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On the Core of Information Graph Games

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Abstract: This paper considers a subclass of minimum cost spanning tree games, called information graph games. It is proved that the core of these games can be described by a set of at most $2n-1$ linear constraints, where n is the number of players. Furthermore, it is proved that each information graph game has an associated concave information graph game, which has the same core as the original game. Consequently, the set of extreme core allocations of an information graph game is characterized as the set of marginal allocation vectors of its associated concave game. Finally, it is proved that all extreme core allocations of an information graph game are marginal allocation vectors of the game itself, though not all marginal allocation vectors need to be core allocations.

1 Introduction

Given is a set of customers N who are all interested in a particular piece of information, e.g. a computer program. A subset Z of N, called the informed set, already possesses this information. Other customers may purchase the information from a central supplier for a fixed price, say 1, or they may share the information with a friendly customer, who already has the information. These friendly relations between customers are stored in an undirected graph $G=(N, E)$, called the information graph. Players i and j can send information to each other if and only if $\{i, j\} \in E$.

Suppose a subset of customers $S \subseteq N$ decides to form a coalition. Assume that the customers in S all need to get informed and that they do not seek the cooperation of customers outside S in order to achieve this goal. Consider the graph $G_{|S}$ that results from G by deleting all vertices (customers) outside S and all edges with at least one endpoint outside S. Customers within one component of $G_{|S}$ can freely share their information. So, if this component contains a customer in the informed set, then the cost to get all customers of this component informed equals 0. Otherwise, one of the customers in this component will have to purchase the information from the central supplier at cost 1 and then share it with the other customers in the component. Thus, the cost to get all players of S informed is equal to the number of components of $G_{|S}$ that have no customer in the informed set. We denote this cost by $c(S)$. The cooperative cost allocation game with player set N and characteristic function c is called an information graph game.

One may also view information graph games as minimum cost spanning tree games, for which the cost to connect two customers or to connect a customer to the central supplier is either 0 or 1.

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It was proved by Granot and Huberman [2] that the core of a minimum cost spanning tree game is never empty and that a core allocation can be easily read from the minimum spanning tree for the grand coalition. However, this core allocation is often considered unfair. E.g., a player who is connected directly to the central supplier has to pay his individual cost $c(\{i\})$ in this solution. Thus, he will not loose any money if he leaves the grand coalition. On the other hand, the other players will loose if he does. It is plausible that this player will use this threat to convince the other players that they should transfer some money to him. It is therefore desirable to have other options for the cost allocation of a minimum cost spanning tree game. This problem was also addressed by Granot and Huberman in [3]. They provided efficient procedures (polynomial in the number of players) to generate other vectors in the core. However, repeated application of these procedures does not exhaust all extreme elements of the core. As far as we know, no such procedure is known at this moment. For the subclass of information graph games the situation is better. In this paper we shall give a description of the core by means of at most $2n-1$ linear constraints and we give a characterization of the extreme elements of the core.

2 Concave Information Graph Games

Let $(N; c)$ be a cooperative cost allocation game. In analogy with the definition of a convex savings game we shall call the game $(N; c)$ *concave* if

$$c(S)+c(T)\geq c(S\cap T)+c(S\cup T)$$

for all $S, T\subseteq N$.

In this section we give necessary and sufficient conditions on the information graph and the informed set, such that the associated information graph game is concave. First we need some preliminary definitions.

Let $G=(V, E)$ be an undirected graph. Two vertices $v, w\in V$ are called *adjacent* if $\{v, w\}\in E$. A sequence of vertices $(v_1, \ldots, v_k)$ is called a *path* from v_1 to v_k if v_i and v_{i+1} are adjacent for $i=1, \ldots, k-1$. v_1 is called the *initial* vertex and v_k is called the *final* vertex of the path. An *elementary path* is a path for which no two vertices coincide. A *circuit* is a path whose initial and final vertex coincide, but for which no other vertices coincide. A circuit that meets only two different vertices is called a *trivial circuit*. Two paths from v to w are called *vertex-disjoint* if v and w are the only vertices that these paths have in common.

