International Journal of Game Theory (1993) 21: 329-338

The Bayesian Formulation of Incomplete Information – The Non-Compact Case

AVIAD HEIFETZ

School of Mathematical Sciences, Raymond and Beverly Sackler Faculty of Exact Sciences, Tel Aviv University, Tel Aviv, Israel

Abstract: In a game with incomplete information, a player may have beliefs about nature, about the other players' beliefs about nature, and so on, in an infinite hierarchy. We generalize a construction of Mertens & Zamir and show, that if nature is any Hausdorff space, and beliefs are regular Borel probability measures, then the space of all such infinite hierarchies of the players is a product of nature and the types of every player, where a type of a player is a belief about nature and the other players' types.

1 Introduction

In a game with incomplete information, describing the space of possible states is a complicated task: first of all, one has to describe the state of nature – the possible values of the various parameters of the game itself, such as the payoffs and the players' utility functions. Then, one has to describe the players' beliefs on the states of nature, which are, say, probability measures on the space of nature states. Afterwards, one has to describe the belief of every player on the beliefs of the other players on the states of nature, and so on. It turns out that in order to describe "the state of the world" in the game, one has to deal with an infinite hierarchy of beliefs for every one of the players. To simplify this state of affairs, Harsanyi [9] suggested describing the states of the world as a "types space" with the following properties:

1) A state of the world consists of the state of nature and the type of every player;
2) A type of a player is a joint probability measure on the states of nature and the types of the other players.

Thus, in every state of the world it is possible to compute the belief of every player on the states of nature, his belief on the other players' beliefs on the states of nature, and so on, as required.

Armbruster & Boge [2], Boge & Eisele [6] and more recently Mertens & Zamir [12] introduced a concrete way of constructing Harsanyi's types space. Assuming the nature states space is *compact,* they formally defined the hierarchies of beliefs, and showed that the space of all such hierarchies (which turns out to be compact) has the above desired properties. Brandenburger & Dekel [7] considered a different version

This work constitutes the main part of my M.Sc. dissertation, and I am grateful to my supervisor, Prof. Dov Samet, for his excellent guidance. I am also indebted to Prof. Shmuel Zamir and Prof. Adam Brandenburger for fruitful discussions.

0020-7276/93/4/329-338 $ 2.50

of the same construction, and showed that it could also be carried out assuming the underlying nature states space is *Polish* (metric, separable and topologically complete).

What happens in more general cases? Suppose, for instance, there is only one parameter in the description of the game upon which the players have uncertainties, and this parameter may have values in the [0,1] interval. Suppose further that before the game begins, the players witness together a random signal, which specifies for them a Borel subset S of [0,1] in which this parameter lies. Generally, S might be neither compact nor Polish, but still this set is the object of beliefs of the players in this case.

This is a motivation to consider the construction of the types space in more general settings. We shall show hereby that such a generalization is possible, *assuming only that the nature states space is Hausdorff, and the players' beliefs are regular Borel probability measures.*

The construction of the types space has also motivated further research. Lipman [11], Vassilakis [16] and Heifetz [10] have treated other hierarchic constructions in similar spirit.

In section 2 we shall present and develop the notions and properties of measures spaces we will be using. In section 3 we shall present the construction itself. We shall use a construction scheme which is close to the one in [13] and [15]. In section 4 we shall bring the proofs to the theorems and propositions of sections 2 and 3.

2 Topologies for Measures Spaces

1. Definition. A regular Borel probability measure on a topological space X is a measure μ s.t.:

a) μ is defined on the Borel field of X.

b) $\mu(X)=1$, μ is σ-additive.

c) for every Borel set $B\subseteq X$ and every $\varepsilon>0$ there is a compact $C\subseteq B$, s.t. $\mu(B\setminus C)<\varepsilon$.

2. Definition. For a topological space X, $\Delta(X)$ is the space of regular Borel probability measures on X, with the topology whose sub-base are the sets of the form

$$\mathcal{O}(\mu_0, V, \varepsilon)=\{\mu\in\Delta(X):\mu(V)>\mu_0(V)-\varepsilon\},$$

where $\mu_0\in\Delta(X)$, $V\subseteq X$ open and $\varepsilon>0$. An open neighborhood of μ_0 may be any union of finite intersections of such sets.

This topology was first introduced by Blau [4], and it is also mentioned in Billingsley [3, appendix III].

