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Analytic Torsion and Holomorphic Determinant Bundles

II. Direct Images and Bott-Chern Forms

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Abstract. In this paper, we derive the main properties of Kähler fibrations. We introduce the associated Levi-Civita superconnection to construct analytic torsion forms for holomorphic direct images. These forms generalize in any degree the analytic torsion of Ray and Singer. In the case of acyclic complexes of holomorphic Hermitian vector bundles, such forms are calculated by means of Bott-Chern classes.

Contents

Introduction

This is the second of a series of three papers devoted to the study of holomorphic determinant bundles and direct images. Parts I and III of this work will be referred

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to as [BGS 1] and [BGS 3]. Also the Introduction of [BGS 1] contains a general description of our results. We will refer to it when necessary.

Let $\pi: M \to B$ be a proper holomorphic map of complex manifolds and let ξ be a complex holomorphic vector bundle on M. For $y \in B$, let $Z_y = \pi^{-1}(y)$ be the fiber over y. Assume that for every $y \in B$, there is a Kähler metric g^{Z_y} on Z_y depending smoothly on y, and let h^ξ be a smooth metric on ξ. If $\ell = \dim Z_y$, let

$$0 \to E_y^0 \xrightarrow[\bar\partial]{} E_y^1 \to \dots \to E_y^\ell \to 0 \tag{0.1}$$

be the $\bar\partial$ complex associated with the restriction of ξ to Z_y.

In Sect. 1, we describe conditions under which the infinite dimensional vector bundles $E^0, \dots, E^\ell$ are infinite dimensional holomorphic Hermitian vector bundles on B (for a special choice of the metric g^Z). This is precisely the case when π is locally Kähler (in the sense of [BGS 1]).

Also, by using Quillen's superconnections [Q 1], higher order analytic torsion forms associated with finite dimensional acyclic complexes of holomorphic Hermitian vector bundles were constructed in [BGS 1], which are the analogues in any degree of the analytic torsion of Ray and Singer [RS 2]. The number operator N of the considered complex was used to construct such forms.

In [B 1], Quillen's superconnections were used in an infinite dimensional context to obtain a local Index Theorem for families. Quite remarkably, the Levi-Civita superconnection – which was introduced in [B 1] to obtain a local version of the families Index Theorem of Atiyah-Singer [AS] – incorporates the algebraic formalism of the double transgression which was described in [BGS 1] to calculate the higher order analytic torsion forms. In particular we show in Sect. 2, that the analogue of the number operator is now the Kähler form of the fibration.

However several difficulties arise. Contrary to [B 1] and [BF 2] where, because of "extraordinary cancellations," the asymptotic expansions as $t \downarrow\downarrow 0$ of the objects which were considered were non-singular, we here have singular expansions like

$$\frac{A_{-1}}{t} + A_0 + O(t). \tag{0.2}$$

Still the "interesting" quantity is A_0.

In Sect. 2, A_0 is calculated by complicated algebraic manipulations on traces, and also by using Brownian motion and anticommuting variables. For greater clarity, we have described some of these manipulations in a finite dimensional context in [BGS 1].

Also in Sect. 2, we obtain several results on secondary characteristic classes for direct images in any degree. In particular an analogue of [BGS 1, Theorem 0.3] is proved in Theorem 2.21 in any degree, and is related to work by Gillet and Soulé [GS 1, 2] on direct images in Arakelov theory. Theorem 2.21 will be used in [BGS 3] to prove [BGS 1, Theorem 0.3].

Our paper is divided into two sections. In Sect. 1, we introduce Kähler fibrations. In Sect. 2, we calculate higher order analytic torsion forms for direct images, and we study their behavior in exact sequences.

Let us point out that the analytic torsion was introduced in Riemannian geometry by Ray and Singer [RS 1], and in complex geometry by the same authors [RS 2]. Several developments in the Riemannian case were also obtained by Cheeger [C] and Müller [M].

We will use the same notations as in [BGS 1] to which the reader is referred.

In particular, if B is a complex manifold, P denotes the set of smooth differential forms on B which are sums of forms of type (p, p) (for $0 \leqq p \leqq \dim_C B$). P' denotes the subset of P which consists of the forms ω in P which can be written in the form $\omega = \partial^B \eta + \bar{\partial}^B \eta'$. If $\omega, \omega' \in P$, we write $\omega \equiv \omega'$ if $\omega - \omega' \in P'$.

If E is a vector bundle on B with connection ∇ and curvature ∇^2, we denote by $\overline{\mathrm{ch}}(E)$ the normalized Chern character cohomology classes which are represented by the forms $\mathrm{Tr}[\exp(-\nabla^2)]$.

If K is a Z_2 graded algebra, if $A, B \in K$, we denote $[A, B]$ the supercommutator of A and B.

Finally the notations Tr and Tr_s are used for traces and supertraces.

The results contained in this paper were announced in [BGS 2].

1. Kähler Fibrations

In this section, we introduce Kähler fibrations, and we derive their main properties.

In fact, let us remember that in the case of smooth fibrations $M \xrightarrow[\pi]{} B$, when the fibers are endowed with a smooth metric, Bismut [B 1] introduced an Euclidean connection ∇^Z on TZ. This connection plays a critical role in the derivation of the local Index Theorem of [B 1].

Here, when M and B are complex Hermitian manifolds, we find conditions under which the connection ∇^Z of [B 1] is holomorphic. This is precisely the Kähler fibration condition described in the Introduction of [BGS 1], which generalizes the standard Kähler condition for complex Hermitian manifolds.

We also calculate the complex geometry of the infinite dimensional vector bundles introduced in [B 1].

This section is organized as follows. In a), we describe the construction of [B 1] of a connection ∇^Z on TZ. In b), the results of [B 1] are slightly extended. In c), we introduce Kähler fibrations, and we derive their main properties. In d), we construct a family of Dirac operators, naturally associated with the family of operators $\bar{\partial}^Z$ acting on the fibers Z. Finally in e), we prove that in a generalized sense, the infinite dimensional vector bundles on B, $E^0, \ldots, E^\ell$, which were considered in (0.1), are holomorphic, and that the family $\bar{\partial}^{Z_y}$ depends holomorphically on $y \in B$.

a) An Euclidean Connection on the Vertical Tangent Space of a Fibration

Let n, n' be positive integers, and let M, B be smooth connected manifolds of dimension $n+n'$, n'.

Let Z be a smooth compact connected manifold of dimension n. Let $\pi: M \xrightarrow[Z]{} B$ be a fibration of M on B, which is modelled on Z: there is an open covering $\mathscr{U}$ of B such that if $U \in \mathscr{U}$, $\pi^{-1}(U)$ is diffeomorphic to $U \times Z$. For $y \in B$, Z_y is the fiber $\pi^{-1}(\{y\})$.

Let $T^H M$ be a smooth subbundle of TM such that

$$TM = T^H M \oplus TZ. \tag{1.1}$$

$T^H M$ and TZ are the horizontal and vertical parts of TM. Let P_H, P_Z be the projections from TM on $T^H M$, TZ.

The map π_* is a linear isomorphism from $T^H_x M$ into $T_{\pi(x)}B$. If $Y \in TB$, $Y^H \in T^H M$ is the horizontal lift of Y in TM so that

$$Y^H \in T^H M, \quad \pi_* Y^H = Y.$$

Let g^B, g^Z be smooth metrics on TB, TZ. The metric g^B lifts to a metric on $T^H M$. Let $g^B \oplus g^Z$ denote the metric on TM which coincides with g^B on $T^H M$, with g^Z on TZ and is such that $T^H M$ and TZ are orthogonal. Let $\langle , \rangle$ be the scalar product for $g^B \oplus g^Z$.

Most of the objects which we construct will ultimately be independent of the metric g^B.

Let ∇^B be the Levi-Civita connection on TB for the metric g^B, and ∇^L the Levi-Civita connection of TM for the metric $g^B \oplus g^Z$. The connection ∇^B lifts to a connection on $T^H M$, which we still note ∇^B.

Definition 1.1. Let ∇^Z be the connection on TZ

$$\nabla^Z = P_Z \nabla^L \tag{1.2}$$

and R^Z the curvature tensor of ∇^Z. Let ∇ be the connection on $TM = T^H M \oplus TZ$ defined by

$$\nabla = \nabla^B \oplus \nabla^Z \tag{1.3}$$

and R the curvature of ∇. Let T be the torsion of ∇, and S the tensor $S = \nabla^L - \nabla$.

Note that ∇^B, ∇^L, ∇^Z, ∇ preserve the corresponding metrics. The tensor S is a one form on TM with values in antisymmetric matrices. Note that if $X, Y, Z \in TM$, by [B 1, Eq. (1.28)]

$$\begin{gathered} S(X)Y - S(Y)X + T(X, Y) = 0, \\ 2\langle S(X)Y, Z\rangle + \langle T(X, Y), Z\rangle + \langle T(Z, X), Y\rangle - \langle T(Y, Z), X\rangle = 0. \end{gathered} \tag{1.4}$$

By [B 1, Theorem 1.9] and [BF 2, Sect. 1c], we know that:

- T takes its values in TZ.
- If $X, Y \in TZ$, $T(X, Y) = 0$.
- ∇^Z, T and the (3,0) tensor $\langle S(\cdot), \cdot\rangle$ do not depend on g_B.
- For any $X \in TM$, $S(X)$ maps TZ in $T^H M$.
- For any $X, Y \in T^H M$, $S(X)Y \in TZ$.
- If $X \in T^H M$, $S(X)X = 0$.

The connection ∇^Z will be called the Levi-Civita connection of Z.

b) Invariance Properties of ∇^Z

We now briefly prove that ∇^Z and part of the (3,0) tensor $\langle S(\cdot)\cdot, \cdot\rangle$ can be calculated using metrics on TM which are not necessarily constant on $T^H M$.

Namely let g be a metric on TM which has the following properties:

- g coincides with g^Z on TZ.
- $T^H M$ and TZ are orthogonal for g.

Let $\langle , \rangle_g$ denote the scalar product for g, $\nabla^{L,g}$ the corresponding Levi-Civita connection on TM. Note that the metric $g^B \oplus g^Z$ is a special case of such a g. $P_Z \nabla^{L,g}$ is an Euclidean connection on (TZ, g^Z).

Theorem 1.2. *One has $\nabla^Z = P_Z \nabla^{L,g}$. Furthermore if X, X' are smooth sections of TZ, A, A' smooth sections of TB, and if*

$$Y = X' + A^H, \qquad Y' = A'^H, \tag{1.5}$$

then

$$\langle \nabla_X^{L,g} Y, Y' \rangle_g - \langle \nabla_X^{L,g} Y', Y \rangle_g = 2\langle S(X)Y, Y' \rangle. \tag{1.6}$$

Proof. Let U be a smooth section of TM and V, W smooth sections of TZ. Classically [KN, IV, Proposition 2.3], we know that

$$2\langle \nabla_U^{L,g} V, W \rangle_g = U\langle V, W \rangle_g + V\langle W, U \rangle_g - W\langle U, V \rangle_g + \langle [U, V], W \rangle_g$$
$$+ \langle [W, U], V \rangle_g - \langle [V, W], U \rangle_g .$$

Since $[V, W] \in TZ$, by using the assumptions which we have done on g, we get:

$$2\langle P_Z \nabla_U^{L,g} V, W \rangle = U\langle V, W \rangle + V\langle W, P_Z U \rangle - W\langle P_Z U, V \rangle + \langle P_Z[U, V], W \rangle$$
$$+ \langle P_Z[W, U], V \rangle - \langle [V, W], P_Z U \rangle .$$

Since $g^B \oplus g^Z$ verifies the same assumptions as g, we find that

$$\langle P_Z \nabla_U^{L,g} V, W \rangle = \langle P_Z \nabla_U^{L} V, W \rangle, \tag{1.7}$$

and so $P_Z \nabla_U^{L,g} = \nabla^Z$.

Since $\nabla^{L,g}$ is torsion free, we have

$$\langle \nabla_X^{L,g} Y, Y' \rangle_g - \langle \nabla_X^{L,g} Y', Y \rangle_g = \langle \nabla_Y^{L,g} X, Y' \rangle_g - \langle \nabla_{Y'}^{L,g} X, Y \rangle_g$$
$$+ \langle [X, Y], Y' \rangle_g - \langle [X, Y'], Y \rangle_g . \tag{1.8}$$

The vector X can be identified with the one form $\tilde{X} : U \in TM \to \langle X, U \rangle_g = \langle X, P_Z U \rangle$, a form independent of g. Since $\nabla^{L,g}$ is torsion free

$$d\tilde{X}(Y, Y') = \langle \nabla_Y^{L,g} X, Y' \rangle_g - \langle \nabla_{Y'}^{L,g} X, Y \rangle_g . \tag{1.9}$$

Moreover one verifies trivially that $[X, Y]$, $[X, Y'] \in TZ$. Since $Y' \in T^H M$, we see that

$$\langle [X, Y], Y' \rangle_g - \langle [X, Y'], Y \rangle_g = -\langle [X, Y'], P_Z Y \rangle . \tag{1.10}$$

Using (1.8)–(1.10), we obtain

$$\langle \nabla_X^{L,g} Y, Y' \rangle_g - \langle \nabla_X^{L,g} Y', Y \rangle_g = d\tilde{X}(Y, Y') - \langle [X, Y'], P_Z Y \rangle . \tag{1.11}$$

Since (1.11) also holds for the metric $g^B \oplus g^Z$, we find that

$$\langle \nabla_X^{L,g} Y, Y' \rangle_g - \langle \nabla_X^{L,g} Y', Y \rangle_g = \langle \nabla_X^L Y, Y' \rangle - \langle \nabla_X^L Y', Y \rangle . \tag{1.12}$$

Also

$$\begin{gathered} \nabla_X^L Y = \nabla_X Y + S(X)Y, \qquad \nabla_X^L Y' = \nabla_X Y' + S(X)Y', \\ \nabla_X Y' = 0, \qquad \nabla_X Y = \nabla_X X' \in TZ . \end{gathered} \tag{1.13}$$

From (1.13), we find that $\langle \nabla_X Y, Y' \rangle = 0$, and so

$$\langle \nabla_X^L Y, Y' \rangle - \langle \nabla_X^L Y', Y \rangle = 2\langle S(X)Y, Y' \rangle . \tag{1.14}$$

Equation (1.6) follows from (1.12), (1.14) □

Remark 1.3. If α is the second fundamental form of a fiber Z in M for the metric g, it follows from (1.6) that if X, $X' \in TZ$, $Y' \in T^H M$, then

$$\langle \alpha(X)X', Y' \rangle_g = \langle S(X)X', Y' \rangle .$$

c) *Kähler Fibrations and the Levi-Civita Connection*

We now assume that n, m are even, so that $n = 2\ell$, $n' = 2\ell'$. We also assume that M, B are complex manifolds of complex dimension ℓ, ℓ' and that π is holomorphic.

Let J and J' be the complex structure on TM, TB. J maps TZ into itself. We also assume that J maps $T^H M$ into itself.

Let $T_C M$ be the complexified tangent space $T_C M = TM \otimes_R \mathbb{C}$. Set

$$T^{(1,0)}M = \{X \in T_C M;\, JX = iX\}, \qquad T^{(0,1)}M = \{X \in T_C M;\, JX = -iX\}.$$

Let T^*M be the vector bundle of real linear forms on TM. Set $T_C^* M = T^*M \otimes_R \mathbb{C}$. If $\tilde{J}$ is the transpose of J which acts on $T_C^* M$, set

$$T^{*(1,0)}M = \{\alpha \in T_C^* M;\, \tilde{J}\alpha = i\alpha\}, \qquad T^{*(0,1)}M = \{\alpha \in T_C^* M;\, \tilde{J}\alpha = -i\alpha\}.$$

$T^{*(1,0)}M$ and $T^{*(0,1)}M$ are the bundles of holomorphic and antiholomorphic one forms on M.

In the same way, we define $T_C B$, $T_C Z$, $T_C^H M$, $T_C^{(1,0)} B$, etc.

The holomorphic bundle $\pi^* T^{(1,0)} B$ is isomorphic to $T^{(1,0)}M / T^{(1,0)}Z$, and we have the exact sequence of holomorphic bundles over M:

$$0 \to T^{(1,0)}Z \to T^{(1,0)}M \to \pi^* T^{(1,0)}B \to 0 .$$

Note that as C^∞ bundles $T^{H(1,0)}M \cong \pi^* T^{(1,0)}B$. However in general, $T^{H(1,0)}M$ is not a holomorphic subbundle of $T^{(1,0)}M$.

Let $\Lambda(T_C^* M)$ be the exterior algebra of $T_C^* M$. For $0 \leqq i \leqq n$, the vector bundle $\Lambda^i(T_C^* M)$ splits into $\Lambda^i(T_C^* M) = \bigoplus_{p+q=i} \Lambda^{(p,q)}(T_C^* M)$, where $\Lambda^{(p,q)}(T_C^* M)$ is the set of forms on M of complex type (p, q).

Definition 1.4. The triple $(\pi, g^Z, T^H M)$ will be said to define a Kähler fibration if there exists a smooth 2-form ω on M of complex type (1, 1), which has the following properties:

a) ω is closed;

b) $T^H M$ and TZ are orthogonal with respect to ω;

c) If X, $Y \in TZ$, then $\omega(X, Y) = \langle X, JY \rangle$.

We say that ω is associated with $(\pi, g^Z, T^H M)$. In the sequel, ω will be fixed once for all.

Properties a) and c) imply that J induces an isometry of TZ, that the fibers (Z, g^Z) are Kähler and that, when restricted to TZ, ω is the Kähler form of the corresponding fiber.

We will denote by ω^H, ω^Z the restrictions of ω to $T^H M$, TZ. We extend ω^H and ω^Z to TM by taking the convention that, if $X \in TZ$ and $Y \in T^H M$, then $i_X \omega^H = 0$ and $i_Y \omega^Z = 0$. Therefore

$$\omega = \omega^H + \omega^Z . \tag{1.15}$$

The pair $(g^Z, T^H M)$ is entirely determined by ω, as we shall see in the following theorem.

Theorem 1.5. *Let ω be a smooth 2-form on M of complex type* $(1,1)$, *which has the following properties:*

a) *ω is closed;*

b) *If $X, Y \in TZ$, $X, Y \to \omega(JX, Y)$ defines a Hermitian product g^Z on TZ.*

For any $x \in M$, let $T_x^H M$ be the subspace of $T_x M$;

$$T_x^H M = \{Y \in T_x M;\ \text{for any } X \in T_x Z,\ \omega(X, Y) = 0\}.$$

Then, $T^H M$ is a smooth subbundle of TM such that: $TM = T^H M \oplus TZ$. $(\pi, g^Z, T^H M)$ is a Kähler fibration, and ω is an associated $(1,1)$*-form.*

Proof. By condition b), it is obvious that:

$$\dim T^H M + \dim TZ = \dim TM, \qquad T^H M \cap TZ = \{0\}.$$

Therefore $T^H M$ is a smooth vector bundle on M and (1.1) holds. Since ω is of type $(1,1)$, it is clear that J maps $T^H M$ into itself. The theorem is now obvious. □

Example 1. Assume that (M, g) is a Kähler manifold, and let Φ be its Kähler form. If $T^H M = (TZ)^\perp$ and if g^Z is the restriction of g to TZ, by Theorem 1.5, $(\pi, g^Z, T^H M)$ is a Kähler fibration.

Example 2. If X is a Kähler manifold, set $M' = X \times B$. If Φ' is the Kähler form of X, if $T^H M' = TB$, by taking $\omega = \Phi'$, we still have a Kähler fibration with constant fiber X.

Remark 1.6. B is locally Kähler. Namely there is an open covering $\mathcal{U}$ of B such that, if $U \in \mathcal{U}$, there is a closed $(1,1)$ form η^U on U which induces a Kähler metric on TB.

If $(\pi, g^Z, T^H M)$ defines a Kähler fibration with associated $(1,1)$ form ω, on $\pi^{-1}(U)$, for any $\lambda > 0$, we can replace ω by $\omega + \lambda \pi^* \eta^U$. Since the fibers Z are compact, for λ large enough, $\omega + \lambda \pi^* \eta^U$ is a Kähler form on $\pi^{-1}(U)$, which induces the metric g^Z on the fibers Z, and is such that $T^H M = (TZ)^\perp$. This implies that locally on B, we are in the situation described in Example 1.

If α is a smooth p form on M, if $Y \in TM$, $\nabla_Y \alpha$ is still a p form. $\nabla^a \alpha$ denotes the $p+1$ form which is the antisymmetrization of $(X_1, \ldots, X_{p+1}) \to \nabla_{X_1} \alpha(X_2, \ldots, X_{p+1})$.

Since T is a 2 form on TM with values in TZ, $i_T \alpha$ will be a $p+1$ form on TM. Also remember that the $(3,0)$ tensor $\langle S(\cdot)\cdot, \cdot\rangle$ does not depend on g^B.