G is called a *forest* if it contains only trivial circuits. It is called a *tree* if it is a connected forest. G is called *complete* if v and w are adjacent for all $v, w\in V$ $(v\neq w)$. G is called *bipartite* if all circuits contain an even number of vertices. G is called *2-connected* if it is connected and if removing one vertex, no matter which one, leaves the remainder of the graph connected.

For convenience we shall say that a set $U\subseteq V$ is connected, 2-connected, complete, etc., if the graph $G_{|U}$, which results from G by deleting all vertices outside U, has this property.

The maximal 2-connected subsets of a set $U \subseteq V$ are called the *blocks* of G.

The following result can be found in many text books on graph theory and is therefore stated without proof (see e.g. Berge [1]).

Lemma 1 (Menger, 1926). A necessary and sufficient condition for a graph to be 2-connected is that each pair of distinct vertices can be joined by two vertex-disjoint paths.

Using this lemma one easily proves the following lemma.

Lemma 2. Let S be a 2-connected subset of N and $i, j, k \in S$. Then there exists an elementary path from i to j that contains k.

Proof: According to lemma 1, there exist two vertex-disjoint paths from i to k. At least one of these paths does not contain j. Thus, there exists a path from i to k not containing j. Furthermore, there exist two vertex-disjoint paths from k to j. It is clear that all of these paths may be taken elementary. Follow the path from i to k until it hits one of the paths from j to k. From here, follow this path until the vertex k is reached. Then finally, follow the second path from j to k backwards. This way, we have constructed an elementary path from i to j, which contains the vertex k. □

Lemma 3. Let $G = (V, E)$ be an undirected graph with the property that each 2-connected subset of V is complete. Then we have: if $S, T \subseteq V$ are connected, then also $S \cap T$ is connected.

Proof: Suppose $v, w \in S \cap T$. Let $(s_0, \ldots, s_n)$ be a path from v to w which uses only vertices in S and let $(t_0, \ldots, t_m)$ be a path from v to w which uses only vertices in T (so $s_0 = t_0 = v$ and $s_n = t_m = w$). Suppose $s_i = t_i$ for $i = 1, \ldots, h$. If $h = n$ or $h = m$ then clearly we have a path from v to w which uses only vertices in $S \cap T$. Therefore, suppose $h < \min(n, m)$. Let k be the smallest index greater than h, for which one of the t_i's equals s_k, say $t_\ell = s_k$. Such indices exist since $s_n = t_m = w$. Then $(s_h, \ldots, s_k)$ and $(t_h, \ldots, t_\ell)$ are two vertex-disjoint paths from $s_h = t_h$ to $s_k = t_\ell$. It is clear that the set of vertices that lie on these two paths form a 2-connected set. Thus, this set is complete, and therefore $\{s_h, s_k\}$ is an edge of G. Now, remove all vertices between s_h and s_k from the path $(s_1, \ldots, s_n)$ and remove all vertices between t_h and t_ℓ from the path $(t_1, \ldots, t_m)$. Then both sequences remain paths from v to w. After renumbering the remaining vertices in both paths we have $s_i = t_i$ for $i = 1, \ldots, h+1$. Repeat this process until both paths are identical. The process ends with a path from v to w, which uses only vertices in $S \cap T$. Since v and w were chosen arbitrarily, it follows that $S \cap T$ is connected. □

Lemma 4. Let $G = (V, E)$ be an undirected graph with the property that each 2-connected subset of V is complete. Furthermore, let $S, T \subseteq V$ with components respectively $S_1, \ldots, S_k$ and $T_1, \ldots, T_\ell$. Let $\mathcal{G}$ be the bipartite graph with node set $\mathcal{V} = \{S_1, \ldots, S_k\} \cup \{T_1, \ldots, T_\ell\}$ and edge set $\mathcal{E} = \{\{S_i, T_j\} \mid S_i \cup T_j$ is connected$\}$. Then $\mathcal{G}$ is a forest.

