First, let G be proper. Denote by $\ell(p)$ the length of the longest path in G ending in node p. Since the graph is proper, $\ell(p)$ is a well-defined, nonnegative integer for any $0 \le p \le d$, and of course $\ell(0) = 0$. Define $(s, t) \in UV$ by

$$s_i = -\ell(p) \text{ if } i \in M_p,\ \forall i \in M;\ t_j = \ell(q) \text{ if } j \in N_q,\ \forall j \in N. \tag{5.8}$$

Consider the payoff $(u', v') = (\bar{u}, \underline{v}) + \beta \cdot (s, t)$ for some $\beta \in \boldsymbol{R}$. The satisfaction of coalition $(i, j) \in (M, N)$ at the new payoff is $f_{ij}(u', v') = f_{ij}(\bar{u}, \underline{v}) + \beta \cdot (s_i + t_j)$. If $(i, j) \in \Delta$ then $p = q$, so $s_i + t_j = -\ell(p) + \ell(p) = 0$, thus β does not alter the satisfaction. If $(i, j) \in \Sigma_= (\neq \emptyset)$ then $p \neq q$ and there is a (p, q) arc in G, implying $\ell(p) + 1 \le \ell(q)$, hence $s_i + t_j = -\ell(p) + \ell(q) \ge 1$. For $\beta > 0$ the satisfaction increases by at least β. Note that there is always an active coalition whose satisfaction increases by exactly β, because there is always an arc involved in a longest path. If $(i, j) \in \Sigma_>$ then $f_{ij}(\bar{u}, \underline{v}) > \alpha$, so for a small enough $\beta > 0$ even $f_{ij}(u', v') \ge \alpha + \beta$ holds. Therefore, it is possible to choose $\beta > 0$ small enough such that $\underset{(i,j)\in\Sigma}{\text{Min}} f_{ij}(u', v') = \alpha + \beta > \alpha$, implying $(u', v') \in X$ and $\alpha^{r+1} > \alpha$.

Secondly, let G be improper. In case of a $(p, 0)$ arc, there is $i \in M_p$ and $j \in N_o$ such that $\bar{u}_i + \underline{v}_j - a_{ij} = \alpha$. Since $(0, j) \in (M_o, N_o) \subset \Delta$, $f_{oj}(u, v) = v_j = c_{oj}$ for all $(u, v) \in X$. Hence $\underline{v}_j = c_{oj}$. On the other hand, $\bar{u}_i \ge u_i$, implying $f_{ij}(u, v) \le f_{ij}(\bar{u}, \underline{v}) = \alpha$ for all $(u, v) \in X$. This shows that $\alpha^{r+1} \le \alpha$. In case of a cycle (p, q), (q, k), (k, p) (for simplicity of length 3 only), there are rows and columns in the related classes such that $f_{i_p j_q}(\bar{u}, \underline{v}) = \alpha$, $f_{i_q j_k}(\bar{u}, \underline{v}) = \alpha$ and $f_{i_k j_p}(\bar{u}, \underline{v}) = \alpha$. While summing them we can replace the sum of the variables with a constant, because the same variables appear in $f_{i_p j_p}(\bar{u}, \underline{v}) + f_{i_q j_q}(\bar{u}, \underline{v}) + f_{i_k j_k}(\bar{u}, \underline{v}) = 0$. (The cycle guarantees that for every node involved there is exactly one row and one column.) Actually, this can be done for any $(u, v) \in X$. Thus, the sum of the satisfactions along the cycle is independent of the payoff. By summing the inequalities $f_{i_p j_q}(u, v) \ge \alpha$, etc. along the cycle, we get $3 \cdot \alpha = \sum_{\text{cycle}} f_{ij}(\bar{u}, \underline{v}) = \sum_{\text{cycle}} f_{ij}(u, v) \ge 3 \cdot \alpha$, implying $f_{i_p j_q}(u, v) = \alpha$, etc. for any $(u, v) \in X$. Therefore, $\alpha^{r+1} \le \alpha$. ●

5.9. Corollary. Let $G(r, \alpha)$ be proper and direction (s, t) be given by (5.8). If $0 \le \beta \le \beta(r, \alpha) := \text{Max}\{\gamma: \underset{(i,j)\in\Sigma}{\text{Min}} f_{ij}((\bar{u}, \underline{v}) + \gamma \cdot (s, t)) = \alpha + \gamma\}$ then $(u', v') = (\bar{u}, \underline{v}) + \beta \cdot (s, t)$ is the u-best corner of $X' = X(r, \alpha + \beta)$.

