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AN ARITHMETIC PROPERTY OF PROFINITE GROUPS

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We intend to generalize a crucial lemma of [4] to prove a somewhat surprising arithmetic property of profinite groups; namely, that a profinite group G has nontrivial p-Sylow-subgroups for only a finite number of primes if and only if this is true for its procyclic subgroups. This will yield as a corollary that every profinite torsion group has finite exponent if and only if this is true for its Sylow-subgroups, a result also contained in [4].

Our main result is:

Theorem. If a profinite group G has nontrivial p-Sylow-subgroups for infinitely many primes p, then it has a procyclic subgroup with the same property.

Throughout the paper we adopt the notation of [4]. Accordingly we say $G \in \pi(\infty)$ if G has nontrivial p-Sylow-subgroups for infinitely many primes p. For the notion of p-Sylow-subgroup see [6]. The proof of our theorem rests on [4] insofar as we assume G to be prosolvable and to have a Sylow-tower. However, the difference in the situation now is that the Sylow-subgroups are not torsion groups. This makes it impossible to apply Thompson's Fixed-Point-Theorem. Unfortunately such powerful results from finite group theory as in F. Gross [2], or T. Berger [1] which bound the solvability length or the Fitting length, respectively, of groups under fixed-point-free action do not seem to yield the

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result immediately. Our proof essentially produces a torsion-free abelian subgroup of rank 2 whose representations on the Frattini-Factors of Sylow-subgroups in the Sylow-tower are studied, yielding an element with a $\pi(\infty)$-centralizer in G. This will be enough to prove the whole theorem.

Lemma 1. *Let $G \in \pi(\infty)$. There exists a subgroup $G_0 :=$ $= P_1 P_2 P_3 \ldots$ of G such that:*

(i) *P_i is a pro-p_i-group (p_i a prime);*

(ii) *$P_i \triangleleft P_i P_j$ for $i > j$;*

(iii) *G_0 is homeomorphic to $\prod_{i=1}^{\infty} P_i$.*

Proof. This is Lemma 8 in [4].

Lemma 2. *The following assertions about a profinite, prosolvable group G are equivalent:*

(a) *Each closed $\pi(\infty)$-subgroup K of G contains an element $x \neq e$ such that $C_K(x) \in \pi(\infty)$.*

(b) *Each closed $\pi(\infty)$-subgroup of G contains an abelian $\pi(\infty)$-subgroup A.*

Proof. This is Lemma 9 in [4].

Because of Lemmata 1,2, in order to prove the Theorem it suffices to show:

<u>Proposition</u> 1. <u>Let</u> G_o <u>be as in Lemma</u> 1; <u>then there exists</u> $x \in G_o \setminus \{e\}$ <u>such that</u> $C_{G_o}(x) \in \pi(\infty)$.

<u>Lemma</u> 3. <u>Let</u> G_o <u>be as in Lemma</u> 1. <u>If there exists an element</u> $x \in G_o \setminus \{e\}$ <u>of finite order then</u> $C_{G_o}(y) \in \pi(\infty)$ <u>for some</u> $y \in G_o \setminus \{e\}$.

Proof. W.l.o.g. let $o(x) = p_1$. Then x acts via conjugation on $G_l := P_l P_{l+1} P_{l+2} \dots$ for $l = 2, 3, \dots$. If, for fixed l, x acts fixed-point-free on G_l, then by the profinite version of Thompson's Fixed-Point-Theorem [4, Theorem 2, p. 405] it follows that $G_l = \prod_{k=l}^{\infty} P_k$ is nilpotent, and for $t \in \prod_{k=l}^{\infty} P_k$ we have $C_{G_o}(t) \in \pi(\infty)$. The other possibility is that x does not act fixed-point-free on infinitely many of the G_l's. Then there exist $x_l \in G_l \setminus \{e\}$, such that $[x, x_l] = e, l = 2, 3, \dots$, and therefore $C_{G_o}(x) \geq \langle x_2, x_3, \dots \rangle \in \pi(\infty)$.

It is enough to assume G_o to be torsion-free.

<u>Definition</u>. <u>For a profinite group</u> G, <u>we say that</u> G <u>is of rank</u> l, <u>if a minimal topologically generating set</u> X <u>for</u> G <u>has cardinality</u> l. <u>The abelian subgroup rank of</u> G <u>is defined to be the maximum of the ranks of the closed abelian subgroups</u>.

<u>Lemma</u> 4. <u>Let</u> G_o <u>be as in Lemma</u> 1; <u>assume that the abelian subgroup rank is one</u>. <u>Then there exists an element</u> $x \in G_o \setminus \{e\}$ <u>such that</u> $C_{G_o}(x) \in \pi(\infty)$.

Proof. Choose $x_1 \in P_1$. Let $j \geq 2$; then, since $|\langle x_1 \rangle| = \infty$, $\langle x_1 \rangle P_j \cong \langle x_1 \rangle \rhd P_j$ (semidirect product) does not have the structure of a profinite Frobenius-group [3, Th. 3.6.], so that there exist elements $a_1 \in \mathbb{N}$ such that $x_1^{p^{a_j}}$ has a fixed-point in P_j. Let b_j be minimal with respect to this property; then $y_j = x_1^{n_j}$, where $n_j = p_1^{b_j - 1}$, induces an automorphism of prime order p_1 acting fixed-point-free on P_j; by [4, Th. 2] P_j is nilpotent and therefore $Z(\Phi(P_j))$ is procyclic so that a profinite version of [5, III, 7.8.c] yields that $\Phi(P_j)$ is procyclic. Here, $\Phi(P_j)$ denotes the Frattini-subgroup of P_j. We note that from P_j torsion-free $\Phi(P_j) \neq \{e\}$ follows. Let $H_o = \Phi(P_2)\Phi(P_3)\ldots$; then $\Phi(P_i) \lhd \lhd \Phi(P_i)\Phi(P_j)$ for $i > j$ and H_o is homeomorphic to $\prod_{i \geq 2} \Phi(P_i)$. So H_o has procyclic Sylow-subgroups and, by Zassenhaus' Theorem [5, p. 420], $H_o = \langle a,b \rangle \trianglerighteq \langle b \rangle \trianglerighteq H_o'$, for suitably chosen elements a,b in H_o follows. Clearly if neither $\langle b \rangle$ nor $\langle a \rangle \in \pi(\infty)$ then $H_o/\langle b \rangle \notin \pi(\infty)$, so that $H_o \notin \pi(\infty)$, a contradiction.