Proof: Suppose that $\mathcal{G}$ is not a forest, i.e. it contains a non-trivial circuit, say $(U_1, \ldots, U_n, U_1)$ (with $n \geq 3$). Choose $u_i \in U_i$ for $i = 1, \ldots, n$. $U_1 \cup U_2$ is connected, so there exists a path from u_1 to u_2 in the graph G. Also, there are paths from u_i to u_{i+1} for $i = 2, \ldots, n-1$ and finally there is a path from u_n to u_1. Thus, after concatenation of these paths we obtain a path visiting a vertex in all sets $U_1, \ldots, U_n$ and which has u_1 as its initial and final vertex. This path need not be a circuit in the graph G, but a circuit can be constructed from it as follows. If the path is not a circuit then it contains a circuit which has less vertices than the path itself. If this circuit visits at least three of the sets $U_1, \ldots, U_n$ then stop. Otherwise, construct a new path by replacing the circuit in this path by its initial (and final) vertex. The resulting path still visits all sets $U_1, \ldots, U_n$. This is trivial if the circuit visits only one of the U_i's. Thus suppose the circuit visits precisely two of the U_i's. Obviously, these sets must be consecutive. Without loss of generality assume that the circuit visits U_1 and U_2 and that its initial and final vertex lies in U_1. After removal of the circuit from the path, and replacing it by its initial vertex, the resulting path trivially still visits U_1. It also visits U_2, since otherwise the path would jump from a vertex in U_1 to a vertex in U_3, which contradicts the fact that $\mathcal{G}$ is bipartite. Repeat the process until a circuit is found that visits at least three of the sets U_i or until the path itself has become a circuit that visits all sets U_i. The vertices that lie on this circuit form a 2-connected set. According to the property of G, this set is complete and therefore any set $U_i \cup U_j$ is connected if both U_i and U_j are visited by the circuit. The circuit is constructed such that it visits at least three of the sets $U_1, \ldots, U_n$. Thus, the graph $\mathcal{G}$ contains a circuit on three of its nodes. This contradicts the fact that $\mathcal{G}$ is bipartite. We conclude that the graph $\mathcal{G}$ can only contain trivial circuits. In other words, $\mathcal{G}$ is a forest. □

The following theorem gives necessary and sufficient conditions on the informed set and the information graph, such that the corresponding information graph game is concave.

Theorem 1. Let $(N; c)$ be an information graph game with information graph $G = (N, E)$ and informed set Z. Then c is concave if and only if G and Z satisfy the following properties.

- *i) Each elementary path with both endpoints in Z is contained in Z.*
- *ii) Each 2-connected coalition $S \subseteq N$, whose intersection with Z contains at most one element, is complete.*

Proof: To prove the 'only if'-part suppose that c is concave. Let $(v_1, \ldots, v_k)$ be an elementary path with $v_1, v_k \in Z$. Let i be such that $1 < i < k$. Define $S = \{v_1, \ldots, v_i\}$ and $T = \{v_i, \ldots, v_k\}$. Then S, T and $S \cup T$ are connected and contain at least one element of Z. Therefore $c(S) = c(T) = c(S \cup T) = 0$. From the concavity of c it follows that $c(S \cap T) = c(\{v_i\}) = 0$ and thus $v_i \in Z$. This proves that Z satisfies property i). Let $U \subseteq N$ be a 2-connected coalition whose intersection with Z contains at most one element. Choose $i, j \in U$. According to lemma 1 there exist two vertex-disjoint paths in U from i to j, say $(i, v_1, \ldots, v_k, j)$ and $(i, w_1, \ldots, w_\ell, j)$. Define $S = \{i, v_1, \ldots, v_k, j\}$ and $T = \{i, w_1, \ldots, w_\ell, j\}$. Consider three possibilities. a) Neither S nor T contains an element of Z. b) Precisely one of the coalitions S and T contains an element of Z. c) Both S and T contain the unique element of $Z \cap U$. One easily verifies that in all three cases we have $c(S \cap T) + c(S \cup T) > c(S) + c(T)$, unless $\{i, j\} \in E$. From the concavity of c it follows that $\{i, j\} \in E$. This proves that G and Z satisfy property ii).

