International Journal of Game Theory (1993) 22: 335–344

# A Conjecture of Shapley and Shubik on Competitive Outcomes in the Cores of NTU Market Games*

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*Abstract:* It is shown that for every NTU *market game,* there is a *market* that *represents* the game whose *competitive payoff* vectors completely fill up the *inner core* of the game. It is also shown that for every NTU market game and for any point in its inner core, there is a market that represents the game and further has the given inner core point as its *unique* competitive payoff vector. These results prove a conjecture of Shapley and Shubik.
*Journal of Economic Literature* Classification Numbers: C71, D51.

## 1 Introduction

The possibility of representing games by markets was studied by Shapley and Shubik (1969). They proved that a TU game is representable by a market if and only if the game is *totally balanced.* The markets they considered were *pure exchange economies* with money in which agents have continuous concave monetary utility functions. For markets with *production* but without money in which agents have continuous concave utility functions, Billera (1974) proved that an NTU game is representable by a market if and only if the game is totally balanced and *compactly convexly* generated.[1]

Games that are representable by markets have come to be called market games. Since a market game may be represented by multiple markets, a question that can be naturally raised is how to compare the cores of market games with the competitive payoff vectors of markets that represent them. This question motivated the work of Shapley and Shubik (1975). They proved that for any TU market game there exists a market that represents the game and whose competitive payoff vectors completely fill up the core. They also proved that for any TU market game and given any point in the core, there exists a market that represents the game and has the given core point as its unique competitive payoff vector. The latter result is in a spirit somewhat similar to that of the second welfare theorem in general equilibrium analysis. For NTU market games a conjecture by them states that the same results hold with respect to the inner core. For discussions about implications of these results, the reader is referred to Shubik (1984, Ch. 11).

* I wish to thank Lloyd Shapley for his guidance. I also thank Rod Garratt and two anonymous referees for helpful comments. Research support is provided by the Academic Senate of University of California at Santa Barbara.

[1] For results on the possibility of representing NTU games by pure exchange economies without money, the reader is referred to Billera and Bixby (1973) and Mas-Colell (1975).

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The purpose of this paper is to provide a proof of the above conjecture. Although markets with production are used in the present paper, by applying an idea due to Rader (1972), one can transform such markets into ones without production. Therefore, the results still hold when restricted to markets without production.

## 2 Games

Let $N$ be a finite set with $n$ members, and let $\mathfrak{R}^N$ be the $n$-dimensional Euclidean space of vectors $u$ with coordinates $u_i$ indexed by $i \in N$. For each $S \subseteq N$, let $\mathfrak{R}^S$ denote the subspace of $\mathfrak{R}^N$ defined by $\mathfrak{R}^S = \{u \in \mathfrak{R}^N \mid u_i = 0,\ i \in N \backslash S\}$, let $\mathfrak{R}^S_+$ denote the nonnegative orthant of $\mathfrak{R}^S$, and let $e^S$ denote the vector in $\mathfrak{R}^S$ defined by $e^S_i = 1$ for $i \in S$. The symbol "$\cdot$" denotes the ordinary inner product. Given any two vectors $x$ and $y$ in $\mathfrak{R}^N$, $x \times y$ denotes the vector in $\mathfrak{R}^N$ whose $i$th coordinate is $x_i y_i$, for $i \in N$. Let $m$ be any finite positive integer and let $\{A^j\}^m_{j=1}$ be any family of subsets of $\mathfrak{R}^N$. Elements $x$ in the Cartesian product $\times^m_{j=1} A^j$ are sometimes denoted by $x = (x^{(1)}, x^{(2)}, \ldots, x^{(m)})$ with $x^{(j)} \in A^j$, $j = 1, \ldots, m$. A subset $X$ of $\mathfrak{R}^S$ is $S$-comprehensive if $X = X - \mathfrak{R}^S_+$ (algebraic subtraction). For $u \in \mathfrak{R}^N$, $u^S$ denotes the projection of $u$ to $\mathfrak{R}^S$.

An NTU game (or simply a game) $\mathscr{G}$ is a pair $(N, V)$, where $V$ is a function from subsets of $N$ to nonempty subsets of $\mathfrak{R}^N$ such that for each $S \subseteq N$, $V(S)$ is a nonempty subset of $\mathfrak{R}^S$ that is $S$-comprehensive. A game $\mathscr{G}$ is compactly (convexly) generated if for every $S \subseteq N$, there exists a *compact* (*convex*) subset $C_S$ of $\mathfrak{R}^S$ such that $V(S) = C_S - \mathfrak{R}^S_+$.