The following theorem is the main mathematical contribution of this paper.

3. Theorem. If X is Hausdorff then $\Delta(X)$ is also Hausdorff.

Clearly, if X is Hausdorff and non-empty, then $\Delta(X)$ is also non-empty, since it contains all the measures concentrated in a finite or countable number of points. For every $x \in X$ denote by δ_x the probability measure concentrated in x.

Define now another topology on $\Delta(X)$:

4. Definition. Let X be a topological space and $\mu_0 \in \Delta(X)$. A weak sub-basic neighborhood of μ_0 is a set of the form

$$\mathscr{W}(\mu, f, \varepsilon) = \{\mu \in \Delta(X) : |\int_X f\, d\mu - \int_X f\, d\mu_0| < \varepsilon\},$$

where f is a continuous bounded real-valued function on X and $\varepsilon > 0$. A weak neighborhood of μ_0 may be any union of finite intersections of such sets.

This topology is called *the weak topology* of $\Delta(X)$. Generally, it might be weaker than the topology in definition 2 above. However, we have the following theorem:

5. Theorem. If X is a compact Hausdorff space or a Polish space, then $\Delta(X)$ is the space of Borel probability measures on X with the weak topology.

From this theorem it will follow that the construction presented hereby is a generalization of the construction in the compact and the Polish cases.

3 Construction of the Universal Beliefs Space Ω

Let S – the space of nature states – be a non-empty Hausdorff space and I the set of players. For every $i \in I$ define inductively two sequences of spaces, $(\Omega_k^i)_{k=1}^{\infty}$ and $(T_k^i)_{k=1}^{\infty}$. Ω_k^i will be i's domain of uncertainty of level k. T_k^i will consist of k-tuples of coherent beliefs on $\Omega_1^i, \ldots, \Omega_k^i$:

$$\Omega_1^i = S \qquad\qquad T_1^i = \Delta(S)$$

and for all $k \geq 1$

$$\Omega_{k+1}^i = S \times \prod_{j \neq i} T_k^j \qquad T_{k+1}^i = \{((\mu_1^i, \ldots, \mu_k^i), \mu_{k+1}^i) \in T_k^i \times \Delta(\Omega_{k+1}^i) : \text{the marginal of } \mu_{k+1}^i \text{ on } \Omega_k^i \text{ is } \mu_k^i\}$$

The motivation for these definitions is the following:
i's domain of uncertainty of level $k+1$ is nature and the other players' beliefs up to level k (player i knows his own beliefs). A belief μ_{k+1}^i of level $k+1$ is a probability measure on this domain of uncertainty. In T_{k+1}^i μ_{k+1}^i appears along with the lower-level beliefs which are coherent with it.

$\forall \ell > k$ denote by $\psi_{\ell k}^i$ the projection from Ω_ℓ^i to Ω_k^i. Clearly, if $(\mu_1^i, \ldots, \mu_k^i, \ldots, \mu_\ell^i) \in T_\ell^i$ then the marginal of μ_ℓ^i on Ω_k^i is μ_k^i, and formally –

$$\mu_k^i = \mu_\ell^i (\psi_{\ell k}^i)^{-1}.$$

The following theorem is not a must for establishing the construction. It was proved by Mertens & Zamir [12] using a non-constructive argument (the Hahn-Banach theorem), and avoided by Brandenburger & Dekel [7]. However, it has an appealing aesthetic taste. It says that at all stages every lower-level belief can be extended to some higher-order belief (actually, many such extensions are possible).

6. Theorem. $\forall k \geq 1$ and $\forall i \in I$ *the projection of* T^i_{k+1} *on* T^i_k *is onto.*

Now, let

$$T^i = \lim_{\leftarrow} T^i_k$$

be the projective limit of the spaces $(T^i_k)^\infty_{k=1}$. This is the space of all the towers of beliefs $(\mu^i_1, \ldots, \mu^i_k, \ldots) \in \prod_{k=1}^{\infty} \Delta(\Omega^i_k)$, for which the beginning k-tuple $(\mu^i_1, \ldots, \mu^i_k)$ belongs to T^i_k $\forall k \geq 1$. Call T^i the set of types of player i.