To prove the 'if'-part suppose that G and Z satisfy properties i) and ii). Suppose that S is a 2-connected coalition, which is not complete. Then it must contain at least 2 vertices in Z, say z_1 and z_2. Let $i \in S$. According to lemma 2, there exists an elementary path from z_1 to z_2 that contains i. And thus, $i \in Z$. It follows that $S \subseteq Z$. Now it is clear that the characteristic function c does not change if an edge is added between two vertices in S. Therefore we may assume that all 2-connected subsets are complete. Let $S, T \subseteq N$. Denote the components of S by $S_1, \ldots, S_k$ and the components of T by $T_1, \ldots, T_\ell$. Construct the graph $\mathscr{G} = (\mathscr{V}, \mathscr{E})$ as in lemma 4, i.e. $\mathscr{V} = \{S_1, \ldots, S_k\} \cup \{T_1, \ldots, T_\ell\}$ and $\mathscr{E} = \{\{S_i, T_j\} \mid S_i \cup T_j \text{ is connected}\}$. According to lemma 4 this graph is a forest. Therefore, the number of edges equals the number of vertices minus the number of components, i.e.,

$$|\mathscr{E}| = k(S) + k(T) - k(S \cup T),$$

where $k(S)$, $k(T)$ and $k(S \cup T)$ denote the number of components of S, T and $S \cup T$ respectively.

Let $\mathscr{G}_Z = (\mathscr{V}_Z, \mathscr{E}_Z)$ be the graph that results from $\mathscr{G}$ when all vertices S_i and T_j are removed that have empty intersection with Z. Clearly, this graph is also a forest and its number of edges therefore equals the number of vertices minus the number of components. The number of vertices of $\mathscr{G}_Z$ is reduced with the amount $c(S) + c(T)$, compared to the graph $\mathscr{G}$. It follows from property i) that within each component of $\mathscr{G}$ the vertices that have a non-empty intersection with Z form a connected set in $\mathscr{G}$. Thus, the number of components in $\mathscr{G}_Z$ is reduced with the amount $c(S \cup T)$ compared to the graph $\mathscr{G}$. It follows that

$$|\mathscr{E}| - |\mathscr{E}_Z| = c(S) + c(T) - c(S \cup T).$$

Furthermore, if $S_i \cap T_j \neq \emptyset$ is connected and if $S_i \cap T_j \cap Z = \emptyset$ then it follows from property i) that $S_i \cap Z = \emptyset$ or $T_j \cap Z = \emptyset$. Using these results the concavity of c now follows from

$$\begin{aligned}
c(S \cap T) &= c\left(\bigcup_{i=1}^{k} S_i \cap \bigcup_{j=1}^{\ell} T_j\right) = \\
&= c\left(\bigcup_{i,j} (S_i \cap T_j)\right) \leq \sum_{i,j} c(S_i \cap T_j) = \\
&= |\{(i,j) \mid S_i \cap T_j \neq \emptyset \text{ and } S_i \cap T_j \cap Z = \emptyset\}| = \\
&= |\{(i,j) \mid S_i \cap T_j \neq \emptyset \text{ and } (S_i \cap Z = \emptyset \text{ or } T_j \cap Z = \emptyset)\}| \leq \\
&\leq |\{(i,j) \mid S_i \cup T_j \text{ is connected and } (S_i \cap Z = \emptyset \text{ or } T_j \cap Z = \emptyset)\}| = \\
&= |\{(i,j) \mid S_i \cup T_j \text{ is connected}\}| - \\
&\quad - |\{(i,j) \mid S_i \cup T_j \text{ is connected and } (S_i \cap Z \neq \emptyset \text{ and } T_j \cap Z \neq \emptyset)\}| = \\
&= |\mathscr{E}| - |\mathscr{E}_Z| = c(S) + c(T) - c(S \cup T).
\end{aligned}$$

□

It is not obvious that the conditions in the above theorem can be verified efficiently. However, it is not difficult to prove that the conditions in theorem 1 are equivalent to the conditions in the following corollary of the theorem.

