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Hierarchical Organization Structures and Constraints on Coalition Formation[1]

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Abstract: This paper studies the constraints in coalition formation that result from a hierarchical organization structure on the class of players in a cooperative game with transferable utilities. If one assumes that the superiors of a certain individual have to give permission to the actions undertaken by the individual, then one arrives at a limited collection of formable or *autonomous* coalitions. This resulting collection is a lattice of subsets on the player set.

We show that if the collection of formable coalitions is limited to a lattice, the core allows for (infinite) exploitation of subordinates. For discerning lattices we are able to generalize the results of Weber (1988), namely the core is a subset of the convex hull of the collection of all attainable marginal contribution vectors plus a fixed cone. This relation is an equality if and only if the game is convex. This extends the results of Shapley (1971) and Ichiishi (1981).

1 Introduction

In social sciences and game theory one regularly addresses the problem of cooperative behavior assuming that every decision maker is an autonomous acting individual. Under this hypothesis one traditionally arrives at a model in which, in principle, every group of individuals has to be regarded as a formable or autonomous coalition. This is the viewpoint as reflected in the traditional definition of the core, the Shapley value, and the bargaining set in a cooperative game with transferable utilities.

If one discards the assumption that individuals are completely free in forming coalitions, one arrives at refinements which incorporate certain *constraints in coalition formation*. Aumann and Dreze (1974) describe a situation in which certain exogenous conditions result in a partition of individuals into a finite number of jurisdictions. These *coalition structures* lead to well specified constraints on coalition formation, namely groups within a jurisdiction can be formed freely, but individuals cannot overstep the boundaries as set by the partition of the individuals into jurisdictions.

Another approach is to derive constraints in coalition formation from certain social defects of the individuals such as limited abilities to communicate with other individuals. This is the starting point of Myerson (1977), who introduces an undirected graph as a description of the limited communication possibilities open to the individuals. Now a coalition can only form if it does not depend on outside individuals

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with respect to its communication, i.e., a coalition can only form if it constitutes a connected subgraph. For an analysis of cooperative behavior in the context of graph theoretic models of limited communication we also refer to Owen (1986).

In this paper we focus on another type of constraints in coalition formation, namely those resulting from situations in which certain individuals can veto cooperation of certain other individuals. In Gilles, Owen and van den Brink (1992) it is remarked that in many decision situations there is an asymmetry between the decision makers in the sense that there exist dominance relations between them. The organization structure in which certain individuals dominate others, in the sense that they have veto power over the cooperative abilities of those other individuals, is called a permission structure. (For a study of similar configurations we also refer to Kalai, Postlewaite and Roberts (1978), Radner (1992), and Winter (1989). For an economic discussion related to these configurations we refer to Grossman and Hart (1986).)

Let $N = \{1, \ldots, n\}$ be a finite set of players. A *permission structure* is formally described by an irreflexive mapping $S : N \to 2^N$, i.e., for every player $i \in N : i \notin S(i)$. The collection of all permission structures on the player set N is denoted by $\mathcal{S}(N)$. Obviously $S \in \mathcal{S}(N)$ describes a hierarchical structure on N, where $j \in S(i)$ is interpreted as that player i *dominates* player j. The players in $S(i)$ are denoted as the (direct) *subordinates* of player i in $S \in \mathcal{S}(N)$. On the other hand the player in

$$S^{-1}(i) := \{j \in N \mid i \in S(j)\}$$

are called the *superiors* of player i in the permission structure S.

Special attention has to be given to *strict* hierarchies in which the domination is "top-down". Such a hierarchy is described by an acyclic permission structure on N. Formally, $S \in \mathcal{S}(N)$ is *acyclic* if for every player $i \in N$ there is no sequence $i_1, \ldots, i_K$ in N such that $i_1 = i_K = i$ and for every $1 \leqq k \leqq K-1$ we have that $i_{k+1} \in S(i_k)$. The collection of acyclic permission structures on N is indicated by $\mathcal{S}_a(N) \subset \mathcal{S}(N)$.

In the analysis of cooperative behavior in the presence of a permission structure one has to take account of the veto power that certain individuals have with respect to the actions or decisions taken by certain other individuals in the organization. In Gilles, Owen and van den Brink (1992) it is assumed that every superior has veto power over all actions undertaken by his subordinates in the permission structure. Thus, every individual decision maker has to get permission for his actions from *all* his superiors. This approach is referred to as the *conjunctive approach* to cooperative behavior in permission structures. In Gilles and Owen (1994) it is asumed that an individual only has to get permission from at least one of his superiors. Thus, within this *disjunctive approach* only the set of all superiors as a collective have veto power over their subordinates.

In this paper we restrict ourselves to the conjunctive approach. In this case a coalition is formable if and only if it contains all the superiors of every member of that coalition, i.e., there is no outside individual, who is able to veto an action of any member of the coalition. Formally, the collection of (conjunctively) *autonomous* coalitions in the permission structure $S \in \mathcal{S}(N)$ is now given by

$$\Omega_S := \{E \subset N \mid E \cap S(N \setminus E) = \emptyset\},$$

where for every $F \subset N$ we define $S(F) = \cup_{i \in F} S(i)$.

In this paper we conclude that the adoption of the strong veto power associated with the conjunctive approach leads to the possibility of an infinite exploitation of subordinates by their superiors. We thus arrive at the conclusion that the relationship between superior and subordinate as described by the conjunctive approach to hierarchies incorporates a strong from of veto power. However, we also remark that the form of relationship as developed in the conjunctive approach reflects properties that form the foundations to the theories of, e.g., Coase (1937) and Marx (1876–1894). Both the theory of the firm as developed by Coase (1937) and the theory of a capitalist society as developed by Marx (1876–1894) essentially have their foundation on the hypothesis that subordinates in hierarchical production organizations do not have an alternative to avoid exploitation by their superiors. As shown in our results this leads to the possibility of infinite exploitation contrary to limited exploitation in case that subordinates can "quit". This is the subject of the final section in which a comparison of both possibilities is analyzed.