Define also

$$\Omega^i = S \times \prod_{j \neq i} T^j.$$

Clearly, Ω^i is the space of all those $(s, (\mu^j_1, \ldots, \mu^j_k, \ldots)_{j \neq i})$ such that $(s, (\mu^j_1, \ldots, \mu^j_k)_{j \neq i}) \in \Omega^i_{k+1} = S \times \prod_{j \neq i} T^j_k$, $\forall k \geq 1$, and therefore $\Omega^i = \lim_{\leftarrow} \Omega^i_k$.

$\forall k \geq 1$ denote by $\varphi^i_{\infty,k}: T^i \to \Delta(\Omega^i_k)$ the coordinate projections, and by $\psi^i_{\infty,k}$ the projection from Ω^i to Ω^i_k.

Theorem 6 implies that T^i is not empty. However, many simple examples of types in T^i could be provided: for instance, the type who believes that a game with complete information takes place with s_0 as the prevailing nature state: $(\delta_{s_0}, \delta_{s_0 \times \prod_{j \neq i} \delta_{s_0}}, \ldots)$. This type is sure that the other players' beliefs are exact analogues of his beliefs, and that it is common knowledge that s_0 is the true state of nature.

Every tower of beliefs in T^i is an example of a mathematical object which we now define:

7. Definition. $((A_k)^\infty_{k=1}, (\nu_k)^\infty_{k=1}, (\rho_{\ell k})_{\ell > k})$ is a projective sequence of regular Borel probability measures if $\forall k \geq 1$ A_k is a Hausdorff space, ν_k is a regular Borel probability measure on A_k, and $\forall \ell > k$, $\rho_{\ell k}: A_\ell \to A_k$ is a continuous projection, s.t. $\forall \ell > m > k$ $\rho_{\ell k} = \rho_{mk} \rho_{\ell m}$ and $\nu_k = \nu_\ell \rho^{-1}_{\ell k}$.

It is clear from the construction that for every tower of beliefs $(\mu^i_k)^\infty_{k=1} \in T^i$ $((\Omega^i_k)^\infty_{k=1}, (\mu^i_k)^\infty_{k=1}, (\psi^i_{\ell k})_{\ell > k})$ is a projective sequence of regular Borel probability measures.

The following theorem is a generalization of the Kolmogoroff consistency theorem. It was proved even in a more general setup by Métivier [14, Theorem III.3.2.], who improved a result of Bochner [5, pp. 118–120].

8. Theorem. Let $((A_k)_{k=1}^{\infty}, (\nu_k)_{k=1}^{\infty}, (\rho_{\ell k})_{\ell>k})$ *be a projective sequence of regular Borel probability measures,* $A = \lim\limits_{\leftarrow} A_k$ *the projective limit of* $\{A_k\}_{k=1}^{\infty}$ *subject to the projections* $\{\rho_{\ell k}\}_{\ell>k}$ *and* $\rho_{\infty,k}: A \to A_k$ *the projection on the coordinate* A_k. *Then there exists a unique regular Borel probability measure* ν *on* A *s.t.* $\forall k \geq 1$, $\nu_k = \nu \rho_{\infty,k}^{-1}$.

It follows from the theorem that to every tower of beliefs $(\mu_k^i)_{k=1}^{\infty} \in T^i$ there corresponds a unique belief $\mu^i \in \Delta(\Omega^i)$ such that $\forall k \geq 1$ the marginal of μ^i on Ω_k^i is μ_k^i:

$$\mu_k^i = \mu^i (\psi_{\infty,k}^i)^{-1}.$$

This is how $\Omega^i = S \times \prod_{j \neq i} T^j$ becomes the terminal domain of uncertainty for player i. The reverse mapping assigns to each belief $\mu \in \Delta(\Omega^i)$ the tower of beliefs $(\mu(\psi_{\infty,k}^i)^{-1})_{k=1}^{\infty} \in T^i$. That is, there is a one to one and onto mapping between T^i and $\Delta(S \times \prod_{j \neq i} T^j)$. Moreover, this mapping is also a homeomorphism:

9. Theorem. T^i *is homeomorphic to* $\Delta(S \times \prod_{j \neq i} T^j)$.

Define now

$$\Omega = S \times \prod_{i \in I} T^i$$

to be the space of the states of the world. It turns out that:
1) A state of the world consists of the nature state and the type of every player.
2) A type of a player is a regular probability measure on the states of nature and the types of the other players.

- as required.