Let $\mathscr{G}$ be a compactly generated game and let $\lambda \in \mathfrak{R}^N_+$. Define a real-valued set function $v_\lambda : 2^N \to \mathfrak{R}$ by

$$v_\lambda(S) = \max\{\lambda \cdot u \mid u \in V(S)\}, \tag{1}$$

and a new game $\mathscr{G}_\lambda = (N, V_\lambda)$ by

$$V_\lambda(S) = \{u \in \mathfrak{R}^S \mid \lambda \cdot u \leq v_\lambda(S)\}. \tag{2}$$

$\mathscr{G}_\lambda$ is called the $\lambda$-*transfer* game of $\mathscr{G}$. Since $V_\lambda(S)$ is homogeneous of degree 0 in $\lambda$, i.e., depends only on the ratios of $\lambda_i$, only those $\lambda$ that are elements of the set $\Delta = \{\lambda \in \mathfrak{R}^N_+ \mid \lambda \cdot e^N = 1\}$ are considered.

For each $\beta \in \mathfrak{R}^N_+$, let $\Gamma(\beta)$ denote the set of all nonnegative vectors $\gamma = (\gamma_S \mid S \subseteq N)$ that satisfy

$$\sum_{S \subseteq N} \gamma_S e^S = \beta. \tag{3}$$

A game $\mathscr{G}=(N, V)$ is *balanced* if the condition

$$\sum_{S\subseteq N} \gamma_S V(S)\subseteq V(N) \tag{4}$$

holds for any $\gamma\in\Gamma(e^N)$; it is totally balanced if for each $T\subseteq N$, the condition

$$\sum_{S\subseteq T} \gamma_S V(S)\subseteq V(T) \tag{5}$$

holds for any $\gamma\in\Gamma(e^T)$.

The core of a game $\mathscr{G}=(N, V)$ is the set of $u\in\Re^N$ such that (i) $u\in V(N)$ and (ii) for every $S\subseteq N$, there does not exist any $u'\in V(S)$ such that $u'_i>u_i$, for all $i\in S$. The inner core of a compactly generated game $\mathscr{G}=(N, V)$ is the set of $u\in\Re^N$ such that $u\in V(N)$ and $u$ is in the core of $\mathscr{G}_\lambda$ for some $\lambda\in\Delta$. Such a $\lambda$ is called a vector of *supporting weights* for $u$.

*Remark 1:* Let $\lambda\in\Delta$ be such that $\lambda_i=0$ for some $i\in N$. Then by (1), $v_\lambda(\{i\})=0$, and by (2), $V_\lambda(\{i\})=\Re^{\{i\}}$. Thus, the coalition consisting of player $i$ alone would then be able to improve upon any utility vector in the $\lambda$-transfer game $\mathscr{G}_\lambda$. Consequently, the core of $\mathscr{G}_\lambda$ is empty. Therefore, the vectors of supporting weights for a utility vector in the inner core must all be strictly positive.

*Remark 2:* By Scarf Theorem (1967), a totally balanced compactly convexly generated game always has a nonempty core. However, the inner cores of such games are not necessarily nonempty (see Example 1 of Qin (1993)).

## 3 Markets

Let $l$ denote a finite positive integer. For each $i\in N$, let $X^i$ be a nonempty closed convex subset of $\Re^l_+$, $Y^i$ a nonempty closed convex subset of $\Re^l$ such that $Y^i\cap\Re^l_+=\{0\}$, $a^i$ an element of $X^i-Y^i$, and $u^i$ a continuous concave function from $X^i$ to the reals. The collection $\mathscr{E}=\{X^i, Y^i, a^i, u^i\}_{i\in N}$ is called a market (see Shapley (1973) and Billera (1974)). When $X^i=X$ and $Y^i=Y$ for all $i\in N$, write $\mathscr{E}=\{X, Y, \{a^i, u^i\}_{i\in N}\}$. Here, $l$ is the number of commodities and for each $i\in N$, $X^i$ and $Y^i$ are, respectively, the *consumption* and *production* sets of $i$, while $a^i$ and $u^i$ are, respectively, his *initial endowment* and *utility function.* The assumption $a^i\in X^i-Y^i$ implies that the initial endowment $a^i$ may not be in $i$'s consumption set, but that there exists at least one production activity by which he can select an element $y^i$ in $Y^i$ so that $a^i+y^i\in X^i$. Given a market $\mathscr{E}$ and given $S\subseteq N$, an *S-allocation* is a $S$-tuple $(x^i)_{i\in S}$ such that $x^i\in X^i$ for each $i\in S$; it is *feasible* if

$$\sum_{i\in S}(x^i-a^i)\in\sum_{i\in S} Y^i.$$

Denote by $F(S)$ the set of all feasible $S$-allocations. A market $\mathscr{E}$ generates a game, denoted by $\mathscr{G}(\mathscr{E})=(N, V(\mathscr{E}))$, in a natural way (see Scarf (1967)): simply define $V(\mathscr{E})(S)$ to be

$$V(\mathscr{E})(S)=\{u\in\Re^S \mid \exists (x^i)_{i\in S}\in F(S),\ u_i\leq u^i(x^i),\ \forall i\in S\}. \tag{6}$$