The setup of the paper is as follows. In Section 2 we develop the notion of a lattice of autonomous coalitions on a given set of individuals. We show that any lattice can be supported as the collection of conjunctively autonomous coalitions in some well chosen permission structure. Furthermore, we give a characterization of the core of a cooperative game with side payments – or, simply, a TU-game – on a permission structure.

In Section 3 we analyze strict hierarchies as represented by acyclic permission structures. We show that the collection of autonomous coalitions in an acyclic permission structure forms a discerning lattice. For such discerning lattices we are able to generalize the result of Weber (1988) that the core of a TU-game is a subset of the convex hull of the attainable marginal contribution vectors.

Section 4 is devoted to convex TU-games on strict hierarchies. Here we generalize the results of Shapley (1971) and Ichiishi (1981), and show that the core of a convex game on a strict hierarchy is equal to the algebraic sum of the convex hull of the attainable marginal contribution vectors and a fixed cone.

The final section discusses the core of TU-game with a permission structure in which players have the option to "leave" the game. We conclude that in such a situation superiors have limited opportunities to exploit their subordinates.

2 Hierarchies and Lattices

In the sequel we denote by $\Omega \subset 2^N$ an arbitrary collection of coalitions in the player set N. In this section we investigate under which conditions on an arbitrary collection Ω there exists a permission structure $S \in \mathcal{S}(N)$ with $\Omega_S = \Omega$. Using the properties on Ω we then are able to analyze the core of an arbitrary cooperative game with transferable utilities given that there are constraints on coalition formation as imposed by a conjunctive permission structure.[2]

Definition 2.1: A collection of coalitions $\Omega \subset 2^N$ is a lattice if

(i) $\emptyset, N \in \Omega$ *and*
(ii) *for every* $E, F \in \Omega : E \cap F, E \cup F \in \Omega$.

Since the lattice is closed for taking finite intersections and N is a finite set it is evident that a lattice is simply a topology on N. For reasons of interpretation we will refer to these structures as lattices rather than topologies. Now for every player $i \in N$ we may define

$$D_i^\Omega := \cap\{E \in \Omega \mid i \in E\} \in \Omega.$$

Thus, the mapping $D^\Omega : N \to \Omega$ assigns to every player i the smallest allowable coalition D_i^Ω in Ω of which i is a member. Observe that for all $i, j \in N$ $D_j^\Omega \subset D_i^\Omega$ whenever $j \in D_i^\Omega$. We remark that if Ω is a lattice the collection

$$\mathcal{D}(\Omega) = \left\{D_i^\Omega \in \Omega \;\middle|\; i \in N\right\}$$

is a basis for Ω, i.e., every coalition in Ω can be written as a union of elements in $\mathcal{D}(\Omega)$. In particular we mention that the collection $\mathcal{D}(\Omega)$ is the *smallest* basis of the lattice Ω. That is, $\mathcal{D}(\Omega)$ is the *unique* minimal basis of Ω.

Proposition 2.2: Let $\Omega \subset 2^N$ be any collection of autonomous coalitions in N. Then the following statements are equivalent:

(i) *Ω is a lattice.*
(ii) *There exists a permission structure $S \in \mathcal{S}(N)$ on N such that $\Omega_S = \Omega$.*

Proof: In Gilles, Owen and van den Brink (1992) it already has been shown that (ii) implies (i). Therefore, we are left to show that (i) implies (ii).
Let $\Omega \subset 2^N$ be a lattice. Define $S : N \to 2^N$ by

$$S(i) := \{j \in N \mid i \in D_j^\Omega\} \setminus \{i\}.$$

Evidently S is a permission structure on N. We are left to show that $\Omega_S = \Omega$.
We already mentioned that $\mathcal{D}(\Omega) = \{D_i^\Omega \mid i \in N\}$ is the unique minimal basis of Ω.
We now show that $\mathcal{D}(\Omega)$ also is a basis of Ω_S, thereby showing the desired assertion.
Let $i \in N$. Now for any $j \notin D_i^\Omega$ it holds that $D_i^\Omega \cap S(j) = \emptyset$, since otherwise for $h \in D_i^\Omega \cap S(j)$ it holds that $j \in D_h^\Omega \subset D_i^\Omega$.

[2] For the mathematical theory of abstract lattices we refer to, e.g., Birkhoff (1948) and Donnellan (1968).

Now take $E \in \Omega_S$. Clearly for every $j \in E$ and $i \in D_j^\Omega$ with $i \neq j$ we have that $j \in S(i)$. Thus, $E \cap S(i) \neq \emptyset$. By definition, therefore, $i \in E$. This implies that for every $j \in E : D_j^\Omega \subset E$ and hence

$$E = \bigcup_{j \in E} D_j^\Omega .$$

This shows that $\mathcal{D}(\Omega)$ indeed is a basis of Ω_S. □

We introduce some notation. A payoff vector on the player set N is represented by $x \in \mathbb{R}^N$. For any $E \subset N$ we write $x(E) := \sum_{i \in E} x_i$. Furthermore, for every $i \in N$ we denote by $e^i \in \mathbb{R}_+^N$ the i-th unit vector.

If players are organized within a certain permission structure, obviously some players have a better position than other players in the sense that they can achieve higher payoffs than those others. This is subject to analysis in Gilles et al. (1992) and Gilles and Owen (1994). Here we investigate the side payment vectors that are attainable in such a situation. In particular we call a side payment vector *stable* if there is no autonomous coalition that makes a loss under these side payments.