Proof. First of all, $\beta(r, \alpha) > 0$ is finite, because the uniform increase $-\ell(p) + \ell(q)$ in the satisfactions of the tied coalitions in block (M_p, N_q) requires the uniform decrease $-\ell(q) + \ell(p)$ in block (M_q, N_p). Obviously $(u', v') \in X'$.

Let $(u, v) \in X'$ be chosen arbitrarily. We claim for all $j \in N$ that $v_j \ge v_j' = \underline{v}_j + \ell(q) \cdot \beta$ if $j \in N_q$. For simplicity, assume $\ell(q) = 2$ and that (p, k), (k, q) is a longest path to node q. There are $i_p \in M_p$, $j_k \in N_k$, $i_k \in M_k$, $j_q \in N_q$ such that $\bar{u}_{i_p} + \underline{v}_{j_k} - a_{i_p j_k} = \alpha$ and $\bar{u}_{i_k} + \underline{v}_{j_q} - a_{i_k j_q} = \alpha$. Subtract these equations from

$u_{i_p}+v_{j_k}-a_{i_pj_k}\geq\alpha+\beta$ and $u_{i_k}+v_{j_q}-a_{i_kj_q}\geq\alpha+\beta$, respectively. Then we have $(u_{i_p}-\bar{u}_{i_p})+(v_{j_k}-\underline{v}_{j_k})\geq\beta$ and $(u_{i_k}-\bar{u}_{i_k})+(v_{j_q}-\underline{v}_{j_q})\geq\beta$. By adding these two and using $u_{i_k}+v_{j_k}=\bar{u}_{i_k}+\underline{v}_{j_k}$ we get $v_{j_q}\geq\underline{v}_{j_q}+2\cdot\beta+(\bar{u}_{i_p}-u_{i_p})\geq\underline{v}_{j_q}+2\cdot\beta=v'_{j_q}$. The last inequality holds because $(\bar{u}, \underline{v})$ is the u-best corner of $X\supset X'\ni(u, v)$. Hence the claim. Similarly, we can show $u_i\leq u'_i$ for all $i\in M$. Therefore, (u', v') is the u-best corner of X'. ●

5.10. Corollary. Let $G(r, \alpha)$ be improper. If arc (p, q) is contained in a cycle or $q=0$ then $(M_p, N_q)\cup(M_q, N_p)\subset\Sigma_{r+1}$.

Proof. Let (p, q) be one of the arcs that make the graph improper. There are $i\in M_p$ and $j\in N_q$ such that $f_{ij}(\bar{u}, \underline{v})=\alpha$, where $(\bar{u}, \underline{v})=(\bar{u}(r, \alpha), \underline{v}(r, \alpha))$. In the proof of Theorem 5.7 we actually showed that $f_{ij}(u, v)=\alpha$ for all $(u, v)\in X=X(r, \alpha^{r+1})=X^{r+1}$. By Theorem 4.4(ii), $f_{ij}-f_{k\ell}\equiv$constant on X for any $(k, \ell)\in(M_p, N_q)$, so $f_{k\ell}\equiv$constant on X, implying $(k, \ell)\in\Sigma_{r+1}$. Therefore, $(M_p, N_q)\subset\Sigma_{r+1}$. The inclusion $(M_q, N_p)\subset\Sigma_{r+1}$ comes from Theorem 4.4(i). ●

6 The Algorithm

In this section we present an algorithm that is a realization of Definition 3.1. The idea is to construct a finite sequence of payoff vectors such that each payoff is the u-best corner of the current feasible set, and the last one is the nucleolus. The problem of finding the next payoff is translated to finding the longest paths in the graph which summarizes the basic relations among the unsettled coalitions. This reformulation has the advantage that it immediately reveals the subset of relations which prevents the further improvement in the guaranteed satisfaction level, if such case happens.

The algorithm consists of iterations, the iterations in turn consist of steps. The iterations are the same as in the definition, so a new iteration starts whenever the settled-unsettled partition (the node set in the graph) has to be changed. In a step the guaranteed satisfaction level is increased by moving along a direction determined by the currently worst-off unsettled coalitions. As soon as a previously passive coalition becomes active (a new arc is added to the graph), the direction has to be changed, and a new step begins. When no improving direction can be found, then the coalitions involved in the deadlock are removed from further consideration and the iteration ends.

First, we present the algorithm in general terms, then we discuss it part by part in more detail.

6.1. Algorithm. Given: σ, an optimal (M, N)-matching on $A=(a_{ij})_{(M,N)}\geq 0$.

Initially, let $r=0$, $\Delta=\sigma$, $\Sigma=(M, N)\backslash\sigma$, $u_i=a_{i\sigma(i)}$ $\forall i\in M$, $v_j=0$ $\forall j\in N$, $f_{ij}=u_i+v_j-a_{ij}$ $\forall(i, j)\in(M, N)$, $\alpha=\operatorname{Min}_{(i,j)\in\Sigma} f_{ij}$.