10. Remark. A counterexample of Andersen & Jessen [1] (see also [8, p. 214]) shows that theorem 8 might be wrong if the measures ν_k are not regular. Furthermore, Bochner and Métivier's proofs rely on the claim that the projective limit of a non-trivial projective sequence of compact sets is compact and non-empty. This claim is true under the assumption that the sets in the sequence are Hausdorff (for a proof see, for instance, [17, p. 257]). Therefore, it does not seem possible to generalize the construction to T_1 spaces, and certainly not to non-regular measures. It is quite surprising that theorem 8, which seems to be of measure theoretic nature, relies essentially on topological properties.

4 Proofs of the Theorems and Propositions

Proof of Theorem 3. Let $\mu_1 \neq \mu_2 \in \Delta(X)$. Then there exists an open $O \subseteq X$ s.t. $\mu_1(O) \neq \mu_2(O)$: otherwise μ_1 and μ_2 would be identical on the open sets, and hence on all the Borel field, since for every Borel $B \subseteq X$ and every $\mu \in \Delta(X)$

$$\mu(B) = \inf\{\mu(O) : B \subseteq O,\ O^c \text{ compact}\}$$

by regularity. With no loss of generality assume then that $\mu_2(O) - \mu_1(O) = \varepsilon > 0$.

Let $K \subseteq X$ be compact s.t. $\mu_1(K) > 1 - \frac{\varepsilon}{8}$, $\mu_2(K) > 1 - \frac{\varepsilon}{8}$ and $\mu_2(O \backslash K) < \frac{\varepsilon}{2}$. Then $K \cap O^c \neq \emptyset$, because otherwise we would have $K \subseteq O$, $\mu_1(O) > 1 - \frac{\varepsilon}{8}$, $\mu_2(O) > 1 - \frac{\varepsilon}{8}$ and hence $|\mu_2(O) - \mu_1(O)| < \frac{\varepsilon}{8}$, contradicting the supposition.

Denote $O^* = K \cap O$. Then

$$\mu_2(O^*) - \mu_1(O^*) \geq \mu_2(O) - \mu_2(O \backslash K) - \mu_1(O) > \varepsilon - \frac{\varepsilon}{2} = \frac{\varepsilon}{2}.$$

Let $C \subseteq O^*$ be compact s.t. $\mu_2(O^* \backslash C) < \frac{\varepsilon}{4}$.

K is a compact subspace of a Hausdorff space, and hence normal. C and $K \backslash O^*$ are closed in K, so by Urysohn's lemma there exists a continuous real-valued function f on K s.t. $K \backslash O^* \prec f \prec C$ ($0 \leq f \leq 1$, f is 0 on $K \backslash O^*$ and 1 on C).

Now, there is a $t \in [0,1)$ s.t. $\mu_1(\{x \in K : f(x) = t\}) = 0$. (There are at most countably many t-s in $[0,1)$ for which it is not so.)

Denote $A = \{x \in K : f(x) > t\}$. Then $C \subseteq A \subseteq O^*$, hence $\mu_2(O^* \backslash A) < \frac{\varepsilon}{4}$. Therefore,

$$\mu_2(A) - \mu_1(A) \geq \mu_2(O^*) - \mu_2(O^* \backslash A) - \mu_1(O^*) > \frac{\varepsilon}{2} - \frac{\varepsilon}{4} = \frac{\varepsilon}{4}.$$

A is open in K s.t. $\mu_1(\overline{A}) = \mu_1(A)$. Let $V \subseteq X$ be open s.t. $A = V \cap K$ and $W = (\overline{V})^c$. Then $\mu_1(K) = \mu_1(V \cap K) + \mu_1(W \cap K)$. Define

$$G_1 = \left\{\mu \in \Delta(X) : \mu(W) > \mu_1(W) - \frac{\varepsilon}{16}\right\}$$

$$G_2 = \left\{\mu \in \Delta(X) : \mu(V) > \mu_2(V) - \frac{\varepsilon}{16}\right\}$$