Formally, let $\Omega \subset 2^N$ be any collection of autonomous coalitions. Then the set of stable side payments with respect to Ω is defined by

$$\mathcal{C}_\Omega := \left\{ x \in \mathbb{R}^N \,\middle\|\, \begin{array}{c} x(N) = 0 \text{ and} \\ \text{for every } E \in \Omega : x(E) \geqq 0 \end{array} \right\}.$$

Players in N can exploit their positions as described by Ω if $\mathcal{C}_\Omega \neq \{0\}$. The standard case is obviously when $\Omega = 2^N$. In that case $\mathcal{C}_\Omega = \{0\}$, which reflects that there is no justified exploitation in the case of free coalition formation.[3] A precise description of $\mathcal{C}_\Omega$ for the case in which Ω is a lattice is given in the next theorem. It shows that in many cases exploitation of subordinates by superiors is possible. This exploitation has an unlimited character.

Theorem 2.3: If $\Omega \subset 2^N$ is a lattice, then the set of stable side payments with respect to Ω is given by

$$\mathcal{C}_\Omega = \operatorname{Cone}\left\{ e^j - e^i \,\middle|\, i \in N \text{ and } j \in D_i^\Omega \right\}.$$

Proof: Let Ω be a lattice on N.
It is obvious that for every $i \in N$ and $j \in D_j^\Omega$ the payoff vector $e^j - e^i$ is in $\mathcal{C}_\Omega$. Furthermore it is evident that $\mathcal{C}_\Omega$ is a cone in $\mathbb{R}^N$.
This leaves us to show that

$$\mathcal{C}_\Omega \subset \operatorname{Cone}\left\{ e^j - e^i \,\middle|\, i \in N \text{ and } j \in D_i^\Omega \right\}.$$

[3] For a detailed study of the collection of stable side payments with respect to classes of formable coalitions we refer to Derks and Reijnierse (1992). There, in particular, certain conditions are given under which the only stable side payment is the null vector.

Let $x \in \mathcal{C}_\Omega$ with $x \not\leqq 0$ and let $j \in N$ be such that $x_j > 0$. In the sequel we construct a player $i \in N$ with $x_i < 0$, $j \in D_i^\Omega$, and there is an $\varepsilon > 0$ with

$$x' = x - \varepsilon(e^j - e^i) \in \mathcal{C}_\Omega.$$

Notice that $x'(E) = x(E) - \varepsilon$ for every $E \in \Omega$ with $i \notin E$ and $j \in E$. Thus, whenever $x(E) = 0$ the situation is not $i \notin E$ and $j \in E$ allowed to occur.
This implies that player i has to be chosen in

$$F := \cap\{E \in \Omega \mid j \in E \text{ and } x(E) = 0\} \in \Omega.$$

Since $N \in \Omega$ and $x(N) = 0$, F is well defined and non-empty. Furthermore, $x(F) = 0$ since for any $G, H \in \Omega$ with $x(G) = x(H) = 0$ we have that $G \cap H$, $G \cup H \in \Omega$. Thus, $x(G \cap H) \geqq 0$ as well as $x(G \cup H) \geqq 0$. Together with

$$x(G \cap H) + x(G \cup H) = x(G) + x(H) = 0$$

this implies that $x(G \cup H) = x(G \cap H) = 0$. Actually we have shown that the collection $\{G \in \Omega \mid x(G) = 0\}$ is a lattice as well.
Since $j \in F$, $x_j > 0$, and $x(F) = 0$, there is a member $i \in F$ with $x_i < 0$. Suppose that for any $i \in F$ with $x_i < 0 : j \notin D_i^\Omega$. Then

$$F' := \cup\{D_i^\Omega \mid i \in F \text{ and } x_i < 0\} \in \Omega.$$

Note that $F' \subset F$. Further, $x(F') \geqq 0$ and therefore

$$0 < x_j \leqq x(F \setminus F') = -x(F') \leqq 0.$$

This is however impossible. Therefore we conclude that there exists $i \in F$ with $x_i < 0$ and $j \in D_i^\Omega$.

Consider the payoff vector given by $x' = x - \varepsilon(e^j - e^i)$ with

$$\varepsilon = \min\left[\{x_j, -x_i\} \cup \{x(E) \mid E \in \Omega \text{ with } j \in E \text{ and } i \notin E\}\right].$$

By the definition of F the choice of i in F it is clear that $x(E) > 0$ for all $E \in \Omega$ with $j \in E$ and $i \notin E$. Therefore, $\varepsilon > 0$.

Next we show that $x' \in \mathcal{C}_\Omega$. First observe that $x'(N) = 0$. For any $E \in \Omega$ we now distinguish the following cases:

1. $i \in E$ and $j \in E$: Then $x'(E) = x(E) \geqq 0$.
2. $i \in E$ and $j \notin E$: This case cannot occur since $j \in D_i^\Omega$.
3. $i \notin E$, $j \in E$, and $x(E) = 0$: This case cannot occur either since $i \in F$.
4. $i \notin E$, $j \in E$, and $x(E) > 0$: Then $x'(E) = X(E) - \varepsilon \geqq 0$ by the choice of ε.
5. $i \notin E$ and $j \notin E$: Then $x'(E) = x(E) \geqq 0$.

We may also conclude that for all $E \in \Omega : x'(E) \leqq x(E)$, since the case that $i \in E$ and

$j \notin E$ does not occur. Thus

$$\#\{E \in \Omega \mid x'(E) = 0\} \geqq \#\{E \in \Omega \mid x(E) = 0\}. \tag{1}$$

Moreover,

$$\#\{h \in N \mid x'_h = 0\} \geqq \#\{h \in N \mid x_h = 0\}. \tag{2}$$

The choice of ε is such that one of the inequalities (1) and (2) has to be strict.