(1) While $\Sigma \neq \emptyset$ do (2)–(12) (*iteration* r)
(2) Build the graph $G := G(r, \alpha)$
(3) While G is proper do (4)–(9) (*step* (r, α))
(4) Find direction (s, t)
(5) Find step size $\beta := \beta(r, \alpha)$
(6) Update arcs in graph, $G := G(r, \alpha + \beta)$
(7) Update payoff, $(u, v) := (u, v) + \beta \cdot (s, t)$
(8) Update satisfactions, $f_{ij} := f_{ij} + \beta \cdot (s_i + t_j)\ \forall (i, j) \in \Sigma$
(9) Update guaranteed satisfaction level, $\alpha := \alpha + \beta$
(10) Find to-be-settled coalitions $\bar{\Sigma} := \Sigma_{r+1}$
(11) Update partition, $\Sigma := \Sigma \setminus \bar{\Sigma}$, $\Delta := \Delta \cup \bar{\Sigma}$
(12) Set $r := r + 1$.

(6.1.1) Is $\Sigma \neq \emptyset$? The answer to this question is equivalent to the answer for "Is $d > 0$ in the decomposition $\Delta = (M_0, N_0) \cup \ldots \cup (M_d, N_d)$?".

(6.1.2) Build the graph $G := G(r, \alpha)$. Given: (Δ, Σ), d, α, $f_{ij}\ \forall (i, j) \in (M, N)$. The set of nodes is $D = \{0, \ldots, d\}$. The set of arcs is $E = \{(p, q) : 0 \leq p \neq q \leq d, (M_p, N_q) \cap \Sigma_{=} \neq \emptyset\}$, where $\Sigma_{=} = \{(i, j) \in \Sigma : f_{ij} = \alpha\}$. In order to guarantee that there is at least one arc in G, set $\alpha := \max(\alpha, \min_{(i,j) \in \Sigma} f_{ij})$ if necessary.

(6.1.3) Is G proper? Given: the directed graph $G = (D, E)$, let us initialize $\bar{\Sigma} := \emptyset$.

(A) Check if there is an incoming arc to node 0. If yes, then melt the tail-node of that arc into node 0 but preserve all other arcs. That is, if $(p, 0) \in E$ then drop arc $(p, 0)$ from E; in case of $(p, q) \in E$ add arc $(0, q)$ and drop (p, q); in case of $(q, p) \in E$ add $(q, 0)$ and drop (q, p). Delete node p. Set $\bar{\Sigma} := \bar{\Sigma} \cup (M_p, N_0) \cup (M_0, N_p)$. Repeat (A) until the answer is no. Proceed with (B).

(B) Check if there is a cycle. If yes, then shrink the cycle into one node which inherits all incoming and outgoing arcs, to and from the cycle. That is, if $C = \{(p_1, p_2), (p_2, p_3), \ldots, (p_k, p_{k+1} = p_1)\} \subset E$, then add a new node p to D; remove all arcs in C from E; for all $i = 1, \ldots, k$ in case of $(p_i, q) \in E$ add (p, q) and drop (p_i, q); for all $i = 1, \ldots, k$ in case of $(q, p_i) \in E$ add (q, p) and drop (q, p_i). Delete all the nodes in C from D. Set $\bar{\Sigma} := \bar{\Sigma} \cup \bigcup_{i=1}^{k} ((M_{p_i}, N_{p_{i+1}}) \cup (M_{p_{i+1}}, N_{p_i}))$. Repeat (B) until the answer is no.

If after completing (A) and (B) the set $\bar{\Sigma}$ is still empty, then G is proper, otherwise G is improper and as the proof of Corollary 4.5 and Corollary 5.10 show, $\bar{\Sigma}$ is exactly the set of to-be-settled coalitions Σ_{r+1}.

(6.1.4) Find direction (s, t). Given: a proper graph $G = (D, E)$. An easy-to-describe algorithm to find the length of the longest paths is the following. Set $k := 0$, $H := G$.

(C) Set $\ell(p) := k$ for all nodes p in H with no incoming arc. Remove from H all such nodes together with all the arcs going out from them. Increase $k := k + 1$. Repeat (C) until no nodes are left.

After completing (C) set (s, t) according to (5.8).