Finally note that, if Y is a smooth vector field on B, the vector field Y^H acts on the fibers Z. If ψ_s is the group of diffeomorphisms of M generated by Y^H, and if β is a smooth section of $\Lambda(T^*Z)$, set

$$L^Z_{Y^H} \beta = \left[\left(\frac{d}{ds} \right) \psi_s^* \beta \right]_{s=0} \in \Lambda(T^*Z). \tag{1.16}$$

Note that in (1.16), β and $L^Z_{Y^H} \beta$ are not considered as elements of $\Lambda(T^*M)$ but only as elements of $\Lambda(T^*Z)$.

One verifies that $Y \in TB \to L^Z_{Y^H} \beta \in \Lambda(T^*Z)$ is a tensor, i.e. does not involve differentiation in Y.

Theorem 1.7. *Assume that $(\pi, g^Z, T^H M)$ defines a Kähler fibration, with associated* $(1,1)$ *form ω. Then:*

a) *The connection ∇^Z on TZ preserves the complex structure of TZ, and induces on $T^{(1,0)}Z$ its holomorphic Hermitian connection.*

b) *For any $X \in TZ$, the 2 form $\langle S(X)\cdot,\cdot\rangle$ on TM is of complex type* $(1,1)$. *If $X \in T^{(1,0)}Z$, $Y \in T^{(0,1)}Z$ and $Y' \in T_C M$, then*

$$\langle S(X)Y, Y'\rangle = \langle S(Y)X, Y'\rangle = 0. \tag{1.17}$$

c) *As a 2 form on TM, the torsion T is of complex type* $(1,1)$.

d) *The following relations hold:*

For any $Y \in TB$, $L^Z_{Y^H}\omega^Z = 0$,

$$\begin{aligned} \nabla^Z_{\cdot}\omega^Z = 0; \quad i_T\omega^Z = 0 \quad \text{on} \quad T^H M \times TZ \times TZ, \\ \nabla^a_{\cdot}\omega^H = 0 \quad \text{on} \quad T^H M \times T^H M \times T^H M, \\ \nabla^a_{\cdot}\omega^H + i_T\omega^Z = 0 \quad \text{on} \quad T^H M \times T^H M \times TZ. \end{aligned} \tag{1.18}$$

e) *A smooth* $(1,1)$ *form ω' on M is associated with the Kähler fibration $(\pi, g^Z, T^H M)$ if and only if there is a smooth closed* $(1,1)$ *form η on B such that: $\omega' - \omega = \pi^*\eta$.*

Proof. Statements a)–c) only need to be proved locally. If U is taken as in Remark 1.6, by restricting ourselves to $\pi^{-1}(U)$, we many and we will *temporarily* assume that ω is a Kähler form over $\pi^{-1}(U)$.

Let ∇^L be the Levi-Civita connection on TM associated with the Kähler form ω. Then J is parallel with respect to ∇^L.

By Theorem 1.2, we know that: $\nabla^Z = P_Z \nabla^L$. Since $[P_Z, J] = 0$, it is clear that ∇^Z preserves the complex structure of TZ. Since $\pi^{-1}(U)$ is Kähler, ∇^L is a holomorphic connection on $T^{(1,0)}M$.

Since $T^{(1,0)}Z$ is a holomorphic subbundle of $T^{(1,0)}M$, $\nabla^Z = P_Z \nabla^L$ is a holomorphic connection on $T^{(1,0)}Z$. Since ∇^Z is Hermitian on $T^{(1,0)}Z$, ∇^Z is the unique holomorphic Hermitian connection on $T^{(1,0)}Z$.

Theorem 1.2 still holds with $X, X' \in T_C Z$, $A, A' \in T_C B$. Using the same notations as in (1.5) if $Y, Y' \in T^{(1,0)}M$, since $\nabla^L_X Y$, $\nabla^L_X Y' \in T^{(1,0)}M$, we have

$$\langle \nabla^L_X Y, Y'\rangle = \langle \nabla^L_X Y', Y\rangle = 0. \tag{1.19}$$

Using Theorem 1.2, we find that $\langle S(X)Y, Y'\rangle = 0$, or equivalently

$$\langle S(X)(X' + A^H), A'^H\rangle = 0. \tag{1.20}$$

Also we have seen in Sect. 1a) that if $X', X'' \in TZ$,

$$\langle S(X)X', X''\rangle = 0. \tag{1.21}$$

By (1.20) and (1.21), we find that $\langle S(X)\cdot,\cdot\rangle$ is of complex type $(1,1)$. Equivalently, $S(X)$ is a complex endomorphism of TM.

If $X \in T^{(1,0)}Z$, $Y \in T^{(0,1)}Z$, since $T(X, Y) = 0$, by (1.4) we know that

$$S(X)Y = S(Y)X. \tag{1.22}$$

Since $S(X)$ is a complex endomorphism, $S(X)Y \in T^{(0,1)}Z$. Similarly, $S(Y)X \in T^{(1,0)}Z$. So by (1.22), we get

$$S(X)Y = S(Y)X = 0. \tag{1.23}$$

Take $U, V \in T^{(1,0)}M$. We will prove that $T(U, V)=0$. Since T vanishes on $TZ \times TZ$, we may, and we will assume that $V \in T^{H(1,0)}M$.

a) If $U \in T^{(1,0)}Z$, then $S(V)U \in T_C^H M$. Using (1.4), we find that if $X \in T_C Z$, then:

$$\langle X, T(U, V)\rangle = -\langle X, S(U)V\rangle = \langle S(U)X, V\rangle .$$

Since $T(U, X) \in T_C Z$, and $S(U)X - S(X)U + T(U, X) = 0$, we find

$$\langle X, T(U, V)\rangle = \langle S(X)U, V\rangle . \tag{1.24}$$

Since $S(X)$ is a complex endomorphism of TM, $\langle S(X)U, V\rangle = 0$. By (1.24), we find that $T(U, V)=0$.

b) If $U \in T^{H(1,0)}M$, we have

$$\langle X, T(U, V)\rangle = -\langle X, S(U)V\rangle + \langle X, S(V)U\rangle = \langle S(U)X, V\rangle - \langle S(V)X, U\rangle .$$

Since $T(U, X)$, $T(V, X) \in T_C Z$, we find

$$\langle X, T(U, V)\rangle = \langle S(X)U, V\rangle - \langle S(X)V, U\rangle = 2\langle S(X)U, V\rangle . \tag{1.25}$$

Since $S(X)$ is a complex endomorphism, we find again that: $T(U, V)=0$.

We have proved that T is of complex type $(1,1)$.

We do not assume any longer that u is ω Kähler form.

Since ∇^Z preserves the metric and the complex structure of TZ, clearly $\nabla^Z \omega^Z = 0$. This shows that $\nabla^a \omega^Z = 0$. On the other hand, it is classical that

$$d = \nabla^a + i_T . \tag{1.26}$$

Since $\omega = \omega^Z + \omega^H$ is closed, we find that:

$$\nabla^a(\omega^Z + \omega^H) + i_T(\omega^Z + \omega^H) = 0 .$$

Since T takes its values in TZ, $i_T \omega^H = 0$, and so, since $\nabla^a \omega^Z = 0$,

$$\nabla^a \omega^H + i_T \omega^Z = 0 . \tag{1.27}$$

On $T^H M \times TZ \times TZ$, $\nabla^a \omega^H = 0$, and on $T^H M \times T^H M \times T^H M$, $i_T \omega^Z = 0$.

All equalities in d) have been proved except the first one.

Take $Y \in TB$. Clearly $i_{Y^H} \omega^Z = 0$. Therefore

$$L^Z_{Y^H} \omega^Z = i_{Y^H} d\omega^Z \quad \text{restricted to} \quad TZ \times TZ . \tag{1.28}$$

Since $\omega^H + \omega^Z$ is closed, $d\omega^Z = -d\omega^H$, and so

$$L^Z_{Y^H} \omega^Z = -i_{Y^H} d\omega^H \quad \text{restricted to} \quad TZ \times TZ . \tag{1.29}$$

One verifies easily that $i_{Y^H} d\omega^H$ vanishes on $TZ \times TZ$ and so

$$L^Z_{Y^H} \omega^Z = 0 . \tag{1.30}$$

Let us prove e). If $\omega' = \omega'^H + \omega^Z$ is another closed $(1,1)$ form associated with $(\pi, g^Z, T^H M)$, we find from d) that:

$$\nabla^a(\omega'^H - \omega^H) = 0 \quad \text{on} \quad T^H M \times T^H M \times TZ .$$

Equivalently if $X \in TZ$, we find that

$$\nabla_X(\omega'^H - \omega^H) = 0 \quad \text{on} \quad T^H M \times T^H M . \tag{1.31}$$

Equation (1.31) exactly means that $\omega'^H - \omega^H = \pi^* \eta$. Since $\omega' - \omega$ is closed, η is also closed.

The theorem is proved. □

Remark 1.8. If $Y \in TB$, by Theorem 1.7 we know that

$$L^Z_{Y^H} \omega^Z = 0; \quad \nabla^Z_{Y^H} \omega^Z = 0. \tag{1.32}$$

On the other hand, we know that when acting on smooth sections of TZ

$$\nabla^Z_{Y^H} = L^Z_{Y^H} + T(Y^H, \cdot). \tag{1.33}$$

We conclude that

$$[T(Y^H, \cdot)] \omega^Z = 0 \tag{1.34}$$

or equivalently

$$i_T \omega^Z = 0 \quad \text{on} \quad T^H M \times TZ \times TZ, \tag{1.35}$$

which was proved in Theorem 1.6.

Note that ω^Z is a symplectic form on Z. The relation $L^Z_{Y^H} \omega_Z = 0$ exactly means that the holonomy group of the fibration preserves the symplectic form ω^Z.

If $Y, Z \in TB$, since ∇^B is torsion free, we find that

$$T(Y^H, Z^H) = -P_Z [Y^H, Z^H]. \tag{1.36}$$

The vertical vector field $T(Y^H, Z^H)$ must therefore preserve the symplectic form ω^Z. The relation

$$\nabla^a \omega^H + i_T \omega^Z = 0 \quad \text{on} \quad T^H M \times T^H M \times TZ$$

exactly means that $T(Y^H, Z^H)$ is a Hamiltonian vector field on Z, associated with the Hamiltonian function $\omega^H(Y^H, Z^H)$.

Remark 1.9. We may ask under what conditions the holonomy group of the fibers Z acts holomorphically on the fibers. This exactly means that, if J^Z is the restriction of J to TZ, if $Y \in TB$, then $L^Z_{Y^H} J^Z = 0$. However since $L^Z_{Y^H} \omega^Z = 0$, we find that $L^Z_{Y^H} g^Z = 0$. In other words, the holonomy group of the fibration must then consist of holomorphic isometries. This is of course a very restrictive assumption.

d) A Family of Dirac Operators

From now on, we assume that the fibration $(\pi, g^Z, T^T M)$ is Kähler, and that $\omega = \omega^H + \omega^Z$ is an associated $(1,1)$ form.

Let $\Lambda T^{*(0,1)} Z$ be the exterior algebra of $T^{*(0,1)} Z$, and $\Lambda^p T^{*(0,1)} Z$ the p forms in $\Lambda T^{*(0,1)} Z$. The vector bundle $\Lambda (T^{*(0,1)} Z)$ is Hermitian, and splits orthogonally as a direct sum

$$\Lambda (T^{*(0,1)} Z) = \bigoplus_{p=0}^{\ell} \Lambda^p (T^{*(0,1)} Z).$$

The bundle $T^{*(0,1)} Z$ is identified to $T^{(1,0)} Z$ by the metric g^Z. Therefore $T^{*(0,1)} Z$ inherits the holomorphic structure of $T^{(1,0)} Z$. ∇^Z induces on $T^{*(0,1)} Z$ the corresponding holomorphic Hermitian connection. $\Lambda (T^{*(0,1)} Z)$ then becomes a holomorphic Hermitian vector bundle on M.

Let ξ be a holomorphic Hermitian vector bundle on M, of complex dimension k. Then $\Lambda(T^{*(0,1)}Z)\otimes\xi$ is a holomorphic Hermitian vector bundle on M.

Definition 1.10. For $0\leq p\leq \ell$, E^p denotes the set of C^∞ sections over M of $\Lambda^p(T^{*(0,1)}Z)\otimes\xi$.

As in [B 1], we will regard E^p as being the set of C^∞ sections over B of an infinite dimensional bundle. For $y\in B$, the corresponding fiber E^p_y is the set of C^∞ sections over Z_y of $\Lambda^p(T^{*(0,1)}Z)\otimes\xi$. Set

$$E^+=\bigoplus_{p\,\text{even}} E^p,\qquad E^-=\bigoplus_{p\,\text{odd}} E^p,\qquad E=E^+\oplus E^-. \tag{1.37}$$

Let dx be the Riemannian volume element in the fiber Z. For any $y\in B$, E_y is endowed with the Hermitian product

$$h,h'\in E_y\to \int_{Z_y}\langle h,h'\rangle(x)dx. \tag{1.38}$$

Let $(z^1=x^1+iy^1,\dots,z^\ell=x^\ell+iy^\ell)$ be a complex system of coordinates in one given fiber Z. Clearly $\dfrac{\partial}{\partial y^j}=J\left(\dfrac{\partial}{\partial x^j}\right)$, $1\leq j\leq\ell$. We assume that TZ is oriented by the base $\left(\dfrac{\partial}{\partial x^1},\dfrac{\partial}{\partial y^1},\dots,\dfrac{\partial}{\partial x^\ell},\dfrac{\partial}{\partial y^\ell}\right)$.

Set

$$\begin{gathered}\frac{\partial}{\partial z^j}=\frac{1}{2}\left(\frac{\partial}{\partial x^j}-i\frac{\partial}{\partial y^j}\right),\qquad \frac{\partial}{\partial \bar z^j}=\frac{1}{2}\left(\frac{\partial}{\partial x^j}+i\frac{\partial}{\partial y^j}\right),\\ dz^j=dx^j+idy^j,\qquad d\bar z^j=dx^j-idy^j.\end{gathered} \tag{1.39}$$

For every $y\in B$, the operator $\bar\partial^{Z_y}$ acts naturally on E_y. By also taking holomorphic coordinates on ξ, $\bar\partial^{Z_y}$ is expressed locally by the formula

$$\bar\partial^{Z_y}=\sum_{j=1}^{\ell} d\bar z^j\wedge\frac{\partial}{\partial \bar z^j}. \tag{1.40}$$

Let $\bar\partial^{Z_y*}$ be the formal adjoint of $\bar\partial^{Z_y}$ with respect to the Hermitian product (1.38). Set

$$\bar\partial_y=\sqrt{2}\,\bar\partial^{Z_y},\qquad \bar\partial^*_y=\sqrt{2}\,\bar\partial^{*Z_y},\qquad D_y=\bar\partial_y+\bar\partial^*_y. \tag{1.41}$$

The operator D_y interchanges E^+_y and E^-_y. Let $D_{\pm,y}$ be the restriction of D_y to $E^\pm_y$. We will write D_y in the form

$$D_y=\begin{bmatrix}0 & D_{-,y}\\ D_{+,y} & 0\end{bmatrix}.$$

By taking a local trivialization of the fibration $M\xrightarrow{\pi}B$, one verifies easily that $\bar\partial_y$, $\bar\partial^*_y$, D_y are first order differential operators whose coefficients depend smoothly on $x\in M$. Also D_y is formally self-adjoint on E_y.

We now turn $\Lambda(T^{*(0,1)}Z)\otimes\xi$ into a $T_C Z$ Clifford module. Namely if $X\in T^{(1,0)}Z$, if $X^*\in T^{*(0,1)}Z$ is the 1 form $Y\in T_C Z\to\langle X,Y\rangle$, we define $c(X)\in\operatorname{End}(\Lambda(T^{*(0,1)}Z)\otimes\xi)$ by the relation

$$c(X)=\sqrt{2}X^*\wedge. \tag{1.42}$$

Similarly if $X' \in T^{(0,1)}Z$, set

$$c(X') = -\sqrt{2}\, i_{X'}. \tag{1.43}$$

The map c extends by linearity to the whole $T_C Z$. Clearly, if $X, X' \in T_C Z$, then

$$c(X)c(X') + c(X')c(X) = -2\langle X, X'\rangle. \tag{1.44}$$

Let ∇^ξ be the unique holomorphic Hermitian connection on ξ. Let L^ξ be the curvature of ∇^ξ. As 2 form, L^ξ is of complex type $(1,1)$. The bundle $\Lambda(T^{*(0,1)}Z)\otimes\xi$ is then naturally endowed with the connection $\nabla^Z\otimes 1 + 1\otimes\nabla^\xi$ which we will note ∇ (there is no risk of confusion with the connection ∇ we had defined on TM).

Let $e_1, \ldots, e_n$ be an orthonormal basis of TZ. $w_1, \ldots, w_\ell$ is an orthonormal basis of $T^{(1,0)}Z$, $\bar{w}_1, \ldots, \bar{w}_\ell$ the conjugate basis in $T^{(0,1)}Z$, $w^1, \ldots, w^\ell$ the dual basis in $T^{*(1,0)}Z$, and $\bar{w}^1, \ldots, \bar{w}^\ell$ the corresponding conjugate basis of $T^{*(0,1)}Z$.

We now define a family of Dirac operators acting on E.

Definition 1.11. For $y \in B$, D'_y denotes the operator:

$$D'_y = \sum_{j=1}^{n} c(e_j)\nabla_{e_j}. \tag{1.45}$$

We first prove the basic simple result.

Proposition 1.12 *For any $y \in B$, $D_y = D'_y$.*

Proof. Clearly

$$D'_y = c(w_j)\nabla_{\bar{w}_j} + c(\bar{w}_j)\nabla_{w_j} = \sqrt{2}\,\bar{w}^j \wedge \nabla_{\bar{w}_j} - \sqrt{2}\, i_{\bar{w}_j}\nabla_{w_j}.$$

Since Z_y is Kähler, we also have

$$\bar{\partial}^{Z_y} = d\bar{z}^j \wedge \nabla_{\frac{\partial}{\partial \bar{z}^j}} = \bar{w}^j \wedge \nabla_{\bar{w}_j}, \qquad \bar{\partial}^{Z_y *} = -i_{\bar{w}_j}\nabla_{w_j}. \tag{1.46}$$

The proposition is proved. □

We must now compare the connection ∇ on $\Lambda(T^{*(0,1)}Z)\otimes\xi$ with the connection on twisted TZ-spinors which is used in [B 1].

If $x \in M$, at least on a neighborhood of $x \in M$, the holomorphic Hermitian bundle $\det T^{(0,1)}Z$ has a holomorphic square root μ, which we endow with the square root metric and the corresponding holomorphic connection ∇^μ.

Set

$$F_+ = \Lambda^{\text{even}} T^{*(0,1)}Z \otimes \mu^{-1}, \qquad F_- = \Lambda^{\text{odd}} T^{*(0,1)}Z \otimes \mu^{-1}. \tag{1.47}$$

By [H, Theorem 2.2], F_+ and F_- can be identified with the (locally defined) Hermitian bundles of spinors over TZ. Also $\Lambda(T^{*(0,1)}Z)$ is a holomorphic vector bundle on M. ∇^Z induces on $\Lambda(T^{*(0,1)}Z)$ the corresponding holomorphic Hermitian connection. Therefore F, F_+, F_- are holomorphic Hermitian bundles, and $\nabla^F = \nabla^Z \otimes 1 + 1 \otimes \nabla^{\mu^{-1}}$ is the corresponding holomorphic Hermitian connection. Tautologically, ∇^F induces on TZ the connection ∇^Z which is holomorphic on $T^{(1,0)}Z$.

Now, by Theorem 1.7, the connection ∇^Z on TZ is exactly the Euclidean connection on TZ which was considered in [B1, Sect. 1]. Also ∇^F is a Spin(n)

connection on F, and so ∇^F is necessarily the unique Spin(n) connection on F which lifts the Euclidean connection ∇^Z on TZ. ∇^F thus coincides with the connection on F which was constructed in [B, Sect. 1e)]. Also, from (1.47), we find that

$$\Lambda^{\mathrm{even}} T^{*(0,1)}Z \otimes \xi = F_+ \otimes \mu \otimes \xi, \qquad \Lambda^{\mathrm{odd}} T^{*(0,1)}Z \otimes \xi = F_- \otimes \mu \otimes \xi. \tag{1.48}$$

Now $\mu \otimes \xi$ is a (locally defined) holomorphic Hermitian bundle endowed with the holomorphic Hermitian connection $\nabla^\mu \otimes 1 + 1 \otimes \nabla^\xi$.

It thus follows that at least locally on M we are *exactly* in the situation of [B 1, Sect. 1]:

- The bundles F_+ and F_- are endowed with the unitary connection considered in [B 1, Sect. 1e)].
- The twisting bundle with metric and connection ξ in [B 1, Sect. 1] is here $\mu \otimes \xi$.

Note that in [B 1], TZ was assumed to be Spin, in which case μ can be globally defined. However, as in Atiyah-Bott-Patodi [ABP], only the existence of a local spin structure on TZ is needed for the results of [B 1] to apply in our situation.