So, we may repeat the procedure as followed above a finitely many times. The outcome is a sequence of payoff vector $x^0 = x, x^1, \ldots, x^m$ with for every $0 \leqq k \leqq m-1$

$$x^{k+1} = x^k - \varepsilon_k(e^{j_k} - e^{i_j}_k)$$

with $j_k \in D^{\Omega}_{i_k}$. The procedure is terminated when $x^m = 0$. Thus,

$$x = \sum_{k=0}^{m-1} \varepsilon_k(e^{j_k} - e^{i_k}) \in \text{Cone}\left\{e^j - e^i \;\middle|\; i \in N \text{ and } j \in D^{\Omega}_i\right\}. \qquad \square$$

Using Proposition 2.2 and Theorem 2.3 we may derive our first conclusion concerning the exploitation of subordinates by superiors in a hierarchy as described by the conjunctive approach to a permission structure $S \in \mathcal{S}(N)$. For that purpose we introduce a cooperative game with transferable utilities, usually referred to as a TU-game, on a permission structure S as a mapping $v : \Omega_S \to \mathbb{R}$ with $v(\emptyset) = 0$. The collection of all TU-games on $S \in \mathcal{S}(N)$ is denoted by $\mathcal{G}^N(S)$. We now introduce the core of a TU-game on some permission structure as those payoff vectors to which no autonomous coalition has an objection.

Definition 2.4: Let $S \in \mathcal{S}(N)$ and $v \in \mathcal{G}^N(S)$. The core of v is defined as

$$\mathcal{C}_S(v) := \left\{ x \in \mathbb{R}^N \;\middle|\; \begin{array}{c} x(N) = v(N) \text{ and} \\ \text{for every } E \in \Omega_S : x(E) \geqq v(E) \end{array} \right\}.$$

We mention that Faigle (1989) studied conditions that imply the nonemptiness of the core with restricted coalition formation. The next consequence of proposition 2.2 and Theorem 2.3 shows that superiors indeed may exploit subordinates in core payoff vectors.

Corollary 2.5: Let $S \in \mathcal{S}(N)$ and $v \in \mathcal{G}^N(S)$. Then

$$\mathcal{C}_S(v) = \mathcal{C}_S(v) + \text{Cone}\left\{e^j - e^i \;\middle|\; i \in N \textit{ and } j \in S^{-1}(i)\right\}.$$

3 Strict Hierarchies and Discerning Lattices

In this section we turn to the analysis of strict hierarchies as represented by acyclic permission structures. These are also the objects of analysis in Gilles and Owen (1994). There it is remarked that strict hierarchies are the appropriate tools to describe a hierarchical organization structure as encountered in the theory of the firm.

Our first objective is to strengthen the assertion as stated in Proposition 2.2. For that purpose we introduce discerning collections of autonomous coalitions.

Definition 3.1: Let $\Omega \subset 2^N$ be some collection of autonomous coalitions. Ω is discerning if for all $i, j \in N$ there exists a coalition $E \in \Omega$ with either $i \in E$ and $j \notin E$ or $i \notin E$ and $j \in E$.

We remark that the requirement, that a collection of coalitions is discerning, is similar to the T_0-separation property on topological spaces. As a consequence we mention that any discerning lattice therefore is a topology satisfying the T_0-separation property. In our game theoretic setting it however represents the situation that any two players can be distinguished socially in the coalitional structure as describing by Ω.

Of particular interest in our analysis of strict hierarchies is the possibility to order the player set such that one describes a growing sequence of autonomous coalitions. This implies that one can compute the marginal contribution of players in that particular order.

Definition 3.2: Let $\Omega \subset 2^N$ be some collection of autonomous coalitions in N. A Weber string in Ω is a permutation $\omega : N \to N$ such that for every $k \in N = \{1, \ldots, n\}$

$$\{\omega(1), \ldots, \omega(k)\} \in \Omega.$$

Another notation for a Weber string in Ω is given by a collection of autonomous coalitions $(W_i)_{i \in N}$, where there exists a permutation $\omega : N \in N$ with for every $i \in N$: $W_i := \{\omega(1), \ldots, \omega(j)\} \in \Omega$, where $j \in N$ is such that $\omega(j) = i$. Thus, in a Weber string $(W_i)_{i \in N}$ the coalition $W_i \in \Omega$ is such that it is the only coalition in the string for which $i \in W_i$ and $W_i \setminus \{i\}$ is also in the string. This implies that the participation of player i in the autonomous coalitions $W_i \setminus \{i\} \in \Omega$ is feasible. The marginal contribution of player i to the coalition $W_i \setminus \{i\}$ is therefore attainable. Thus, to every Weber string there corresponds a vector of attainable marginal contributions.

Proposition 3.3: Let $\Omega \subset 2^N$ be a lattice of autonomous coalitions in N. Then the following statements are equivalent:

(i) *Ω is discerning.*
(ii) *For all $i, j \in N : D_i^\Omega \neq D_j^\Omega$.*
(iii) *For every non-empty coalition $E \in \Omega$ there exists $i \in E$ with $E \setminus \{i\} \in \Omega$.*
(iv) *There exists a Weber string in Ω.*

(v) *For every player* $i \in N : D_i^{\Omega} \setminus \{i\} \in \Omega$.
(vi) C_{Ω} *does not contain a non-trivial linear subspace as a subset.*
(vii) *There exists an acyclic permission structure* $S \in \mathcal{S}_a(N)$ *such that* $\Omega_S = \Omega$.

Proof:

(i) implies (ii)
Let $i, j \in N$. Without loss of generality we may assume that there is an $E \in \Omega$ with $i \in E$ and $j \notin E$. Then $D_i^{\Omega} \subset E$ and $D_j^{\Omega} \not\subset E$. Thus $D_i^{\Omega} \neq D_j^{\Omega}$.

(ii) implies (iii)
Suppose by contradiction that there exists $E \in \Omega$ with for every $i \in E : E \setminus \{i\} \notin \Omega$. The union $\bigcup_{j \in E, j \neq i} D_j^{\Omega}$ is by definition an element of Ω containing $E \setminus \{i\}$ and is contained in E. So, $E = \bigcup_{j \in E, j \neq i} D_j^{\Omega}$. Thus for every $i \in E$ there exists a $j \in E \setminus \{i\}$ with $i \in D_j^{\Omega}$.
Since E is finite there exists a sequence $i_1, \ldots, i_K$ in E with $i_{k+1} \in D_{i_k}^{\Omega}$, $1 \leqq k \leqq K-1$, and $i_K \in D_{i_1}^{\Omega}$. By definition therefore $D_{i_1}^{\Omega} = D_{i_2}^{\Omega} = \ldots = D_{i_K}^{\Omega}$. This is a contradiction of (ii).