(6.1.5) Find the step size $\beta := \beta(r, \alpha)$. Given: $G=(D, E)$ with $\ell(p), p \in D$, direction (s, t), f_{ij}, $(i, j) \in \Sigma$, and α. Determine those blocks of tied coalitions whose satisfaction does not increase along direction (s, t), i.e. $E^- := \{(p, q): 0 \leq p \neq q \leq d, -\ell(p)+\ell(q) \leq 0\}$. Clearly $E^- \cap E = \emptyset$. Calculate $\beta := \text{Min}\{\beta_{ij}; \beta_{ij}=(f_{ij}-\alpha)/(1-s_i-t_j)\ \forall (i, j) \in (M_p, N_q)$ such that $(p, q) \in E^-\}$. We call any such coalition where the above minimum is attained a *threshold coalition*.

Claim. $\beta=\beta(r, \alpha)$ as defined in Corollary 5.9. For a threshold coalition $(i, j) \in \Sigma$ we have $s_i+t_j \leq 0$ and $f_{ij} > \alpha$. Further, $\gamma=\beta_{ij}$ is the threshold value where $f_{ij}+\gamma \cdot (s_i+t_j)=\alpha+\gamma$. Namely, the nonincreasing satisfaction of the coalition reaches the increasing level of satisfaction guaranteed for all unsettled coalitions. Since for $(i, j) \in \Sigma$ with $s_i+t_j \geq 1$ such restriction does not apply, so really $\beta=\text{Max}\{\gamma: \min_{(i,j) \in \Sigma} f_{ij}((u, v)+\gamma \cdot (s, t))=\alpha+\gamma\}$. Recall from the proof of Theorem 5.7 that there is always an active coalition whose satisfaction increases by exactly γ along the direction found in (6.1.4).

(6.1.6) Update arcs in graph, $G := G(r, \alpha+\beta)$. Given: the proper graph $G=G(r, \alpha)=(D, E)$ with $\ell(p), p \in D$, and E^- as defined in (6.1.5). The node set does not change. Keep arc $(p, q) \in E$ if $-\ell(p)+\ell(q)=1$; drop arc $(p, q) \in E$ if $-\ell(p)+\ell(q) \geq 2$; add arc $(p, q) \in E^-$ if $\beta_{ij}=\beta(r, \alpha)$ for some $(i, j) \in (M_p, N_q)$, i.e. β_{ij} is a minimal threshold, i.e. $(i, j) \in \Sigma$ is a passive coalition that first became active. Since we eliminate the redundant arcs (the ones not involved in any longest path), the updated G is $G(r, \alpha+\beta)$.

(6.1.10) Find to-be-settled coalitions $\bar{\Sigma} := \Sigma_{r+1}$. Done in (6.1.3).

The other parts of the algorithm are self-explanatory.

Next we show that the algorithm is a realization of Definition 3.1.

6.2. Theorem. In Algorithm 6.1 at the beginning of iteration r, $\Delta=\Delta^r$, $\Sigma=\Sigma^r$, $\alpha=\alpha^r$, $(u, v)=(\bar{u}^r, \underline{v}^r)$ (the u-best corner of X^r), $f_{ij}=f_{ij}(\bar{u}^r, \underline{v}^r)\ \forall (i, j) \in (M, N)$.

Proof. We use induction on r while $\Sigma \neq \emptyset$.

For $r=0$ the claim is trivially true.

Let us assume that the claim is true up to some $r \geq 0$ and $\Sigma \neq \emptyset$. We show that at the end of iteration r (i.e. at the beginning of iteration $r+1$ if Σ is still nonempty) $\Delta=\Delta^{r+1}$, $\Sigma=\Sigma^{r+1}$, $\alpha=\alpha^{r+1}$, $(u, v)=(\bar{u}^{r+1}, \underline{v}^{r+1})$, $f_{ij}=f_{ij}(\bar{u}^{r+1}, \underline{v}^{r+1})\ \forall (i, j) \in (M, N)$.