Given a game $\mathscr{G}$, a market $\mathscr{E}$ such that $\mathscr{G}(\mathscr{E})=\mathscr{G}$ is said to represent the game $\mathscr{G}$. A game that is representable by a market is called a market game.

A *competitive equilibrium* for $\mathscr{E}$ is a triple $((\hat{x}^i)_{i\in N}, (\hat{y}^i)_{i\in N}, \hat{p})$ such that (i) $\hat{p}$, a competitive equilibrium price vector, is a vector in $\Re^l_+$; (ii) $\sum_{i\in N}\hat{x}_i=\sum_{i\in N}(\hat{y}^i+a^i)$; (iii) for each $i\in N$, $\hat{y}^i$ solves $\max_{y^i\in Y^i}\hat{p}\cdot y^i$; and (iv) for each $i\in N$, $\hat{x}^i$ is maximal with respect to $u^i$ in the *budget set* $\{x^i\in X^i \mid \hat{p}\cdot x^i\leq\hat{p}\cdot a^i+p\cdot\hat{y}^i\}$. With each such triple is associated a vector $\hat{u}$ of competitive payoffs, $\hat{u}=(u^i(\hat{x}^i))_{i\in N}$.

## 4 Results

Let $\mathscr{G}$ be a market game. By Theorem 2.3 and Proposition 2.1 of Billera (1974), $\mathscr{G}$ is totally balanced compactly convexly generated. Thus, for any $S\subseteq N$, $V(S)=C_S-\Re^S_+$ for some nonempty compact convex subset $C_S$ of $\Re^S$. By Proposition 2.2 of Billera and Bixby (1973), we may assume, without loss of generality, that for any $S\subseteq N$,

$$C_S\subset\Re^S_+ \text{ and } C_S\cap\Re^S_{++}\neq\emptyset. \tag{7}$$

The *induced market* by $\mathscr{G}=(N, V)$ is the market, $\mathscr{E}(\mathscr{G})=\{X, Y, \{a^i, u^i\}_{i\in N}\}$, where

$$X=\Re^N_+\times\{0\}, \tag{8}$$

$$Y=convexcone\left[\bigcup_{S\subseteq N}(C_S\times\{-e^S\})\right], \tag{9}$$

and for each $i\in N$,

$$a^i=(0, e^{\{i\}}), \tag{10}$$

$$u^i(x)=x^{(1)}_i,\ \forall x\in\Re^N_+\times\{0\}. \tag{11}$$

Here, the convex cone of a set is the set of all nonnegative linear combinations of elements of the set, and "0" denotes the origin in $\Re^N$.

*Remark 3:* Since $Y$ is a cone, $\sum_{i\in S}Y=Y$ for any $S\subseteq N$, and $\max_{y\in Y}\hat{p}\cdot y=0$ for any competitive equilibrium price vector $\hat{p}$.

*Lemma 1:* $\mathscr{G} = \mathscr{G}(\mathscr{E}(\mathscr{G}))$.

*Proof:* See Billera[2] (1974, Theorem 3.3).

*Theorem 1: The inner core of a market game $\mathscr{G}$ coincides with the set of competitive payoff vectors of the induced market by $\mathscr{G}$.*

*Proof:* Let $u^*$ be in the inner core of $\mathscr{G}$ and let $\lambda^*$ be a vector of supporting weights for $u^*$. Let $\hat{p} \in \Re^N_+ \times \Re^N_+$ be such that $\hat{p}^{(1)} = \lambda^*$ and $\hat{p}^{(2)} = \lambda^* \times u^*$. Then, for any $S \subseteq N$ and for any $u \in C_S$, $\hat{p} \cdot (u, -e^S) \leq 0$, and hence $\max_{y \in Y} \hat{p} \cdot y = 0$. Let $\hat{x}^i = (u^* \times e^{\{i\}}, 0)$ and $\hat{y}^i = \dfrac{1}{n}(u^*, -e^N)$, $\forall i \in N$. Then, by (9) and (11),

$$\max \{u^i(x^i) \mid x^i \in X^i, \hat{p} \cdot x^i \leq \lambda_i^* u_i^* + \hat{p} \cdot \hat{y}^i\}$$
$$= \max \{x_i^{(1)i} \mid x^i \in X^i, \hat{p} \cdot x^i \leq \lambda_i^* u_i^*\} = u_i^*.$$