Then G_1 and G_2 are disjoint open neighborhoods of μ_1 and μ_2, respectively, because had there been $\mu \in G_1 \cap G_2$ we would have

$$1=\mu(X)\geq\mu(V)+\mu(W)>\mu_2(V)-\frac{\varepsilon}{16}+\mu_1(W)-\frac{\varepsilon}{16}\geq$$
$$\geq\mu_2(V\cap K)+\mu_1(W\cap K)-\frac{\varepsilon}{8}=$$
$$=[\mu_2(V\cap K)-\mu_1(V\cap K)]+[\mu_1(V\cap K)+\mu_1(W\cap K)]-\frac{\varepsilon}{8}>\frac{\varepsilon}{4}+\mu_1(K)-\frac{\varepsilon}{8}>$$
$$>\frac{\varepsilon}{4}+\left(1-\frac{\varepsilon}{8}\right)-\frac{\varepsilon}{8}=1,$$

a contradiction. ■

Proof of Theorem 5. If X is a compact Hausdorff space or a Polish space then X is a normal space and every Borel probability measure on X is regular. Thus Theorem 5 follows from the following theorem, which was proved in [4]:

Theorem 5. Let X be a topological space and $\mu_0\in\Delta(X)$. Then*
a) *every weak sub-basic neighborhood of μ_0 contains an open neighborhood of μ_0;*
b) *if X is normal, then every sub-basic neighborhood of μ_0 contains a weak neighborhood of μ_0.*
Particularly, if X is normal the 2 topologies coincide. ■

For proving Proposition 6 we shall need the following lemma:

Lemma 6. If $g:A\to B$ is continuous then $h:\Delta(A)\to\Delta(B)$ defined by $h(\mu)=\mu g^{-1}$ is well defined and continuous.*

Proof: To prove that h is well defined we have to show that $\forall\mu\in\Delta(A)$, μg^{-1} is a regular measure on B. And indeed, if $F\subseteq B$ is measurable (i.e. a Borel set) then $E=g^{-1}(F)$ is measurable in A. So $\forall\varepsilon>0$ there exists a compact $C\subseteq E$ such that $\mu(E\setminus C)<\varepsilon$. $K=g(C)$ is compact since g is continuous, $C\subseteq g^{-1}(K)$ and hence

$$\mu g^{-1}(F\setminus K)=\mu(g^{-1}(F)\setminus g^{-1}(K))\leq\mu(E\setminus C)<\varepsilon,$$

as required. Now, to prove that h is continuous it suffices to show that the inverse image of every sub-basic open set in $\Delta(B)$ is open in $\Delta(A)$. And indeed, if

$$\mathcal{O}=\{v\in\Delta(B):v(V)>v_0(V)-\varepsilon\},$$

$v_0\in\Delta(B)$, $V\subseteq B$ open and $\varepsilon>0$, then

$$h^{-1}(\mathcal{O})=\{\mu\in\Delta(A):\mu(g^{-1}(V))>v_0(V)-\varepsilon\},$$

which is open: if $g^{-1}(V)$ is empty, then $h^{-1}(\mathcal{O})$ is either $\Delta(A)$ or empty, according to whether $v_0(V)-\varepsilon<0$ or not; and if $g^{-1}(V)\neq\emptyset$ then we can write $v_0(V)-\varepsilon=\delta_a(g^{-1}(V))-\varepsilon'$, where $a\in g^{-1}(V)$ and $\varepsilon'=1-v_0(V)+\varepsilon>0$, so that

$$h^{-1}(\mathcal{O})=\{\mu\in\Delta(A):\mu(g^{-1}(V))>\delta_a(g^{-1}(V))-\varepsilon'\}$$

- a (sub-basic) open set in $\Delta(A)$.

Proof of Theorem 6. For every $(\mu_1^i, \ldots, \mu_k^i) \in T_k^i$ we have to find a $\mu_{k+1}^i \in \Delta(\Omega_{k+1}^i)$ such that $(\mu_1^i, \ldots, \mu_k^i, \mu_{k+1}^i) \in T_{k+1}^i$. We shall actually prove the existence of a continuous

$$f_k^i : \Delta(\Omega_k^i) \to \Delta(\Omega_{k+1}^i)$$

such that $(\mu_1^i, \ldots, \mu_k^i, f_k^i(\mu_k^i)) \in T_{k+1}^i$ (or, in other words, the marginal of $f_k^i(\mu_k^i)$ on Ω_k^i is μ_k^i).

This will be accomplished by showing the existence of a continuous

$$F_k^i : \Omega_k^i \to \Omega_{k+1}^i$$

such that $\psi_{k+1,k}^i F_k^i : \Omega_k^i \to \Omega_k^i$ is the identity on Ω_k^i.