These considerations permit us to use all the results of [B 1] without further comments.

e) A Holomorphic Hermitian Connection on Infinite Dimensional Bundles

We now define connections on the infinite dimensional bundles E^p as in [B 1, Sect. 1f)] and in [BF 2, Sect. 1e)].

Definition 1.13. For $0 \leqq p \leqq \ell$, let $\tilde{\nabla}$ be the connection on E^p such that if h is C^∞ section of E^p and if $Y \in TB$,

$$\tilde{\nabla}_Y h = \nabla_{Y^H} h. \tag{1.49}$$

Since the curvature tensor R^Z of ∇^Z takes its values in the complex endomorphisms of TZ, R^Z acts naturally on $\Lambda T^{*(0,1)}Z$ and preserves the grading of $\Lambda T^{*(0,1)}Z$.

In the sequel, $(y^1, \ldots, y^{\ell'})$ is a complex coordinate system in B. $\left(\frac{\partial}{\partial y^1}, \ldots, \frac{\partial}{\partial y^{\ell'}}\right)$ is the corresponding basis of $T^{(1,0)}B$, $\left(\frac{\partial}{\partial \bar{y}^1}, \ldots, \frac{\partial}{\partial \bar{y}^{\ell'}}\right)$ the conjugate basis in $T^{(0,1)}B$, $(dy^1, \ldots, dy^{\ell'})$ and $(d\bar{y}^1, \ldots, d\bar{y}^{\ell'})$ the dual bases in $T^{*(1,0)}B$ and in $T^{*(0,1)}B$. Furtherjore $e_1, \ldots, e_n, w_1, \ldots, w_\ell, \ldots$ will be taken as in Sect. 1d).

We will use $\alpha, \beta \ldots$ as indices for horizontal variables $\frac{\partial}{\partial y^\alpha}$, and $i, j \ldots$ as indices for vertical variables like e_i.

We identify $\frac{\partial}{\partial y^\alpha}$ and $\left(\frac{\partial}{\partial y^\alpha}\right)^H$, dy^α and $\pi^* dy^\alpha$ etc.

Theorem 1.14. *The connection $\tilde{\nabla}$ preserves the Hermitian product of E. The curvature $(\tilde{\nabla})^2$ of $\tilde{\nabla}$ is such that if $Y, Y' \in TB$,*

$$(\tilde{\nabla})^2(Y, Y') = R^Z(Y^H, Y'^H) \otimes 1 + 1 \otimes L^\xi(Y^H, Y'^H) - \nabla_{T(Y^H, Y'^H)}. \tag{1.50}$$

For any $Y, Y' \in TB$, $(\tilde{\nabla})^2(Y, Y')$ is a skew-adjoint element of End E. *As a 2-form on TB, $(\tilde{\nabla})^2$ is of complex type* (1, 1). *Finally, if $U \in T^{(1,0)}B$, $V \in T^{(0,1)}B$, then*

$$\tilde{\nabla}_V \bar{\partial} = 0, \qquad \tilde{\nabla}_U \bar{\partial}^* = 0. \tag{1.51}$$

Proof. Set $k=-\frac{1}{2}\sum_1^n S(e_i)e_i$. Then, $k\in T^HM$. In [BF 2, Proposition 1.4], it is proved that the connection $\tilde{\nabla}^u$ on E, which is such that, if $Y\in TB$, $\tilde{\nabla}^u_Y=\tilde{\nabla}_Y+\langle k, Y^H\rangle$, is unitary. We claim that $k=0$. In fact

$$k=-\tfrac{1}{2}[S(w_i)\bar{w}_i+S(\bar{w}_i)w_i]. \tag{1.52}$$

By Theorem 1.7, we know that $S(v_i)\bar{v}_i=S(\bar{v}_i)v_i=0$. It follows that $\tilde{\nabla}=\tilde{\nabla}^u$, i.e. $\tilde{\nabla}$ is unitary.

Equivalently, one can say that $\dfrac{(-\omega^Z)^\ell}{\ell!}$ is the volume form of Z. By [BF 2, Proposition 1.4], if $Y\in TB$,

$$\langle k, Y^H\rangle(\omega^Z)^\ell=\tfrac{1}{2}L^Z_{Y^H}(\omega^Z)^\ell. \tag{1.53}$$

By Theorem 1.7, $L^Z_{Y^H}\omega^Z=0$, and so $\langle k, Y^H\rangle=0$.

By [B 1, Proposition 1.11], the curvature $(\tilde{\nabla})^2$ is given by (1.50). Since $\tilde{\nabla}$ is unitary, for $Y, Y'\in TB$, $(\tilde{\nabla})^2(Y,Y')$ is necessary skew-adjoint.

Since ∇^Z is holomorphic and Hermitian on $T^{(1,0)}Z$, R^Z is of complex type (1,1), and so is L^ξ. By Theorem 1.7, T is also of complex type (1,1). It is now obvious that $(\tilde{\nabla})^2$ is of complex type (1,1).

We identify $\tilde{\nabla}D$ to the element of $\Lambda^1(T^*_CB)\otimes\operatorname{End}E$,

$$\tilde{\nabla}D=dy^\alpha\nabla_{\frac{\partial}{\partial y^\alpha}}D+d\bar{y}^\alpha\nabla_{\frac{\partial}{\partial\bar{y}^\alpha}}D.$$

By [B 1, Theorem 2.5], we know that $\tilde{\nabla}D$ is given by

$$\begin{aligned}\tilde{\nabla}D=&\,dy^\alpha c(w_i)\left[R^Z\left(\frac{\partial}{\partial y^\alpha},\bar{w}_i\right)\otimes 1+1\otimes L^\xi\left(\frac{\partial}{\partial y^\alpha},\bar{w}_i\right)-\nabla_{T\left(\frac{\partial}{\partial y^\alpha},\bar{w}_i\right)}\right]\\&+d\bar{y}^\alpha c(\bar{w}_i)\left[R^Z\left(\frac{\partial}{\partial\bar{y}^\alpha},w_i\right)\otimes 1+1\otimes L^\xi\left(\frac{\partial}{\partial\bar{y}^\alpha},w_i\right)-\nabla_{T\left(\frac{\partial}{\partial\bar{y}^\alpha},w_i\right)}\right].\end{aligned} \tag{1.54}$$

Note that in (1.54), we have used the fact that R^Z, L^ξ, and T are of type (1,1), and so we eliminated terms like $T\left(\dfrac{\partial}{\partial\bar{y}^\alpha},\bar{w}_i\right)$.

Also since $D=\bar{\partial}+\bar{\partial}^*$, trivially

$$\tilde{\nabla}D=\tilde{\nabla}\bar{\partial}+\tilde{\nabla}\bar{\partial}^*. \tag{1.55}$$

Also $\tilde{\nabla}$ preserves the grading in E, and so $\tilde{\nabla}\bar{\partial}$ increases the degree in E by 1, while $\tilde{\nabla}\bar{\partial}^*$ decreases the degree by 1. Also R^Z, L^ξ and ∇_T do not change the grading in E. We immediately derive from (1.54) and (1.55) that

$$\tilde{\nabla}\bar{\partial}=dy^\alpha c(w_i)\left[R^Z\left(\frac{\partial}{\partial y^\alpha},\bar{w}_i\right)\otimes 1+1\otimes L^\xi\left(\frac{\partial}{\partial y^\alpha},\bar{w}_i\right)-\nabla_{T\left(\frac{\partial}{\partial y^\alpha},\bar{w}_i\right)}\right],$$

$$\tilde{\nabla}\bar{\partial}^*=d\bar{y}^\alpha c(\bar{w}_i)\left[R^Z\left(\frac{\partial}{\partial\bar{y}^\alpha},w_i\right)\otimes 1+1\otimes L^\xi\left(\frac{\partial}{\partial\bar{y}^\alpha},w_i\right)-\nabla_{T\left(\frac{\partial}{\partial\bar{y}^\alpha},w_i\right)}\right].$$

Equation (1.51) is proved. □

Remark 1.15. If E' is a finite dimensional complex Hermitian vector bundle on B, endowed with a Hermitian connection ∇ whose curvature is of complex type $(1,1)$, it is a well-known consequence of the Newlander-Nirenberg theorem (see [AHS, Theorem 5.1]) that there is a unique holomorphic structure on E' such that ∇ is the corresponding unique holomorphic Hermitian connection.

Here E is an infinite dimensional complex Hermitian vector bundle, which is endowed with a unitary connection whose curvature is of complex type $(1,1)$. However since E is infinite dimensional, the result of [AHS, Theorem 5.1] is unapplicable in our situation.

Also the condition $\tilde{\nabla}\bar{\partial}=0$ means that $\bar{\partial}$ is a "holomorphic" section of the "holomorphic" vector bundle $\operatorname{End} E$.

However the fact that, at least formally, E is a holomorphic vector bundle will be of utmost importance when defining a genuine holomorphic structure on the determinant bundle associated with the family $\bar{\partial}$.

2. Double Transgression for Direct Images and the Heat Equation

In this section, we consider a chain complex of holomorphic Hermitian vectors bundles on M,

$$0 \to \xi_0 \underset{v}{\to} \xi_1 \underset{v}{\to} \cdots \underset{v}{\to} \xi_m \to 0.$$

By considering the Dolbeault resolutions of the $\bar{\partial}$ complexes associated with $\xi_0 \ldots \xi_m$ restricted to Z_y, we obtain a family of infinite dimensional complexes $(E_y, \bar{\partial}_y + v_y)$.

Using the Levi-Civita superconnection and the local index formula of [B 1], we obtain Chern character forms on B for this family. The purpose of this section is to double transgress these Chern character forms by imitating formally what has been done in [BGS 1] for finite dimensional complexes.

In a), we briefly describe the Levi-Civita superconnection of [B 1]. In b), we prove that in our situation, the Chern character forms of [B 1] associated with the Levi-Civita superconnection A_u – which depends on a parameter $u>0$ – are in the space P considered in [BGS 1].

In c), we prove that the form $\omega=\omega^H+\omega^V$ plays the role of a number operator associated with the $\bar{\partial}$ complexes. In particular, we find that this formal number operator together with the Levi-Civita superconnection verify the algebraic identities which were proved in [BGS 1] in a finite dimensional context.

In d), and imitating [BGS 1], we double transgress infinitesimally the local index forms of [B 1]. However, contrary to the situation considered in [BGS 1], certain asymptotic expansions (for $u\downarrow\downarrow 0$) have singular terms. Before obtaining the integrated double transgression of the local Chern forms, we need to understand the structure of such expansions.

Thus in e), and extending Bismut-Freed [BF 2, Sect. 3], we prove in full generality that the first transgressed forms are non-singular as $u\downarrow\downarrow 0$.

In f), and by a formal transfer of the results of [BGS 1], we obtain various identities with anticommuting variables, and we establish a generalized Lichnerowicz formula.

If N_u is our generalized number operator – which also depends on $u>0$, we calculate in g) the asymptotic expansion of $\mathrm{Tr}_s[N_u(\exp - A_u^2)]$ which is of the form

$$\mathrm{Tr}_s[N_u(\exp - A_u^2)] = \frac{C_{-1}}{u} + C_0 + O(u). \tag{2.1}$$

C_{-1} and C_0 are explicitly calculated using the identities established in f). Understanding the structure of C_0 will be essential in establishing [BGS 1, Theorems 0.1, 0.2, and 0.3].

In h), when $(E, \bar{\partial}+v)$ is acyclic, we obtain a double transgression formula in P for our Chern character local forms. This formula is of an essentially analytic nature. It is obtained as a generalized analytic torsion in the sense of Ray-Singer [RS 1, 2].

In i), we prove in Theorem 2.21 that when the chain complex ξ is acyclic, the double transgressed forms constructed in h) are equal in P/P' to the differential form appearing in [BGS 1, Eq. (0.6)]. Such a result is a double transgressed version of the Atiyah-Singer Index theorem for families [AS, B 1], since it equals an expression constructed by analytic methods, i.e. a generalized analytic torsion, to a local expression obtained via secondary characteristic classes.

The proof is technically difficult due to the fact that P' is in general not closed for any reasonable topology. However in degree 0, P' is irrelevant. The proof of Theorem 2.21 in degree 0 becomes much simpler, and most of the technicalities disappear.

In application to determinants in [BGS 3], we only use Theorem 2.21 in degree 0. So the reader interested in determinants may well skip most of the technicalities of the proof of Theorem 2.21.

Finally, observe that in degree 0. Theorem 2.21 exactly says that the Ray-Singer analytic torsion of a certain infinite dimensional complex is given by a local formula.

Note that as in [BGS 1], the notation $[A,B]$ will always represent the supercommutator of A and B.

a) Kähler Fibrations and the Levi-Civita Superconnection

We now suppose that the assumptions of Sect. 1c) are verified. The fibration $(\pi, g^Z, T^H M)$ is Kähler with associated $(1,1)$ form $\omega = \omega^H + \omega^Z$.

The bases (e_i), (w_i)... are taken as in Sect. 1d).

Let

$$0 \to \xi_0 \xrightarrow{v} \xi_1 \xrightarrow{v} \cdots \xrightarrow{v} \xi_m \to 0 \tag{2.2}$$

be a holomorphic chain complex of finite dimensional holomorphic Hermitian vector bundles on M. Set

$$\xi_+ = \bigoplus_{j\,\mathrm{even}} \xi_j, \qquad \xi_- = \bigoplus_{j\,\mathrm{odd}} \xi_j, \qquad \xi = \xi_+ \oplus \xi_- . \tag{2.3}$$

Then $\xi_\pm$, ξ are also holomorphic Hermitian vector bundles, and ξ is naturally Z_2 graded.

Let ∇^{ξ_j} be the unique holomorphic Hermitian connection on ξ_j, whose curvature we denote by L^{ξ_j}. Therefore $\nabla^{\xi} = \oplus \nabla^{\xi_j}$ is the unique holomorphic Hermitian connection on ξ and $L^{\xi} = \oplus L^{\xi_j}$ the corresponding curvature.

Let v^* be the adjoint of v. Set

$$V = v + v^*. \tag{2.4}$$

For $0 \leq j \leq m$, we can do the various constructions of Sect. 1 (with $\xi = \xi_j$). E_j^p, $E_j^\pm$, E_j denote the corresponding infinite dimensional Hermitian vector bundles on B which we endow with the (unlabelled) "holomorphic" Hermitian connection $\tilde{\nabla}$.

Also the unlabelled families of operators $\bar{\partial}$, $\bar{\partial}^*$, D act on E_j as well as the vertical Clifford multiplication operators $c(e_i)$.

Let τ be the involution defining the grading on E_j, i.e. $\tau = \pm 1$ on $E_j^\pm$. We will take the convention that v, v^*, V act on E_j like $\tau(1 \otimes v)$. Therefore v, v^*, V anticommute with the vertical Clifford multiplication operators $c(e_i)$.

We thus have an infinite dimensional "holomorphic" double chain complex of infinite dimensional vector bundles on B,

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & & 0 & \\
 & & \uparrow & & \uparrow & & & \uparrow & \\
0 & \to & E_0^\ell & \underset{v}{\to} & E_1^\ell & \to \dots & \underset{v}{\to} & E_m^\ell & \to 0 \\
0 & \to & \uparrow\bar{\partial} & & & & & \uparrow\bar{\partial} & \\
 & & \vdots & & & & & \vdots & \\
 & & \uparrow\bar{\partial} & & \uparrow\bar{\partial} & & & \uparrow\bar{\partial} & \\
0 & \to & E_0^0 & \underset{v}{\to} & E_1^0 & \to \dots & \underset{v}{\to} & E_m^0 & \to 0 \\
 & & \uparrow & & \uparrow & & & \uparrow & \\
 & & 0 & & 0 & & & 0 &
\end{array}$$

As in [BGS 1, Sect. 1d)], the double complex has a horizontal, a vertical, and a total Z grading. Set

$$E = \bigoplus_{j,p} E_j^p, \qquad E_+ = \bigoplus_{j+p\,\text{even}} E_j^p, \qquad E_- = \bigoplus_{j+p\,\text{odd}} E_j^p.$$

The operators $\bar{\partial}$, $\bar{\partial}^*$, D, v, v^*, V are odd in $\operatorname{End} E$. Since v is holomorphic, we have

$$[\bar{\partial}, v] = [\bar{\partial}^*, v^*] = 0, \qquad (\bar{\partial} + v)^2 = (\bar{\partial}^* + v^*)^2 = 0. \tag{2.5}$$

Also $\tilde{\nabla}$ splits into $\tilde{\nabla} = \tilde{\nabla}' + \tilde{\nabla}''$, where $\tilde{\nabla}'$, $\tilde{\nabla}''$ are the holomorphic and antiholomorphic parts of $\tilde{\nabla}$. By Theorem 1.14,

$$\tilde{\nabla}''(\bar{\partial} + v) = 0, \qquad \tilde{\nabla}'(\bar{\partial}^* + v^*) = 0. \tag{2.6}$$

For $u > 0$, $\tilde{\nabla} + \sqrt{u}(D + V)$ is a superconnection on E. This superconnection is the natural extension of the superconnection of [BGS 1, Sect. 1c)] in an infinite dimensional situation. However, due to the results of Bismut [B 1], we know it is not the right choice of a superconnection to obtain a local form of the Theorem of Atiyah-Singer for families.

So we define the Levi-Civita superconnection introduced in [B 1, Sect. 3].

Definition 2.1. For $u > 0$, the Levi-Civita superconnection A_u on E is given by

$$A_u = \tilde{\nabla} + \sqrt{u}(D + V) - \left(\frac{1}{4\sqrt{u}}\right) dy^\alpha d\bar{y}^\beta c\left(T\left(\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial \bar{y}^\beta}\right)\right). \tag{2.7}$$

Instead of (2.7), we will also the notation

$$A_u = \tilde{\nabla} + \sqrt{u}(D+V) - \frac{c(T)}{4\sqrt{u}}. \tag{2.8}$$

When $V=0$, by [BF 2, Proposition 1.18], A_u is exactly the Levi-Civita superconnection of [B 1, Sect. 3].

b) Construction of Chern Character Forms in P

We now prove a first basic result concerning the superconnections $\tilde{\nabla} + \sqrt{u}(D+V)$ and A_u.

In all the formulas where characteristic classes appear, R^Z will be considered as the curvature tensor of $T^{(1,0)}Z$.

Let us recall that the ad-invariant Todd polynomial on complex (ℓ, ℓ) matrices is characterized by the fact that if B is diagonal with diagonal entries $y_1, \ldots, y_\ell$, then

$$Td(B) = \prod_1^\ell \frac{y_j}{1-e^{-y_j}}.$$

Theorem 2.2. *For any $u>0$, the smooth differential forms on B, $\mathrm{Tr}_s[\exp-(\tilde{\nabla} + \sqrt{u}(D+V))^2]$ and $\mathrm{Tr}_s[\exp-(A_u)^2]$, are elements of P. They are closed and they are in the same cohomology class, which does not depend on $u>0$.*

Also, uniformly on compact sets in B

$$\lim_{u \downarrow\downarrow 0} \mathrm{Tr}_s[\exp - A_u^2] = \left(\frac{1}{2\pi i}\right)^\ell \int_Z Td(-R^Z)\,\mathrm{Tr}_s[\exp(-L^\xi)], \tag{2.9}$$

and the differential form in the right hand-side of (2.9) is also in the same cohomology class as $\mathrm{Tr}_s[\exp - A_u^2]$.

If B is compact, let $T_j \in K(B)$ be given by

$$T_j = \mathrm{Ker}(D_+ | E_j) - \mathrm{Ker}(D_- | E_j).$$

The differential forms considered above represent in cohomology $\overline{\mathrm{ch}}(T_0 - T_1 + T_2 \ldots)$.

Proof. The proof that $\mathrm{Tr}_s[\exp-(\tilde{\nabla} + \sqrt{u}(D+V))^2]$ is in P is the infinite dimensional analogue of the proof of [BGS 1, Theorem 1.9]. We here use instead the relation (2.6).

Also one verifies that

$$[c(T), V] = 0. \tag{2.10}$$

We thus find that

$$A_u^2 = \left(\tilde{\nabla} + \sqrt{u}D - \frac{c(T)}{4\sqrt{u}}\right)^2 + \sqrt{u}(\tilde{\nabla}' v + \tilde{\nabla}'' v^*) + u([\bar{\partial}, v^*] + [\bar{\partial}^*, v]) + u(vv^* + v^*v). \tag{2.11}$$

The operators $[\bar{\partial}, v^*]$, $[\bar{\partial}^*, v]$, vv^*, v^*v preserve the total grading in E.

$\tilde{\nabla}' v$ is of type $(1,0)$ and increases the total degree in E by 1, while $\tilde{\nabla}'' v$ is of type $(0,1)$ and decreases the total degree in E by 1.

