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A Value for Games with n Players and r Alternatives

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Abstract: We study value theory for a class of games called games with n players and r alternatives. In these games, each of the n players must choose one and only one of the r alternatives. A linear, efficient value is obtained using three characterizations, two of which are axiomatic. This value yields an a priori evaluation for each player relative to each alternative.

1 Introduction

There are n players and r alternatives. Let $N=\{1, 2, \ldots, n\}$ be the set of players. Each of the n players must choose one of the r alternatives. Let $C(j)$ be the set of players who choose alternative j and let $|C(j)|$ be the cardinality of $C(j)$. The vector $(C(1), \ldots, C(r))$ is called an arrangement of the n players among the r alternatives. Let Γ be such an arrangement. If $S \in \Gamma$, we call (S, Γ) an embedded coalition (ECL).

If with each arrangement Γ there is associated an r-tuple of real numbers, then we interpret the i^{th} coordinate as the "worth" of $C(i)$ with respect to the arrangement Γ and we write $v(C(i), (C(1), \ldots, C(r)))$ for the worth of $C(i)$ with respect to the arrangement $(C(1), \ldots, C(r))$. The triple (N, r, v) will be called a game on N with r alternatives provided $v(T, \Gamma)=0$ whenever $T=\phi$. For brevity, we also say that v is an (N, r) game.

Example 1: The United Nations Security Council has five permanent members and 10 nonpermanent members. Each of the five permanent members individually has veto power and any coalition of 7 of the nonpermanent members has veto power. In addition, at least 9 affirmative votes are needed to pass a motion. A member can vote "yes", "no" or "abstain". To decide if a motion passes, one must know how many of the permanent members and how many of the others choose each of the three alternatives. For instance, if one permanent member abstains and all other members vote "yes", the motion passes; whereas, if that member votes "no" and all others vote "yes", the motion fails. (In November, 1990, a motion passed even though a permanent member, China, abstained.) Let Y, N, A, be the voters who vote "yes", "no", "abstain", respectively, and let $\Gamma=(Y, N, A)$. We set $v(Y, \Gamma)=1$ if the motion passes and $v(Y, \Gamma)=0$ otherwise. In this example, it might not make sense to define $v(A, \Gamma)$. (It should be noted that the U. N. Security Council game is often erroneously modeled as a 2-alternative, namely "yes" or "no", game in which

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an issue passes if and only if it receives "yes" votes from all five permanent members and at least 4 nonpermanent members.)

Example 2: A Simplified Unemployment Benefit/Welfare System. There are n players. If player i is employed, player i can earn (annually) w_i dollars. There are four categories of players. Category 1 consists of those who are working. Category 2 consists of those who are currently unemployed but actively seeking employment. If player i is in this category, then i receives, as an unemployment benefit, 70% of his or her potential income, namely $0.7w_i$. Category 3 consists of those who are unemployed and not seeking employment. If player i is in Category 3, then i receives, as a welfare payment, 30% of his or her potential income. Category 4 consists of those who will leave the system and receive no benefits. The problem is to decide how much a working player should contribute to support the members of categories 2 and 3. We can model this "game" as an (N, r) game by setting

$$v(\Gamma_1, \Gamma) = \sum_{i \in \Gamma_1} w_i - 0.7 \cdot \sum_{i \in \Gamma_2} w_i - 0.3 \cdot \sum_{i \in \Gamma_3} w_i$$

Then $v(\Gamma_1, \Gamma)$ can be thought of as the net amount remaining of the employed players' wages after those in categories 2 and 3 receive their unemployment or welfare payments.

Example 3: $N = \{1, 2\}$ and $r = 3$. If both players choose alternative j, the joint payoff is $2j$. If player 1 chooses j while player 2 chooses k, then 1 receives j and 2 receives $|k-j|$. For example, $v(\{1,2\}, (\phi, \phi\{1,2\})) = 6$, $v(\{2\}, (\{2\}, \{1\}, \phi)) = 1$, $v(\{2\}, (\{2\}, \phi, \{1\})) = 2$, etc. It seems that the players should each choose alternative 3 if they can agree on how to share the joint payoff of 6 units. Otherwise, player 1 should choose alternative 3 and player 2 will most likely choose alternative 1.

The major goal of this paper will be to extend the Shapley value to games with r alternatives. In the case of voting games (such as the U.N. Security Council Game) in which each voter has more than two choices, this value can be used to measure a voter's a priori voting power. In example 3 above, the value can be used to measure the a priori worth of each player. For example 2, the value could be used to suggest how much each working player should retain of his or her salary. (See example 7 in section 5.)

2 An Efficient Value for (N, r) Games

Definition: An (N, r) game v will be called alternative-symmetric if $v(T_i, (T_1, T_2, \ldots, T_r)) = v(T_i, (T_{j_1}, T_{j_2}, \ldots, T_{j_r}))$ for all $r!$ arrangements $(j_1, j_2, \ldots, j_r)$ of $\{1, 2, \ldots, r\}$. Such games are equivalent to partition function games. See Lucas/Thall [1963].

There are apparently many ways to extend the Shapley value to games in partition function form. (See Myerson [1977], Bolger [1987], McCaulley [1990], Merki [1991].) In Bolger [1987], there is presented an infinite family of values for partition function games which satisfy efficiency, dummy, linearity, and symmetry axioms.

The question then arose as to whether one could get a unique "natural" extension of the Shapley value to games in partition function form by extending to such games the notions of "restricted game", "reduced game", and "consistent value" defined by Hart and Mas-Colell [1989]. Here again, there is apparently no unique way to extend these notions to games in partition function form.