(iii) implies (iv)
Let $E_1 = N \in \Omega$. Then by (iii) there is an $i_1 \in E_1$ with $E_2 := E_1 \setminus \{i_1\} \in \Omega$. Let $E_k \in \Omega$, then by (iii) there is $i_{k+1} \in E_k$ such that $E_{k+1} := E_k \setminus \{i_{k+1}\} \in \Omega$. Thus we derive an ordering $i_1, \ldots, i_n$ of N, which corresponds to the desired Weber string.

(iv) implies (i)
This immediately follows from the definition.
We now have established equivalence of (i) – (iv). Next we show that (v) – (vii) are equivalent to (ii).

(ii) implies (v)
$j \in D_i^{\Omega} \setminus \{i\}$ implies that $D_j^{\Omega} \subset D_i^{\Omega}$. Now $i \notin D_j^{\Omega}$, since otherwise $D_i^{\Omega} \subset D_j^{\Omega}$ implying equality, and, hence, contradicting (ii). Therefore,

$$D_i^{\Omega} \setminus \{i\} = \bigcup_{\substack{j \in D_i^{\Omega} \\ j \neq i}} D_j^{\Omega} \in \Omega.$$

(v) implies (vi)
Suppose there exists a $y \in \mathbb{R}^N$ with $\{y, -y\} \subset C_{\Omega}$. Then $y(N) = -y(N) = 0$ and for every $E \in \Omega : y(E) \geqq 0$ as well as $-y(E) \geqq 0$. Thus for any $i \in N : y(D_i^{\Omega}) = 0$ and by (v): $y(D_i^{\Omega} \setminus \{i\}) = 0$. Therefore for any $i \in N : y_i = 0$.

(vi) implies (ii)
If $D_i^{\Omega} = D_j^{\Omega}$, then by Theorem 2.3 both $e^i - e^j$ and $e^j - e^i$ are in C_{Ω}. This contradicts (vi).

(ii) is equivalent to (vii)
By Proposition 2.2 there exists a permission structure $S \in \mathcal{S}(N)$ such that $\Omega_S = \Omega$.

Gilles and Owen (1994) show that S is acyclic if and only if $D_i^{\Omega_S} \neq D_j^{\Omega_S}$ for any $i \neq j$. This implies the equivalence. □

If $X \subset \mathbb{R}^N$ is some set of payoff vectors, then by co X we denote its *convex hull*, i.e., the smallest convex subset in $\mathbb{R}^N$ that contains X. For any permission structure and any TU-game on that permission structure we now may define the Weber set as the convex hull of all attainable marginal contribution vectors corresponding to the Weber strings in the discerning lattice of autonomous coalitions.

Definition 3.4: Let $S \in \mathcal{S}_a(N)$ be an acyclic permission structure and let $v \in \mathcal{G}^N(S)$ be a TU-game on S. The Weber set of v is given by the set

$$\mathcal{W}_S(v) := \text{co}\{x^\omega \in \mathbb{R}^N \mid \omega \text{ is a Weber string in } \Omega_S\},$$

where for any Weber string ω in Ω_S we define x^ω by

$$x_i^\omega := v(W_i) - v(W_i \setminus \{i\})$$

with $W_i := \{\omega(1), \ldots, \omega(j)\} \in \Omega_S$, where $j \in N$ is such that $\omega(j) = i$.

The Weber set turns out to have some appealing properties. Let $S \in \mathcal{S}_a(N)$ be an acyclic permission structure. For every autonomous coalition $E \in \Omega_S$ we define the *S-unanimity game* belonging to E by $u_E^S \in \mathcal{G}^N(S)$:

$$u_E^S(F) = \begin{cases} 1 & E \subset F,\ F \in \Omega_S \\ 0 & E \not\subset F,\ F \in \Omega_S \end{cases}.$$

It can easily be deduced that the Weber set of u_E^S is given by

$$\mathcal{W}_S(u_E^S) = \text{co}\left\{e^i \;\middle|\; i \in \partial_S(E)\right\} \text{ where } \partial_S(E) = \{i \in E \mid S(i) \cap E = \emptyset\}.$$

Hence, in the Weber set of a unanimity game the payoff is distributed completely over the players at the lowest level in the hierarchical structure restricted to the coalition E.

Let $v \in \mathcal{G}^N(S)$ be a TU-game on S. Following the analysis of Gilles, Owen and van den Bring (1992) and Derks and Peters (1993) the game v can be expressed in the S-unanimity basis of $\mathcal{G}^N(S)$ by

$$v = \sum_{E \in \Omega_S} \Delta_v^S(E) \cdot u_E^S,$$

where recursively the hierarchical dividends are given by $\Delta_v^S(\emptyset) = 0$ and for every $E \in \Omega_S \setminus \{\emptyset\}$

$$\Delta_v^S(E) := v(E) - \sum_{F \subset E,\, F \neq E} \Delta_v^S(F).$$

From this and the additivity of the marginal contribution vectors we deduce that

$$\mathcal{W}_S(v) \subset \sum_{E \in \Omega_S} \Delta_v^S(E) \cdot \mathcal{W}_S(u_E^S).$$

Hence, we conclude that the allocations in the Weber set of v as much as possible allocate payoffs to the subordinates in the hierarchy.

The next result generalizes the inclusion result of Weber (1988) that the core of a TU-game is a subset of the convex hull of marginal contribution vectors. The proof of this result follows the line as developed in Derks (1992).