We now calculate $\left(\tilde{\nabla}+\sqrt{u}D-\frac{c(T)}{4\sqrt{u}}\right)^2$ acting on E_k. Remember that by the results of Sect. 1d), we can use the results of [B 1], with $\xi=\mu\otimes\xi_k$, where μ is any locally defined square root of $\det T^{(1,0)}Z$ endowed with the corresponding holomorphic Hermitian connection. The curvature L'^{ξ_k} of $\mu\otimes\xi_k$ is given by

$$L'^{\xi_k}=\tfrac{1}{2}\operatorname{Tr}[R^Z]I+L^{\xi_k}. \tag{2.12}$$

Let K be the scalar curvature of Z. By Theorem 1.7, we know that $\langle S(e_j)\cdot,\cdot\rangle$ is a 2-form of complex type $(1,1)$. Using [B 1, Theorem 3.6], we find that on E_k

$$\begin{aligned}\left(\tilde{\nabla}+\sqrt{u}D-\frac{c(T)}{4\sqrt{u}}\right)^2=&-u\left(\nabla_{e_i}+\frac{1}{2\sqrt{u}}\left\langle S(e_i)w_j,\frac{\partial}{\partial\bar{y}^\alpha}\right\rangle c(\bar{w}_j)d\bar{y}^\alpha\right.\\&\left.+\frac{1}{2\sqrt{u}}\left\langle S(e_i)\bar{w}_j,\frac{\partial}{\partial y^\alpha}\right\rangle c(w_j)dy^\alpha+\frac{1}{2u}\left\langle S(e_i)\frac{\partial}{\partial y^\alpha},\frac{\partial}{\partial\bar{y}^\beta}\right\rangle dy^\alpha d\bar{y}^\beta\right)^2\\&+\frac{uK}{4}+\frac{u}{2}[c(w_i)c(\bar{w}_j)\otimes L'^{\xi_k}(\bar{w}_i,w_j)+c(\bar{w}_i)c(w_j)\otimes L'^{\xi_k}(w_i,\bar{w}_j)]\\&+\sqrt{u}\left[c(w_i)dy^\alpha\otimes L'^{\xi_k}\left(\bar{w}_i,\frac{\partial}{\partial y^\alpha}\right)+c(\bar{w}_i)d\bar{y}^\alpha\otimes L'^{\xi_k}\left(w_i,\frac{\partial}{\partial\bar{y}^\alpha}\right)\right]\\&+dy^\alpha d\bar{y}^\beta\otimes L'^{\xi_k}\left(\frac{\partial}{\partial y^\alpha},\frac{\partial}{\partial\bar{y}^\beta}\right).\end{aligned} \tag{2.13}$$

One again verifies that the terms on the right-hand side of (2.13) are of three kinds:

- The terms which preserve the grading of E_k and which are of type $(0,0)$ or $(1,1)$ in the Grassmann variables in T_C^*B;
- The terms which increase the degree in E_k by 1 and are of type $(1,0)$ in the variables in T_C^*B;
- The terms which decrease the degree in E_k by 1 and are of type $(0,1)$ in the variables in T_C^*B.

Using [BGS 1, Proposition 1.8], we find that $\operatorname{Tr}_s[\exp-(A_u^2)]$ is also in P.

When $v=0$, (2.9) is a consequence of [B 1, Theorems 4.12 and 4.16]. When $v\neq 0$, exactly the same methods permit us again to prove (2.9). In particular, because it has the weight u, the 0 order operator $[\bar{\partial}^*,v]+[\bar{\partial},v^*]$ does not contribute to the limit.

When $v=0$, the final part of the Theorem is proved in [B 1, Theorem 3.4]. Replacing D by $D+bV$ $(0\leqq b\leqq 1)$ and using the fact that the cohomology class of the corresponding forms does not change with b – see in particular [B 1, Remark 2.3], the end of the Theorem holds in full generality. □

Remark 2.3. It is a consequence of Bismut-Freed [BF 2, Theorem 1.19] that in degree 0 and $(1,1)$, $\operatorname{Tr}_s[\exp-(\nabla+\sqrt{u}(D+V))^2]$ and $\operatorname{Tr}_s[\exp-(A_u^2)]$ coincide.

In general, $\operatorname{Tr}_s[\exp-(\nabla+\sqrt{u}(D+V))^2]$ does not converge as $u\downarrow\downarrow 0$, except in degree 0 and $(1,1)$. This explain why we need to use the Levi-Civita superconnection to study higher degree characteristic classes.

c) Number Operator and the Levi-Civita Superconnection

The double complex E has a horizontal and a vertical grading. Let N_H, N_V be the number operators corresponding to these two gradings. N_H and N_V act on E_j^k by multiplication by j and k. $N=N_H+N_V$ is thus the total grading number operator.

These N_H, N_V, N are the right choice of number operators if we use the superconnection $\tilde{\nabla}+\sqrt{u}(D+V)$. We can thus reproduce formally what has been done in [BGS 1] to double transgress the Chern forms $\mathrm{Tr}_s[\exp -(\tilde{\nabla}+\sqrt{u}(D+V))^2]$.

However because of (2.9), the right forms to consider are $\mathrm{Tr}_s[\exp(-A_u^2)]$. The number operators have to be changed in order that certain basic commutation relations are still verified.

We first evaluate the number operator N_V in terms of the vertical Kähler form ω^Z. Note that $\omega^Z \in \Lambda^2(TZ)$. The element of the Clifford algebra $c(TZ)$ which corresponds to ω^Z identified with the antisymmetric matrix J^Z is $\omega^{Z,c}$ given by [B 1, Eq. (1.2)]

$$\omega^{Z,c}=-\tfrac{1}{4}\omega^Z(w_j,\bar{w}_j)\,[c(\bar{w}_j)c(w_j)-c(w_j)c(\bar{w}_j)]\,. \tag{2.14}$$

Proposition 2.4. *We have the following identity*

$$N_V=-i\omega^{Z,c}+\frac{\ell}{2}. \tag{2.15}$$

Proof. By (1.41),

$$\omega^{Z,c}=\frac{i}{2}[-i_{\bar{w}_j}\bar{w}^j\wedge+\bar{w}^j\wedge i_{\bar{w}_j}]=i\left[\bar{w}^j\wedge i_{\bar{w}_j}-\frac{\ell}{2}\right].$$

Also $\sum_{j=1}^{\ell}\bar{w}^j\wedge i_{\bar{w}_j}$ acts on E_j^k by multiplication by k. Equation (2.15) is proved. □

We now will define a new vertical number operator, which is an element of $\Lambda(T_C^*B)\hat{\otimes}\mathrm{End}\,E$, and depends on the parameter $u>0$.

Definition 2.5. For $u>0$, the operators $N'_{V,u}$, N_u are given by

$$N'_{V,u}=-i\omega^{Z,c}+i\omega^H\wedge/2u+\frac{\ell}{2},\qquad N_u=N'_{V,u}+N_H. \tag{2.16}$$

We now prove a family of commutation relations which exactly extends [BGS 1, Eq. (1.24)].

Theorem 2.6. *The following relations hold:*

$$\begin{gathered}
[\tilde{\nabla},N_u]=0,\qquad [\tilde{\nabla}'',v]=[\tilde{\nabla}',v^*]=0,\\
[\tilde{\nabla}'',\bar{\partial}]=[\tilde{\nabla}',\bar{\partial}^*]=0,\qquad [\bar{\partial},\omega^H]=ic(T^{(1,0)});\qquad [\bar{\partial}^*,\omega^H]=-ic(T^{(0,1)}),\\
[\tilde{\nabla}'',c(T^{(1,0)})]=[\nabla',c(T^{(0,1)})]=0,\qquad [\bar{\partial},c(T^{(1,0)})]=[\bar{\partial}^*,c(T^{(0,1)})]=0,\\
\left[\sqrt{u}(\bar{\partial}+v)-\frac{c(T^{(1,0)})}{4\sqrt{u}},N_u\right]=-\sqrt{u}(\bar{\partial}+v)-\frac{c(T^{(1,0)})}{4\sqrt{u}},\\
\left[\sqrt{u}(\bar{\partial}^*+v^*)-\frac{c(T^{(1,0)})}{4\sqrt{u}},N_u\right]=\sqrt{u}(\bar{\partial}^*+v^*)+\frac{c(T^{(0,1)})}{4\sqrt{u}}.
\end{gathered} \tag{2.17}$$

In particular

$$A_u=\left(\tilde{\nabla}''+\sqrt{u}(\bar{\partial}+v)-\frac{c(T^{(1,0)})}{4\sqrt{u}}\right)+\left(\tilde{\nabla}'+\sqrt{u}(\bar{\partial}^*+v^*)-\frac{c(T^{(0,1)})}{4\sqrt{u}}\right),$$

$$\left(\sqrt{u}(\bar{\partial}+v)-\frac{c(T^{(1,0)})}{4\sqrt{u}}\right)^2=\left(\sqrt{u}(\bar{\partial}^*+v^*)-\frac{c(T^{(0,1)})}{4\sqrt{u}}\right)^2=0,$$

$$\left(\tilde{\nabla}''+\sqrt{u}(\bar{\partial}+v)-\frac{c(T^{(1,0)})}{4\sqrt{u}}\right)^2=\left(\tilde{\nabla}'+\sqrt{u}(\bar{\partial}^*+v^*)-\frac{c(T^{(0,1)})}{4\sqrt{u}}\right)^2=0,$$

$$A_u^2=\left[\tilde{\nabla}''+\sqrt{u}(\bar{\partial}+v)-\frac{c(T^{(1,0)})}{4\sqrt{u}},\tilde{\nabla}'+\sqrt{u}(\bar{\partial}^*+v^*)-\frac{c(T^{(0,1)})}{4\sqrt{u}}\right],$$

$$\left[\tilde{\nabla}''+\sqrt{u}(\bar{\partial}+v)-\frac{c(T^{(1,0)})}{4\sqrt{u}},A_u^2\right]=\left[\tilde{\nabla}'+\sqrt{u}(\bar{\partial}^*+v^*)-\frac{c(T^{(0,1)})}{4\sqrt{u}},A_u^2\right]=0,$$

$$[A_u,N_u]=\frac{2u\partial}{\partial u}\left[-\left(\sqrt{u}(\bar{\partial}+v)-\frac{c(T^{(1,0)})}{4\sqrt{u}}\right)+\left(\sqrt{u}(\bar{\partial}^*+v^*)-\frac{c(T^{(0,1)})}{4\sqrt{u}}\right)\right]. \tag{2.18}$$

Proof. The number operator N_V is parallel for the connection $\tilde{\nabla}$. Therefore by Proposition 2.4, we find that

$$[\tilde{\nabla},\omega^{Z,c}]=0. \tag{2.19}$$

Of course, (2.19) reflects the fact that $\tilde{\nabla}$ is unitary and preserves the Kähler form ω^Z.

Also, by Theorem 1.7,

$$\nabla^a\cdot\omega^H=0 \quad \text{on} \quad T^HM\times T^HM\times T^HM. \tag{2.20}$$

It follows that

$$[\tilde{\nabla},\omega^H]=0. \tag{2.21}$$

We have thus proved that $[\tilde{\nabla},N'_{V,u}]=[\tilde{\nabla},N_u]=0$. Clearly

$$[\bar{\partial},N_V]=-\bar{\partial}. \tag{2.22}$$

Also since $c(T^{(1,0)})$ increases the vertical degree by 1, we also have

$$[-c(T^{(1,0)}),N_V]=c(T^{(1,0)}). \tag{2.23}$$

Also trivially

$$[c(T^{(1,0)}),\omega^H]=0. \tag{2.24}$$

By Theorem 1.7, we know that

$$\nabla^a\omega^H+i_T\omega^Z=0 \quad \text{on} \quad T^HM\times T^HM\times TZ. \tag{2.25}$$

Also

$$[\bar{\partial},\omega^H]=c(w_i)\nabla_{\bar{w}_i}\omega^H; \qquad [\bar{\partial}^*,\omega^H]=c(\bar{w}_i)\nabla_{w_i}\omega^H. \tag{2.26}$$

Using (2.25), we find that

$$\begin{aligned}[\bar\partial, \omega^H] &= -c(w_i)\omega^Z(T, \bar w_i) = ic(T^{(1,0)}),\\ [\bar\partial^*, \omega^H] &= -c(\bar w_i)\omega^Z(T, w_i) = -ic(T^{(0,1)}).\end{aligned} \tag{2.27}$$

Using (2.6), (2.21) and (2.27), we find that

$$[\tilde V'', c(T^{(1,0)})] = [\tilde V', c(T^{(0,1)})] = 0. \tag{2.28}$$

Also

$$[\bar\partial, [\bar\partial, \omega^H]] = [[\bar\partial, \bar\partial], \omega^H] - [\bar\partial, [\bar\partial, \omega^H]]. \tag{2.29}$$

Since $\bar\partial^2 = 0$, we find that: $[\bar\partial, [\bar\partial, \omega^H]] = 0$, and so using (2.27), we get

$$[\bar\partial, c(T^{(1,0)})] = 0. \tag{2.30}$$

In the same way, we can prove that

$$[\bar\partial^*, c(T^{(0,1)})] = 0. \tag{2.31}$$

Using (2.22)–(2.31), it is now easy to prove the final equalities in (2.17).

Since $(c(T^{(1,0)}))^2 = (c(T^{(0,1)}))^2 = 0$, the second series of equalities in (2.18) is a consequence of (2.17).

Since $(\tilde V')^2 = (\tilde V'')^2 = 0$, we find the third series of equalities in (2.18) also hold. The fourth and fifth equalities in (2.18) are now obvious. The sixth equality is a consequence of (2.17). □

Remark 2.7. Theorem 2.6 should shed some light on the result of Theorem 2.2 which asserts that $\mathrm{Tr}_s[\exp - A_u^2]$ is in P.

In fact by Theorem 2.6, $\sqrt{u}(\bar\partial + v) - \dfrac{c(T^{(1,0)})}{4\sqrt{u}}$ is a "holomorphic" function of $y \in B$; it increases the degree in E by 1, and its square vanishes, while $\sqrt{u}(\bar\partial^* + v^*) - \dfrac{c(T^{(0,1)})}{4\sqrt{u}}$ is "antiholomorphic," decreases the degree by 1, and also has a square which vanishes. The situation is then formally identical to what was done in [BGS 1].

$N'_{V,u}$, N_u will play the role of vertical number operators and of total number operators. In this respect, the final equality in (2.18) is of critical importance, since it shows that N_u incorporates the basic features of a number operator, as used in [BGS 1].

It follows from (2.17) that

$$(\tilde V'' + \bar\partial)^2 = 0. \tag{2.32}$$

We now give an intepretation of (2.32) which will be very useful in [BGS 3]. Because of the splitting $T^{(1,0)}M = T^{(1,0)}Z \oplus T^{H(1,0)}M$, we have the identification:

$$\Lambda(T^{*(0,1)}M) = \Lambda(T^{*(0,1)}B) \hat\otimes \Lambda(T^{*(0,1)}Z).$$

$\tilde V'' + \bar\partial^Z$ acts naturally on the smooth sections of $\Lambda(T^{*(0,1)}B) \hat\otimes \Lambda(T^{*(0,1)}Z) \otimes \xi$.

So $\tilde{\nabla}'' + \bar{\partial}^Z$ acts naturally on the smooth sections of $\Lambda(T^{*(0,1)}M) \otimes \xi$.

Also $\bar{\partial}^M$ acts naturally on smooth sections of $\Lambda(T^{*(0,1)}M) \otimes \xi$.

Theorem 2.8. *We have the equality of operators acting on smooth sections of* $\Lambda(T^{*(0,1)}M) \otimes \xi$,

$$\bar{\partial}^M = \tilde{\nabla}'' + \bar{\partial}^Z . \tag{2.33}$$

Proof. Equality (2.33) is clearly local. Therefore, if U is any open set in B, we only need to prove (2.33) on $\pi^{-1}(U)$.

Recall that in Sect. 1 a), we were free to choose any metric g^B on TB. Therefore, we can assume that U is small enough so that g^B induces a Kähler metric on U. We now will work on $\pi^{-1}(U)$.

The connection $\nabla = \nabla^B \oplus \nabla^Z$ is complex, i.e. induces a connection on $T^{(1,0)}M$.

The operator ∇^a, which acts on smooth sections of $\Lambda(T_C^* M)$, was defined in Sect. 1 c). We extend ∇^a into a operator acting on smooth sections of $\Lambda(T_C^* M) \otimes \xi$ in the obvious way.

By (1.26), we know that we have the equality of operators acting on smooth sections of $\Lambda(T_C^* M) \otimes \xi$,

$$\nabla^\xi = \nabla^a + i_T . \tag{2.34}$$

Also ∇^a splits into $\nabla^a = \nabla^{a'} + \nabla^{a''}$, where $\nabla^{a'}$, $\nabla^{a''}$ are the holomorphic and antiholomorphic parts of ∇^a.

The connection ∇^B is complex and torsion free, and also, when restricted to one given fiber, the connection ∇^Z is complex and torsion free. We thus find that

$$\nabla^{a''} = \tilde{\nabla}'' + \bar{\partial}^Z . \tag{2.35}$$

By Theorem 1.7, T vanishes on $T_C Z \times T_C Z$ and is of complex type $(1,1)$. Therefore

$$i_T = d\bar{z}^i \wedge dy^\alpha i_{T\left(\frac{\partial}{\partial \bar{z}^i}, \frac{\partial}{\partial y^\alpha}\right)} + dz^i \wedge d\bar{y}^\alpha i_{T\left(\frac{\partial}{\partial z^i}, \frac{\partial}{\partial \bar{y}^\alpha}\right)} + dy^\alpha \wedge d\bar{y}^\beta i_{T\left(\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial \bar{y}^\beta}\right)} . \tag{2.36}$$

Clearly $\bar{\partial}^M = \nabla^{\xi''}$. Moreover $\bar{\partial}^M$, $\tilde{\nabla}''$ and $\bar{\partial}^Z$ map forms of type $(0,p)$ into forms of type $(0,p+1)$. Due to (2.36), we see that i_T maps forms of type $(0,p)$ into forms of type $(1,p)$.

From (2.34)–(2.36), we obtain (2.33). □

d) Double Transgression of the Chern Character: The Infinitesimal Form

We now prove the natural analogue of [BGS 1, Theorem 1.15] in an infinite dimensional context.

Theorem 2.9. *For any $u > 0$, the smooth differential form* $\mathrm{Tr}_s[N_u \exp - A_u^2]$ *is in P. Also*

$$\left(\frac{\partial}{\partial u}\right) \mathrm{Tr}_s[\exp(-A_u^2)] = \left(\frac{-1}{2u}\right) \left((\partial^B + \bar{\partial}^B)\, \mathrm{Tr}_s\left[\left(\sqrt{u}(D+V) + \frac{c(T)}{4\sqrt{u}}\right) \exp(-A_u^2)\right]\right),$$
$$\mathrm{Tr}_s\left[\left(\sqrt{u}(D+V) + \frac{c(T)}{4\sqrt{u}}\right) \exp(-A_u^2)\right] = (\partial^B - \bar{\partial}^B)\, \mathrm{Tr}_s[N_u \exp(-A_u^2)] . \tag{2.37}$$

In particular

$$\left(\frac{\partial}{\partial u}\right)\mathrm{Tr}_s[\exp(-A_u^2)] = \left(\frac{-1}{u}\right)\bar{\partial}^B\partial^B\,\mathrm{Tr}_s[N_u\exp(-A_u^2)]\,. \tag{2.38}$$

Proof. To differentiate traces or supertraces, we must proceed rigorously as in [B 1, Sect. 2], i.e. use the fact that since D^2 is fiberwise elliptic, $\exp(-A_u^2)$ is given by a fiberwise smooth kernel depending smoothly on $y\in B$. Ultimately the manipulations of [B 1, Sect. 2] show that formally, in this situation, we can use the same commutation rules as in finite dimensions.

The first line of (2.37) is then a simple consequence of the superconnection algebra. The second line of (2.37) can be proved by the same arguments as in the proofs of [BGS 1, Theorems 1.9 and 1.15], simply using the commutation rules of Theorem 2.6 instead of [BGS 1, Proposition 1.6]. □

Before giving an integrated version of Eq. (2.38), we will study the behavior of the various quantities appearing in (2.37)–(2.38) as $u\downarrow\downarrow 0$.

In Theorem 2.2, we have shown that as $u\downarrow\downarrow 0$, $\mathrm{Tr}_s[\exp - A_u^2]$ is non-singular because of certain cancellations obtained in [B 1, Sect. 4]. This implies that related cancellations occur in right-hand side of (2.37), 2.38).

We will study these cancellations, and also calculate explicitly certain terms in the corresponding asymptotic expansions.

e) Asymptotic Behavior of the First Transgressed Forms

We first study the behavior of $\mathrm{Tr}_s\left[\left(\sqrt{u}(D+V)+\dfrac{c(T)}{4\sqrt{u}}\right)\exp(-A_u^2)\right]$ as $u\downarrow\downarrow 0$. The result which we will obtain generalizes the result obtained in Bismut-Freed [BF 2, Theorem 3.4] which was only concerned with the degree one part of this expression.

The result which we will prove is true in full generality for any family of Dirac operators of the kind considered in [B 1] and has nothing to do with complex geometry. It will be formulated in complex geometric terms for simplicity.

Here du denotes the odd Grassmann variable corresponding to $u\in R^+$.

Recall that L'^{ξ_k} has been introduced in (2.12).