On the other hand, this author believes that there is a "natural" unique extension of the Shapley value to games with r alternatives. This value will be obtained in this section and will be characterized axiomatically in the succeeding two sections.

For an arbitrary (N, r) game, we wish to assign an a priori value, θ_i^j, for player i relative to alternative j. We shall use the notation $v(N; j)$ for the worth of the grand coalition if it chooses alternative j. θ_i^j may be thought of as player i's share of $v(N; j)$.

Definition: A value, θ^j, is called a "j-efficient" value if for each (N, r) game v,

$$\sum_{i=1}^{n} \theta_i^j(v) = v(N; j)$$

In this section, we shall assume that the value for player i depends linearly on the marginal contribution of player i to $C(j)$.

More precisely, we first assume that θ_i^j has the form

$$\theta_i^j(v) = \sum_{\substack{\Gamma \\ i \in \Gamma_j}} \sum_{\substack{T \in \Gamma \\ T \neq \Gamma_j}} f(|\Gamma_j|, n, r)[v(\Gamma_j, \Gamma) - v(\Gamma_j - \{i\}, \alpha_{iT}(\Gamma))] \tag{1}$$

where $\alpha_{iT}(\Gamma)$ is the arrangement of N obtained from Γ by moving player i to the set T of Γ, Γ_j is the j^{th} coordinate of Γ and the summation is over all arrangements Γ of N among the r alternatives in which i belongs to the j^{th} coordinate.

Definition: Let π be a permutation of N. The game πv is defined by

$$\pi v(T, \Gamma) = v(\pi T, \pi\Gamma)$$

The following lemma is easy to prove.

Lemma 1: If θ^j is of form (1), then θ^j is symmetric, that is $\theta_i^j(\pi v) = \theta_{\pi i}^j(v)$ for each game v.

Definition: Let Γ be an arrangement and let $T \in \Gamma$, $T \neq \phi$. Then $v^{T,\Gamma}$ shall denote the (N, r) game in which $v^{T,\Gamma}(T, \Gamma) = 1$ and $v^{T,\Gamma}(T^*, \Gamma^*) = 0$ for all other ECL's (T^*, Γ^*).

Lemma 2: The collection $\{v^{T,\Gamma}\}$ of all such games serves as a basis for the vector space of all (N, r) games. Indeed, if v is any (N, r) game, we may write

$$v = \sum_{(T,\Gamma)} v(T, \Gamma)\, v^{T,\Gamma}.$$

Theorem 1: If θ_i^j is of form (1), then θ^j is j-efficient if and only if

$$f(|\Gamma_j|, n, r) = \frac{(|\Gamma_j| - 1)!\,(n - |\Gamma_j|)!}{n!\,(r-1)^{n-|\Gamma_j|+1}} \tag{2}$$

(Actually, as we shall see later, one can get the same result if the function f is allowed to depend on the sizes of each component of Γ.)

Proof: Assume first that θ^j is j-efficient. Fix the value of t and let v be the game in which

$$v(T_j, (T_1, \ldots, T_j, \ldots, T_r)) = 1 \text{ if } |T_j| \geq t$$

and $v(T, \Gamma) = 0$ otherwise.

Then, by j-efficiency, $\sum_{i=1}^{n} \theta_i^j(v) = 1$. Moreover, if π is any permutation of the player set N, then $\pi v = v$. Then, from lemma 1, each player has the same θ^j-value. It follows that $\theta_i^j(v) = 1/n$ for each i.

As for the right hand side of (1), note that if $|T_j| = t$, and if $i \in T_j$, then $v(T_j, (T_1, \ldots, T_j, \ldots, T_r)) = 1$ and $v(T_j - \{i\}, \alpha_{iT}(\Gamma)) = 0$ for $T \neq T_j$. It remains to count the number of ECL's in which $|T_j| = t$ and $i \in T_j$. This number equals

$$\binom{n-1}{t-1} \cdot (r-1)^{n-t}$$

Thus the right hand side of (1) equals

$$\binom{n-1}{t-1} \cdot (r-1)^{n-t} \cdot f(t, n, r) \cdot (r-1).$$

The result follows immediately.

Conversely, suppose f is given by (2) above. Let Γ be an arrangement and let $T \in \Gamma$, $T \neq \phi$. Let $v = v^{T,\Gamma}$. By lemma 2, it is sufficient to prove that θ^j is j-efficient for each $v^{T,\Gamma}$. If $T = N$, the efficiency is obvious, so assume $T \neq N$.

Case 1: $T = \Gamma_j$. For $i \in T$,

$$\theta_i^j(v) = \frac{(t-1)!\,(n-t)!}{n!\,(r-1)^{n-t}}$$

whereas for $k \notin T$,

$$\theta_k^j(v) = -\frac{t!\,(n-1-t)!}{n!\,(r-1)^{n-t}}$$

Then

$$t\cdot\theta_i^j(v)+(n-t)\cdot\theta_k^j(v)=0.$$

Case 2: $T\neq\Gamma_j$. Then, for all $i\in N$, $\theta_i^j(v)=0$.