Thus, the triple $((\hat{x}^i)_{i \in N}, (\hat{y}^i)_{i \in N}, \hat{p})$ is a competitive equilibrium for the induced market $\mathscr{E}(\mathscr{G})$ and $u^* = (u^i(\hat{x}^i))_{i \in N}$. This shows that the inner core is contained in the set of competitive payoff vectors.

Let the triple $((\hat{x}^i)_{i \in N}, (\hat{y}^i)_{i \in N}, \hat{p})$ be any competitive equilibrium for $\mathscr{E}(\mathscr{G})$. By (11), for each $i \in N$, $u^i(x) > u^i(y)$ whenever $x_i^{(1)} > y_i^{(1)}$. Thus, $\hat{p}^{(1)} \gg 0$. Define $\hat{\lambda} \in \Delta$ by $\hat{\lambda}_i = \dfrac{\hat{p}_i^{(1)}}{\hat{p}^{(1)} \cdot e^N}$, $\forall i \in N$. To prove that the competitive payoff vector $(u^i(\hat{x}^i))_{i \in N}$ is in the inner core, it suffices to prove that it is in the core of $\mathscr{G}_{\hat{\lambda}}$. Suppose not. Then by Lemma 1 and (6), there exists a nonempty coalition $S$ for which there exists an $S$-allocation $(\bar{x}^i)_{i \in S}$ in $F(S)$ such that

$$\sum_{i \in S} \hat{\lambda}_i u^i(\bar{x}^i) > \sum_{i \in S} \hat{\lambda}_i u^i(\hat{x}^i). \tag{12}$$

Together, (11) and (12) imply $\sum_{i \in S} \hat{\lambda}_i \bar{x}_i^{(1)i} > \sum_{i \in S} \hat{\lambda}_i \hat{x}_i^{(1)i}$, and hence

$$\sum_{i \in S} \hat{p} \cdot \bar{x}^i > \sum_{i \in S} \hat{p}_i^{(1)} \hat{x}_i^{(1)i}. \tag{13}$$

Since $\max_{y \in Y} \hat{p} \cdot y = 0$, $\hat{p}_i^{(1)} \hat{x}_i^{(1)i} = \hat{p}^{(2)} \cdot e^{\{i\}}$, $\forall i \in S$. Therefore, $\sum_{i \in S} \hat{p}_i^{(1)} \hat{x}_i^{(1)i} = \hat{p}^{(2)} \cdot e^S$, and by (13),

$$\sum_{i \in S} \hat{p} \cdot \bar{x}^i > \hat{p}^{(2)} \cdot e^S. \tag{14}$$

[2] In Billera (1974), the common production set of the induced market by game $\mathscr{G}$ is the convex hull of $\bigcup_{S \subseteq N} (C_S \times \{-e^S\})$. However, as he remarked (Billera (1974, pp. 136)), his Theorem 3.3 still holds when the convex hull is replaced by the convex cone of the set.

Since $(\bar{x}^i)_{i\in S}\in F(S)$, there exists $\bar{y}\in Y$ such that

$$\sum_{i\in S}\bar{x}^i=(0,\,e^S)+\bar{y}. \tag{15}$$

Together, (14) and (15) imply that $\hat{p}\cdot\bar{y}>0$, which is a contradiction. Thus, the set of competitive payoff vectors is contained in the inner core. Q.E.D.

For the rest of this section, let $u^*$ be any given point in the inner core of $\mathcal{G}$ and let $\lambda^*$ be any vector of supporting weights for $u^*$. By Remark 1, $\lambda^*\gg 0$. By (7), $u^*\gg 0$. Choose $t$ in the open interval (0, 1) and for each $i\in N$, define $f_i:\Re\to\Re$ by

$$\forall u_i\in\Re,\ f_i(u_i)=\begin{cases}u_i, & \text{if } u_i\le u_i^*;\\ tu_i+(1-t)\,u_i^*, & \text{if } u_i>u_i^*.\end{cases}$$

Clearly, $f_i$ is continuous, concave, onto, and strictly monotonically increasing. Let $f=(f_i)_{i\in N}$.