Let L'^{ξ} denote the corresponding curvature tensor associated with ξ, i.e. $L'^{\xi} = \frac{1}{2}\mathrm{Tr}[R^Z]I + L^{\xi}$.

Proposition 2.10. *For any $u>0$, we have the equality*

$$\begin{aligned}
A_u^2 - du\left(\sqrt{u}(D+V)+\frac{c(T)}{4\sqrt{u}}\right) &= -u\left(\nabla_{e_i} + \frac{1}{2\sqrt{u}}\left\langle S(e_i)e_j, \frac{\partial}{\partial y^\alpha}\right\rangle c(e_j)dy^\alpha\right.\\
&\quad\left. + \frac{1}{2\sqrt{u}}\left\langle S(e_i)e_j, \frac{\partial}{\partial \bar{y}^\alpha}\right\rangle c(e_j)d\bar{y}^\alpha + \frac{1}{2u}\left\langle S(e_i)\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial \bar{y}^\beta}\right\rangle dy^\alpha d\bar{y}^\beta - \frac{c(e_i)du}{2\sqrt{u}}\right)^2\\
&\quad + \frac{uK}{4} + \left(\frac{u}{2}\right)c(e_i)c(e_j)\otimes L'^{\xi}(e_i,e_j) + \sqrt{u}c(e_i)dy^\alpha\otimes L'^{\xi}\left(e_i,\frac{\partial}{\partial y^\alpha}\right)\\
&\quad + \sqrt{u}c(e_i)d\bar{y}^\alpha\otimes L'^{\xi}\left(e_i,\frac{\partial}{\partial \bar{y}^\alpha}\right) + dy^\alpha d\bar{y}^\beta\otimes L'^{\xi}\left(\frac{\partial}{\partial y^\alpha},\frac{\partial}{\partial \bar{y}^\beta}\right)\\
&\quad + \sqrt{u}\nabla V + u[D,V] + uV^2 - du\sqrt{u}V.
\end{aligned} \tag{2.39}$$

Proof. When $du=0$, (2.39) is exactly [B 1, Theorem 3.6]. Also

$$-\left(\frac{\left\langle S(e_i)e_j, \frac{\partial}{\partial y^\alpha}\right\rangle}{4}\right)[c(e_j)dy^\alpha c(e_i)du + c(e_i)duc(e_j)dy^\alpha]$$

$$=\left(\frac{\left\langle S(e_i)e_j, \frac{\partial}{\partial y^\alpha}\right\rangle}{4}\right) dy^\alpha du[c(e_j)c(e_i)-c(e_i)c(e_j)]$$

$$=\tfrac{1}{4}\left\langle S(e_i)e_j - S(e_j)e_i, \frac{\partial}{\partial y^\alpha}\right\rangle dy^\alpha duc(e_j)c(e_i)$$

$$=-\tfrac{1}{4}\left\langle T(e_i,e_j), \frac{\partial}{\partial y^\alpha}\right\rangle dy^\alpha duc(e_j)c(e_i). \tag{2.40}$$

Since $T(e_i, e_j)=0$, (2.40) is 0. Of course, (2.40) is still 0 when replacing $\frac{\partial}{\partial y^\alpha}$, dy^α by $\frac{\partial}{\partial \bar{y}^\alpha}$, $d\bar{y}^\alpha$. Also, by (1.4)

$$\frac{1}{2\sqrt{u}}\left\langle S(e_i)\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial \bar{y}^\beta}\right\rangle duc(e_i)dy^\alpha d\bar{y}^\beta = du\left(\frac{c(T)}{4\sqrt{u}}\right).$$

It is now easy to obtain (2.39). □

Theorem 2.11. *There exist C^∞ even differential forms $A_0, A_1, \dots$ in P, and C^∞ odd differential forms $B_1, B_2, \dots$, such that for any $k \in N$,*

$$\begin{aligned}\mathrm{Tr}_s[\exp(-A_u^2)] &= \sum_0^k A_j u^j + o(u^k),\\ \mathrm{Tr}_s\left[\left(\sqrt{u}(D+V)+\frac{c(T)}{4\sqrt{u}}\right)\exp(-A_u^2)\right] &= \sum_1^k B_j u^j + o(u^k),\end{aligned} \tag{2.41}$$

and the various $o(u^k)$ are uniform on the compact sets in B. Also

$$A_0 = \left(\frac{1}{2\pi i}\right)^{\ell} \int_Z Td(-R^Z)\,\mathrm{Tr}_s[\exp(-L^\xi)], \quad (\partial^B + \bar{\partial}^B)B_j = -2jA_j; \quad j>0. \tag{2.42}$$

Proof. By Greiner [Gr, Theorem 1.6.1], we know that for any $k' \in N$,

$$\mathrm{Tr}_s\left[\exp\left\{-u\left(\left(\tilde{V}+D+V-\frac{c(T)}{4}\right)^2 - du\left(D+V+\frac{c(T)}{4}\right)\right)\right\}\right] = \sum_{-\ell}^{k'} E_j u^j + o(u^{k'}),$$

where $o(u^{k'})$ is uniform over the compact sets of B. We now rescale dy^α, $d\bar{y}^\alpha$, du into $\frac{dy^\alpha}{\sqrt{u}}, \frac{d\bar{y}^\alpha}{\sqrt{u}}, \frac{du}{\sqrt{u}}$. We thus find that for any $k \in N$,

$$\mathrm{Tr}_s\left[\exp\left(-A_u^2 + du\left(\sqrt{u}(D+V)+\frac{c(T)}{4\sqrt{u}}\right)\right)\right] = \sum_{-(\ell+\ell')}^{k} E'_j u^j + o(u^k). \tag{2.43}$$

Also by Duhamel's formula, the left-hand side of (2.43) is given by

$$\mathrm{Tr}_s[\exp(-A_u^2)+du\,\mathrm{Tr}_s\left[\left(\sqrt{u}(D+V)+\frac{c(T)}{4\sqrt{u}}\right)\exp(-A_u^2)\right].$$

We thus deduce from (2.43) that

$$\mathrm{Tr}_s[\exp(-A_u^2)]=\sum_{-(\ell+\ell')}^{k} A_j u^j+o(u^k),$$

$$\mathrm{Tr}_s\left[\left(\sqrt{u}(D+V)+\frac{c(T)}{4\sqrt{u}}\right)\exp(-A_u^2)\right]=\sum_{-(\ell+\ell')}^{k} B_j u^j+o(u^k).$$

To prove (2.41), we will show that $E'_j=0, j<0$ and that E'_0 does not contain du.

By Proposition 2.10, we find that the right-hand side of (2.39) is *exactly* of the same form as the corresponding formula of [B 1, Theorem 3.6] for A_u^2, with the exception that $\frac{c(e_i)du}{2\sqrt{u}}$ appears.

We can thus use formally the results of [B 1, Sect. 4], which already show that, for $j<0$, $E'_j=0$. Let us prove that E'_0 does not contain du.

Note the commutation relations

$$[c(e_j)dy^\alpha, c(e_i)du]=2\delta_i^j dy^\alpha du. \tag{2.44}$$

Take $x\in Z_y$. Let w'^1 be a Brownian bridge in T_xZ_y, with $w_0'^1=w_1'^1=0$, and let P_1 be the law of w'^1 on $\mathscr{C}([0,1];\,T_xZ_y)$.

By proceeding as in [B 1, Theorem 4.12] and using the commutation relations (2.44), we find that as $u\downarrow\downarrow 0$, the left-hand side of (2.43) has a limit and that the only term where du appears is given by

$$\begin{aligned}\int\exp\Bigg\{&\ldots\tfrac{1}{4}\int_0^1\left\langle S(w'^1)dw'^1-S(dw'^1)w'^1,\frac{\partial}{\partial y^\alpha}\right\rangle dy^\alpha du\\ &+\tfrac{1}{4}\int_0^1\left\langle S(w'^1)dw'^1-S(dw'^1)w'^1,\frac{\partial}{\partial \bar{y}^\alpha}\right\rangle d\bar{y}^\alpha du\ldots\Bigg\}dP_1(w'^1).\end{aligned} \tag{2.45}$$

Since w'^1, $dw'^1\in TZ$, we have

$$S(w'^1)dw'^1-S(dw'^1)w'^1=-T(w'^1,dw'^1)=0. \tag{2.46}$$

It thus follows from (2.45)–(2.46) that E'_0 does not contain du.

The explicit expression of A_0 has already been found in (2.9). Using (2.37), we obtain the second expression in (2.42) □

Remark 2.12. The asymptotics as $u\downarrow\downarrow 0$ of the first line of Eq. (2.37) is now fully understood.

We will study the asymptotics of $\mathrm{Tr}_s[N_u\exp(-A_u^2)]$. The presence of the diverging term $\frac{i\omega^H}{2u}$ in N_u already indicates that it will be more difficult. In order to solve these difficulties, we now will establish certain formulas which are the infinite dimensional analogues of the results given in [BGS 1].

f) On Certain Identities Verified by the Levi-Civita Superconnection

We now establish the infinite dimensional analogues of [BGS 1, Theorems 1.10, 1.12, 1.13].

Let $da, d\bar{a}$ be two odd Grassmann variables. We still use the convention that if $\eta \in \Lambda(T_C^* B) \hat{\otimes} C(da, d\bar{a})$, if η is written in the form,

$$\eta = \eta_0 + da\eta_1 + d\bar{a}\eta_2 + dad\bar{a}\eta_3 (\eta_i \in \Lambda(T_C^* B), 0 \leqq i \leqq 3),$$

then, we set

$$[\eta]^{dad\bar{a}} = \eta_3 . \tag{2.47}$$

Theorem 2.13. *For any $u > 0$, $b \geqq 0$,*

$$\begin{aligned} & bu \operatorname{Tr}_s\left[\left(\sqrt{u}(D+V) + \frac{c(T)}{4\sqrt{u}}\right)\exp(-A_u^2 + buN_u)\right] \\ & \quad = (\partial^B - \bar{\partial}^B)\operatorname{Tr}_s[\exp(-A_u^2 + buN_u)] . \end{aligned} \tag{2.48}$$

Let $\theta_u \in P$ be given by

$$\begin{aligned} \theta_u = \operatorname{Tr}_s\Bigg[\exp\Bigg(& -A_u^2 - da\left(\sqrt{u}(\bar{\partial} + v) + \frac{c(T^{(1,0)})}{4\sqrt{u}}\right) \\ & - d\bar{a}\left(\sqrt{u}(\bar{\partial}^* + v^*) + \frac{c(T^{(0,1)})}{4\sqrt{u}}\right) - dad\bar{a} i\omega^{Z,c} + buN_u\Bigg)\Bigg]^{dad\bar{a}} . \end{aligned} \tag{2.49}$$

Then, for any $u > 0$, $b \geqq 0$,

$$\begin{aligned} & \frac{\partial}{\partial u}\operatorname{Tr}_s[\exp(-A_u^2 + buN_u)] \\ & = -\left(\frac{1}{2u}\right) d^B \operatorname{Tr}_s\left[\left(\sqrt{u}(D+V) + \frac{c(T)}{4\sqrt{u}}\right)\exp(-A_u^2 + buN_u)\right] \\ & \quad + b\left(\theta_u + \operatorname{Tr}_s\left[\left(N_H + \frac{\ell}{2}\right)\exp(-A_u^2 + buN_u)\right]\right); \end{aligned} \tag{2.50}$$

or equivalently for $u > 0$, $b > 0$,

$$\begin{aligned} & \left(\frac{\partial}{\partial u}\right)\operatorname{Tr}_s[\exp(-A_u^2 + buN_u)] = -\left(\frac{1}{bu^2}\right)\bar{\partial}^B \partial^B \operatorname{Tr}_s[\exp(-A_u^2 + buN_u)] \\ & \quad + b\left(\theta_u + \operatorname{Tr}_s\left[\left(N_H + \frac{\ell}{2}\right)\exp(-A_u^2 + buN_u)\right]\right). \end{aligned} \tag{2.51}$$

Proof. Using the commutation relations of Theorem 2.6, the proof of (2.48) is formally identical to the proof of [BGS 1, Theorem 1.10]. Also the proof of (2.50) is formally identical to the proof of [BGS 1, Theorem 1.12]. Note that

$$\left(\frac{\partial}{\partial u}\right)[uN_u] = -i\omega^{Z,c} + N_H + \frac{\ell}{2},$$

and this explains why ω^H does not appear in the final term in the right-hand side of (2.50).

Equation (2.51) is a consequence of (2.48) and (2.50). □

As in [BGS 1, Theorem 1.12], we now differentiate (2.48), (2.50), and (2.51) at $b=0$.

If $\eta' \in A(T_C^* B) \hat{\otimes} C(du)$, η' can be expanded as

$$\eta' = \eta'_0 + du\,\eta'_1, \qquad \eta'_0, \eta'_1 \in A(T_C^* B).$$

Set:

$$[\eta']^{du} = \eta'_1.$$

Theorem 2.14. *Let σ_u, $\sigma'_u \in A(T_C^* B)$ be given by*

$$\sigma_u = \left\{ \mathrm{Tr}_s \left[\exp\left(-A_u^2 - da\left(\sqrt{u}(\bar{\partial} + v) + \frac{c(T^{(1,0)})}{4\sqrt{u}} \right) \right.\right.\right.$$
$$\left.\left.\left. - d\bar{a}\left(\sqrt{u}(\bar{\partial}^* + v^*) + \frac{c(T^{(0,1)})}{4\sqrt{u}} \right) - da\, d\bar{a}\, i\omega^{Z,c} \right)\right]\right\}^{da d\bar{a}}, \tag{2.52}$$

$$\sigma'_u = \mathrm{Tr}_s \left[N_u \exp\left(-A_u^2 + du\left(\sqrt{u}(D+V) + \frac{c(T)}{4\sqrt{u}} \right) \right) \right]^{du}.$$

Then

$$\sigma'_u = (\partial^B - \bar{\partial}^B) \tfrac{1}{2} \left(\frac{\partial^2}{\partial b^2} \right) \mathrm{Tr}_s[\exp(-A_u^2 + bN_u)]_{b=0}. \tag{2.53}$$

Moreover

$$\left(\frac{\partial}{\partial u} \right) (u\, \mathrm{Tr}_s[N_u \exp(-A_u^2)]) = \sigma_u + \mathrm{Tr}_s \left[\left(N_H + \frac{\ell}{2} \right) \exp(-A_u^2) \right] - d^B \left(\frac{\sigma'_u}{2} \right); \tag{2.54}$$

or equivalently

$$\left(\frac{\partial}{\partial u} \right) (u\, \mathrm{Tr}_s[N_u \exp(-A_u^2)])$$
$$= \sigma_u + \mathrm{Tr}_s \left[\left(N_H + \frac{\ell}{2} \right) \exp(-A_u^2) \right] - \bar{\partial}^B \partial^B \tfrac{1}{2} \left(\frac{\partial^2}{\partial b^2} \right) \mathrm{Tr}_s[\exp(-A_u^2 + bN_u)]_{b=0}. \tag{2.55}$$

Proof. One immediately verifies that

$$\sigma'_u = \left(\frac{\partial}{\partial b} \right) \mathrm{Tr}_s \left[\left(\sqrt{u}(D+V) + \frac{c(T)}{4\sqrt{u}} \right) \exp(-A_u^2 + bN_u) \right]_{b=0}. \tag{2.56}$$

Dividing both sides of (2.48) by b and replacing in the right-hand side of (2.48) $\frac{\partial}{\partial b}$ by $\frac{1}{2}\left(\frac{\partial^2}{\partial b^2} \right) b$, we obtain (2.53).

Differentiating both sides of (2.50) at $b=0$, we obtain (2.54). (2.55) follows from (2.53) and (2.54). □

The second key step in establishing the asymptotics of $\mathrm{Tr}_s[N_u \exp(-A_u^2)]$ is the following remarkable formula:

Theorem 2.15. *For $u>0$, $1 \leqq i \leqq \ell$, let Q^i_u, Q'^i_u be the differential operators*

$$\begin{aligned} Q^i_u &= \nabla_{w_i} + \sum_{j,\alpha} \left(\frac{1}{2u}\right) \left\langle S(w_i)w_j, \frac{\partial}{\partial \bar{y}^\alpha}\right\rangle \sqrt{u}c(\bar{w}_j)d\bar{y}^\alpha \\ &\quad + \sum_{\alpha,\beta} \left(\frac{1}{2u}\right) \left\langle S(w_i)\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial \bar{y}^\beta}\right\rangle dy^\alpha d\bar{y}^\beta + \left(\frac{1}{2u}\right)\sqrt{u}c(w_i)da, \\ Q'^i &= \nabla_{\bar{w}_i} + \sum_{j,\alpha} \left(\frac{1}{2u}\right) \left\langle S(\bar{w}_i)\bar{w}_j, \frac{\partial}{\partial y^\alpha}\right\rangle \sqrt{u}c(w_j)dy^\alpha \\ &\quad + \sum_{\alpha,\beta} \left(\frac{1}{2u}\right) \left\langle S(\bar{w}_i)\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial \bar{y}^\beta}\right\rangle dy^\alpha d\bar{y}^\beta + \left(\frac{1}{2u}\right)\sqrt{u}c(\bar{w}_i)d\bar{a}. \end{aligned} \tag{2.57}$$

Then

$$\begin{aligned} &A_u^2 + da\left(\sqrt{u}(\bar{\partial}+v) + \frac{c(T^{(1,0)})}{4\sqrt{u}}\right) + d\bar{a}\left(\sqrt{u}(\bar{\partial}^* + v^*) + \frac{c(T^{(0,1)})}{4\sqrt{u}}\right) + da d\bar{a} i\omega^{Z,c} \\ &= -u\sum_{i=1}^{\ell} (Q^i_u Q'^i_u + Q'^i_u Q^i_u) + \left(\frac{uK}{4}\right) + u([\bar{\partial}^*, v] + [\bar{\partial}, v^*] + V^2) \\ &\quad + \left(\frac{u}{2}\right)[c(\bar{w}_i)c(w_j) \otimes L'^\xi(w_i, \bar{w}_j) + c(w_i)c(\bar{w}_j) \otimes L'^\xi(\bar{w}_i, w_j)] \\ &\quad + da\sqrt{u}v + d\bar{a}\sqrt{u}v^* + \sqrt{u}\tilde{\nabla} V + \sqrt{u}\left[c(w_i)dy^\alpha \otimes L'^\xi\left(\bar{w}_i, \frac{\partial}{\partial y^\alpha}\right)\right. \\ &\quad \left. + c(\bar{w}_i)d\bar{y}^\alpha \otimes L'^\xi\left(w_i, \frac{\partial}{\partial \bar{y}^\alpha}\right)\right] + dy^\alpha d\bar{y}^\beta \otimes L'^\xi\left(\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial \bar{y}^\beta}\right). \end{aligned} \tag{2.58}$$

Proof. We will prove (2.58) when $v=0$, the extension to the general case being trivial.

If $da = d\bar{a} = 0$, (2.58) is exactly formula (2.13), where we have used the base $(w_i, \bar{w}_i)$ instead of (e_i). Note that here we also use the fact that by Theorem 1.7, $S(w_i)\bar{w}_j = S(\bar{w}_i)w_j = 0$.

In general, the extra contributions of da, $d\bar{a}$ to $-u\sum_{i=1} (Q^i_u Q'^i_u + Q'^i_u Q^i_u)$ is given by

$$\begin{aligned} &da\sqrt{u}\partial + d\bar{a}\sqrt{u}\partial^* + \frac{da}{2\sqrt{u}} \left\langle S(\bar{w}_i)\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial \bar{y}^\beta}\right\rangle dy^\alpha d\bar{y}^\beta \\ &+ \frac{d\bar{a}}{2\sqrt{u}} \left\langle S(w_i)\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial \bar{y}^\beta}\right\rangle dy^\alpha d\bar{y}^\beta \\ &- \tfrac{1}{4}\left\langle S(w_i)w_j, \frac{\partial}{\partial \bar{y}^\alpha}\right\rangle (c(\bar{w}_i)d\bar{a}c(\bar{w}_j)d\bar{y}^\alpha + c(\bar{w}_j)d\bar{y}^\alpha c(\bar{w}_i)d\bar{a}) \\ &- \tfrac{1}{4}\left\langle S(\bar{w}_i)\bar{w}_j, \frac{\partial}{\partial y^\alpha}\right\rangle (c(w_i)dac(w_j)dy^\alpha + c(w_j)dy^\alpha c(w_i)da) \\ &- \tfrac{1}{4}(c(w_i)dac(\bar{w}_i)d\bar{a} + c(\bar{w}_i)d\bar{a}c(w_i)da). \end{aligned} \tag{2.59}$$

By (1.4), we know that

$$\left\langle S(\bar{w}_i)\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial \bar{y}^\beta}\right\rangle = \frac{\left\langle T^{(1,0)}\left(\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial \bar{y}^\beta}\right), \bar{w}_i\right\rangle}{2},$$
$$\left\langle S(w_i)\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial \bar{y}^\beta}\right\rangle = \frac{\left\langle T^{(0,1)}\left(\frac{\partial}{\partial y^\alpha}, \frac{\partial}{\partial \bar{y}^\beta}\right), w_i\right\rangle}{2}. \tag{2.60}$$

Also

$$c(\bar{w}_i)d\bar{a}c(\bar{w}_j)d\bar{y}^\alpha + c(\bar{w}_j)d\bar{y}^\alpha c(\bar{w}_i)d\bar{a} = 0, \tag{2.61}$$

and a similar relation holds for the conjugate quantities.