Corollary 2. Let $(N; c)$ be an information graph game with information graph G and informed set Z. Then c is concave if and only if G and Z satisfy the following properties.

i) For each component K of the information graph G we have that $K \cap Z$ is connected.
ii) Each block of G is contained in Z or it has at most one element in common with Z.
iii) Each block, whose intersection with Z contains at most one element, is complete.

There exist algorithms to determine all blocks of a graph, which are linear in the number of edges of the graph (see Tarjan [4]). Therefore, it is also possible to verify the conditions in corollary 2 efficiently.

3 A Description of the Core by Means of Linear Constraints

Let $(N; c)$ be a cooperative cost game. A vector $x \in \mathbb{R}^N$ is called a core allocation if it satisfies the following linear restrictions.

$$\begin{cases} \sum_{i \in N} x_i = c(N) \\ \sum_{i \in S} x_i \leq c(S) \text{ for all } S \subseteq N. \end{cases}$$

The set of all core allocations is called the core. It is denoted by $\text{Core}(c)$. In the following we shall use the notation $x(S)$ to denote $\sum_{i \in S} x_i$.

Let $(N; c)$ be an information graph game with information graph $G = (N, E)$ and informed set Z. Assume for the moment that the graph G is connected. Let G_i denote the graph that results from G if vertex i and all edges with i as an endpoint are deleted. Furthermore, let $K_{i1}, \ldots, K_{ik_i}$ denote the vertices of the components of G_i. Of course, $\text{Core}(c)$ is contained in the set $C(c)$ described by the following linear constraints.

$$\begin{cases} x(N) = c(N) \\ x(K_{ij}) \leq c(K_{ij}) \text{ for all } i: 1 \leq i \leq n \text{ and for all } j: 1 \leq j \leq k_i. \end{cases}$$

In fact we have the following theorem.

Theorem 3. The set $C(c)$ equals $\text{Core}(c)$. Furthermore, the number of constraints in the description of $C(c)$ is at most $2n - 1$.

Let us first prove that the number of constraints that describe $C(c)$ is bounded by $2n-1$. Suppose that we remove an edge from the graph G, such that the resulting graph remains connected. Clearly, this can only increase the number of components of the graphs G_i. Thus, we get the maximum number of restrictions when no edge of G can be deleted without disturbing its connectedness, i.e. when G is a tree. Therefore, in deriving an upper bound for the number of restrictions, we may assume that G is a tree. Consequently, the graphs G_i are all forests. Let us denote the number of edges in G_i by e_i. k_i denotes the number of components of G_i and the number of vertices of G_i is $n-1$. Thus we have

$$k_i = n-1-e_i.$$

Suppose that $\{i_1, i_2\}$ is an edge of the graph G. It is clear that this edge is an edge in all graphs G_i, except in G_{i_1} and in G_{i_2}. Thus,

$$\sum_{i=1}^{n} e_i = (n-2)(n-1),$$

since each edge of G is counted $n-2$ times and there are $n-1$ edges in G. So it follows that

$$\sum_{i=1}^{n} k_i = n(n-1) - \sum_{i=1}^{n} e_i = 2(n-1).$$

I.e. the number of inequalities in the description of $C(c)$ is precisely $2(n-1)$ when G is a tree. We have one extra equality $x(N)=c(N)$, which makes a total of $2n-1$ constraints.

Now we have to prove that $\text{Core}(c)=C(c)$, i.e. we have to prove that each constraint of the form $x(S) \leq c(S)$ is implied by the constraints of $C(c)$.

Let $S \subseteq N$. First suppose that S is connected. Let us say that a vertex $v \notin S$ separates the vertex $u \neq v$ from S, if all paths with one endpoint in S and the other endpoint equal to u contain the vertex v. Equivalently, the vertex u and the set S lie in different components of the graph G_v. Define

$$U := \{u \in N \backslash S \mid \text{no vertex in } N \backslash S \text{ can separate } u \text{ from } S\}.$$

We then have the following lemma.