*Remark 4:* $f(0)=0$, $f(u)\le u$, $\forall u\in\Re^N$, and $f(u)=u^*$ if and only if[3] $u=u^*$.

*Remark 5:* Since $f$ is continuous concave monotonically increasing, $\tilde{\mathcal{G}}$ is also a market game.

Let $\tilde{\mathcal{G}}=(N,\,\tilde{V})$ denote the game where $\tilde{V}$ is given by

$$\tilde{V}(S)=\{u\in\Re^S\,|\,\exists u'\in V(S),\ u\le f^S(u')\}. \tag{16}$$

For every $S\subseteq N$, let $\tilde{C}_S=\tilde{V}(S)\cap\Re^S_+$. Then, $\tilde{C}_S$ is nonempty compact convex and for any $S\subseteq N$, $\tilde{V}(S)=\tilde{C}_S-\Re^S_+$. By (7) and (16), $\tilde{C}_S\cap\Re^S_{++}\neq\emptyset$.

For each $S\subseteq N$, let $A^1_S$, $A^2_S$, $A^3_S$ be subsets of $\times_{j=1}^5\Re^N_+$ defined, respectively, by

$$A^1_S=\{(u_S,\ -e^S,\ -e^S,\ -e^S,\ 0)\,|\,u_S\in\tilde{C}_S\},$$
$$A^2_S=\{(u_S,\ 0,\ -e^S,\ 0,\ -e^S)\,|\,u_S\in\tilde{C}_S\},\ \text{and}$$
$$A^3_S=\{(u_S,\ 0,\ 0,\ -e^S,\ -e^S)\,|\,u_S\in\tilde{C}_S\}.$$

Let $\tilde{\mathcal{E}}=\{\tilde{X},\ \tilde{Y},\ \{\tilde{a}^i,\ \tilde{u}^i\}_{i\in N}\}$ be the market where

$$\tilde{X}=\Re^N_+\times\{(0,\,0,\,0)\}\times\Re^N_+, \tag{17}$$

$$\tilde{Y}=convexcone\left[\bigcup_{S\subseteq N}(A^1_S\cup A^2_S\cup A^3_S)\right], \tag{18}$$

$$\tilde{a}^i=(0,\ e^{\{i\}},\ e^{\{i\}},\ e^{\{i\}},\ e^{\{i\}}),\ i\in N, \tag{19}$$

$$\tilde{u}^i(x^i)=\min\left\{x_i^{(1)i},\ \frac{\lambda^*\times u^*\cdot x^{(5)i}}{\lambda_i^*}\right\}. \tag{20}$$

[3] For any $u\in\Re^N$, $f(u)=(f_i(u_i))_{i\in N}$ and for any $S\subseteq N$, $f^S(u)$ denotes the projection of $f(u)$ to $\Re^S$.

Since each $A_S^k$ is convex, $y \in \tilde{Y}$ if and only if for every $S \subseteq N$, there exist $\gamma_S^1$, $\gamma_S^2$, $\gamma_S^3 \geq 0$ and $u_S^1$, $u_S^2$, $u_S^3 \in \tilde{C}_S$ such that

$$y = \sum_{S \subseteq N} \left( \sum_{k=1}^{3} \gamma_S^k u_S^k,\ -\gamma_S^1 e^S,\ -(\gamma_S^1+\gamma_S^2)e^S,\ -(\gamma_S^1+\gamma_S^3)e^S,\ -(\gamma_S^2+\gamma_S^3)e^S \right). \quad (21)$$

Define $\pi: \times_{j=1}^{5} \Re_{++}^N \to \Re$ by

$$\pi(p) = \sup_{y \in \tilde{Y}} p \cdot y.$$

*Lemma 2: $u^*$ is in the inner core of $\tilde{\mathscr{G}}$ and $\lambda^* \cdot u < \lambda^* \cdot u^*$ whenever $u \in \tilde{V}(N)$ is such that $u \neq u^*$.*

*Proof:* Let $S$ be any nonempty coalition. For each $u \in \tilde{V}(S)$, there exists $u' \in V(S)$ such that $u \leq f^S(u')$. Since $f^S(u') \leq u'^S$, $\max_{u \in \tilde{V}(S)} \lambda^* \cdot u \leq v_{\lambda^*}(S) \leq \lambda^* \cdot u^{*S}$. This shows that $u^*$ is in the inner core of $\tilde{\mathscr{G}}$. For any $u \in \tilde{V}(N)$ with $u \neq u^*$, there exists $u' \in V(N)$ such that $u \leq f(u')$. Set $T(u') = \{i \in N \mid u'_i > u_i^*\}$. If $u' \leq u^*$, then $u < u^*$. Thus, $\lambda^* \cdot u < \lambda^* \cdot u^*$. If, however, $u' \nleq u^*$, then $T(u') \neq \emptyset$. Thus, $\lambda^* \cdot u \leq \lambda^* \cdot u' - (1-t) \sum_{i \in T(u')} \lambda_i^* (u'_i - u_i^*) < \lambda^* \cdot u^*$. Q.E.D.