<u>Lemma</u> 5. <u>Let</u> G_o <u>be as in Lemma</u> 1, <u>torsion-free and having abelian subgroup rank larger than one</u>. <u>There exists a non-trivial element in</u> G_o <u>having a</u> $\pi(\infty)$-<u>centralizer</u>.

Proof: W.l.o.g. we may assume that there exists $A = \langle x,y \rangle \cong \cong \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_1}$, $A \leq P_1$. We first claim that in every P_j, $j \geq 2$, there exist finitely generated A-invariant subgroups S_j. Fix $j \geq 2$, since $\langle y \rangle$ does not act elementwise fixed-point-free on P_j by conjugation, there exists $r \geq 0$, such that y^m, $m = p^r$, has a fixed point $z \in P_j$; since $z^{y^m x} = z^x$ the group B generated by all conjugates of z with elements in $\langle x, y^m \rangle$ consists of fixed-points for y^m only. In B we pick t such that for some $s > 0$ and $n = p_1{}^s, t^x = t$. Obviously S_j, the subgroup generated by all A-conjugates of t, is a finitely generated A-invariant subgroup of P_j.

In order to finish the proof it suffices to find a nontrivial element $a \in A$ having fixed points in infinitely many of the S_j's $(j \geq 2)$. Since for each $a \in A$ the fixed-points of the induced automorphism $\bar{a}$ on the (finite) Frattini-factor $S_j/\Phi(S_j)$ can be lifted to fixed-points of a in S_j by [4, Lemma 1], it suffices to find $a \in A$ such that a induces a non-fixed-point-free action on infinitely many of the $V_j := S_j/\Phi(S_j)$ which are, respectively, finite vector spaces over $GF(p_j)$. Note that conjugation by elements of A induces a representation ρ_j of A on V_j. Put $\bar{V}_j := V_j \otimes_{GF(p_j)} \overline{GF(p_j)}$, where $\overline{GF(p_j)}$ denotes the algebraic closure of $GF(p_j)$; then ρ_j may be viewed in the canonical way as a representation of A on $\bar{V}_j$. Then, by the foregoing, for an element $e \neq a \in A$ having a fixed-point in S_j means that either $a \in \ker \rho_j$ or that $\rho_j(a)$ has eigenvalue 1. Assume, for an infinite subset I of $\mathbb{N}$, that $a \in \bigcap_I \ker \rho_j$ is nontrivial; then we are done. We also note for later use that $\rho_j(a)$ has a fixed-point in V_j if and only if it has one in $\bar{V}_j$. So for every infinite index set $I \subseteq N$ one has $\bigcap_I \ker \rho_j = \{e\}$. W.l.o.g. we may assume that $d_j := o(\rho_j(x)) \geq o(\rho_j(y))$ for $j \in M$, an infinite subset of $\mathbb{N}$. Since ρ_j splits over $\bar{V}_j$ one can find $u_j \in \overline{GF(p_j)}$ such that $\rho_j(x) = \text{diag}(u_j^{\alpha_{ij}}/i = 1,2,\ldots,t_j)$ and $\rho_j(y) = = \text{diag}(u_j^{\beta_{ij}}/i = 1,2,\ldots,t_j)$ where $t_j = \dim \bar{V}_j$ and where $\alpha_{ij}, \beta_{ij} \in \mathbb{Z}/p_j^{d_j}\mathbb{Z}$. W.l.o.g. one may assume $\alpha_{ij} = 1$ for $j \in L$, where L is an infinite subset of M. Then for $j \in L$ either $\rho_j(yx^{-\beta_{ij}})$ has a fixed-point in $\bar{V}_j$ or $yx^{-\beta_{ij}}$ is in $\ker \rho_j$. If the latter happens for an infinite subset $J \subseteq L$ then $\rho_j(A)$ is cyclic and because of $\bigcap_{j \in J} \ker \rho_j = \{e\}$ we conclude $y \in \langle x \rangle$, i.e. A is procyclic, a contradiction. Therefore there exists an infinite subset $K \subseteq L$ such that for $j \in K$ and the p_1-adic exponents γ_{1j} one has $yx^{-\gamma_{1j}} \notin \ker \rho_j$ and where $\rho_j(yx^{-\gamma_{1j}})$ has fixed-points in $\bar{V}_j$. Let b be any cluster-point of the set $\{\gamma_{ij}/j \in K\} \subseteq \mathbb{Z}_{p_1}$; then $\rho_j(yx^{-b}) = \rho_j(yx^{-\gamma_{1j}})$ has a fixed-point in V_j, for all $j \in K$; finally, let $a := xy^{-b}$. This proves the Theorem.

As an immediate consequence we state without proof.

Corollary. *Let G be a profinite torsion group. Then it has finite exponent if and only if its Sylow-subgroups have finite exponent.*

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