Corollary: If θ^j is j-efficient and of form (1) then,

$$\theta_i^j(v)=\sum_{\substack{\Gamma\\ i\in\Gamma_j}}\frac{(|\Gamma_j|-1)!\,(n-|\Gamma_j|)!}{n!\,(r-1)^{n-|\Gamma_j|}}v(\Gamma_j,\Gamma)-\sum_{\substack{\Gamma\\ i\notin\Gamma_j}}\frac{(|\Gamma_j|)!\,(n-1-|\Gamma_j|)!}{n!\,(r-1)^{n-|\Gamma_j|}}v(\Gamma_j,\Gamma).$$

Proof:

$$\sum_{\substack{\Gamma\\ i\in\Gamma_j}}\sum_{\substack{T\in\Gamma\\ T\neq\Gamma_j}}\frac{(|\Gamma_j|-1)!\,(n-|\Gamma_j|)!}{n!\,(r-1)^{n-|\Gamma_j|+1}}[v(\Gamma_j,\Gamma)-v(\Gamma_j-\{i\},\alpha_{iT}(\Gamma))]$$

$$=\sum_{\substack{\Gamma\\ i\in\Gamma_j}}\sum_{\substack{T\in\Gamma\\ T\neq\Gamma_j}}\frac{(|\Gamma_j|-1)!\,(n-|\Gamma_j|)!}{n!\,(r-1)^{n-|\Gamma_j|+1}}v(\Gamma_j,\Gamma)-$$

$$\sum_{\substack{\Gamma\\ i\in\Gamma_j}}\sum_{\substack{T\in\Gamma\\ T\neq\Gamma_j}}\frac{(|\Gamma_j|-1)!\,(n-|\Gamma_j|)!}{n!\,(r-1)^{n-|\Gamma_j|+1}}v(\Gamma_j-\{i\},\alpha_{iT}(\Gamma))$$

$$=\sum_{\substack{\Gamma\\ i\in\Gamma_j}}\frac{(|\Gamma_j|-1)!\,(n-|\Gamma_j|)!}{n!\,(r-1)^{n-|\Gamma_j|+1}}(r-1)\,v(\Gamma_j,\Gamma)-$$

$$\sum_{\substack{\Gamma\\ i\in\Gamma_j}}\sum_{\substack{T\in\Gamma\\ T\neq\Gamma_j}}\frac{(|\Gamma_j|-1)!\,(n-|\Gamma_j|)!}{n!\,(r-1)^{n-|\Gamma_j|+1}}v(\Gamma_j-\{i\},\alpha_{iT}(\Gamma))$$

We note that player i does not belong to $\Gamma_j-\{i\}$. Also, each arrangement $\hat{\Gamma}$ with $i\notin\hat{\Gamma}_j$ can be obtained from an arrangement Γ in which $i\in\Gamma_j$ by moving player i. (Actually, $\Gamma=\alpha_{i\hat{\Gamma}_j}(\hat{\Gamma})$.). Each arrangement $\hat{\Gamma}$ with $i\notin\hat{\Gamma}_j$ contributes one term to the last double sum above. Further, if $i\in\Gamma_j$ and if we let $\hat{\Gamma}_j=\Gamma_j-\{i\}$, then $|\Gamma_j|-1=|\hat{\Gamma}_j|$ and $n-|\Gamma_j|=n-|\hat{\Gamma}_j|-1$. It follows that the last double sum above can be written in the form:

$$\sum_{\substack{\hat{\Gamma}\\ i\notin\hat{\Gamma}_j}}\frac{(|\hat{\Gamma}_j|)!\,(n-1-|\hat{\Gamma}_j|)!}{n!\,(r-1)^{n-|\hat{\Gamma}_j|}}v(\hat{\Gamma}_j,\hat{\Gamma})$$

This completes the proof of the corollary.

Remark: Suppose $r=2$. Let w be the n-person coalitional game defined by $w(S)=v(S,(S,N-S))$. Let $\phi_i(w)=\theta_i^1(v)$ where $\theta_i^1(v)$ is the value in the above corollary. Then $\phi_i(w)=$

$$\sum_{\substack{\Gamma \\ i\in\Gamma_1}} \frac{(|\Gamma_1|-1)!\,(n-|\Gamma_1|)!}{n!} v(\Gamma_1,\Gamma) - \sum_{\substack{\Gamma \\ i\notin\Gamma_1}} \frac{(|\Gamma_1|)!\,(n-1-|\Gamma_1|)!}{n!} v(\Gamma_1,\Gamma)$$
$$= \sum_{\substack{S \\ i\in S}} \frac{(|S|-1)!\,(n-|S|)!}{n!} w(S) - \sum_{\substack{S \\ i\notin S}} \frac{(|S|)!\,(n-1-|S|)!}{n!} w(S).$$

This latter expression is one form of the (Shapley) value for n-person coalitional games introduced by Shapley [1953].

Example 4: In Example 3 above, it seemed that the players should choose alternative 3. The value of Theorem 1 yields $\theta_1^3(v)=3.75$ and $\theta_2^3(v)=2.25$.

Example 5: In the U. N. Security Council game, a tedious computation shows that the value of a permanent member relative to the "yes" alternative is 0.1632 and the value of a nonpermanent member is 0.0184. Thus, a permanent member has about 9 times as much voting power as a nonpermanent member.

3 An Axiomatic Approach

Definition: Player i is a j-dummy in the (N, r) game v if for each arrangement Γ with $i\in\Gamma_j$, we have $v(\Gamma_j,\Gamma)=v(\Gamma_j-\{i\},\Gamma^*)$ whenever Γ^* is an arrangement obtained from Γ by moving i to some other set in Γ.

Axiom 1: If player i is a j-dummy in v, then $\theta_i^j(v)=0$.

Axiom 2 (Linearity): If v and w are (N, r) games and c is a real number, then

$$\theta^j(v+w)=\theta^j(v)+\theta^j(w) \text{ and } \theta^j(cv)=c\theta^j(v)$$

Axiom 3 (Symmetry): $\theta_i^j(\pi v)=\theta_{\pi i}^j(v)$.