Start by choosing an arbitrary $s_0 \in S$ and define $F_1^i : \Omega_1^i \to \Omega_2^i$ by

$$F_1^i(s) = (s, (\delta_{s_0})_{j \neq i})$$

$\forall s \in S$. F_1^i is clearly continuous, because each of its components is continuous (the identity or a constant function), and $\psi_{2,1}^i(F_1^i(s)) = s\ \forall s \in S$.

Suppose now, by induction, that we have already defined a continuous $F_k^i : \Omega_k^i \to \Omega_{k+1}^i$ such that $\psi_{k+1,k}^i F_k^i : \Omega_k^i \to \Omega_k^i$ is the identity on Ω_k^i. Define $f_i^k : \Delta(\Omega_k^i) \to \Delta(\Omega_{k+1}^i)$ by

$$f_k^i(\mu_k^i) = \mu_k^i (F_k^i)^{-1}.$$

($f_k^i(\mu_k^i)$ will be a distribution on Ω_{k+1}^i according to which i thinks that all the others think that all the others think ... (k times) that s_0 occurred.) By Lemma 6* f_k^i is well defined and continuous. Since $\psi_{k+1,k}^i F_k^i$ is the identity on Ω_k^i,

$$f_k^i(\mu_k^i)(\psi_{k+1,k}^i)^{-1} = \mu_k^i (F_k^i)^{-1} (\psi_{k+1,k}^i)^{-1} = \mu_k^i (\psi_{k+1,k}^i F_k^i)^{-1} = \mu_k^i$$

which means that the marginal of $f_k^i(\mu_k^i)$ on Ω_k^i is μ_k^i.

To finish the inductive definition, define $F_{k+1}^i : \Omega_{k+1}^i \to \Omega_{k+2}^i$ by

$$F_{k+1}^i(s, (\mu_1^j, \ldots, \mu_k^j)_{j \neq i}) = (s, (\mu_1^j, \ldots, \mu_k^j, f_k^j(\mu_k^j))_{j \neq i}).$$

F_{k+1}^i is continuous since the f_k^j-s are continuous, and clearly $\psi_{k+2,k+1}^i F_{k+1}^i$ is the identity on Ω_{k+1}^i. ■

Proof of Theorem 9. The topology on T^i is its relative topology as a subset of the product $\prod_{k=1}^{\infty} \Delta(\Omega_k^i)$. The topology on $\Delta(S \times \prod_{j \neq i} T^j) = \Delta(\Omega^i)$ is the topology of a space of regular probability measures of definition 2. We shall show that the two topologies coincide: let $\mu^i \in \Delta(\Omega^i)$ correspond to $(\mu_k^i)_{k=1}^{\infty} \in T^i$ by theorem 8. A sub-basic neighborhood of $(\mu_k^i)_{k=1}^{\infty}$ is a set of the form $(\varphi_{\infty,k}^i)^{-1}(\mathcal{O}_k)$,

$$\mathcal{O}_k = \{\mu_k \in \Delta(\Omega_k^i) : \mu_k(V_k) > \mu_k^i(V_k) - \varepsilon\},$$

$V_k \subseteq \Omega_k^i$ open and $\varepsilon > 0$. The corresponding set to $(\varphi_{\infty,k}^i)^{-1}(\mathcal{O}_k)$ in $\Delta(\Omega^i)$ is

$$\{\mu \in \Delta(\Omega^i) : \mu((\psi_{\infty,k}^i)^{-1}(V_k)) > \mu^i((\psi_{\infty,k}^i)^{-1}(V_k)) - \varepsilon\},$$

which is a (sub-basic) open neighborhood of μ^i in $\Delta(\Omega^i)$, since $(\psi_{\infty,k}^i)^{-1}(V_k)$ is open in Ω^i.

Conversely, a sub-basic neighborhood of $\mu_0 \in \Delta(\Omega^i)$ is a set of the form

$$\mathcal{O} = \{\mu \in \Delta(\Omega^i) : \mu(G) > \mu_0(G) - \varepsilon\},$$