Theorem 3.5: Let $S \in \mathcal{S}_a(N)$ be an acyclic permission structure. Then for any TU-game $v \in \mathcal{G}^N(S)$ on Ω_S

$$\mathcal{C}_S(v) \subset \mathcal{W}_S(v) + \mathcal{C}_{\Omega_S}.$$

Proof: Let $\Omega := \Omega_S$. Then by Proposition 3.3 Ω is a discerning lattice and contains at least one Weber string. Furthermore, let $v : \Omega \to \mathbb{R}$ be a TU-game on S.
Suppose that there exists a core allocation $x \in \mathcal{C}_S(v)$ such that $x \notin \mathcal{W}_S(v) + \mathcal{C}_\Omega$. By Theorem 2.3 the set $\mathcal{W}_S(v) + \mathcal{C}_\Omega$ is polyhedral and by Proposition 3.3 (vi) it does not contain a nontrivial linear subspace. Thus x is an extreme element of

$$\mathrm{co}\Big[\{x\} \cup (\mathcal{W}_S(v) + \mathcal{C}_\Omega)\Big].$$

The normals of supporting hyperplanes in an extreme element of a polyhedral set are well known to form a full dimensional cone. Therefore, there exists a normal, say $p \in \mathbb{R}^N$, with non-equal coefficients such that for each $y \in \mathcal{W}_S(v)$ and $y' \in \mathcal{C}_\Omega$

$$p \cdot x < p \cdot (y + y').$$

Now by Theorem 2.3 for each $i \in N$, $j \in D_i^\Omega$, and $M \geqq 0$ it holds that for every $y \in \mathcal{W}_S(v)$:

$$\begin{aligned} p \cdot x &< p \cdot y + Mp \cdot (e^j - e^i) \\ &= p \cdot y + M(p_j - p_i) \end{aligned} \tag{3}$$

Thus, we conclude that for every $i \in N$ and $j \in D_i^\Omega$:

$$p_j \geqq p_i. \tag{4}$$

Label the players in $N = \{1, \ldots, n\}$ such that $p_1 > p_2 > \ldots p_n$. Next consider for every $k \in N$ the coalition $W_k := \{1, \ldots, k\}$. If $i \in W_k$, then $p_i \geqq p_k$. Using (4) we have that $p_j \geqq p_k$ for all $j \in D_i^\Omega$. Hence, $D_i^\Omega \subset W_k$ for all $i \in W_k$. This implies that

$$W_k = \bigcup_{i \in W_k} D_i^\Omega \in \Omega.$$

Hence, we conclude that the collection $(W_k)_{k \in N}$ represents a Weber string in Ω. The corresponding marginal contribution vector $y \in \mathbb{R}^N$ belonging to v is therefore an element of $\mathcal{W}_S(v)$. Thus,

$$\begin{aligned} p \cdot x &< p \cdot y \\ &= \sum_{i=1}^{n} \left[v(W_i) - v(W_{i-1}) \right] p_i \\ &= v(N) p_n + \sum_{i=1}^{n-1} v(W_i)(p_i - p_{i+1}). \end{aligned} \tag{5}$$

Since $x \in \mathcal{C}_S(v)$ and $W_i \in \Omega$, $i \in N$, we have $x(N) = v(N)$ and $x(W_i) \geqq v(W_i)$, $i \in N \setminus \{n\}$. Also, for any $i \in N \setminus \{n\} : p_i - p_{i+1} \geqq 0$. Therefore, (5) may be transformed to

$$\begin{aligned} p \cdot x &< \sum_{j=1}^{n} x_j p_n + \sum_{i=1}^{n-1} \sum_{j=1}^{i} x_j (p_i - p_{i+1}) \\ &= \sum_{i=1}^{n} \sum_{j=1}^{i} x_j p_i - \sum_{i=2}^{n} \sum_{j=1}^{i-1} x_j p_i \\ &= x_1 p_1 + \sum_{i=2}^{n} p_i x_i = p \cdot x. \end{aligned}$$

This is impossible. Thus we conclude that $x \in \mathcal{W}_S(v) + \mathcal{C}_\Omega$. □

As a corollary to Theorem 3.5 we immediately have the original result by Weber (1988). To state this corollary we introduce the *trivial permission structure* as $S_0 \in \mathcal{S}_a(N)$ given by $S_0(i) = \emptyset$, $i \in N$. Obviously, $\Omega_{S_0} = 2^N$. Hence, $C_{\Omega_{S_0}} = \{0\}$ and any permutation ω of N is a Weber string in Ω_{S_0}. From this we conclude that $\mathcal{G}^N(S_0)$ is the collection of all TU-games $v : 2^N \to \mathbb{R}$ and for any TU-game v we have that $\mathcal{C}_{S_0}(v)$ is the traditional core and $\mathcal{W}_{S_0}(v)$ is the traditional Weber set.

Corollary 3.6: For any TU-game $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$ the core is a subset of the Weber set, i.e., $\mathcal{C}_{S_0}(v) \subset \mathcal{W}_{S_0}(v)$.

4 Convex Games

Shapley (1971) introduced the collection of convex TU-games as a class of games of which the core has certain appealing geometric properties. Here we generalize this definition to convexity within the collection of all TU-games with respect to some permission structure.

Definition 4.1: Let $S \in \mathcal{S}(N)$ be some permission structure on the player set N. A TU-game $v \in \mathcal{G}^N(S)$ is convex on Ω_S if for every $E, F \in \Omega_S$:

$$v(E) + v(F) \leqq v(E \cup F) + v(E \cap F).$$

In the sequel we denote by $\mathcal{G}_c^N(S)$ the collection of all convex TU-games on some permission structure $S \in \mathcal{S}(N)$. The definition of convexity as given above is clearly a generalization of the definition as given in Shapley (1971). Namely, Shapley introduced convexity essentially within the context of the trivial permission structure $S_0 \in \mathcal{S}_a(N)$.

The next theorem generalizes the results of Shapley (1971) and Ichiishi (1981).