It is not hard to show that the j-efficient value in the corollary to theorem 1 satisfies axioms 1, 2, and 3. For $n=r=3$, another j-efficient value satisfying axioms 1, 2, and 3 is given by:

$$\psi_i^j(v)=\sum_{\substack{\Gamma \\ i\in\Gamma_j}} A^j(|\Gamma_1|,|\Gamma_2|,\ldots,|\Gamma_r|)\cdot v(\Gamma_j,\Gamma)-$$
$$\sum_{\substack{\Gamma \\ i\notin\Gamma_j}} \frac{|\Gamma_j|}{3-|\Gamma_j|} A^j(|\Gamma_1|,|\Gamma_2|,\ldots,|\Gamma_r|)\cdot v(\Gamma_j,\Gamma)$$

where

$$A^j(t_1, t_2, t_3) = \begin{cases} 1/3 & \text{if } t_j = 3 \\ 1/12 & \text{if } t_j = 2. \\ (1/24) + (t_1\, t_2\, t_3)/12 & \text{if } t_j = 1 \end{cases}$$

It can then be seen that every convex combination of θ^j and ψ^j is also a j-efficient value satisfying axioms 1, 2, and 3 (for $n = r = 3$). Consequently, there are infinitely many such values. In this section, we introduce an additional axiom to get a unique value. It is easy to motivate this axiom in the context of monotonic simple games.

Definition: An (N, r) game v is called simple if $v(S, \Gamma) = 0$ or 1 for each ECL (S, Γ). For a simple game we say that S wins with respect to Γ if $v(S, \Gamma) = 1$; otherwise S is losing.

Definition: A simple game v is called monotonic if whenever S_j wins with respect to the arrangement $(S_1, \ldots, S_r)$, then $S_j \cup T$ wins with respect to the arrangement

$$(S_1 - T, \ldots, S_{j-1} - T, S_j \cup T, S_{j+1} - T, \ldots, S_r - T).$$

Definition: Let (S, Γ) be an ECL and let $i \in S$. Let Γ^* be the arrangement obtained by moving i to some other member T of Γ. The mapping α_{iT} from Γ to Γ^* defined by

$$\alpha_{iT}(S) = S - \{i\}$$
$$\alpha_{iT}(T) = T \cup \{i\}$$
$$\alpha_{iT}(X) = X \text{ for all other } X \text{ in } \Gamma$$

is called a move for player i. If v is a monotonic simple game, such a move is called a pivot move for i if S wins with respect to Γ and $S - \{i\}$ loses with respect to Γ^*.

A simple monotonic game can be used to model a voting situation in which a set of n voters is to choose precisely one of r alternatives according to some election rule.

Axiom 4: Let v and w be (N, r) games. If for each arrangement Γ with $i \in \Gamma_j$,

$$\sum_{\substack{T \in \Gamma \\ T \neq \Gamma_j}} [v(\Gamma_j, \Gamma) - v(\Gamma_j - \{i\}, \alpha_{iT}(\Gamma))]$$
$$= \sum_{\substack{T \in \Gamma \\ T \neq \Gamma_j}} [w(\Gamma_j, \Gamma) - w(\Gamma_j - \{i\}, \alpha_{iT}(\Gamma))],$$

then $\theta_i^j(v) = \theta_i^j(w)$.

If v and w are monotonic simple games, the above axiom states that player i has the same value in both games if relative to each arrangement Γ with $i \in \Gamma_j$, player i has the same number of pivot moves in v as in w.

Theorem 2: If θ^j satisfies axioms 1 through 4, then

$$\theta_i^j(v) = \sum_{\substack{\Gamma \\ i\in\Gamma_j}} \sum_{\substack{T\in\Gamma \\ T\neq\Gamma_j}} \frac{\theta_i^j(v^{\Gamma_j,\Gamma})}{r-1}\,[v(\Gamma_j,\Gamma)-v(\Gamma_j-\{i\},\alpha_{iT}(\Gamma))] \tag{3}$$

where the summation is over all arrangements Γ of N among the r alternatives for which i belongs to the j^{th} coordinate.

Proof: If $v=v^{N;j}$, then the right hand side of (3) equals

$$\frac{1}{r-1}\,\theta_i^j(v^{N;j})\cdot(r-1)$$

Next, let Γ be an arrangement and let $v=v^{\Gamma_j,\Gamma}$. For $i\in\Gamma_j$, the right hand side equals $\frac{1}{r-1}\,\theta_i^j(v)\cdot(r-1)$. Now let $i\in\Gamma_k\neq\Gamma_j$. Define the game w by

$$w=v^{\Gamma_j\cup\{i\},\,\alpha_{i\Gamma_j}(\Gamma)}+\sum_{\substack{p=1\\p\neq j}}^{r} v^{\Gamma_j,\alpha_{i\Gamma_p}(\Gamma)} \tag{4}$$

We shall first show that player i is a j-dummy in this game. To do so, let $\hat{\Gamma}$ be an arrangement with $i\in\hat{\Gamma}_j$ and let Γ^* be an arrangement obtained from $\hat{\Gamma}$ by moving player i to some other coordinate set of $\hat{\Gamma}$. If $\hat{\Gamma}=\alpha_{i\Gamma_j}(\Gamma)$, then $w(\hat{\Gamma}_j,\hat{\Gamma})=1$ and $w(\hat{\Gamma}_j-\{i\},\Gamma^*)=1$. On the other hand, if $\hat{\Gamma}\neq\alpha_{i\Gamma_j}(\Gamma)$, then $w(\hat{\Gamma}_j,\hat{\Gamma})=0$ and $w(\hat{\Gamma}_j-\{i\},\Gamma^*)=0$. Then, since player i is a j-dummy in w, $\theta_i^j(w)=0$. Equation (4) yields:

$$0=\theta_i^j(v^{\Gamma_j\cup\{i\},\,\alpha_{i\Gamma_j}(\Gamma)})+\sum_{\substack{p=1\\p\neq j}}^{r}\theta_i^j(v^{\Gamma_j,\,\alpha_{i\Gamma_p}(\Gamma)}).$$