To this end, we argue that the algorithm generates a sequence (r, α^r), ..., (r, α), $(r, \alpha+\beta(r, \alpha))$, ... of intermediate steps such that (i) the sequence is finite; (ii) the last one is (r, α^{r+1}); (iii) at the beginning of each one $(u, v)=(\bar{u}(r, \alpha), \underline{v}(r, \alpha))$ (the u-best corner of $X(r, \alpha)$) and $f_{ij}=f_{ij}(\bar{u}(r, \alpha), \underline{v}(r, \alpha))\ \forall (i, j) \in (M, N)$. Finiteness comes from the fact that at the end of a step a new arc, which is related to a block with nonincreasing satisfactions, must be added to the graph, but once the satisfactions in a block become increasing they remain increasing in subsequent steps. Theorem 5.7 shows that when the sequence ends (r, α^{r+1}) is reached. Since (iii) is true at the first step (r, α^r) (by the inductive hypothesis), so the repeated quote of Corollary 5.9 and $f_{ij}(u, v)+\gamma\cdot(s_i+t_j)=f_{ij}((u, v)+\gamma\cdot(s, t))$ proves (iii). Now, when (r, α^{r+1}) is reached, the set $\bar{\Sigma}$ is found in (6.1.3), which is exactly the set of to-be-settled coalitions Σ_{r+1}. Hence after (6.1.11), $\Delta=\Delta^r\cup\Sigma_{r+1}=\Delta^{r+1}$, $\Sigma=\Sigma^r\backslash\Sigma_{r+1}=\Sigma^{r+1}$. Obviously, $X(r, \alpha^{r+1})=X^{r+1}$, so at the end of iteration r, $(u, v)=(\bar{u}(r, \alpha^{r+1}), \underline{v}(r, \alpha^{r+1}))=(\bar{u}^{r+1}, \underline{v}^{r+1})$. Thus $f_{ij}=f_{ij}(\bar{u}^{r+1}, \underline{v}^{r+1})\ \forall(i, j)\in(M, N)$, as we claimed. ●

6.3. Corollary. Algorithm 6.1 stops after iteration $r=\rho$. The final (u, v) is the nucleolus and $f_{ij}=c_{ij}\ \forall(i, j)\in(M, N)$.

Proof. The theorem implies that at the beginning of iteration $r=\rho$, $\Sigma=\Sigma^\rho\neq\emptyset$ and in that iteration $\bar{\Sigma}=\Sigma_{\rho+1}$ is removed from Σ. Since $\Sigma^\rho=\Sigma_{\rho+1}$, at the end of iteration $r=\rho$ we have $\Sigma=\emptyset$. Hence the algorithm stops. From the above proof it also follows that the final (u, v) is the u-best corner of $X^{\rho+1}$, which is, by Theorems 3.8 and 3.10, a singleton containing only the nucleolus. Therefore the final payoff is the nucleolus and $f_{ij}=c_{ij}\ \forall(i, j)\in(M, N)$. ●

Although the finiteness of the algorithm is already implied by the previous arguments, next we give upper bounds for the number of iterations and steps.

6.4. Theorem. In Algorithm 6.1 the number of

(i) iterations is at most m;
(ii) steps is at most $1/2\cdot m(m+3)$;

where $m=|\bar{M}|=\min(|\bar{M}|, |\bar{N}|)$.

Proof. (i) From (6.1.3) it follows that each iteration decreases the number of nodes in the graph by at least one. Initially there are $m+1$ nodes, and the algorithm stops when only the settled node 0 is left.

To prove (ii), at the beginning of each step let us color every unsettled block (M_p, N_q), $p\neq q$, and every coalition $(i, j)\in(M_p, N_q)$ by blue if there is a path from p to q in the current graph; by red if $q=0$ or (M_q, N_p) is blue; by green otherwise. If (i, j) is blue then $s_i+t_j=-\ell(p)+\ell(q)\geq 1$, so at the end of the step the passive coalitions (may be only one) which become active are red or green. If only green coalitions become active, let us call the step a green step. In such a step the graph remains proper, and the threshold coalitions, together with all the tied ones in their block(s), turn blue in the next step. Hence after a green step the number of blue coalitions is strictly increased. If at least one red coalition becomes active, then we call the step a red step. In this case the graph becomes improper and the settled-unsettled partition has to be updated. Since this is done by joining smaller blocks into larger ones, it might happen that a smaller green block turns blue or red, but no blue or red coalition can turn green. Hence after a red step the number of blue coalitions is not decreasing. Since there are always at least as many red coalitions as blue ones, the number of green steps is at most $1/2\cdot m(m+1)$. The number of red steps is at most m, because each one completes an iteration. Therefore, the number of steps in the algorithm is at most $1/2 \cdot m(m+1)+m = 1/2 \cdot m(m+3)$. ●

The key factor in a practical implementation of the algorithm is a routine which, for a directed graph, either determines the length of the longest path to each node if the graph is acyclic, or exhibits the arcs that form the cycle(s) otherwise. For one such routine which requires at most $O(d^2)$ elementary operations for a directed graph with d nodes, see Noltemeier (1975). Since generating s_i+t_j, calculating β_{ij}, updating f_{ij} for all $(i,j)\in\Sigma$ can be done in $O(m\cdot n)$ elementary operations and $d\le m\le n$, so each step can be carried out in at most $O(m\cdot n)$ elementary operations.