Finally, using (2.14), we find that

$$-\tfrac{1}{4}(c(w_i)dac(\bar{w}_i)d\bar{a} + c(\bar{w}_i)d\bar{a}c(w_i)da)$$
$$= \tfrac{1}{4}dad\bar{a}(c(w_i)c(\bar{w}_i) - c(\bar{w}_i)c(w_i)) = dad\bar{a}i\omega^{Z,c}. \tag{2.62}$$

Equation (2.58) follows from (2.59)–(2.62). □

g) The Asymptotics of $\mathrm{Tr}_s[N_u \exp(-A_u^2)]$

We now establish several explicit results concerning the asymptotics of $\mathrm{Tr}_s[N_u \exp(-A_u^2)]$ as $u \downarrow\downarrow 0$.

I denote the identity map on $T^{(1,0)}Z$.

Theorem 2.16. *There exist smooth differential forms* $C_{-1}, C_0, \ldots, D_{-2}, D_{-1}, \ldots$ *in* P *and smooth differential forms on* B $E_0, E_1, \ldots$ *such that as* $u \downarrow\downarrow 0$, *for any* $k \in N$,

$$\mathrm{Tr}_s[N_u \exp(-A_u^2)] = \sum_{j=-1}^{k} C_j u^j + o(u^k),$$
$$\sigma'_u = \sum_{j=0}^{k} E_j u^j + o(u^k), \tag{2.63}$$
$$\tfrac{1}{2}\left(\frac{\partial^2}{\partial b^2}\right)\mathrm{Tr}_s[\exp(-A_u^2 + bN_u)]_{b=0} = \sum_{j=-2}^{k} D_j u^j + o(u^k),$$

and the various $o(u^k)$ *are uniform on compact sets on* B.

D_{-2}, D_{-1} *are closed, and for* $j \geqq 0$, $(\partial^B - \bar{\partial}^B)D_j = E_j$. C_{-1}, C_0 *are closed differential forms given by*

$$C_{-1} = \left(\frac{1}{2\pi i}\right)^{\ell} \int_Z \frac{i\omega}{2} Td(-R^Z)\mathrm{Tr}_s[\exp - L^\xi],$$
$$C_0 = \left(\frac{1}{2\pi i}\right)^{\ell} \int_Z \frac{\partial}{\partial b}[Td(-R^Z - bI)]_{b=0}\mathrm{Tr}_s[\exp - L^\xi]$$
$$+ \ell\left(\frac{1}{2\pi i}\right)^{\ell} \int_Z Td(-R^Z)\mathrm{Tr}_s[\exp - L^\xi]$$
$$+ \left(\frac{1}{2\pi i}\right)^{\ell} \int_Z Td(-R^Z)\mathrm{Tr}_s[N_H \exp - L^\xi] - \frac{d^B E_0}{2}. \tag{2.64}$$

If $E_0 = E_0(v)$, *then*

$$E_0(v) = E_0(0) + \left(\frac{1}{2\pi i}\right)^{\ell} \int_Z \frac{i\omega}{2} Td(-R^Z)\left[\left(\frac{1}{\sqrt{u}}\right)\mathrm{Tr}_s(V\exp-(\nabla+\sqrt{u}V)^2)\right]_{u=0}, \tag{2.65}$$

and so

$$d^B E_0(v) = d^B E_0(0) - \left(\frac{1}{2\pi i}\right)^{\ell} \int_Z i\omega\, Td(-R^Z)\left[\left(\frac{\partial}{\partial u}\right)\mathrm{Tr}_s(\exp-(\nabla+\sqrt{u}V)^2)\right]_{u=0}. \tag{2.66}$$

Finally

$$(\partial^B - \bar{\partial}^B)C_j = B_j, \quad (\bar{\partial}^B\partial^B)C_j = -jA_j \quad (j>0). \tag{2.67}$$

Proof. The existence of the expansions (2.63), and the corresponding cancellations can be proved easily by the methods of the proof of Theorem 2.11.

Using (2.53), we find that D_{-2}, D_{-1} are closed and that for $j \geq 0$, $(\partial^B - \bar{\partial}^B)D_j = E_j$. From (2.37) and (2.41), we also find that C_{-1}, C_0 are closed and that for $j>0$, $(\partial^B - \bar{\partial}^B)C_j = B_j$. Using (2.42), we find that for $j>0$, $(\bar{\partial}^B\partial^B)C_j = -jA_j$.

We now calculate C_{-1} and C_0.

Clearly

$$C_{-1} = \lim_{u \downarrow\downarrow 0} \mathrm{Tr}_s[uN_u \exp(-A_u^2)]. \tag{2.68}$$

Also

$$uN_u = -iu\omega^{Z,c} + \frac{i\omega^H}{2} + u\left(N_H + \frac{\ell}{2}\right).$$

Note that $\omega^{Z,c}$ has length 2 in $c(TZ)$. In uN_u, each vertical Clifford variable has the weight $\sqrt{u}$. It is then easy to adapt the proof of [B 1, Theorems 4.12 and 4.16] and obtain the first line of (2.64).

We now calculate C_0. Clearly

$$\mathrm{Tr}_s[uN_u \exp(-A_u^2)] = C_{-1} + C_0 u + \ldots + C_k u^{k+1} + o(u^{k+1}). \tag{2.69}$$

One verifies easily that we can differentiate (2.69) so that as $u \downarrow\downarrow 0$,

$$\left(\frac{\partial}{\partial u}\right)\mathrm{Tr}_s[uN_u \exp(-A_u^2)] = C_0 + \sum_1^k (j+1)C_j u^j + o(u^k). \tag{2.70}$$

We now will use (2.54). By the methods of [B 1, Sect. 4], one finds that

$$\lim_{u \downarrow\downarrow 0} \mathrm{Tr}_s\left[\left(N_H + \frac{\ell}{2}\right)\exp(-A_u^2)\right] = \left(\frac{1}{2\pi i}\right)^{\ell} \int_Z Td(-R^Z)\,\mathrm{Tr}_s\left[\left(N_H + \frac{\ell}{2}\right)\exp(-L^{\xi})\right]. \tag{2.71}$$

We now will prove that $\lim_{u\downarrow\downarrow 0} \sigma_u$ exists and we calculate this limit. Let O_u be the operator appearing in (2.58). We have

$$\sigma_u = \mathrm{Tr}_s[\exp - O_u]^{da d\bar{a}}. \tag{2.72}$$

To go back to the formulas in [B 1, Sect. 4], we scale the Grassmann variables dy^α, $d\bar{y}^\alpha$, da, $d\bar{a}$ by the factor $\frac{1}{\sqrt{2}}$, and replace u by $\frac{u}{\sqrt{2}}$. O_u is changed into $\frac{O_u}{2}$. We now study the limit as $u\downarrow\downarrow 0$ of $\mathrm{Tr}_s\left[\exp - \frac{O_u}{2}\right]$.

Formula (2.58) shows that we can use the methods of [B 1, Theorem 4.12] since it has the same structure as the formula of [B 1, Theorem 3.6], given in (2.13). So we already know that $\mathrm{Tr}_s\left[\exp - \frac{O_u}{2}\right]$ has a limit as $u\downarrow\downarrow 0$.

For the same reason as before, v does not contribute to the limit. Also L^{ξ} contributes to the limit in a trivial way. So in order to simplify the argument, we will temporarily assume that $v=0$, $\xi=\mu^{-1}$ and so $L^{\xi}=0$.

We now adopt without further reference the notation of [B 1, Sect. 4] to which the reader is referred, except that t in [B 1] is now u.

Take $x_0 \in Z_{y_0}$. Let $w_h'^1 (0 \leqq h \leqq 1)$ be a Brownian bridge in $T_{x_0}Z_{y_0}$ with $w_0'^1 = w_1'^1 = 0$. Let P_1 be the law of w'^1 on $\mathscr{C}([0,1]; T_{x_0}Z_{y_0})$. We can split $w_h'^1$ as a sum

$$w_h'^1 = \eta_h^1 + \bar{\eta}_h^1; \qquad \eta_h^1 \in T_{x_0}^{(1,0)} Z_{y_0}, \qquad \bar{\eta}_h^1 \in T_{x_0}^{(0,1)} Z_{y_0}.$$

Let $x_h^u (0 \leqq h \leqq 1)$ be the Riemannian Brownian bridge in Z_{y_0} with $x_0^u = x_1^u = x_0$ associated with $\sqrt{u}w'^1$ as in [B 1, Sect. 4]. $\tau_0^{h,u}$ denotes the parallel transport operator from fibers over x_h^u into fibers at x_0 along x^u. Set

$$\gamma_h^u = \int_0^h \tau_0^{h',u} dx_{h'}^u, \tag{2.73}$$

and decompose γ_h^u in the form

$$\gamma_h^u = \varepsilon_h^u + \bar{\varepsilon}_h^u; \qquad \varepsilon_h^u \in T_{x_0}^{(1,0)} Z_{y_0}, \qquad \bar{\varepsilon}_h^u \in T_{x_0}^{(0,1)} Z_{y_0}. \tag{2.74}$$

Let $e_1, \ldots, e_n$ be an orthonormal real base of $T_{x_0}Z_{y_0}$, (f_α) a real base of $T_{y_0}B$, dy'^α the corresponding dual base in $T_{y_0}^*B$.

Equation (2.58) shows that instead of the equation in [B 1, Definition 4.1], we now consider the solution U_h^u of the equation

$$\begin{aligned} dU_h^u = U_h^u \Bigg[& \left(\frac{1}{2u}\right) \langle \tau_0^{h,u} S(dx_h^u) e_i, f_\alpha \rangle \sqrt{u}\, c(e_i) dy'^\alpha \\ & + \left(\frac{1}{4u}\right) \langle S(dx_h^u) f_\alpha, f_\beta \rangle dy'^\alpha dy'^\beta + \left(\frac{\sqrt{u}}{2u}\right) (c(d\varepsilon_h^u) da + c(d\bar{\varepsilon}_h^u) d\bar{a}) \Bigg], \\ & U_0 = I_{\Lambda_{x_0}(T_C^* Z) \otimes \mu^{-1}}. \end{aligned} \tag{2.75}$$

By proceeding as in [B 1, Theorem 4.1], we can integrate Eq. (2.75) explicitly. We obtain

$$\begin{aligned}U_1=\exp\Big\{&\left(\frac{1}{2u}\right)\int_0^1 \langle \tau_0^{h,u} S(dx_h^u)e_i, f_\alpha\rangle \sqrt{u}\,c(e_i)dy'^\alpha \\ &+\left(\frac{1}{4u}\right)\int_0^1 \langle S(dx_h^u)f_\alpha, f_\beta\rangle dy'^\alpha dy'^\beta+\left(\frac{1}{2u}\right)\int_0^1 \sqrt{u}(c(d\varepsilon_h^u)da+c(d\bar\varepsilon_h^u)d\bar a) \\ &+(\tfrac{1}{4})\int_{0\leqq h\leqq h'\leqq 1}\left\langle \frac{P_Z(S(dx_h^u)f_\alpha)}{\sqrt{u}}, \frac{P_Z(S(dx_{h'}^u)f_\beta)}{\sqrt{u}}\right\rangle dy'^\alpha dy'^\beta \\ &+\left(\frac{1}{4u}\right)\int_{0\leqq h\leqq h'\leqq 1}(\langle \tau_0^{h,u} S(dx_h^u)d\varepsilon_{h'}^u, f_\alpha\rangle-\langle \tau_0^{h',u} S(dx_{h'}^u)d\varepsilon_h^u, f_\alpha\rangle)dy'^\alpha da \\ &+\left(\frac{1}{4u}\right)\int_{0\leqq h\leqq h'\leqq 1}(\langle \tau_0^{h} S(dx_h^u)d\bar\varepsilon_{h'}^u, f_\alpha\rangle-\langle \tau_0^{h',u} S(dx_{h'}^u)d\bar\varepsilon_h^u, f_\alpha\rangle)dy'^\alpha d\bar a \\ &+\left(\frac{1}{4u}\right)\int_0^1(\langle \varepsilon_h^u, d\bar\varepsilon_h^u\rangle-\langle \bar\varepsilon_h^u, d\varepsilon_h^u\rangle)da\,d\bar a\Big\}. \end{aligned} \tag{2.76}$$

[B 1, Theorem 4.2] remains formally verified when replacing $\exp\left(-\frac{I^{L,t}}{2}\right)$ by $\exp\left(-\frac{O_u}{2}\right)$. If $P_1^u(x_0,x_0')$ is the kernel of $\exp\left(-\frac{O_u}{2}\right)$, the asymptotic evaluation of $\mathrm{Tr}_s[P_1^u(x_0,x_0)]$ can be done as in [B 1, Theorem 4.12], to which we refer for the main arguments.

With the notations of [B 1], we know that

$$\gamma_h^u=\sqrt{u}w_h'^1+hv^2(\sqrt{u}w'^1). \tag{2.77}$$

Since $w_1'^1=0$, we have $\gamma_1^u=v^2(\sqrt{u}w'^1)$. By [B 3, Eq. (4.178)] (with $b'=0$), we find that as $u\downarrow\downarrow 0$, $\frac{v^2(\sqrt{u}w'^1)}{u}\to 0$, or equivalently

$$\frac{\varepsilon_1^u}{u}\to 0, \quad \frac{\bar\varepsilon_1^u}{u}\to 0. \tag{2.78}$$

The same argument as in [B 1, Eq. (4.43)–(4.45)] shows that

$$\begin{aligned}&\left(\frac{1}{4u}\right)\int_{0\leqq h\leqq h'\leqq 1}(\langle \tau_0^{h,u} S(dx_h^u)d\varepsilon_{h'}^u, f_\alpha\rangle-\langle \tau_0^{h',u} S(dx_{h'}^u)d\varepsilon_h^u, f_\alpha\rangle) \\ &\to\tfrac{1}{4}\int_0^1 \langle S_{x_0}(w'^1)d\eta-S(dw'^1)\eta, f_\alpha\rangle. \end{aligned} \tag{2.79}$$

By Theorem 1.14, $S(w'^1)d\eta=S(\eta)d\eta$, $S(dw'^1)\eta=S(d\eta)\eta$, and so the right-hand side of (2.79) is exactly

$$-\tfrac{1}{4}\int_0^1 \langle T(\eta,d\eta), f_\alpha\rangle=0. \tag{2.80}$$

A similar analysis can be done on the conjugate term. Finally, using (2.77), we find that

$$\left(\frac{1}{4u}\right)\int_0^1 (\langle \varepsilon_h^u, d\bar{\varepsilon}_h^u\rangle - \langle \bar{\varepsilon}_h^u, d\varepsilon_h^u\rangle) \to \tfrac{1}{4}\int_0^1 (\langle \eta_h, d\bar{\eta}_h\rangle - \langle \bar{\eta}_h, d\eta_h\rangle)$$
$$= \left(-\frac{i}{4}\right)\int_0^1 \langle J^Z w'^1, dw'^1\rangle. \tag{2.81}$$

Using [B 1, Theorems 4.12 and 4.14] and returning to the initial scaling, we find that

$$\lim_{u\downarrow\downarrow 0} \mathrm{Tr}_s[\exp(-O_u)] = \left(\frac{1}{2\pi i}\right)^{\ell} \int_Z\int \exp^{\wedge}\left\{\tfrac{1}{2}\int_0^1 \langle (R^Z - iJ^Z da d\bar{a})w'^1, dw'^1\rangle\right\} dP_1(w'^1). \tag{2.82}$$

In (2.82), $\exp^{\wedge}$ indicates the exponential in the algebra $A(T_C^*M)\hat{\otimes} C(da, d\bar{a})$.

Let A' be the complex Hirzebruch polynomial. If B is a (ℓ, ℓ) complex matrix with diagonal entries $y_1, \ldots, y_\ell$, we have

$$A'(B) = \prod \left[\frac{\left(\frac{y_j}{2}\right)}{\sinh\left(\frac{y_j}{2}\right)}\right].$$

Using a formula of P. Lévy as in [B 1, Theorem 4.16], we find that

$$\lim_{u\downarrow\downarrow 0} \mathrm{Tr}_s[\exp(-O_u)] = \left(\frac{1}{2\pi i}\right)^{\ell} \int_Z A'(R^Z - iJ^Z da d\bar{a}). \tag{2.83}$$

With a general ξ, we obtain

$$\lim_{u\downarrow\downarrow 0} \mathrm{Tr}_s[\exp(-O_u)] = \left(\frac{1}{2\pi i}\right)^{\ell} \int_Z A'(R^Z - iJ^Z da d\bar{a})$$
$$\times \exp(-\tfrac{1}{2}\mathrm{Tr}\, R^Z)\, \mathrm{Tr}_s[\exp(-L^\xi)]. \tag{2.84}$$

Using (2.84), we find that

$$\lim_{u\downarrow\downarrow 0} \sigma_u = \left(\frac{1}{2\pi i}\right)^{\ell} \int \left(\frac{\partial}{\partial b}\right) [A'(R^Z - ibJ^Z)]_{b=0}$$
$$\times \exp(-\tfrac{1}{2}\mathrm{Tr}\, R^Z)\, \mathrm{Tr}_s[\exp(-L^\xi)]. \tag{2.85}$$

Using (2.54), (2.70), (2.71), (2.85), and the fact that on $T^{(1,0)}Z$, $J^Z = iI$, we get

$$C_0 = \left(\frac{1}{2\pi i}\right)^{\ell} \int_Z \left(\frac{\partial}{\partial b}\right) [A'(R^Z + bI)]_{b=0} \exp(-\tfrac{1}{2}\mathrm{Tr}\, R^Z)\, \mathrm{Tr}_s[\exp(-L^\xi)]$$
$$+ \left(\frac{1}{2\pi i}\right)^{\ell} \int_Z Td(-R^Z)\, \mathrm{Tr}_s\left[\left(N_H + \frac{\ell}{2}\right)\exp(-L^\xi)\right] - \frac{d_B E_0}{2}. \tag{2.86}$$

If B is a (ℓ, ℓ) matrix, we have

$$Td(-B) = A'(B)\exp(-\tfrac{1}{2}\mathrm{Tr}\, B);$$

and so

$$Td(-B-bI)=A'(B+bI)\exp(-\tfrac{1}{2}\operatorname{Tr}B)\exp\left(-\frac{b\ell}{2}\right),$$

which implies that

$$\left[\left(\frac{\partial}{\partial b}\right)Td(-B-bI)\right]=\left[\frac{\partial}{\partial b}A'(B+bI)\right]_{b=0}\exp(-\tfrac{1}{2}\operatorname{Tr}B)-\left(\frac{\ell}{2}\right)Td(-B). \tag{2.87}$$

From (2.86), (2.87), we obtain (2.64).

We now study the dependence of E_0 on ξ, V^ξ and v.

When scaling the Grassmann variables $dy^\alpha, d\bar{y}^\alpha, du$ into $\frac{dy^\alpha}{\sqrt{2}}, \frac{d\bar{y}^\alpha}{\sqrt{2}}, \frac{du}{\sqrt{2}}$, we must study the constant term in the expansion as $u\downarrow\downarrow 0$ of

$$\operatorname{Tr}_s\left[\left(-i\omega^{Z,c}+\frac{i\omega^H}{2u}+N_H+\frac{\ell}{2}\right)\exp\left\{\tfrac{1}{2}\left(-A_u^2+du\left(\sqrt{u}(D+V)+\frac{c(T)}{4\sqrt{u}}\right)\right)\right\}\right]^{du}. \tag{2.88}$$

By proceeding as in Theorem 2.11, we see easily that $N_H+\frac{\ell}{2}$ does not contribute to this term.

We now use Proposition 2.10 to obtain a probabilistic representation of the kernel of

$$\left(-i\omega^{Z,c}+\frac{i\omega^H}{2u}\right)\exp\left\{\tfrac{1}{2}\left(-A_u^2+du\left(\sqrt{u}(D+V)+\frac{c(T)}{4\sqrt{u}}\right)\right)\right\}.$$

As $u\downarrow\downarrow 0$, in (2.88), a first sort of term will come from the factor $\frac{c(e_i)du}{2\sqrt{u}}$. Since v contributes by terms which are factor of u, the same argument as in (2.45)–(2.46) shows that v does not appear in this part of the constant term in the expansion of (2.88). After rescaling, we obtain $E_0(0)$.

A second sort of term in the expansion of (2.88) comes from $du\sqrt{u}V$. Then $\sqrt{u}Vv$ and $\sqrt{u}[D, V]$ necessarily contribute to the constant term, while uV^2 does not ultimately appear. We obtain after rescaling

$$\left(\frac{1}{2\pi i}\right)^\ell \int_Z \left(\frac{i\omega}{2}\right)Td(-R^Z)\left[\left(\frac{1}{\sqrt{u}}\right)\operatorname{Tr}_s(V\exp(-V+\sqrt{u}V)^2)\right]_{u=0}. \tag{2.89}$$

We thus obtain (2.65).