By axiom 4, the above equation can be written

$$0=\theta_i^j(v^{\Gamma_j\cup\{i\},\,\alpha_{i\Gamma_j}(\Gamma)})+(r-1)\,\theta_i^j(v^{\Gamma_j,\Gamma})$$

or

$$\theta_i^j(v^{\Gamma_j,\Gamma})=-\frac{1}{r-1}\,\theta_i^j(v^{\Gamma_j\cup\{i\},\,\alpha_{i\Gamma_j}(\Gamma)}).$$

We have now shown that (3) holds for $v=v^{\Gamma_j,\Gamma}$. Next, let $v=v^{\Gamma_p,\Gamma}$ where $\Gamma_p\neq\phi$ and $p\neq j$. Let $i\in N$. Then, $v(\Gamma_j,\Gamma)=0$ and $v(\Gamma_j-\{i\},\alpha_{iT}(\Gamma))=0$ since $\Gamma_j\neq\Gamma_p\neq\Gamma_j-\{i\}$. Thus, each player is a j-dummy in $v^{\Gamma_p,\Gamma}$, and both sides of (3) are 0.

The final step is to observe that the right hand side of (3) can be used to define a linear value ψ on the class of all (N, r) games. Since ψ agrees with θ^j on the basis games $\{v^{\Gamma_k, \Gamma}\}$, ψ must be identical with θ^j.

The next result will show that if we further require that θ^j be a j-efficient value, we get a unique such value, namely the value in the corollary to Theorem 1.

Theorem 3: Let θ^j be a value for (N, r) games which is j-efficient and satisfies axioms 1 through 4 above. Then,

$$\theta_i^j(v) = \sum_{\substack{\Gamma \\ i \in \Gamma_j}} \sum_{\substack{T \in \Gamma \\ T \neq \Gamma_j}} \frac{(|\Gamma_j| - 1)!\,(n - |\Gamma_j|)!}{n!\,(r-1)^{n-|\Gamma_j|+1}} [v(\Gamma_j, \Gamma) - v(\Gamma_j - \{i\}, \alpha_{iT}(\Gamma))] \tag{5}$$

Proof: Consider a fixed r-tuple $(t_1, t_2, \ldots, t_r)$ of nonnegative integers whose sum is n. Let $T_1 = \{1, 2, \ldots, t_1\}$, $T_2 = \{t_1 + 1, \ldots, t_1 + t_2\}, \ldots, T_r = \{t_1 + t_2 + \ldots t_{r-1} + 1, \ldots, n\}$ where it is understood that $T_k = \phi$ if $t_k = 0$. Let $\Gamma = (T_1, \ldots, T_r)$ and define for $i \in T_j$,

$$a_j(t_1, \ldots, t_r) = \frac{\theta_i^j(v^{T_j, \Gamma})}{r-1}.$$

Note that for the special case $T_j = N$, we have $t_j = n$ and

$$a_j(0, 0, \ldots, 0, n, 0, \ldots, 0) = \frac{1}{(r-1)\cdot n}.$$

In general, it follows from (3) that for $i \in T_p \neq T_j$,

$$\theta_i^j(v^{T_j, \Gamma}) = -a_j(t_1, t_2, \ldots, t_{j-1}, t_j + 1, t_{j+1}, \ldots, t_{p-1}, t_p - 1, t_{p+1}, \ldots, t_r).$$

Using j-efficiency, we get

$$(r-1)\cdot t_j \cdot a_j(t_1, \ldots, t_r) = \sum_{p \neq j} t_p \cdot a_j(t_1, t_2, \ldots, t_{j-1}, t_j + 1, t_{j+1}, \ldots, t_{p-1}, t_p - 1, t_{p+1}, \ldots, t_r). \tag{6}$$

Now assume (using backwards induction on t_j) that whenever the value of the j^{th} coordinate is greater than t_j,

$$a_j(s_1, \ldots, s_r) = \frac{(s_j - 1)!\,(n - s_j)!}{n!\,(r-1)^{n-s_j+1}}$$

The recursion relation (6) then determines the value of the function a_j when the j^{th} coordinate equals t_j. It follows that equation (5) holds for the basis elements $v^{T, \Gamma}$. By linearity, (5) holds for each game v.

4 Restricted Games and Pairwise Consistency

It seems appropriate to ask if the value, θ^j, in Theorem 3 has a "consistency" property similar to that introduced for coalitional games by Hart and Mas-Colell [1989]. We shall show that θ^j has a "pairwise-consistency" property similar to that in Hart and Mas-Colell [1989] and that θ^j is the only j-efficient, pairwise-consistent value satisfying the symmetry, j-dummy, and linearity axioms. (Thus, axiom 4 can be replaced by a "pairwise consistency" axiom.)