*Lemma 3: $\tilde{\mathscr{E}}$ represents $\tilde{\mathscr{G}}$.*

*Proof:* By Lemma 2, for any $u$ in $\tilde{C}_S$, $\lambda^* \cdot u \leq \lambda^* \cdot u^{*S}$. Thus, there exists $\theta \in \Re_+^S$ such that $\sum_{i \in S} \theta_i = 1$ and $u_i \leq \frac{\theta_i \lambda^* \cdot u^{*S}}{\lambda_i^*}$, $\forall i \in S$. Let $x^i = (u \times e^{\{i\}}, 0, 0, 0, \theta_i e^S)$, $\forall i \in S$, and $y = (u, -e^S, -e^S, -e^S, 0)$. Then, $\forall i \in S$, $x^i \in \tilde{X}$, $y \in \tilde{Y}$, and $\sum_{i \in S} x^i = \sum_{i \in S} \tilde{a}^i + y$. Thus, $(x^i)_{i \in S} \in F(S)$, the set of feasible $S$-allocations. Moreover, for each $i \in S$, $\tilde{u}^i(x^i) = \min\left\{u_i, \frac{\theta_i \lambda^* \cdot u^{*S}}{\lambda_i^*}\right\} = u_i$. This shows that $\tilde{C}_S \subseteq V(\tilde{\mathscr{E}})(S)$, and therefore, $\tilde{V}(S) \subseteq V(\tilde{\mathscr{E}})(S)$.

Let $u \in V(\tilde{\mathscr{E}})(S)$. Then, by (6), there exists an $S$-tuple $(x^i)_{i \in S} \in F(S)$ such that

$$u_i \leq \tilde{u}_i(x^i),\ i \in S. \quad (22)$$

Since $(x^i)_{i \in S} \in F(S)$, there exists a $y \in \tilde{Y}$ such that

$$\sum_{i \in S} x_i = (0, e^S, e^S, e^S, e^S) + y, \quad (23)$$

and by (21), for each $T \subseteq N$, there exist $\gamma_T^1$, $\gamma_T^2$, $\gamma_T^3 \geq 0$ and $u_T^1$, $u_T^2$, $u_T^3 \in \tilde{C}_T$ such that

$$y = \sum_{T \subseteq N} \left( \sum_{j=1}^{3} \gamma_T^j u_T^j,\ -\gamma_T^1 e^T,\ -(\gamma_T^1+\gamma_T^2)\, e^T,\ -(\gamma_T^1+\gamma_T^3)\, e^T,\ -(\gamma_T^2+\gamma_T^3)\, e^T \right). \quad (24)$$

By (17), $x^{(2)i}=x^{(3)i}=x^{(4)i}=0$, $\forall i\in S$, and hence, it follows from (23) and (24)

$$\sum_{T\subseteq N} \gamma_T^1 e^T = e^S, \tag{25}$$

$$\sum_{T\subseteq N} (\gamma_T^1+\gamma_T^2)\, e^T = e^S, \tag{26}$$

$$\sum_{T\subseteq N} (\gamma_T^1+\gamma_T^3)\, e^T = e^S. \tag{27}$$

Clearly, $\gamma_T^2=\gamma_T^3=0$, $\forall T\subseteq N$, and $\gamma_T^1=0$, $\forall T\not\subseteq S$. Therefore, by (25), $(\gamma_T^1 | T\subseteq N) \in\Gamma(e^S)$. Since $\mathscr{G}$ is a market game and therefore is totally balanced, $\sum_{T\subseteq S}\gamma_T^1 u_T^1 \in \tilde{V}(S)$. From (20) and (22)–(25), it follows that $u\leq\sum_{T\subseteq S}\gamma_T^1 u_T^1$. Therefore, $\tilde{V}(\mathscr{E})(S)\subseteq\tilde{V}(S)$. Q.E.D.