Theorem 4.2: Let $S \in \mathcal{S}_a(N)$ be an acyclic permission structure on N. A game $v \in \mathcal{G}^N(S)$ is convex on Ω_S if and only if

$$\mathcal{C}_S(v) = \mathcal{W}_S(v) + \mathcal{C}_{\Omega_S}.$$

The proof of Theorem 4.2 is developed through a sequence of intermediate results, stated in the following lemmas. These lemmas involve a further study of discerning lattices. The outline of the proof closely follows the proof of the original result by Shapley (1971).

Lemma 4.3: Let $\Omega \subset 2^N$ be a discerning lattice on the player set N and let $E \in \Omega \setminus \{\emptyset\}$. Then $\Omega(E) := \Omega \cap 2^E$, respectively $\Omega' := \{N \setminus F \mid F \in \Omega\}$, are discerning lattices on the player set E, respectively the player set N.

Proof: Both $\Omega(E)$ and Ω' are obviously lattices with respect to E and N respectively. Using Theorem 3.5 (iv) it is straightforward to see that $\Omega(E)$ possess a Weber string, implying that $\Omega(E)$ is a discerning lattice on E.

Furthermore, for every $i, j \in N$ we may without loss of generality assume that there exists a coalition $E \in \Omega$ with $i \in E$ and $j \notin E$. Hence, $i \notin N \setminus E$ and $j \in N \setminus E$. Thus, Ω' is discerning as well. □

Lemma 4.4: Let Ω be a discerning lattice on the player set N. For each coalition $E \in \Omega$ there is a Weber string in Ω of which E is one of its members.

Proof: Let $E \in \Omega$ where Ω is a discerning lattice. Then there exist Weber strings in $\Omega(E)$ and $\Omega'(N \setminus E)$, say $(W_i')_{i \in E}$ and $(W_j'')_{j \in N \setminus E}$ respectively. Then $(W_i)_{i \in N}$ is Weber string in Ω, where

$$W_i = \begin{cases} W_i' & i \in E \\ \{i\} \cup (N \setminus W_i'') & i \notin E \end{cases}.$$

Clearly, if $i \in E$ then $W_i \in \Omega$; furthermore, $i \in W_i' = W_i$ and for player i such that $W_i = W_i' \neq \{i\}$ there is a player $j \in E$ with $W_i \setminus \{i\} = W_i' \setminus \{i\} = W_j' = W_j$. Now if $i \notin E$ then either $W_i = \{i\} \cup N \setminus W_i'' = N \setminus (W_i'' \setminus \{i\}) = N \setminus W_j''$ for a player $j \notin E$, or

W_i equals $N \setminus \emptyset = N$, and sine $(\Psi')'$ equals Ψ for each lattice Ψ we must have $W_i \in \Omega$ here. Furthermore, $i \in W_i$ and

- if $W_i'' = N \setminus E$ then $W_i \setminus \{i\} = N \setminus W_i'' = E = W_j' = W_j$ for some $j \in E$;
- if $W_i'' \neq N \setminus E$ then there is a player $k \notin E$ with $W_i'' = W_k'' \setminus \{k\}$; hence, $W_i \setminus \{i\} = N \setminus W_i'' = N \setminus (W_k'' \setminus \{k\}) = \{k\} \cup N \setminus W_k'' = W_k$.

We conclude that for all players $i \in N$ we have $W_i \in \Omega$ and if $W_i \neq \{i\}$ there is a player j such that $W_i \setminus \{i\} = W_j$. □

Using Lemma 4.3 and Lemma 4.4 we immediately derive the following conclusion.

Lemma 4.5: Let $E_1, \ldots, E_k$ be elements in some discerning lattice Ω on the player set N such that $E_1 \subset \cdots \subset E_k$. Then there exists a Weber string in Ω of which $E_1, \ldots, E_k$ are members.

We are now in the position to proof the main result of this section.

Proof of Theorem 4.2:

Only if:
Suppose that $v \in \mathcal{G}_c^N(S)$. By Theorems 2.3 and 3.5 we only have to show that every Weber allocation $x \in \mathcal{W}_S(v)$ belongs to $\mathcal{C}_S(v)$.
So, let $x \in \mathcal{W}_S(v)$ correspond to some Weber string $(W_i)_{i \in N}$ with for every $i \in N$

$$x_i = v(W_i) - v(W_i \setminus \{i\}).$$

Since v is convex on $\Omega = \Omega_S$ for every $E \in \Omega$ and $i \in E$ we have that

$$v(W_i) + v(E \cap (W_i \setminus \{i\})) \geqq v(E \cap W_i) + v(W_i \setminus \{i\}).$$

Thus, for each $i \in E$:

$$x_i \geqq v(E \cap W_i) - v(E \cap W_i \setminus \{i\})).$$

Without loss of generality we may assume that N is labeled such that players in E are labeled first and $\{1, \ldots, k\} \subset W_k$ for each $k \in E$. Then

$$\begin{aligned} \sum_{i \in E} x_i &= \sum_{k=1}^{|E|} x_k \geqq \sum_{k=1}^{|E|} v(E \cap W_k) - v(E \cap (W_k \setminus \{k\})) \\ &= \sum_{k=1}^{|E|} v(\{1, \ldots, k\}) - \sum_{k=1}^{|E|-1} v(\{1, \ldots, k\}) \\ &= v(\{1, \ldots, |E|\}) = v(E). \end{aligned}$$

Since $E \in \Omega$ is chosen arbitrarily we may conclude that $x \in \mathcal{C}_S(v)$.

If:
Suppose that $\mathcal{C}_S(v) = \mathcal{W}_S(v) + \mathcal{C}_\Omega$, where $\Omega = \Omega_S$. Take $E, F \in \Omega$. Since Ω is discerning by application of Lemma 4.5 there exists a Weber string in Ω, say $(W_i)_{i \in N}$, such that it contains the coalitions $E \cup F$ and $E \cap F$. Let $x \in \mathcal{W}_S(v)$ be the corresponding Weber allocation – or marginal contribution vector. It is evident that $x(E \cap F) = v(E \cap F)$ as well as $x(E \cup F) = v(E \cup F)$. By hypothesis $x \in \mathcal{C}_S(v)$, which implies that

$$\begin{aligned} v(E) + v(F) &\leqq \sum_{i \in E} x_i + \sum_{i \in F} x_i \\ &= \sum_{i \in E \cap F} x_i + \sum_{i \in E \cup F} x_i = v(E \cap F) + v(E \cup F). \end{aligned}$$

This proves the convexity of v. □

5 Cores with Limited Exploitation

In this section we take a certain permission structure $S \in \mathcal{S}(N)$. The lattice of conjunctively autonomous coalitions is indicated by $\Omega := \Omega_S$. Finally, by $v \in \mathcal{G}^N(S)$ we indicate a TU-game on Ω.