By [BGS 1, Theorem 1.15], we know that

$$d^B\left[\left(\frac{1}{\sqrt{u}}\right)\operatorname{Tr}_s(V\exp(-(V+\sqrt{u}V)^2))\right]=\left(-2\frac{\partial}{\partial u}\right)\operatorname{Tr}_s\exp(-V+\sqrt{u}V)^2. \tag{2.90}$$

Equation (2.66) follows from (2.65) and (2.90). The theorem is proved. □

Remark 2.17. It is elementary to verify directly that C_{-1} and C_0 are closed.

In the case of finite dimensional complexes, we saw in [BGS 1, Sect. 1 c)] that the analogue of C_0 is the derived Euler characteristic, which is naturally a closed differential form. In particular, $C_0^{(0)}$ is an integer.

In our infinite dimensional context, the closed form C_0 plays formally the role of a derived Euler characteristic, but in general $C_0^{(0)}$ is no longer an integer. It is remarkable that in cohomology C_0 is given by characteristic classes, i.e. C_0 has a topological interpretation.

Finally observe that $\int_Z \left(\frac{\partial}{\partial b}\right)[Td(-R^Z - bI)]_{b=0} \operatorname{Tr}_s[\exp(-L^\xi)]$ will be interpreted as a secondary characteristic class in [BGS 3], when we study the variation of the analytic torsion with respect to the metric.

We finally state a consequence of Theorem 2.16.

Theorem 2.18. *For $u>0$, let σ''_u be the differential form in P,*

$$\sigma''_u = \operatorname{Tr}_s\left[\exp\left(-A_u^2 - da\left(\sqrt{u}(\bar{\partial}^* + v) + \frac{c(T^{(1,0)})}{4\sqrt{u}}\right)\right.\right.$$
$$\left.\left.- d\bar{a}\left(\sqrt{u}(\bar{\partial}^* + v^*) + \frac{c(T^{(0,1)})}{4\sqrt{u}}\right)\right)\right]^{da d\bar{a}}. \quad (2.91)$$

There exist smooth differential forms in $P, F_{-1}, F_0, \ldots$, such that for any $k \in N$, as $u \downarrow\downarrow 0$,

$$\sigma''_u = \frac{F_{-1}}{u} + F_0 + F_1 u + \ldots + o(u^k). \quad (2.92)$$

Also

$$F_{-1} = \left(\frac{1}{2\pi i}\right)^{\ell} \int_Z \left(-\frac{i\omega^Z}{2}\right) Td(-R^Z) \operatorname{Tr}_s[\exp(-L^\xi)], \qquad F_0^{(0)} = 0. \quad (2.93)$$

If $\omega^H = 0$, $F_0 = \dfrac{d^B E_0}{2}$.

Proof. Using formula (2.54), we know that

$$\sigma''_u = \left(\frac{\partial}{\partial u}\right) u \operatorname{Tr}_s[N_u \exp(-A_u^2)]$$
$$- \operatorname{Tr}_s\left[\left(-i\omega^{Z,c} + N_H + \frac{\ell}{2}\right)\exp(-A_u^2)\right] + \frac{d_B \sigma'_u}{2}. \quad (2.94)$$

It is now easy to prove that σ''_u has the expansion (2.92). By using the methods of [B 1, Sect. 4], we find that

$$\lim_{u \downarrow\downarrow 0} \operatorname{Tr}_s[iu\omega^{Z,c} \exp(-A_u^2)]$$
$$= \left(\frac{1}{2\pi i}\right)^{\ell} \int_Z \left(-\frac{i\omega^Z}{2}\right) Td(-R^Z) \operatorname{Tr}_s[\exp(-L^\xi)]. \quad (2.95)$$

Using (2.63), (2.94) and (2.95), we obtain the first line of (2.93). If $\omega^H=0$, (2.94) is equivalent to

$$\begin{aligned}\sigma''_u &= \left(\frac{\partial}{\partial u}\right)[u\,\mathrm{Tr}_s[N_u\exp(-A_u^2)]]-\mathrm{Tr}_s[N_u\exp(-A_u^2)]+\frac{d_B\sigma'_u}{2}\\ &= u\left(\frac{\partial}{\partial u}\right)[\mathrm{Tr}_s[N_u\exp(-A_u^2)]]+\frac{d_B\sigma'_u}{2}.\end{aligned}\tag{2.96}$$

Since $u\left(\frac{\partial}{\partial u}\right)[\mathrm{Tr}_s[N_u\exp(-A_u^2)]]$ does not contain a constant term in its asymptotic expansion, we find that $F_0=\frac{d_B E_0}{2}$.

In degree 0 in $\Lambda(T_C^*B)$, we can always neglect ω^H, i.e. assume that $\omega^H=0$. The theorem is proved. □

h) Double Transgression of the Chern Character Forms

By Theorem 2.16, we know the asymptotic expansion as $u\downarrow\downarrow 0$ or $\mathrm{Tr}_s[N_u\exp(-A_u^2)]$. We are thus ready to imitate [BGS 1, Sect. 1 c)] in order to calculate the double transgression of the Chern character forms $\mathrm{Tr}_s[\exp(-A_u^2)]$.

We do the basic assumption that the double complex $(E,\bar\partial+v)$ is acyclic.

It is then not difficult to show that as $u\uparrow\uparrow+\infty$, $\mathrm{Tr}_s[\exp(-A_u^2)]$, $\mathrm{Tr}_s\left[\left(\sqrt{u}D+\frac{c(T)}{4\sqrt{u}}\right)\exp(-A_u^2)\right]$, $\mathrm{Tr}_s[N_u\exp(-A_u^2)]$ decay exponentially uniformly on compact sets in B.

In fact, one can show that A_1^2 is a small enough perturbation of D^2 and that $\mathrm{Tr}_s[\exp(-uA_1^2)]$ decays exponentially. By rescaling the Grassmann variables in T^*B, we therefore obtain the exponential decay of $\mathrm{Tr}_s[\exp(-A_u^2)]$. A similar argument also works for the other considered quantities.

Definition 2.19. For $s\in C$, $\mathrm{Re}(s)>1$, $\zeta_E(s)\in P$ is defined by the relation

$$\zeta_E(s)=\left(-\frac{1}{\Gamma(s)}\right)\int_0^{+\infty}u^{s-1}\,\mathrm{Tr}_s[N_u\exp(-A_u^2)]\,du.\tag{2.97}$$

Because of the expansion (2.63) $\zeta_E(s)$ is indeed well defined for $\mathrm{Re}(s)>1$. It extends into a meromorphic function on C with simple poles, which is holomorphic at $s=0$. In particular

$$\begin{aligned}\zeta_E(0)&=-C_0,\\ \zeta'_E(0)&=-\int_0^1\left(\mathrm{Tr}_s[N_u\exp(-A_u^2)]-\frac{C_{-1}}{u}-C_0\right)\frac{du}{u}\\ &\quad-\int_1^{+\infty}\mathrm{Tr}_s[N_u\exp(-A_u^2)]\frac{du}{u}+C_{-1}+\Gamma'(1)C_0.\end{aligned}\tag{2.98}$$

If $a\in C^*$, we can change $\sqrt{u}(\bar\partial+v)-\frac{c(T^{(1,0)})}{4\sqrt{u}}$ into $\sqrt{ua}(\bar\partial+v)-\frac{c(T^{(1,0)})}{4\sqrt{u\bar a}}$, $\sqrt{u}(\bar\partial^*+v^*)-\frac{c(T^{(0,1)})}{4\sqrt{u}}$ into $\sqrt{u\bar a}(\bar\partial^*+v^*)-\frac{c(T^{(0,1)})}{4\sqrt{ua}}$ and N_u into $N_{|a|^2u}$. A_u^2 is changed into $A^2_{|a|^2 u}$ and $\zeta_E(s)$ into $|a|^{-2s}\zeta_E(s)$. It follows in particular that $\zeta'_E(0)$ is changed into $\zeta'_E(0) - 2\,\mathrm{Log}(|a|)\zeta_E(0)$.

Theorem 2.20. *If the chain complex $(E, \bar\partial + v)$ is acyclic, then*

$$\begin{aligned} \int_0^{+\infty} \mathrm{Tr}_s\left[\left(\sqrt{u}(D+V) + \frac{c(T)}{4\sqrt{u}}\right)\exp(-A_u^2)\right]\frac{du}{u} &= -(\partial^B - \bar\partial^B)\zeta'_E(0), \\ \left(\frac{1}{2\pi i}\right)^{\ell} \int_Z Td(-R^Z)\,\mathrm{Tr}_s \exp[-L^\xi] &= -\bar\partial^B\partial^B\zeta'_E(0). \end{aligned} \tag{2.99}$$

Proof. Observe that by Theorem 2.11, the left-hand side is indeed well defined since as $u \downarrow\downarrow 0$

$$\left(\frac{1}{u}\right)\mathrm{Tr}_s\left[\left(\sqrt{u}(D+V) + \frac{c(T)}{4\sqrt{u}}\right)\exp(-A_u^2)\right] = O(1).$$

By Theorem 2.9, we find that

$$-(\partial^B - \bar\partial^B)\zeta_E(s) = \left(\frac{1}{\Gamma(s)}\right)\int_0^{+\infty} u^{s-1}\,\mathrm{Tr}_s\left[\left(\sqrt{u}(D+V) + \frac{c(T)}{4\sqrt{u}}\right)\exp(-A_u^2)\right]du.$$

Using Theorem 2.11, we immediately obtain the first line in (2.99). The second line follows from Theorems 2.2 and 2.9. □

i) The Case where (ξ, v) is Acyclic

We now do the assumption that (ξ, v) is everywhere acyclic. Hence $(E, \bar\partial + v)$ is also everywhere acyclic.

Recall that $\zeta_\xi(s)$ has been defined in [BGS 1, Definition 1.16] (where ξ was instead denoted E). In particular

$$\bar\partial^M\partial^M\zeta'_\xi(0) = -\mathrm{Tr}_s[\exp(-L^\xi)]. \tag{2.100}$$

Also, if $\alpha, \alpha' \in P$, we write $\alpha \equiv \alpha'$ if $\alpha - \alpha' \in P'$.

We now state the basic result of this section.

Theorem 2.21. *If (ξ, v) is acyclic, then*

$$\zeta'_E(0) \equiv \left(\frac{1}{2\pi i}\right)^{\ell}\int_Z Td(-R^Z)\zeta'_\xi(0). \tag{2.101}$$

Proof. We briefly explain the two main steps of the proof. The first step is to show that if for $t > 0$, $\zeta_{E,t}(s)$ is the zêta function associated with the chain complex $(E, \sqrt{t}\bar\partial + v)$, then $\zeta'_{E,t}(0)$ is constant in P/P'. The second step will consist in proving that as $t \downarrow\downarrow 0$,

$$\zeta'_{E,t}(0) + \frac{A}{t} \to \left(\frac{1}{2\pi i}\right)^{\ell}\int_Z Td(-R^Z)\zeta'_\xi(0).$$

Note that since P' is generally not closed in P, we will have to be careful in the convergence arguments. However in degree 0, (2.101) is simply an equality of numbers, and P' is irrelevant. The argument is much simpler in this case.

Only the degree 0 part of (2.101) will be used when we study determinant bundles in [BGS 3].

Step 1. For $t \geqq 0$, we scale $\bar{\partial}$, $\bar{\partial}^*$ by the factor $\sqrt{t}$. Namely, let A_u^t be the superconnection over B

$$A_u^t = \tilde{\nabla} + \sqrt{u}(\sqrt{t}D + V) - \frac{c(T)}{4\sqrt{ut}}. \tag{2.102}$$

If $A_u = A_u(v)$, we have the obvious

$$A_u^t = A_{ut}\left(\frac{v}{\sqrt{t}}\right). \tag{2.103}$$

The total number operator corresponding to A_u^t will of course be N_{ut}.
Let $\tilde{\zeta}_{E,t}(s) \in P$ be defined by

$$\tilde{\zeta}_{E,t}(s) = \left(-\frac{1}{\Gamma(s)}\right) \int_0^{+\infty} u^{s-1} \operatorname{Tr}_s[N_{ut} \exp(-A_u^t)^2]\, du. \tag{2.104}$$

By Theorem 2.20, we find easily that

$$\bar{\partial}^B \partial^B \tilde{\zeta}'_{E,t}(0) = -\left(\frac{1}{2\pi i}\right)^{\ell} \int_Z Td(-R^Z) \operatorname{Tr}_s[\exp - L^{\xi}]. \tag{2.105}$$

Definition 2.22. For $t > 0$, $s \in C$ and $\operatorname{Re}(s)$ large enough, set

$$\alpha_t(s) = \left(\frac{1}{\Gamma(s)}\right) \int_0^{+\infty} u^{s-1} \operatorname{Tr}_s\left[N_{ut} \exp\left(-(A_u^t)^2 + du\left(\sqrt{ut}D + \frac{c(T)}{4\sqrt{ut}}\right)\right)\right]^{du} du,$$

$$\begin{aligned}\beta_t(s) = \left(\frac{1}{\Gamma(s)}\right) \int_0^{+\infty} u^{s-1} \operatorname{Tr}_s\Bigg[N'_{V,ut} \exp\Bigg(&-(A_u^t)^2 + du\Bigg(-\sqrt{ut}\bar{\partial} \\ &- \frac{c(T^{(1,0)})}{4\sqrt{ut}} - \sqrt{u}v\Bigg) + du\left(\sqrt{ut}\bar{\partial}^* + \frac{c(T^{(0,1)})}{4\sqrt{ut}} + \sqrt{u}v^*\right)\Bigg)\Bigg]^{du} du.\end{aligned} \tag{2.106}$$

By proceeding as in the proof of Theorem 2.11, we find easily that for a given $t > 0$, as $u \downarrow\downarrow 0$, the expressions appearing in the integrals which define $\alpha_t(s)$ and $\beta_t(s)$ have asymptotic expansions where only integer powers of u appear. In particular $\alpha_t(s)$ and $\beta_t(s)$ are meromorphic functions of s, which extend holomorphically at $s = 0$.

Recall that $E_0(0)$ was defined in Theorem 2.16.

Theorem 2.23. *For any $t > 0$, the following identity holds:*

$$\begin{aligned}\frac{\partial}{\partial t}\tilde{\zeta}'_{E,t}(0) = {} & \frac{1}{2t}(\bar{\partial}^B + \partial^B)\alpha'_t(0) - \frac{1}{2t}(\bar{\partial}^B - \partial^B)\beta'_t(0) \\ & + \frac{1}{t}\left\{\left(\frac{1}{2\pi i}\right)^{\ell} \int_Z \frac{\partial}{\partial b}[Td(-R^Z - bI)]_{b=0} \operatorname{Tr}_s[\exp(-L^{\xi})] \right. \\ & \left. + \ell\left(\frac{1}{2\pi i}\right)^{\ell} \int_Z Td(-R^Z) \operatorname{Tr}_s[\exp(-L^{\xi})]\right\} - \frac{1}{2t}(\bar{\partial}^B + \partial^B)E_0(0) \\ & - \frac{1}{2t^2}(\bar{\partial}^B + \partial^B)\left[\left(\frac{1}{2\pi i}\right)^{\ell} \int_Z \frac{i\omega}{2} Td(-R^Z) \right. \\ & \left. \times \left\{\frac{1}{\sqrt{u}} \operatorname{Tr}_s[V \exp -(\nabla + \sqrt{u}V)^2]\right\}_{u=0}\right].\end{aligned} \tag{2.107}$$

In particular, for any $t>0$, $\frac{\partial}{\partial t}\zeta'_{E,t}(0)$ *is an element of* P'.

Proof. For Re(s) large enough, we have

$$\begin{aligned}\frac{\partial}{\partial t}\zeta_{E,t}(s)=\left(-\frac{1}{\Gamma(s)}\right)\int_0^{+\infty} u^{s-1}\Bigg\{&\mathrm{Tr}_s\left[\frac{\partial}{\partial t}N_{ut}\exp(-(A_u^t)^2)\right]\\ &+\frac{\partial}{\partial b}\left[\mathrm{Tr}_s\left[N_{ut}\exp\left(-(A_u^t)^2-b\left[A_u^t,\frac{\partial}{\partial t}A_u^t\right]\right)\right]\right]_{b=0}\Bigg\}du.\end{aligned}\tag{2.108}$$

By proceedings as in [BGS 1, Eq. (1.106)] we find that

$$\begin{aligned}&\frac{\partial}{\partial b}\left[\mathrm{Tr}_s\left[N_{ut}\exp\left(-(A_u^t)^2-b\left[A_u^t,\frac{\partial}{\partial t}A_u^t\right]\right)\right]\right]_{b=0}\\ &=-\frac{\partial}{\partial b}\left[\mathrm{Tr}_s\left[\left[A_u^t,\frac{\partial}{\partial t}A_u^t\right]\exp(-(A_u^t)^2+bN_{ut})\right]\right]_{b=0}\\ &=-d^B\frac{\partial}{\partial b}\left[\mathrm{Tr}_s\left[\left(\frac{\partial}{\partial t}A_u^t\right)\exp(-(A_u^t)^2+bN_{ut})\right]\right]_{b=0}\\ &\quad-\frac{\partial}{\partial b}\left[\mathrm{Tr}_s\left[\left(\frac{\partial}{\partial t}A_u^t\right)\exp(-(A_u^t)^2+b[A_u^t,N_{ut}])\right]\right]_{b=0}.\end{aligned}\tag{2.109}$$

As in Theorem 2.6, we split A_u^t into a holomorphic and a antiholomorphic part so that

$$A_u^t=A_u^{t\prime}+A_u^{t\prime\prime};\quad (A_u^{t\prime})^2=(A_u^{t\prime\prime})^2=0;\quad (A_u^t)^2=[A_u^{t\prime},A_u^{t\prime\prime}].\tag{2.110}$$

By formula (2.18) in Theorem 2.6, we find that

$$\begin{aligned}&[A_u^t,N_{ut}]=2u\frac{\partial}{\partial u}(-A_u^{t\prime\prime}+A_u^{t\prime}),\\ &\frac{1}{2t}[-A_u^{t\prime\prime}+A_u^{t\prime},N'_{V,ut}]=\frac{\partial}{\partial t}A_u^t.\end{aligned}\tag{2.111}$$

We thus find that

$$\begin{aligned}&\frac{\partial}{\partial b}\left\{\mathrm{Tr}_s\left[\left(\frac{\partial}{\partial t}A_u^t\right)\exp(-(A_u^t)^2+b[A_u^t,N_{ut}])\right]\right\}_{b=0}\\ &=-\frac{1}{2t}(\bar\partial^B-\partial^B)\frac{\partial}{\partial b}\{\mathrm{Tr}_s[N'_{V,ut}\exp(-(A_u^t)^2+b[A_u^t,N_{ut}])]\}_{b=0}\\ &\quad+\frac{1}{2t}\frac{\partial}{\partial b}\left\{\mathrm{Tr}_s\left[N'_{V,ut}\exp\left(-(A_u^t)^2+2bu\left[A_u^{t\prime\prime}-A_u^{t\prime},\frac{\partial}{\partial u}(-A_u^{t\prime\prime}+A_u^{t\prime})\right]\right)\right]\right\}_{b=0}.\end{aligned}\tag{2.112}$$

Using (2.110), we know that

$$\left[A_u^{t\prime\prime}-A_u^{t\prime},\frac{\partial}{\partial u}(-A_u^{t\prime\prime}+A_u^{t\prime})\right]=\frac{\partial}{\partial u}(A_u^t)^2.\tag{2.113}$$

The second term in the right-hand side of (2.112) is then given by

$$-\frac{u}{t}\frac{\partial}{\partial u}(\mathrm{Tr}_s[N'_{V,ut}\exp(-(A_u^t)^2)])+\mathrm{Tr}_s\left[\frac{u}{t}\frac{\partial}{\partial u}(N'_{V,ut})\exp(-(A_u^t)^2)\right]. \quad (2.114)$$

Observe that

$$\frac{\partial}{\partial t}N_{ut}=\frac{u}{t}\frac{\partial}{\partial u}(N'_{V,ut})=\frac{-i\omega^H}{2ut^2}. \quad (2.115)$$

From (2.108)–(2.115), we find that

$$\frac{\partial}{\partial t}\tilde{\zeta}_{E,t}(s)=\frac{1}{2t}(\bar{\partial}^B+\partial^B)\alpha_t(s)-\frac{1}{2t}(\bar{\partial}^B-\partial^B)\beta_t(s)$$
$$-\frac{1}{t\Gamma(s)}\int_0^{+\infty}u^s\frac{\partial}{\partial u}\mathrm{Tr}_s[N'_{V,ut}\exp(-(A_u^t)^2)]\,du. \quad (2.116)$$

For Re(s) large enough, we can integrate by parts in the last integral in the right-hand side of (2.116), and so we obtain

$$\frac{\partial}{\partial t}\tilde{\zeta}_{E,t}(s)=\frac{1}{2t}(\bar{\partial}^B+\partial^B)\alpha_t(s)-\frac{1}{2t}(\bar{\partial}^B-\partial^B)\beta_t(s)$$
$$+\frac{1}{t}\frac{s}{\Gamma(s)}\int_0^{+\infty}u^{s-1}\,\mathrm{Tr}_s[N'_{V,ut}\exp(-(A_u^t)^2)]\,du. \quad (2.117)$$

In Theorem 2.16, C_0 depends explicitly on v through E_0. We will now write C_0^v instead of C_0. Set

$$C_{0,V}^v=C_0^v-\left(\frac{1}{2\pi i}\right)^{\ell}\int_Z Td(-R^Z)\,\mathrm{Tr}_s[N_H\exp(-L^\xi)].$$

Using Theorem 2.16 and Eq. (2.71), we find that for $t>0$, as $u\downarrow\downarrow 0$, we have the asymptotic expansion,

$$\mathrm{Tr}_s[N'_{V,ut}\exp(-A_u^t)^2]=\frac{C_{-1}}{ut}+C_{0,V}^{v/\sqrt{t}}+O(u). \quad (2.118)$$

From (2.117)–(2.118), we find that

$$\frac{\partial}{\partial t}\tilde{\zeta}'_{E,t}(0)=\frac{1}{2t}(\bar{\partial}^B+\partial^B)\alpha'_t(0)-\frac{1}{2t}(\bar{\partial}^B-\partial^B)\beta'_t(0)+\frac{1}{t}C_{0,V}^{v/\sqrt{t}}. \quad (2.119)$$

Equation (2.107) is proved.