Lemma 5. For each vertex $v \in N \backslash (U \cup S)$ there is precisely one vertex $u \in U$ that separates v from S.

Proof: Suppose $v \in N \backslash (U \cup S)$. We first prove that at least one vertex $u \in U$ separates v from S. By definition there exists a vertex $u_0 \in N \backslash S$ that separates v from S. If $u_0 \in U$ then the existence part of the proof is finished. If not, choose $u_1 \in N \backslash S$ that separates u_0 from S. Obviously, u_1 also separates v from S. Continu this process until finally a vertex $u_k \in U$ is found, which separates v (and $u_0, \ldots, u_{k-1}$) from S.

We shall prove now that at most one vertex $u \in U$ separates v from S. Suppose that u, $w \in N \backslash S$ ($u \neq w$) both separate v from S. Then each path from v to S contains both u and w. On such a path the vertices u and w are always visited in a fixed order, since otherwise it would be possible to construct a path from v to S with only one of the vertices u or w. Assume that w is always visited before u. Then u separates w from S and therefore w cannot be an element of U. It follows that U contains at most one element that separates v from S. The existence of such an element was already proved, which shows that there is precisely one element in U that separates v from S. □

For each $u \in U$ let K_u denote the component of G_u that contains S and let $x \in \mathbb{R}^N$. Using lemma 5, we obtain the following equality.

$$\sum_{u \in U} x(K_u) = (|U| - 1)x(N) + x(S).$$

This is explained as follows. Let i be a player in S. Then by definition $i \in K_u$ for all $u \in U$. Thus, the variable x_i is counted $|U|$ times on the left hand side of the equation. Trivially, x_i is also counted $|U|$ times on the right hand side.

Now suppose that $i \in U$. Trivially, i is not an element of K_i and since i cannot be separated from S it is an element of K_u for all $u \neq i$. Thus the variable x_i is counted $|U| - 1$ times on the left hand side and also $|U| - 1$ times on the right hand side of the equation.

Finally, suppose $i \notin U \cup S$. According to lemma 5, $i \in K_u$ for all $u \in U$ except one. Thus the variable x_i is counted $|U| - 1$ times on the left hand side and also $|U| - 1$ times on the right hand side. We conclude that the equation is correct.

The constraints $x(K_u) \leq c(K_u)$ are all constraints which are used in the description of $C(c)$. Together with the constraint $x(N) = c(N)$, they imply

$$x(S) = \sum_{u \in U} x(K_u) - (|U| - 1)\, x(N) \leq \sum_{u \in U} c(K_u) - (|U| - 1)\, c(N).$$

It remains to prove that

$$\sum_{u \in U} c(K_u) - (|U| - 1)\, c(N) \leq c(S).$$

We distinguish three cases.

i) The informed set Z is empty.
ii) S contains an element of Z.
iii) The remaining case, i.e. $Z \neq \emptyset$ and $Z \cap S = \emptyset$.

Let us first consider case i). If the informed set is empty, then all connected sets have a cost equal to 1. Thus,

$$\sum_{u \in U} c(K_u) - (|U| - 1)\, c(N) = |U| - (|U| - 1) = 1 = c(S).$$

Case ii): In this case all connected sets containing an element of Z have a cost equal to 0. Thus,

$$\sum_{u \in U} c(K_u) - (|U| - 1)\, c(N) = 0 = c(S).$$

Case iii): Let $z \in Z$. Since $z \notin S$, it follows from lemma 5 that z is an element of precisely $|U| - 1$ of the sets K_u. These $|U| - 1$ components therefore have a cost equal to 0. The remaining component can have a cost of at most 1. Thus,

$$\sum_{u \in U} c(K_u) - (|U| - 1|)\, c(N) \leq 1 = c(S).$$

We have proved that the constraints of $C(c)$ imply all constraints of the form $x(S) \leq c(S)$ for connected coalitions S. Let us suppose therefore that S is not connected. Let $S_1, S_2, \ldots, S_k$ denote the components of S. We have just proved that the constraints $x(S_i) \leq c(S_i)$ are all implied by the constraints of $C(c)$. And thus,

$$x(S) = \sum x(S_i) \leq \sum c(S_i) = c(S)$$

is also implied by these constraints.