In the previous sections of this paper we implicitly assumed that the subordinates in the hierarchy S cannot "leave" the game v, i.e., there is no reservation value for each player. This hypothesis forms the foundation of the definition of the core $\mathcal{C}_S(v)$ of the game v, since in that definition it is assumed implicitly that a non-formable coalition $E \notin \Omega$ has been assigned the value $-\infty$. That is, a non-formable coalition $E \notin \Omega$ cannot be assigned any value.

In Gilles et al. (1992) it is however assumed that non-formable coalitions are assigned the value 0. Thus, it is assumed implicitly that every player $i \in N$ is able to leave the game without any loss or gain. Application of the assumption that players may "leave" the permission structure S and the game v leads to a different core-like solution concept. In order to analyze this core-like concept we introduce for every coalition $E \subset N$ its sovereign part by

$$\sigma(E) := \cup\{F \in \Omega \mid F \subset E\} \in \Omega$$

as the largest autonomous subcoalition of E with respect to the permission structure S. Clearly, E is autonomous if and only if $\sigma(E) = E$.

Now we are in the position to formulate the hypothesis that players $i \in N$ may leave the game $v \in \mathcal{G}^N(S)$ by assigning to any arbitrary coalition $E \subset N$ the worth of its sovereign part. Formally, the TU game $\omega : 2^N \to \mathbb{R}$ is the *conjunctive extension* of $v \in \mathcal{G}^N(S)$ if for every $E \subset N : \omega(E) = v(\sigma(E))$. The conjunctive extension treats a non-formable coalition as if it can form partially. Since $\emptyset \in \Omega$ it is clear that each player $i \in N$, who is subordinate to some other player in S, has a reservation value of $\omega(\{i\}) = v(\emptyset) = 0$. This prevents the infinite exploitation of i by his superiors. This is reflected in the following definition.

Definition 5.1: Let $S \in \mathcal{S}(N)$ and $v \in \mathcal{G}^N(S)$. The core with limited exploitation – or L-core – of v is given by

$$\mathcal{C}_S^\ell(v) := \left\{ x \in \mathbb{R}^N \,\middle\|\, \begin{array}{c} x(N) = v(N) \text{ and} \\ \text{for every } E \subset N : x(E) \geqq v(\sigma(E)) \end{array} \right\}.$$

It may be obvious that the L-core of a TU-game v on some permission structure S is the traditional core of its conjunctive extension. Hence, if ω is the conjunctive extension of v, then $\mathcal{C}_S^\ell(v) = \mathcal{C}_{S_0}(\omega)$. The next proposition gives a complete characterization of the L-core.

Proposition 5.2: $S \in \mathcal{S}(N)$ and $v \in \mathcal{G}^N(S)$. The L-core of v on the permission structure S is given by

$$\mathcal{C}_S^\ell(v) = \mathcal{C}_S(v) \cap (\mathbb{R}^+_{N \setminus B_S} \times \mathbb{R}_{B_S}),$$

where $B_S = \{i \in N \mid S^{-1}(i) = \emptyset\}$.

The proof of Proposition 5.2 is a direct consequence of the fact that the inequalities in the definition of $\mathcal{C}_S^\ell(v)$ are precisely given by the inequalities as given in the definition of $\mathcal{C}_S(v)$ and the requirements that for every $i \in N \setminus B_S : x(\{i\}) \geqq v(\emptyset) = 0$.

For certain TU-games we may raise some doubts concerning the usefulness of the L-core as a useful solution concept. For that purpose consider $N := \{1, 2, 3, 4\}$ and $S \in \mathcal{S}_a(N)$ given by $S(1) = \{2, 3\}$, $S(2) = S(3) = \{4\}$, and $S(4) = \emptyset$. From this it immediately follows that the collection of autonomous coalitions is given by

$$\Omega_S = \{N, \{1, 2, 3\}, \{1, 2\}, \{1, 3\}, \{1\}, \emptyset\}.$$

Next consider the following game $v \in \mathcal{G}^N(S)$ on Ω_S given by $v(\emptyset) = v(1) = 0$ and $v(12) = v(13) = v(123) = v(N) = 1$. Note that v is not convex.
We conclude that the following properties hold:

(i) $\mathcal{W}_S(v) = \{(0, \lambda, 1 - \lambda, 0) \mid 0 \leqq \lambda \leqq 1\}$
(ii) $\mathcal{C}_{\Omega_S} = \text{Cone}\{(1, -1, 0, 0), (1, 0, -1, 0), (0, 1, 0, -1), (0, 0, 1, -1)\}$
(iii) $\mathcal{C}_S(v) = \text{Co}\{(1, 0, 0, 0), (0, 1, 1, -1)\} + \mathcal{C}_{\Omega_S}$
(iv) $\mathcal{C}_S^\ell(v) = \{(1, 0, 0, 0)\}$
(v) $\mathcal{C}_S(v) \cap \mathcal{W}_S(v) = \emptyset$

These properties show that in this case the L-core does not allow that the productive players 2 and 3 obtain any payoff. The discrepancy between the Weber set, which proposes allocations of the total output between players 2 and 3 only, and the L-core is therefore significant. The core of this game, however, describes a compromise between both the L-core and the Weber set. This shows that observing the L-core only is not sufficient for a proper insight in the coalitional possibilities in the game v.

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