To compute the so called fair outcome (the midpoint of the long axis of the core suggested by Thompson (1981)) we need both the u-best and the v-best corner of the core. The u-best corner can be found in the first iteration of the algorithm as presented above, by computing the payoffs at $\alpha=0$. The v-best corner can be determined by simply changing the roles of the rows and columns, i.e. by finding the u-best corner of the core of the game on the transposed profit matrix. The modifications in the algorithm necessary to accomodate the case $|M|\ge|N|$ are obvious.

Finally, we note that the algorithm (which requires an optimal assignment) can be modified to find an optimal assignment itself. We can start with any full assignment pretending it is optimal and we are looking for the nucleolus. The difference comes only when an iteration ends. Then we can improve the assignment along the cycle or the path to the settled node. The current assignment is optimal when α becomes nonnegative. In this way a primal, non-simplex algorithm can be obtained, which generates primal feasible (but not necessarily basic) solutions and maintains complementary slackness while trying to reach dual feasibility. The method of Balinski and Gomory (1964) has similar features.

7 An Illustrative Example

Let us consider the (4, 5)-person assignment game with augmented profit-matrix

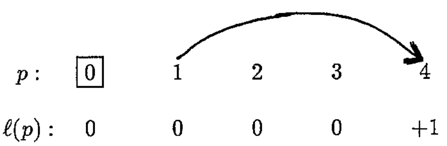
$$A = \left[\begin{array}{c|ccccc} \boxed{0} & \boxed{0} & 0 & 0 & 0 & 0 \\ \hline 0 & 6 & \boxed{7} & 4 & 5 & 9 \\ 0 & 4 & 3 & \boxed{7} & 8 & 3 \\ 0 & 0 & 1 & 3 & \boxed{6} & 4 \\ 0 & 2 & 2 & 5 & 7 & \boxed{8} \end{array}\right] = (a_{ij})_{\substack{i=0,\dots,4\cdot\\ j=0,\dots,5}}$$

The (unique) optimal (M, N)-matching $\sigma=\{(0, 0), (0, 1), (1, 2), (2, 3), (3, 4), (4, 5)\}$ is denoted by the boxes around the entries. Initially they are the only settled coalitions, $\Delta=\sigma$. Throughout the algorithm these optimally matched players share the exact profit they make (see Corollary 2.6). Initially everything is given to the row players and nothing to the columns, i.e. we start with payoffs $u_1=7$, $u_2=7$, $u_3=6$, $u_4=8$, and $v_j=0$, $j=1, \ldots, 5$. Remember that always $u_0=v_0=0$. Since here $(0, 1)\in\sigma$, v_1 remains 0. The satisfaction matrix $f_{ij}=(u_i+v_j-a_{ij})_{i=0,\ldots,4;\, j=0,\ldots,5}$ is used and updated. Recall that $u_i=f_{i0}$ and $v_j=f_{0j}$, i.e. the current payoffs are in the 0-th row and column.

Iteration $r=0$ begins. The satisfaction matrix is the following:

p:		0	0	1	2	3	4	s_i:
	j:	0	1	2	3	4	5	
	i:							
0	0	$\boxed{0}$	$\boxed{0}$	0_0	0_0	0_0	0_{+1}	0
1	1	7_0	1_0	$\boxed{0}$	3_0	2_0	-2^{*}_{+1}	0
2	2	7_0	3_0	4_0	$\boxed{0}$	$-1^{\diamond}_{0}$	4_{+1}	0
3	3	6_0	6_0	5_0	3_0	$\boxed{0}$	2_{+1}	0
4	4	8_{-1}	6_{-1}	6_{-1}	3_{-1}	1_{-1}	$\boxed{0}$	-1
t_j:		0	0	0	0	0	+1	$\beta=1$

The index p of the node a player belongs to is indicated on the left or top margin of the satisfaction matrix next to the index i or j of the player himself. The initial guaranteed satisfaction level is $\alpha=\alpha^0=-2$. It is attained at coalition $(1, 5)\in(M_1, N_4)$. We build the graph



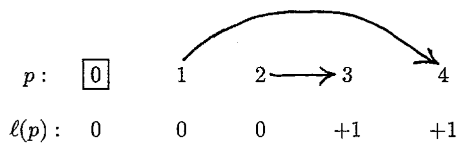
with nodes related to the equivalence classes of tied players (each one is a singleton here except for the settled node), and with arcs related to active coalitions marked by a star (*) in the matrix (here $\Sigma_= =\{(1, 5)\}$, so $(M_1, N_4)\cap\Sigma_= \neq\emptyset$, hence the only arc $(1, 4)\in E$). The graph is clearly proper, so we start