We conclude that $\mathrm{Core}(c) = C(c)$. Up to this moment it was assumed that the information graph G was connected. If this is not the case then one can deal with each component of G separately, because the core of the game associated with G is the Cartesian product of the cores associated with the components of G.

4 The Extreme Elements of the Core

In this section we shall use the results of the previous two sections to prove that each information graph game has an associated concave information graph game, which has the same core as the original game. This result is implied by two theorems, that tell us how to adjust the information graph and the informed set of an information graph game without changing its core. After a finite number of adjustments we end up with a concave information game which has the same core as the original game. This gives us a nice characterization of the extreme elements of the core of an information graph game, since these are precisely the marginal allocation vectors of the associated concave game. Also, every extreme core allocation is a marginal allocation vector in the original game, though not all marginal allocation vectors need to be core allocations.

Theorem 4. Let $(N; c)$ be an information graph game with information graph $G = (N, E)$ and informed set Z. Suppose there is a player $i \in N \backslash Z$ and an elementary path from one vertex in Z to another vertex in Z that contains player i. Let $\hat{c}$ be the *information graph game with the same information graph G and informed set* $\hat{Z}=Z\cup\{i\}$. *Then* $Core(c)=Core(\hat{c})$.

Proof: Suppose K is a component of the graph G_j $(j\in N)$. If $i\notin K$ then trivially $c(K)=\hat{c}(K)$. Thus suppose $i\in K$. There exists an elementary path with both endpoints in Z containing player i. It is obvious that at least one of these endpoints also lies in K, so $c(K)=\hat{c}(K)=0$. We also have $c(N)=\hat{c}(N)$, so, using theorem 3, it follows directly that

$$\mathrm{Core}(\hat{c})=C(\hat{c})=C(c)=\mathrm{Core}(c). \qquad \square$$

Also, the information graph of an information graph game can be adjusted without changing the core. This is expressed in the following theorem.

Theorem 5. Let $(N; c)$ *be an information graph game with information graph* $G=(N, E)$ *and informed set Z. Suppose* $i, j\in N$ *are non-adjacent players contained in a 2-connected subset of N. Let* $\hat{c}$ *be the information graph game with information graph* $\hat{G}=(N, E\cup\{\{i, j\}\})$ *and informed set Z. Then* $Core(\hat{c})=Core(c)$.

Proof: Let $u\in N$. We shall first prove that the components of G_u and $\hat{G}_u$ are the same. To this end we show that v and w lie in one component of $\hat{G}_u$ if and only if they lie in one component of G_u. To prove the 'if'-part suppose that v and w lie in one component of G_u. Then apparently there exists a path from v to w in the graph G that does not use the vertex u. This path also exists in $\hat{G}$, since the edge set of $\hat{G}$ contains the edge set of G as a subset. Therefore, v and w lie in one component of the graph $\hat{G}_u$.

To prove the 'only if'-part suppose that v and w lie in one component of $\hat{G}_u$. Then there exists a path from v to w in $\hat{G}$ that does not use the vertex u. Suppose this path uses the extra edge $\{i, j\}$. Since the vertices i and j lie in a 2-connected set of G there exist two vertex-disjoint paths from i to j in the graph G. At least one of these paths does not use the vertex u. Use this path to get from i to j instead of the edge $\{i, j\}$. Thus, we have constructed a path from v to w that does not use vertex u in the graph G. Therefore, v and w lie in one component of G_u. It follows that the components of $\hat{G}_u$ and G_u are the same.