Suppose θ^j is a j-efficient value for (N, r) games which satisfies the symmetry, j-dummy, and linearity axioms. Let $\Gamma=(T_1, \ldots, T_r)$ be a fixed arrangement of N and for $1 \leq q \leq r$, let $v_q = v^{T_q, \Gamma}$. Let $k \in T_q$ and set $A^j_q(t_1, \ldots, t_r) = \theta^j_k(v_q)$. Not let $i \in T_p \neq T_q$ and set

$$B^j_{p,q}(t_1, \ldots, t_r) = -\theta^j_i(v_q)$$

It will be convenient to write $A_j(n;j)$ for $A^j_j(0, \ldots, 0, n, 0, \ldots, 0)$ where the n is in the j^{th} position.

Lemma 3: $A_j(n;j) = 1/n$.

Proof: By symmetry and j-efficiency, $\theta^j_i(v^{N;j}) = 1/n$ for each i.

Lemma 4: If $q \neq j$, then $A^j_q(t_1, \ldots, t_r) = B^j_{p,q}(t_1, \ldots, t_r) = 0$.

Proof: If $k \neq j$, then every player is a j-dummy in $v^{\Gamma_k, \Gamma}$.

Since the notion of pairwise-consistency depends on the notion of the "restricted" game, our next goal is to define the restriction of a (N, r) game to the reduced player set $N-\{i\}$.

Definition: Let Γ be an arrangement of N. For $i \notin \Gamma_j$,

$$(N-\{i\}, v^{\Gamma_j, \Gamma})(\Gamma_j, \Gamma-\{i\}) = 1/(r-1)$$

whereas

$$(N-\{i\}, v^{\Gamma_j, \Gamma})(T^*, \Gamma^*) = 0$$

for all other ECL's T^*, Γ^*. (Here, Γ^* is an arrangement of $N-\{i\}$.). Further, for $i \in \Gamma_j$, $(N-\{i\}, v^{\Gamma_j, \Gamma}) \equiv 0$.

Lemma 5: Suppose $i \notin \Gamma_j$. For $k \in \Gamma_j$, $\theta^j_k(N-\{i\}, v^{\Gamma_j, \Gamma}) =$

$$A^j_j(|\Gamma_1 - \{i\}|, |\Gamma_2 - \{i\}|, \ldots, |\Gamma_r - \{i\}|)/(r-1)$$

whereas, for $k \in \Gamma_p \neq \Gamma_j$, $\theta^j_k(N-\{i\}, v^{\Gamma_j, \Gamma}) =$

$$-B^j_{p,j}(|\Gamma_1 - \{i\}|, |\Gamma_2 - \{i\}|, \ldots, |\Gamma_r - \{i\}|)/(r-1)$$

So far, we have only defined the restricted game for the basis games.

Definition: Let v be any (N, r) game and write $v = \sum_{(T,\Gamma)} v(T, \Gamma)\, v^{T,\Gamma}$.
We define $(N-\{i\}, v) = \sum_{(T,\Gamma)} v(T, \Gamma)\,(N-\{i\}, v^{T,\Gamma})$.

When $r=2$, this notion of the restricted game is equivalent to the coalitional game restricted to subsets of $N-\{i\}$.

Definition: A linear j-efficient value θ^j is called pairwise consistent if for each i and k and each game v,

$$\theta_k^j(v) - \theta_i^j(v) = \theta_k^j(N-\{i\}, v) - \theta_i^j(N-\{k\}, v)$$

When $r=2$, this agrees with the notion of pairwise consistency for n-person coalitional games in Hart/Mas-Colell [1989] and is related to the notion of "balanced-contribution" in Myerson [1980].

Theorem 4: A j-efficient value θ^j which satisfies the symmetry, j-dummy, and linearity axioms is pairwise consistent if and only if

$$A_j^j(t_1, \ldots, t_r) = \frac{(t_j-1)!\,(n-t_j)!}{n!\,(r-1)^{n-t_j}} \tag{7}$$

and

$$B_{p,j}^j(t_1, \ldots, t_r) = (t_j/(n-t_j))\, A_j^j(t_1, \ldots, t_r) \tag{8}$$

Proof: Suppose first that (7) and (8) are true. Let θ^j be the j-efficient value which satisfies the symmetry, j-dummy, and linearity axioms and satisfies (7) and (8). Then a direct computation verifies that this θ^j is pairwise consistent.

Conversely, suppose that θ^j is a pairwise consistent, j-efficient value satisfying the symmetry, j-dummy, and linearity axioms. Let $v = v^{T_j,\Gamma}$ where $\Gamma = \{T_1, \ldots, T_r\}$. Let $k \in T_j$ and $i \in T_p \neq T_j$. Then, from pairwise consistency,

$$A_j^j(t_1, \ldots, t_r) + B_{p,j}^j(t_1, \ldots, t_r) = \frac{A_j^j(t_1, t_2, \ldots, t_p-1, \ldots, t_r)}{r-1} \tag{9}$$

Let $m = \sum_{q \neq j} t_q$. We proceed by induction on m. For $m=1$, we have $t_p = 1$ and (9) becomes

$$\begin{aligned} &A_j^j(0, \ldots, 0, n-1, 0, \ldots, 0, 1, 0, \ldots, 0) + \\ &B_{p,j}^j(0, \ldots, 0, n-1, 0, \ldots, 0, 1, 0, \ldots, 0) \\ &= A_j^j(0, \ldots, 0, n-1, 0, \ldots, 0)/(r-1). \end{aligned}$$

By j-efficiency and symmetry,

$$(n-1)A_j^j(0,\ldots,0,n-1,0,\ldots,0,1,0,\ldots,0)= B_{p,j}^j(0,\ldots,0,n-1,0,\ldots,0,1,0,\ldots,0)$$