Step $r=0$, $\alpha=-2$. We determine the length $\ell(p)$ of a longest path to each node p, and direction (s, t) according to (5.8). The values s_i and t_j appear on the right and bottom margins. For every currently unsettled coalition $(i, j)\in\Sigma$ we calculate the change index s_i+t_j (the numbers at the lower right corner of such entries), the rate of change in the satisfaction if the payoff is changed in the direction (s, t). For every unsettled coalition with nonpositive change index we find the threshold β_{ij} as in (6.1.5). For example, for coalition (4, 4) we have $f_{44}=1$ and $s_4+t_4=-1$, so $\beta_{44}=3/2$. The step size β is the minimum of these numbers. Here $\beta=1$ and β_{24} is the only minimal threshold. Thus, coalition (2, 4) (marked by a diamond ($\diamond$) in the matrix) is the only threshold coalition, i.e. a passive coalition that first becomes active. Hence we add arc (2, 3) to the graph, since $(2, 4)\in(M_2, N_3)$. Updating the satisfactions and the guaranteed satisfaction level (here $\alpha=-2+1=-1$) ends the step.

Step $r=0$, $\alpha=-1$. The updated satisfaction matrix is

p:		0	0	1	2	3	4	s_i:
	i: \ j:	0	1	2	3	4	5	
0	0	$\boxed{0}$	$\boxed{0}$	$0_0^{\diamond}$	$0_0^{\diamond}$	0_{+1}	1_{+1}	0
1	1	7_0	1_0	$\boxed{0}$	3_0	2_{+1}	-1^*_{+1}	0
2	2	7_0	3_0	4_0	$\boxed{0}$	-1^*_{+1}	5_{+1}	0
3	3	6_{-1}	6_{-1}	5_{-1}	3_{-1}	$\boxed{0}$	3_0	-1
4	4	7_{-1}	5_{-1}	5_{-1}	2_{-1}	$0_0^{\diamond}$	$\boxed{0}$	-1
t_j:		0	0	0	0	+1	+1	$\beta=1$

The obviously proper updated graph with the current lengths is

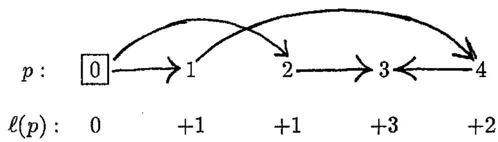


The step size $\beta=1$ is attained at coalitions $(0, 2)\in(M_0, N_1)$, $(0, 3)\in(M_0, N_2)$, and $(4, 4)\in(M_4, N_3)$, so the arcs (0, 1), (0, 2), and (4, 3) have to be added.

Step $r=0$, $\alpha=0$. The nonnegative guaranteed satisfaction level indicates that the core is reached. In fact, the current payoff vector is the row-best corner of the core.

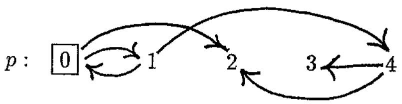
p:		0	0	1	2	3	4	s_i:
	i : j:	0	1	2	3	4	5	
0	0	$\boxed{0}$	$\boxed{0}$	0^*_{+1}	0^*_{+1}	1_{+3}	2_{+2}	0
1	1	7_{-1}	$1^{\diamond}_{-1}$	$\boxed{0}$	3_0	3_{+2}	0^*_{+1}	−1
2	2	7_{-1}	3_{-1}	4_0	$\boxed{0}$	0^*_{+2}	6_{+1}	−1
3	3	5_{-3}	5_{-3}	4_{-2}	2_{-2}	$\boxed{0}$	3_{-1}	−3
4	4	6_{-2}	4_{-2}	4_{-1}	$1^{\diamond}_{-1}$	0^*_{+1}	$\boxed{0}$	−2
t_j:		0	0	+1	+1	+3	+2	$\beta = 1/2$

The related graph is still proper.



The step size $\beta = 1/2 = \beta_{11} = \beta_{43}$ is attained in blocks (M_1, N_0) and (M_4, N_2), so the new arcs (1, 0) and (4, 2) must be added. On the other hand, arc (2, 3) must be dropped, since the change index in the active block (M_2, N_3) (containing only the active coalition (2, 4)) exceeds 1, hence the satisfaction there will increase faster than the guaranteed minimum level.

Iteration $r = 0$ *ends.* We realize that adding arc (1, 0) makes the graph improper.