$G \subseteq \Omega^i$ open and $\varepsilon > 0$. G may be any union of finite intersections of open cylinders $G_k^* = (\psi_{\infty,k}^i)^{-1}(G_k)$, whose base $G_k \subseteq \Omega_k^i$ is open. Since $\forall \ell > k$ $G_k^* = (\psi_{\infty,\ell}^i)^{-1}((\psi_{\ell k}^i)^{-1}(G_k))$, G_k^* is also a cylinder with an open base $(\psi_{\ell k}^i)^{-1}(G_k)$ in Ω_ℓ^i. Hence a finite intersection of open cylinders is an open cylinder (just intersect the open bases in Ω_ℓ^i with ℓ large enough). So let $G = \bigcup_\alpha G_\alpha$, where the G_α are open cylinders. Define

$$V_k = \bigcup \{\psi_{\infty,k}^i(G_\alpha) : \text{the base of } G_\alpha \text{ is in } \Omega_k^i\},$$

and $V_k^* = (\psi_{\infty,k}^i)^{-1}(V_k)$. Then $V_k \subseteq \Omega_k^i$ is open as a union of open bases, and $\{V_k^*\}_{k=1}^{\infty}$ is an increasing sequence s.t. $G = \bigcup_{k=0}^{\infty} V_k^*$. Hence $\exists k$ s.t. $\mu_0(G \setminus V_k^*) < \frac{\varepsilon}{2}$. The set

$$W_k = \left\{\mu \in \Delta(\Omega^i) : \mu(V_k^*) > \mu_0(V_k^*) - \frac{\varepsilon}{2}\right\}$$

corresponds to

$$(\varphi_{\infty,k}^i)^{-1}\left(\left\{\mu_k \in \Delta(\Omega_k^i) : \mu_k(V_k) > (\mu_0(\psi_{\infty,k}^i)^{-1})(V_k) - \frac{\varepsilon}{2}\right\}\right),$$

which is an open neighborhood of the tower of marginal beliefs $(\mu_0(\psi_{\infty,k}^i)^{-1})_{k=1}^{\infty}$ in the topology of T^i, and W_k is contained in $\mathcal{O}$, since $\forall \mu \in W_k$

$$\mu_0(G) - \mu(G) = [\mu_0(G) - \mu_0(V_k^*)] + [\mu_0(V_k^*) - \mu(V_k^*)] + [\mu(V_k^*) - \mu(G)] <$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} + 0 = \varepsilon. \qquad \blacksquare$$

References

[1] Andersen ES & Jessen B, On the Introduction of Measures in Infinite Product Sets, Danske Vid. Selsk. Mat-Fys. Medd 25 (1948), 8 pp.

[2] Armbruster W & Boge W, Bayesian Game Theory, in O Moeschlin & D Pallaschke (eds.), Game Theory and Related Topics, Amsterdam: North Holland (1979).

[3] Billingsley P, Convergence of Probability Measures, John Wiley & Sons (1968).

[4] Blau JH, The space of Measures on a Given Set, Fund. Math. 38 (1951), pp. 23–34.

[5] Bochner S, Harmonic Analysis and the Theory of Probability, University of California Press (1960).

[6] Boge W & Eisele T, On Solutions of Bayesian Games, Int. Jour. Game Theory 8 (1979), pp. 193–215.

[7] Brandenburger A & Dekel E, Hierarchies of Beliefs and Common Knowledge, Working Paper, Harvard Business School (1990).

[8] Halmos P, Measure Theory, Springer-Verlag (1950).

[9] Harsanyi JC, Games with Incomplete Information Played by Bayesian Players, parts I, II, & III, Man. Sc. 14 (1967–68), pp. 159–182, 320–334, 486–502.

[10] Heifetz A, Hierarchic Constructions of Incomplete Information Models, preprint, School of Mathematical Sciences, Tel Aviv University (1990).

[11] Lipman BL, How to Decide How to Decide How to ...: Modeling limited Rationality, Carnegie Mellon University (1988).

[12] Mertens JF & Zamir S, Formulation of Bayesian Analysis for Games with Incomplete Information, Int. Jour. Game Theory 14 (1985), pp. 1–29.

[13] Mertens JF, Sorin S & Zamir S, Repeated Games, preprint.

[14] Métivier M, Limites Projectives de Mesures, Martingales, Applications, Ann. Mat. Pura Appl. (4) 63 (1963), pp. 225-352.

[15] Myerson RB, Bayesian Equilibrium and Incentive-Compatibility: an Introduction, in L Hurwicz, D Shmeidler & H Sonnenschein (eds.), Social Goals and Social Organization, Essays in Memory of Elisha A. Pazner, Cambridge University Press (1985).

[16] Vassilakis S, Economic Data Types, Working Paper no. 252, Econ. Dept., University of Pittsburgh (1989).

[17] Willard S, General Topology, Addison Wesley (1970).

Received October 1990
Revised version December 1991