It follows immediately that

$$A_j^j(0,\ldots,0,n-1,0,\ldots,0,1,0,\ldots,0)=\frac{1}{n(n-1)(r-1)}.$$

Assume the result is true for $\sum_{q\neq j} t_q<m$ and consider $A_j^j(t_1,\ldots,t_r)$ for $\sum_{q\neq j} t_q=m$. From (9), we get

$$A_j^j(t_1,\ldots,t_r)+B_{p,j}^j(t_1,\ldots,t_r)=\frac{(t_j-1)!\,(n-1-t_j)!}{(n-1)!\,(r-1)^{n-t_j}}.$$

Note that $B_{p,j}^j$ is therefore independent of p. It then follows from symmetry and j-efficiency that

$$t_jA_j^j(t_1,\ldots,t_r)=(n-t_j)\,B_{p,j}^j(t_1,\ldots,t_r)$$

Solving for $B_{p,j}^j$ and substituting in the previous equation yields the desired result.

Remark: The reader may easily show that the j-efficient value of Theorem 4 is identical to the j-efficient value of Theorem 3. We thus have three characterizations of this j-efficient value.

5 Final Arrangements and the Induced Subgames

If θ^j is a j-efficient value, then $\theta_i^j(v)$ is player i's share of $v(N;j)$ provided the grand coalition forms and chooses alternative j. Suppose the grand coalition does not form but instead the "final" arrangement is $(\Gamma_1,\Gamma_2,\ldots,\Gamma_r)$? How much should each of the players in Γ_j get of the total available to Γ_j, namely $v(\Gamma_j,\Gamma)$?

In the case of coalitional games (i.e. cooperative games in characteristic function form), a final coalition S induces a subgame, (S,v), with player set S in which $(S,v)(T)$ is defined to be $v(T)$ for each $T\subseteq S$. One may then use player i's value in this subgame as player i's share of $v(S)$.

We wish to define, for games with r alternatives, the notion of a game induced by a final arrangement Γ. For ease of notation we shall assume in the remainder of this section that $j=1$.

Definition: Let $\Gamma=(\Gamma_1, \Gamma_2, \ldots, \Gamma_r)$ be an arrangement of the player set N. Let v be an (N, r) game. Let Γ^* be an arrangement of Γ_1. We define a new game (Γ, v) (relative to alternative 1) with player set Γ_1 by:

$$(\Gamma, v)(\Gamma_1^*, \Gamma^*)=v(\Gamma_1^*, \hat{\Gamma})$$

where $\hat{\Gamma}=(\Gamma_1^*, \Gamma_2^* \cup \Gamma_2, \ldots, \Gamma_r^* \cup \Gamma_r)$. The game (Γ, v) is called the subgame of v induced by Γ.

Example 6: Let $N=\{1,2,3\}$ and let $r=3$. Let v be the game defined by: $v(\{1,2,3\}, (\{1,2,3\}, \phi, \phi))=v(\{1,2\},(\{1,2\},\phi, \{3\}))=v(\{1\},(\{1\},\{2\},\{3\}))=1; v(T, \Gamma)=0$ for all other ECL's. Let Γ be the final arrangement $\Gamma=(\{1,2\}, \phi, \{3\}))$. The induced subgame (Γ, v) is the game with player set $\{1,2\}$ in which $(\Gamma, v)(\{1,2\}, (\{1,2\}, \phi, \phi))=(\Gamma, v)\ (\{1\}, (\{1\}, \{2\}, \phi))=1$ whereas $(\Gamma, v)(T^*, \Gamma^*)=0$ for all ECL's (T^*, Γ^*) where Γ^* is an arrangement of $\{1,2\}$.

We can now use, for $i\in\Gamma_1$, $\theta_i^1(\Gamma, v)$ as the value of player i in the induced game (Γ, v), that is, $\theta_i^1(\Gamma, v)$ is player i's share of $v(\Gamma_1, \Gamma)$, assuming the final arrangement is Γ.

The induced subgame can also be used in situations where some of the players are forced to "choose" specific alternatives. These players would be placed in the appropriate alternative sets at the outset of the game and would remain there until the final arrangement is determined.

Example 7: In example 2, section 1, suppose that there are four potential wage earners who can earn \$10000, \$20000, \$30000, and \$42000, respectively. Suppose further that the final arrangement is $(\{3,4\}, \{2\}, \{1\}, \phi)$, so that players 3 and 4 are employed, player 2 is seeking employment and player 1 is content to receive welfare. Players 1 and 2 are to receive \$3000 and \$14000, respectively, from players 3 and 4. We shall use the value $\theta_3^1(\Gamma, v)$ to suggest how much of the remaining \$55000 should go to player 3. By direct computation, we find that $\theta_3^1(\Gamma, v)=\$19500$ and $\theta_4^1(\Gamma, v)=\$35500$. It is interesting to note that the higher wage earner pays less "tax" than the lower wage earner. This may be attributed to the fact that the higher wage earner is the most valuable member of category 1 in the sense that player 4 would be the most costly if she were in category 2 or 3.

6 Dummy Independence

In this final section, we show that the addition (or removal) of a dummy player, d, has no effect on the values of the other players. Throughout this section, θ^j refers to the value in Theorem 3.