*Theorem 2: $u^*$ is the unique competitive payoff vector for $\tilde{\mathscr{E}}$.*

*Proof:* Let $((\hat{x}^i)_{i\in N}, (\hat{y}^i)_{i\in N}, \hat{p})$ be any competitive equilibrium for $\tilde{\mathscr{E}}$. Then

$$\sum_{i\in N} \hat{x}^i = (0, e^N, \ldots, e^N) + \sum_{i\in N} \hat{y}^i, \tag{28}$$

and $\pi(\hat{p})=0$. By (21) for each $i\in N$ and for each $S\subseteq N$, there exist $\hat{\gamma}_S^{i1}, \hat{\gamma}_S^{i2}, \hat{\gamma}_S^{i3}\geq 0$ and $\hat{u}_S^{i1}, \hat{u}_S^{i2}, \hat{u}_S^{i3}\in\tilde{C}_S$ such that

$$\hat{y}^i = \sum_{S\subseteq N}\left(\sum_{j=1}^{3}\hat{\gamma}_S^{ij}\hat{u}_S^{ij}, -\hat{\gamma}_S^{i1} e^S, -(\hat{\gamma}_S^{i1}+\hat{\gamma}_S^{i2})e^S, -(\hat{\gamma}_S^{i1}+\hat{\gamma}_S^{i3})e^S, -(\hat{\gamma}_S^{i2}+\hat{\gamma}_S^{i3})e^S\right). \tag{29}$$

By (17), $\hat{x}^{(2)i}=\hat{x}^{(3)i}=\hat{x}^{(4)}=0$, for all $i\in N$. Thus, from (28) and (29), it follows that a similar argument as in the proof of Lemma 3 (see (25)–(27)) would show that

$$\hat{\gamma}_S^{i2}=\hat{\gamma}_S^{i3}=0, \ \forall i\in N, \ \forall S\subseteq N. \tag{30}$$

*Claim: $\hat{p}^{(5)}\gg 0$.*

Suppose not. Then, $\hat{p}_i^{(5)}=0$ for some $i\in N$. By (20), the maximality of $\hat{x}^j$ with respect to $\tilde{u}^j$ in $j$'s budget set, $\{x\in\tilde{X} | \hat{p}\cdot x\leq\hat{p}\cdot\tilde{a}^j+\hat{p}\cdot\hat{y}\}$, would then imply that $\hat{x}_k^{(5)j}=0$ for all $j$, $k\in N$ such that $k\neq i$, $\hat{p}_k^{(5)}>0$. However, by (28)–(30), $\sum_{j\in N}\hat{x}^{(5)j}=e^N$. Thus, $\hat{p}^{(5)}=0$. Since $u^j(x)>u^j(y)$ whenever $x_j^{(1)}>y_j^{(1)}$ and $x^{(5)}>y^{(5)}$, $\hat{p}_j^{(1)}$ must be positive. Therefore, the condition $\hat{p}^{(5)}=0$ implies that $\hat{p}^{(1)}\gg 0$. Since $\sum_{S\subseteq N}(\sum_{j\in N}\hat{\gamma}_S^{j1})\, e^S=e^N$, it follows that $\hat{\gamma}_S^{i1}>0$ for some $i\in N$ and for some $S\subseteq N$. Since $\tilde{C}_S\cap\Re_{++}^S\neq\emptyset$, it follows from the maximality of $\hat{y}^i$ that $\hat{p}^{(1)}\cdot\hat{u}_S^{i1}>0$. Since $\pi(\hat{p})=0$, it must be true that

$$\hat{p}^{(1)}\cdot\hat{u}_S^{i1}-\hat{p}^{(2)}\cdot e^S-\hat{p}^{(3)}\cdot e^S-\hat{p}^{(4)}\cdot e^S=0. \tag{31}$$

For any $j \in N$, choose $u \in \tilde{C}_{\{j\}} \cap \Re^{\{j\}}_{++}$ and choose $\gamma > 0$. Then,

$$(\gamma u,\ 0,\ -\gamma e^{\{j\}},\ 0,\ -\gamma e^{\{j\}}) \in \tilde{Y}$$

and

$$\hat{p} \cdot (\gamma u,\ 0,\ -\gamma e^{\{j\}},\ 0,\ -\gamma e^{\{j\}}) = \gamma(\hat{p}_j^{(1)} u_j - \hat{p}_j^{(3)}).$$

Since $p^{(1)} \gg 0$, $\hat{p}_j^{(3)}$ must be positive in order for $\pi(\hat{p})$ to be 0. This shows that $\hat{p}^{(3)} \gg 0$. Similarly, $\hat{p}^{(4)} \gg 0$. Therefore, from (31) it follows that

$$\hat{p}^{(1)} \cdot \hat{u}_S^{i1} - \hat{p}^{(3)} \cdot e^S > 0.$$

This contradicts to the condition that $\pi(\hat{p}) = 0$. Hence, $\hat{p}^{(5)} \gg 0$.