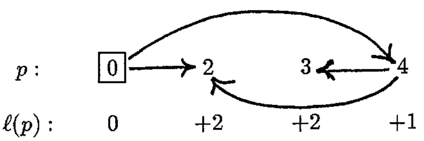
We melt node 1 into the settled node 0 such that all arcs are inherited (except possible multiple arcs). All the coalitions in blocks (0, 1) and (1, 0) becomes settled, hence are left out from any further calculation. The new graph (see the next one below) is proper, so

Iteration $r = 1$ *starts.*

Step $r = 1$, $\alpha = 1/2 = \alpha^1$. The updated satisfaction matrix is

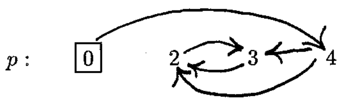
p:		0	0	0	2	3	4	s_i:
	i: \ j:	0	1	2	3	4	5	
0	0	0	0	1/2	$1/2^*_{+2}$	$5/2_{+2}$	3_{+1}	0
0	1	13/2	1/2	0	3_{+2}	4_{+2}	$1/2^*_{+1}$	0
2	2	$13/2_{-2}$	$5/2_{-2}$	4_{-2}	$\boxed{0}$	$1^{\diamond}_{0}$	$13/2_{-1}$	−2
3	3	$7/2_{-2}$	$7/2_{-2}$	3_{-2}	$1^{\diamond}_{0}$	$\boxed{0}$	$5/2_{-1}$	−2
4	4	5_{-1}	3_{-1}	$7/2_{-1}$	$1/2^*_{+1}$	$1/2^*_{+1}$	$\boxed{0}$	−1
t_j:		0	0	0	+2	+2	+1	$\beta = 1/2$

The related graph is now proper.



The step size $\beta = 1/2 = \beta_{24} = \beta_{33}$ is attained in blocks (M_2, N_3) and (M_3, N_2). Arcs (2, 3) and (3, 2) must be added. Since arc (0, 2) is not part of any longest path, the change index exceeds 1, thus it must be dropped.

Iteration $r = 1$ *ends.* The two new arcs form a cycle {(2, 3), (3, 2)}.



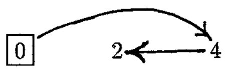
Node 3 is melted into node 2 keeping all arcs (except the parallel ones). The coalitions in blocks (M_2, N_3) and (M_3, N_2) become settled. After eliminating the above cycle the graph is proper again (see the one below).

Iteration $r = 2$ *starts.*

Step $r = 2$, $\alpha = 1 = \alpha^2$. The updated satisfaction matrix is

p:			0	0	0	2	2	4	s_i:
		j:	0	1	2	3	4	5	
	i:								
0	0		0	0	1/2	$3/2_{+2}$	$7/2_{+2}$	$7/2_{+1}$	0
0	1		13/2	1/2	0	4_{+2}	5_{+2}	1^*_{+1}	0
2	2		$11/2_{-2}$	$3/2^{\diamond}_{-2}$	3_{-2}	0	1	6_{-1}	−2
2	3		$5/2_{-2}$	$5/2_{-2}$	2_{-2}	1	0	2_{-1}	−2
4	4		$9/2_{-1}$	$5/2_{-1}$	3_{-1}	1^*_{+1}	1^*_{+1}	[0]	−1
t_j:			0	0	0	+2	+2	+1	$\beta=1/6$

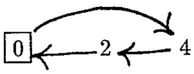
The new graph is proper.

p: 

$\ell(p)$: 0 +2 +1

The step size $\beta=1/6=\beta_{21}$ is attained in block (M_2, N_0), so we add this arc. No arc has to be dropped here.

Iteration $r=2$ *ends.* The new arc makes the graph improper.

p: 

Node 2 is melted into node 0. The graph now is

p: 

still improper, so node 4 is melted into node 0. Only the settled node is left, so the

Algorithm stops with $r=3$, $\alpha=7/6=\alpha^3$. The final matrix containing the satisfactions at the nucleolus is

0	0	1/2	11/6	23/6	11/3
13/2	1/2	0	13/3	16/3	7/6
31/6	7/6	8/3	0	1	35/6
13/6	13/6	5/3	1	0	11/6
13/3	7/3	17/6	7/6	7/6	0

The nucleolus itself is the final payoff vector, located in the 0-th row and column, i.e. $\nu = (13/2, 31/6, 13/6, 13/3; 0, 1/2, 11/6, 23/6, 11/3)$.

From Step$(r=0, \alpha=0)$ we know that the u-best corner of the core is $(\bar{u}, \underline{v}) = (7, 7, 5, 6; 0, 0, 0, 1, 2)$. With the obvious modifications in the algorithm we computed the v-best corner of the core in three steps and found $(\underline{u}, \bar{v}) = (6, 4, 0, 2; 0, 1, 3, 6, 6)$. Thus, the so called fair outcome is the payoff vector (13/2, 11/2, 5/2, 4; 0, 1/2, 3/2, 7/2, 4).

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