Definition: Let $\Gamma=(T_1, T_2, \ldots, T_r)$ be an arrangement of N. Let, for fixed p, $v=v^{T_p, \Gamma}$. The dummy extension, v^d, of v to $N \cup \{d\}$ is defined by

$v^d(T_p \cup \{d\}, (T_1, \ldots, T_p \cup \{d\}, \ldots, T_r)) = 1$
$v^d(T_p, (T_1, \ldots, T_q \cup \{d\}, \ldots, T_r)) = 1$ for $1 \leq q \leq r$, $q \neq p$.
$v^d(T^*, \Gamma^*) = 0$ for all other (T^*, Γ^*)

Lemma 6: Let $v = v^{T_p, \Gamma}$. For each $i \in N$, $\theta_i^j(v^d) = \theta_i^j(v)$.

Proof: If $p \neq j$, then $\theta_i^j(v^d) = \theta_i^j(v) = 0$. So, suppose $p = j$ and let $i \in T_j$. Then $\theta_i^j(v^d) = \sum_{q=1}^{r} A_j^j(t_1, t_2, \ldots, t_q + 1, \ldots, t_r)$ and using (7), this reduces to $A_j^j(t_1, t_2, \ldots, t_r)$. A similar result follows from (8) if $i \notin T_j$.

Now let v be any (N, r) game and write

$$v = \sum_{(T,\Gamma)} v(T, \Gamma) \cdot v^{T,\Gamma}.$$

We then define

$$v^d = \sum_{(T,\Gamma)} v(T, \Gamma) \cdot (v^{T,\Gamma})^d.$$

(Note that the sum is over ECL's for N whereas the domains of $(v^{T,\Gamma})^d$ and v^d are the ECL's for $N \cup \{d\}$.)

Using the lemma, the following result is immediate.

Theorem 5. Let v be any (N, r) game. For each $i \in N$,

$$\theta_i^j(v) = \theta_i^j(v^d).$$

7 Conclusion

In this paper, we have shown that, although there are, for r-alternative games, infinitely many j-efficient values satisfying symmetry, dummy, and linearity axioms, there is a unique "natural" extension of the Shapley value to games with r alternatives. Moreover, this value satisfies a "pairwise consistency" property similar to that in Hart and Mas-Colell [1989].

Appendix

Calculations for Examples 3 and 5 (UNSC)

Example 3: $N=\{1,2\}$ and $r=3$. If both players choose alternative j, the joint payoff is $2j$. If player 1 chooses j while player 2 chooses k, then 1 receives j and 2 receives $|k-j|$. For example, $v(\{1,2\},(\phi,\phi,\{1,2\})=6$, $v(\{2\},(\{2\},\{1\},\phi))=1$, $v(\{2\},(\{2\},\phi,\{1\}))=2$, etc. In order to compute $\theta_1^3(v)$, we need the arrangements Γ for which $1\in\Gamma_3$. These arrangements are $(\phi,\phi,\{1,2\})$, $(\{2\},\phi,\{1\})$, and $(\phi,\{2\},\{1\})$. Then,

$$\begin{aligned}\theta_1^3(v)&=\frac{(2-1)!\,0!}{2!\,2^1}[v(\{1,2\},(\phi,\phi,\{1,2\}))-v(\{2\},(\{1\},\phi,\{2\}))\\&+v(\{1,2\},(\phi,\phi,\{1,2\}))-v(\{2\},(\phi,\{1\},\{2\}))]\\&+\frac{0!\,1!}{2!\,2^2}[v(\{1\},(\{2\},\phi,\{1\}))-v(\phi,(\{1,2\},\phi,\phi))\\&+v(\{1\},(\{2\},\phi,\{1\}))-v(\phi,(\{2\},\{1\},\phi))]\\&+\frac{0!\,1!}{2!\,2^2}[v(\{1\},(\phi,\{2\},\{1\}))-v(\phi,(\{1\},\{2\},\phi))\\&+v(\{1\},(\phi,\{2\},\{1\}))-v(\phi,(\phi,\{1,2\},\phi))]\\&=\frac{6-2+6-1}{4}+\frac{3+3+3+3}{8}=3.75.\end{aligned}$$

Example 5: The United Nations Security Council has five permanent members and 10 nonpermanent members. Each of the five permanent members individually has veto power and any coalition of 7 of the nonpermanent members has veto power. In addition, at least 9 affirmative votes are needed to pass a motion. A member can vote "yes", "no" or "abstain". To decide if a motion passes, one must know how many of the permanent members and how many of the others choose each of the three alternatives. Suppose player #6 is the first nonpermanent member. We shall compute $\theta_6^y(v)$. We need consider only those arrangements Γ which yield pivot moves for player 6. (See page 10 for the definition of "pivot move".) Player 6 can pivot by changing her vote to "no" or "abstain" in those arrangements for which:

(a) player 6 votes "yes"

and

(b) for $j=0,1,\ldots,5$, exactly j of the permanent members vote "yes"; the other 5-j permanent members vote "abstain"; and, exactly 8-j of the nonpermanent members (other than player 6) vote "yes".

To count the number of such arrangements, we note that, for fixed j, the j permanent members to vote "yes" can be chosen in $\binom{5}{j}$ ways; the 8-j nonpermanent members to vote "yes" (along with player 6) can be chosen in $\binom{9}{8-j}$ ways; and, the remaining $j+1$ nonpermanent members can then vote either "no" or "abstain". Consequently, for fixed j, the number of arrangements satisfying (a) and (b) equals $\binom{5}{j} \cdot \binom{9}{8-j} \cdot 2^{j+1}$. Then, $\theta_6^y(v) = \sum_{j=0}^{5} \binom{5}{j} \cdot \binom{9}{8-j} \cdot 2^{j+1} \cdot \frac{8!\,6!}{15!\,2^7} \cdot 2 = 0.0184$. By y-efficiency, the value of a permanent member with respect to the "yes" alternative is 0.1632.

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