Obviously, also the costs of these components is in both cases the same. Using theorem 3 it follows immediately that

$$\mathrm{Core}(\hat{c})=C(\hat{c})=C(c)=\mathrm{Core}(c). \qquad \square$$

From theorem 1, 4 and 5 we get

Corollary 6. Let $(N; c)$ *be an information graph game with information graph* $G=(N, E)$ *and informed set Z. Let* $\hat{Z}$ *be the set of all vertices that lie on an elementary path with both endpoints in Z and let* $\hat{E}$ *be the set of all pairs* $\{i, j\}$, *such that there is a 2-connected set* $S\subseteq N$ *with* $\{i, j\}\subseteq S$. *Let* $\hat{c}$ *be the information graph game* *with information graph $\hat{G}=(N, \hat{E})$ and informed set $\hat{Z}$. Then $\hat{c}$ is concave and $Core(\hat{c}) = Core(c)$.*

The extreme elements of the core of an information graph game can now be characterized as being precisely the marginal allocation vectors of its associated concave information graph game. It follows that the extreme core allocations of an information graph game are integer, since the marginal allocation vectors are integer. This observation is helpful in the proof of

Theorem 7. Let $(N; c)$ be an information game with information graph $G=(N, E)$ and informed set Z. Then the set of marginal allocation vectors of c contains the set of extreme core allocations of c as a subset.

Proof: Without loss of generality we assume that G is a connected graph. Let x be an extreme core allocation. We shall show that there exist coalitions $S_1 \subseteq S_2 \subseteq \ldots \subseteq S_n$ with the property $x(S_k)=c(S_k)$ and $|S_k| = k$ for $k=1, 2, \ldots, n$. We shall provide a proof using an inductive argument.

First we show that there exists a player $i \in N$ with $x_i = c(\{i\})$. Clearly, $x_i \leq 1$ for all $i \in N$. If $x_i = 1$ for some $i \in N$, then apparently $i \notin Z$ and $x_i = 1 = c(\{i\})$. Therefore, suppose $x_i < 1$ for all $i \in N$. From the integrality of x it follows that $x_i \leq 0$ for all $i \in N$ and thus $x(N) \leq 0$. This can only be the case if $Z \neq \emptyset$ and $x_i = 0$ for all $i \in N$. Thus, for an arbitrary player $i \in Z$ we have $x_i = 0 = c(\{i\})$.

Let $S \neq N$ be a coalition such that $x(S) = c(S)$. We shall prove that there exists a player $i \in N \setminus S$ such that $x(S \cup \{i\}) = c(S \cup \{i\})$. Define for all $i \in N \setminus S$ the number q_i as the number of components of S that contain a player adjacent to player i and that contain no player in Z. Furthermore, define $d_i = 0$ if $i \in Z$ or if there exists a component of S that contains both a player in Z and a player adjacent to i. Otherwise $d_i = 1$.

It is not hard to see that $c(S \cup \{i\}) = c(S) + d_i - q_i$. Clearly, $x_i \leq d_i - q_i$ for all $i \in N \setminus S$. Suppose that $x_i < d_i - q_i$ for all $i \in N \setminus S$. It then follows from the integrality of x that $x_i \leq d_i - q_i - 1$ for all $i \in N \setminus S$. And thus, $x(N) \leq c(S) + d(N \setminus S) - q(N \setminus S) - |N \setminus S|$.

If $S \neq N$ then each component of S has at least one adjacent player in $N \setminus S$. (Here we use the fact that G is connected.) Thus, $q(N \setminus S) \geq c(S)$. Furthermore, notice that $d(N \setminus S) = |N \setminus S|$ if $Z = \emptyset$ and that $d(N \setminus S) < |N \setminus S|$ if $Z \neq \emptyset$.

It follows that $x(N) \leq c(S) + d(N \setminus S) - q(N \setminus S) - |N \setminus S| \leq 0$ if $Z = \emptyset$ and that $x(N) \leq c(S) + d(N \setminus S) - q(N \setminus S) - |N \setminus S| < 0$ if $Z \neq \emptyset$. In both cases we have $x(N) < c(N)$, a contradiction. We conclude that there is a player $i \in N \setminus S$ satisfying $x_i = d_i - q_i$, and consequently $x(S \cup \{i\}) = c(S \cup \{i\})$. □

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