Since $u^* \gg 0$, $\lambda^* \gg 0$, and $\hat{p}^{(5)} \gg 0$, to maximize $\tilde{u}^i$ in $i$'s budget set $\hat{x}^i$ must satisfy

$$\tilde{u}^i(\hat{x}^i) = \frac{\lambda^* \times u^* \cdot \hat{x}^{(5)i}}{\lambda_i^*}.$$

Multiplying both sides of the above equation by $\lambda_i^*$ and summing over $i \in N$,

$$\lambda^* \cdot u(\hat{x}) = \lambda^* \cdot u^*.$$

By Lemma 3, $\tilde{u}(\hat{x}) \in \tilde{V}(N)$, and hence, $\tilde{u}(\hat{x}) = u^*$, by Lemma 2.

To complete the proof, we only need to prove that there exists at least one competitive equilibrium for $\tilde{\mathscr{E}}$. Define $((\hat{x}^i)_{i \in N}, (\hat{y}^i)_{i \in N}, \hat{p})$ by

$$\hat{x}^i = (u^* \times e^{\{i\}},\ 0,\ 0,\ 0,\ e^{\{i\}}),\ i \in N,$$

$$\hat{y}^i = \frac{1}{n}(u^*,\ -e^N,\ -e^N,\ -e^N,\ 0),\ i \in N, \text{ and}$$

$$\hat{p} = \left(\lambda^*, \frac{\lambda^* \times u^*}{3}, \frac{\lambda^* \times u^*}{3}, \frac{\lambda^* \times u^*}{3}, \lambda^* \times u^*\right).$$

Then, for each $i \in N$, $(\hat{x}^i, \hat{y}^i) \in \tilde{X} \times \tilde{Y}$, $\hat{p} \cdot \hat{y}^i = 0$, and $\sum_{i \in N} \hat{x}^i = \sum_{i \in N} a^i + \sum_i \hat{y}^i$. Furthermore, for any $S \subseteq N$ and for any $u \in \tilde{C}_S$

$$\hat{p} \cdot (u,\ -e^S,\ -e^S,\ -e^S,\ 0) = \lambda^* \cdot u - \lambda^* \cdot u^{*S} \leq 0,$$

$$\hat{p} \cdot (u,\ 0,\ -e^S,\ 0,\ -e^S) = \lambda^* \cdot u - \frac{4}{3}\lambda^* \cdot u^{*S} < 0, \text{ and}$$

$$\hat{p} \cdot (u,\ 0,\ 0,\ -e^S,\ -e^S) = \lambda^* \cdot u - \frac{4}{3}\lambda^* \cdot u^{*S} < 0.$$

Thus, $\pi(\hat{p}) = 0$, and therefore, the triple $((\hat{x}^i)_{i \in N}, (\hat{y}^i)_{i \in N}, \hat{p})$ is easily seen to be a competitive equilibrium. Q.E.D.

*Theorem 3: There is a market that represents $\mathscr{G}$ and has $u^*$ as its unique competitive payoff vector.*

*Proof:* Let $\mathscr{E} = \{\tilde{X}, \tilde{Y}, \{\tilde{a}^i, u^i\}_{i \in N}\}$ where $u^i = f_i^{-1} \circ \tilde{u}^i$, $\forall i \in N$. Here, $f_i^{-1}$ denotes the inverse of $f_i$ and $f_i^{-1} \circ \tilde{u}^i$ denotes the composition of $f_i^{-1}$ and $\tilde{u}^i$. Since $\tilde{\mathscr{E}}$ represents $\tilde{\mathscr{G}}$, it follows that $\mathscr{E}$ also represents $\mathscr{G}$. Because the set of competitive equilibria does not change when strictly monotonic utility transformations are applied to players' utility functions, the set of competitive equilibria for $\mathscr{E}$ coincides with that for $\tilde{\mathscr{E}}$. Therefore, a utility vector is competitive for $\mathscr{E}$ if and only if $f(u)$ is competitive for $\tilde{\mathscr{E}}$. Since $f(u) = u^*$ if and only if $u = u^*$, by Theorem 2, $u^*$ is the unique competitive payoff vector of $\mathscr{E}$. Q.E.D.

*Remark 6:* As mentioned earlier, using an idea due to Rader (1972), one can transform a market with production into one without production. Therefore, Theorems 1–3 still hold when restricted to markets without production.

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Received September 1992
Revised version June 1993