Using (2.100), we know that the differential form $\mathrm{Tr}_s[\exp(-L^\xi)]$ is exact. It is now clear that $\frac{\partial}{\partial t}\tilde{\zeta}'_{E,t}(0)$ is in P'. The theorem is proved. □

Step 2. We know that $E_0=E_0(v)$ is the constant term in the asymptotic expansion of σ'_u. Also we saw in the proof of Theorem 2.16 that N_H does not contribute to $E_0(v)$.

Therefore $E_0(v)$ is the constant term in the asymptotic expansion as $u\downarrow\downarrow 0$ of

$$\mathrm{Tr}_s\left[N'_{V,u}\exp\left(-A_u^2+du\left(\sqrt{u}(D+V)+\frac{c(T)}{4\sqrt{u}}\right)\right)\right]^{du}.$$

In particular $E_0(0)$ is the constant term in the asymptotic expansion as $u\downarrow\downarrow 0$ of

$$\mathrm{Tr}_s\left[N'_{V,u}\exp\left(-\left(\tilde{\nabla}+\sqrt{u}D-\frac{c(T)}{4\sqrt{u}}\right)^2+du\left(\sqrt{u}D+\frac{c(T)}{4\sqrt{u}}\right)\right)\right]^{du}.$$

We now slightly generalize the definition of $E_0(0)$.

Definition 2.24. $E_0^v(0)$ denotes the constant term in the asymptotic expansion as $u\downarrow\downarrow 0$ of

$$\mathrm{Tr}_s\left[N'_{V,u}\exp\left(-\left(\tilde{\nabla}+\sqrt{u}D+V-\frac{c(T)}{4\sqrt{u}}\right)^2+du\left(\sqrt{u}D+\frac{c(T)}{4\sqrt{u}}\right)\right)\right]^{du}. \tag{2.120}$$

Of course the existence of the asymptotic expansion of (2.120) can be proved as in Theorem 2.11. Incidentally, note that the proof of Theorem 2.11 shows that (2.120) is non-singular as $u\downarrow\downarrow 0$, so that $E_0^v(0)$ is the limit of (2.120) as $u\downarrow\downarrow 0$. Also one verifies easily that for $h\geqq 0$, $E_0^{\sqrt{h}v}(0)$ is a smooth function of h.

Definition 2.25. For $u\geqq 0$, set

$$\begin{aligned}\tilde{C}_{-1}(u)&=\left(\frac{1}{2\pi i}\right)^{\ell}\int_Z\left(\frac{i\omega}{2}\right)Td(-R^Z)\,\mathrm{Tr}_s[\exp-(\nabla+\sqrt{u}V)^2],\\ \tilde{C}_0(u)&=\left(\frac{1}{2\pi i}\right)^{\ell}\int_Z\left(\frac{\partial}{\partial b}\right)[Td(-R^Z-bI)]_{b=0}\,\mathrm{Tr}_s[\exp-(\nabla+\sqrt{u}V)^2]\\ &\quad+\ell\left(\frac{1}{2\pi i}\right)^{\ell}\int_Z Td(-R^Z)\,\mathrm{Tr}_s[\exp-(\nabla+\sqrt{u}V)^2]\\ &\quad+\left(\frac{1}{2\pi i}\right)^{\ell}\int_Z Td(-R^Z)\,\mathrm{Tr}_s[N_H\exp-(\nabla+\sqrt{u}V)^2]-\tfrac{1}{2}d^BE^{\sqrt{u}v}(0).\end{aligned}\tag{2.121}$$

For $\mathrm{Re}(s)>1$, set

$$\begin{aligned}\lambda_0(s)&=\left(-\frac{1}{\Gamma(s)}\right)\int_0^{+\infty}u^{s-1}\tilde{C}_0(u)\,du,\\ \lambda_{-1}(s)&=\left(-\frac{1}{\Gamma(s)}\right)\int_0^{+\infty}u^{s-1}\tilde{C}_{-1}(u)\frac{du}{u}.\end{aligned}\tag{2.122}$$

Note that in Theorem 2.16, we have

$$C_{-1}=\tilde{C}_{-1}(0),\qquad C_0=\tilde{C}_0(0)+\tilde{C}'_{-1}(0). \tag{2.123}$$

Also it is obvious that λ_0, λ_1 extend into meromorphic functions on C, which are holomorphic at $s=0$.

Theorem 2.26. *There is $\beta \in P$ such that as $t \downarrow\downarrow 0$,*

$$\tilde{\zeta}'_{E,t}(0) = \left(\frac{\lambda'_{-1}(0)}{t}\right) + \lambda'_0(0) + \beta t + o(t), \tag{2.124}$$

and $o(t)$ is uniform over compact sets in B.

Proof. Using (2.98) and (2.123), we have

$$\begin{aligned}\tilde{\zeta}'_{E,t}(0) = &-\int_0^1 \left(\mathrm{Tr}_s[N_{ut}\exp-(A^t_u)^2] - \frac{\tilde{C}_{-1}(0)+\tilde{C}'_{-1}(0)u}{ut} - \tilde{C}_0(0)\right)\frac{du}{u} \\ &- \int_1^{+\infty} \mathrm{Tr}_s[N_{ut}\exp-(A^t_u)^2]\frac{du}{u} + \left(\frac{\tilde{C}_{-1}(0)}{t}\right) \\ &+ \Gamma'(1)\left(\tilde{C}_0(0) + \left(\frac{\tilde{C}'_{-1}(0)}{t}\right)\right).\end{aligned} \tag{2.125}$$

- *Expansion as $t \downarrow\downarrow 0$ of* $\mathrm{Tr}_s[N_{ut}\exp-(A^t_u)^2]$.

We claim that for $u>0$, as $t \downarrow\downarrow 0$, we have the asymptotic expansion

$$\mathrm{Tr}_s[N_{ut}\exp-(A^t_u)^2] = \left(\frac{\tilde{C}_{-1}(u)}{ut}\right) + \tilde{C}_0(u) + O_u(ut), \tag{2.126}$$

and $O(ut)$ is uniform as u is bounded.

Equivalently, we must prove that as $u' \downarrow\downarrow 0$,

$$\mathrm{Tr}_s\left[N_{u'}\exp-\left(\tilde{\nabla} + \sqrt{u'}D + \sqrt{u}\,V - \left(\frac{c(T)}{4\sqrt{u'}}\right)\right)^2\right] = \left(\frac{\tilde{C}_{-1}(u)}{u'}\right) + \tilde{C}_0(u) + O_u(u').$$

One verifies easily that as $u' \downarrow\downarrow 0$,

$$\begin{aligned}&\mathrm{Tr}_s\left[N_H\exp-\left(\nabla + \sqrt{u'}D + \sqrt{u}V - \left(\frac{c(T)}{4\sqrt{u'}}\right)\right)^2\right] \\ &\to \left(\frac{1}{2\pi i}\right)^{\ell} \int_Z Td(-R^Z)\,\mathrm{Tr}_s[N_H\exp-(\nabla + \sqrt{u}V)^2].\end{aligned} \tag{2.127}$$

One is thus led to study the behavior as $u' \downarrow\downarrow 0$ of

$$\mathrm{Tr}_s\left[N'_{V,u'}\exp-\left(\nabla + \sqrt{u'}D + \sqrt{u}V - \left(\frac{c(T)}{4\sqrt{u'}}\right)\right)^2\right].$$

Observe that $[N'_{V,u'}, v] = [N_{V,u'}, v^*] = 0$. It is then not difficult to adapt the methods of [BGS 1, Theorem 1.12] and of Theorem 2.13, in order to obtain a formula for

$$\left(\frac{\partial}{\partial u'}\right) u'\,\mathrm{Tr}_s[N'_{V,u'}\exp-\left(\nabla + \sqrt{u'}D + \sqrt{u}V - \left(\frac{c(T)}{4\sqrt{u'}}\right)\right)^2]$$

in which $\sqrt{u}V$ plays only the role of a "parameter."

We find that

$$\left(\frac{\partial}{\partial u'}\right)(u' \operatorname{Tr}_s[N'_{V,u'} \exp -(A_u^{u'/u})^2])$$
$$= \operatorname{Tr}_s\left[\exp\left(-(A_u^{u'/u})^2 - da\left(\sqrt{u'}\bar{\partial} + \frac{c(T^{(1,0)})}{4\sqrt{u'}}\right)\right.\right.$$
$$\left.\left. - d\bar{a}\left(\sqrt{u'}\bar{\partial}^* + \frac{c(T^{(0,1)})}{4\sqrt{u'}}\right) - dad\bar{a}i\omega^{Z,c}\right)\right]^{dad\bar{a}} + \left(\frac{\ell}{2}\right)\operatorname{Tr}_s[\exp -(A_u^{u'/u})^2]$$
$$-\tfrac{1}{2} d^B \operatorname{Tr}_s\left[N'_{V,u'} \exp\left(-(A_u^{u'/u})^2 + du'\left(\sqrt{u'}D + \frac{c(T)}{4\sqrt{u'}}\right)\right)\right]^{du'}. \tag{2.128}$$

Using (2.128), we can proceed as in Theorem 2.16 and obtain (2.126).

Note that quite naturally, $\tilde{C}'_{-1}(0)$ does not appear in (2.126).

- *Uniform estimates as* $u \uparrow +\infty$. We claim that for any compact set K in B, there are constants $c_K > 0$, $\mu_K > 0$ such that for any $u \geqq 1$, $t > 0$, $y \in K$

$$t|\operatorname{Tr}_s[N_{ut} \exp -(A_u^t)^2]| \leqq c_K \exp(-\mu_K u). \tag{2.129}$$

First note that the factor t on the left-hand side of (2.129) kills the divergence of $\operatorname{Tr}_s[N_{ut} \exp -(A_u^t)^2]$ as $t \downarrow\downarrow 0$. A result similar to (2.129) was proved in [B 2, Theorem 1.3]. The proof of [B 2, Theorem 1.3] uses essentially the fact that V is invertible, and can be easily adapted in our situation.

Also if $O_u(ut)$ is taken as in (2.126), for $u \geqq 1$, we find that $|O_u(ut)| \leqq Cut$. Therefore, for $t > 0$,

$$\left|\int_0^1 O_u(ut) \frac{du}{u}\right| \leqq Ct, \tag{2.130}$$

and so using (2.126), we get

$$\int_0^1 \left(\operatorname{Tr}_s[N_{ut} \exp -(A_u^t)^2] - \frac{\tilde{C}_{-1}(0) + \tilde{C}'_{-1}(0)u}{ut} - \tilde{C}_0(0)\right)\frac{du}{u}$$
$$= \left(\frac{1}{t}\right)\int_0^1 \left(\frac{(\tilde{C}_{-1}(u) - \tilde{C}_{-1}(0) - \tilde{C}'_{-1}(0)u)}{u}\right)\frac{du}{u} + \int_0^1 (\tilde{C}_0(u) - \tilde{C}_0(0))\frac{du}{u} + O(t). \tag{2.131}$$

Using (2.126), (2.129) and the dominated convergence Theorem, we find that, as $t \downarrow\downarrow 0$,

$$\int_1^{+\infty} \operatorname{Tr}_s[N_{ut} \exp -(A_u^t)^2]\frac{du}{u} = \left(\frac{1}{t}\right)\left(\int_1^{+\infty}\left(\frac{\tilde{C}_{-1}(u)}{u}\right)\frac{du}{u} + o(1)\right). \tag{2.132}$$

From (2.125), (2.131) and (2.132), we get

$$\zeta'_{E,t}(0) = \left(\frac{1}{t}\right)\left[-\int_0^1\left(\frac{(\tilde{C}_{-1}(u) - \tilde{C}_{-1}(0) - \tilde{C}'_{-1}(0)u)}{u}\right)\frac{du}{u}\right.$$
$$\left. - \int_1^{+\infty}\left(\frac{\tilde{C}_{-1}(u)}{u}\right)\frac{du}{u} + \tilde{C}_{-1}(0) + \Gamma'(1)\tilde{C}'_{-1}(0)\right] + o\left(\frac{1}{t}\right), \tag{2.133}$$

or equivalently

$$\tilde{\zeta}'_{E,t}(0) = \left(\frac{\lambda'_{-1}(0)}{t}\right) + o\left(\frac{1}{t}\right). \tag{2.134}$$

Also for $t>0$, $\mathrm{Tr}_s[N_{ut}\exp-(A_u^t)^2]$ decays exponentially as $u\uparrow+\infty$ and this uniformly as t stays bounded away from 0. From (2.125) we deduce that

$$\begin{aligned}\left(\frac{\partial}{\partial t}\right)(t\tilde{\zeta}'_{E,t}(0)) = &-\int_0^1 \left(\frac{\partial}{\partial t}(t\,\mathrm{Tr}_s[N_{ut}\exp-(A_u^t)^2]) - \tilde{C}_0(0)\right)\frac{du}{u} \\ &- \int_1^{+\infty} \frac{\partial}{\partial t}(t\,\mathrm{Tr}_s[N_{ut}\exp-(A_u^t)^2]^2\frac{du}{u} + \Gamma'(1)\tilde{C}_0(0).\end{aligned} \tag{2.135}$$

Using (2.128) and proceeding as in Theorem 2.16, we find that as $t\downarrow\downarrow 0$, $\left(\frac{\partial}{\partial t}\right)(t\,\mathrm{Tr}_s[N_{ut}\exp-(A_u^t)^2])$ has an asymptotic expansion, which is given by

$$\left(\frac{\partial}{\partial t}\right)(t\,\mathrm{Tr}_s[N_{ut}\exp-(A_u^t)^2]) = \tilde{C}_0(u) + O_u(ut). \tag{2.136}$$

On the other hand, by using formula (2.129), and by proceeding as in the proof of [B 2, Theorem 3.1], we find that for any compact set K in B, there are constants $c'_K>0$ and $\mu'_K>0$ such that for any $u\geqq 1$, $t>0$, $y\in K$, then

$$\left|\left(\frac{\partial}{\partial t}\right)(t\,\mathrm{Tr}_s[N_{ut}\exp-(A_u^t)^2])\right| \leqq c'_K\exp(-\mu'_K u). \tag{2.137}$$

Using (2.135)–(2.137) and the dominated convergence theorem, we find that as $t\downarrow\downarrow 0$,

$$\left(\frac{\partial}{\partial t}\right)(t\tilde{\zeta}'_{E,t}(0)) = -\int_0^1 (\tilde{C}_0(u) - \tilde{C}_0(0))\frac{du}{u} - \int_1^{+\infty} \tilde{C}_0(u)\frac{du}{u} + \Gamma'(1)\tilde{C}_0(0) + o(1), \tag{2.138}$$

or equivalently

$$\left(\frac{\partial}{\partial t}\right)(t\tilde{\zeta}'_{E,t}(0)) = \lambda'_0(0) + o(1). \tag{2.139}$$

More generally, by calculating one term more in the asymptotic expansion of $\left(\frac{\partial}{\partial t}\right)(t\tilde{\zeta}'_{E,t}(0))$ – this is possible by the uniform bounds in [B 2, Theorem 1.3] – we find that there exists $\beta\in P$ such that

$$\left(\frac{\partial}{\partial t}\right)(t\tilde{\zeta}'_{E,t}(0)) = \lambda'_0(0) + 2\beta t + o(t). \tag{2.140}$$

Integrating (2.140) and using (2.134), we find that (2.124) holds. The theorem is proved. □

We now complete the proof of Theorem 2.21. By proceeding as in Theorem 2.11, 2.16, and 2.26, we find easily that as $t\downarrow\downarrow 0$, $\alpha'_t(0)$ and $\beta'_t(0)$ have asymptotic expansions similar to the expansion (2.124) for $\tilde{\zeta}'_{E,t}(0)$, and that moreover, the operators ∂^B and $\bar{\partial}^B$ can be applied to these expansions. Using Theorem 2.23, we find that there are differential forms η_{-2}, η_{-1} in P', and also smooth differential forms κ_t, κ'_t on B such that

$$\frac{\partial}{\partial t}\tilde{\zeta}'_{E,t}(0)=\frac{\eta_{-2}}{t^2}+\frac{\eta_{-1}}{t}+\partial^B\kappa_t+\bar{\partial}^B\kappa'_t, \tag{2.141}$$

and moreover, κ'_t and κ''_t depend continuously on t together with their derivatives, and have a limit together with their derivatives as $t \downarrow\downarrow 0$.

Integrating (2.141) and comparing with (2.124), if we identify the constant terms in the asymptotic expansion of $\tilde{\zeta}'_{E,t}(0)$, we find that

$$\lambda'_0(0)-\tilde{\zeta}'_E(0)\in P'. \tag{2.142}$$

Using [BGS 1, Theorems 1.15 and 1.17], we find easily that for any $u>0$,

$$\tilde{C}_0(u)\equiv\left(\frac{1}{2\pi i}\right)^{\ell}\int_Z Td(-R^Z)\operatorname{Tr}_s[N_H\exp-(\nabla+\sqrt{u}V)^2].$$

Proceeding as in [BGS 1, Eq. (1.72)], we get

$$\lambda'_0(0)-\left(\frac{1}{2\pi i}\right)^{\ell}\int_Z Td(-R^Z)\zeta'_\xi(0)\in P'. \tag{2.143}$$

The theorem is proved. $\square$

Remark 2.27. A by-product of the proof of Theorem 2.21 is that the logarithmic singularity which should appear when integrating the right-hand side of (2.107) vanishes identically. Also observe that when integrating the coefficient of $\frac{1}{t^2}$ in the right-hand side of (2.107), we obtain the coefficient of $\frac{1}{t}$ in the expansion of $\tilde{\zeta}'_{E,t}(0)$. The fact that this coefficient coincides with $\lambda'_{-1}(0)$ can be verified directly.

In a preliminary version of this paper, we gave a proof of Theorem 2.21 based on a slightly different principle.

Remark 2.28. In Gillet-Soulé [GS 1, 2], a group $\hat{K}_0(X)$ was introduced, whose generators are triples (E,h,η), where E is a holomorphic vector bundle on the complex manifold X, h a smooth Hermitian metric on E, and η a class in $\frac{P}{P'}$. These are submitted to the relation

$$(E,h,\eta'+\eta'')=(S,h',\eta')+(Q,h'',\eta'')+(0,0,\widetilde{\operatorname{ch}}(\mathscr{E})) \tag{2.144}$$

for every exact sequence $\mathscr{E}: 0\to S\to E\to Q\to 0$, and choice of metrics h', h, h'' on S, E, Q, and forms $\eta', \eta''\in\frac{P}{P'}$. Here $\widetilde{\operatorname{ch}}(\mathscr{E})$ is the element of $\frac{P}{P'}$ defined in [BGS 1, Eqs. (1.124)].

Let Y be a Kähler manifold and $f: X\times Y\to X$ the first projection. In [GS 1], a direct image morphism $f_!:\hat{K}_0(X\times Y)\to\hat{K}_0(X)$ was introduced using a notion of higher analytic torsion similar to $\zeta'_E(0)$. If $\xi=(\xi_j)_{0\leq j\leq m}$ is an acyclic complex of holomorphic Hermitian bundles on $X\times Y$, the following relation holds in $\hat{K}_0(X\times Y)$,

$$\sum_i (-1)^{j+1}(\xi_j,0)=(0,\widetilde{\mathrm{ch}}(\xi)). \tag{2.145}$$

Theorem 2.21 means that this relation is respected by $f_!$. The same will hold for an arbitrary smooth projective map $\pi: M\to B$.

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