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# Comparisons between Model Equations for Long Waves

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**Summary.** Considered here are model equations for weakly nonlinear and dispersive long waves, which feature general forms of dispersion and pure power nonlinearity. Two variants of such equations are introduced, one of Korteweg–de Vries type and one of regularized long-wave type. It is proven that solutions of the pure initial-value problem for these two types of model equations are the same, to within the order of accuracy attributable to either, on the long time scale during which nonlinear and dispersive effects may accumulate to make an order-one relative difference to the wave profiles.

**Key words.** long wave models, Korteweg–de Vries-type equations, regularized long-wave equations, dispersion relations

## 1. Introduction

This paper is concerned with model equations for long waves that incorporate the competing effects of nonlinearity and dispersion. The prototypical equation in view is that due to Korteweg and de Vries, but a general class of such models will be considered here. Many articles have been written about such models since the mid-1960s, some of which will be mentioned presently.

The issue that will be the focus of our efforts here concerns the comparison between the two different forms

$$\eta_t + \eta_x + \eta^p \eta_x - (M\eta)_x = 0 \tag{1.1}$$

and

$$\zeta_t + \zeta_x + \zeta^p \zeta_x + (M\zeta)_t = 0 \tag{1.2}$$

of these models. In (1.1) and (1.2), $\eta$ and $\zeta$ are functions of the two real variables $x$ and $t$, $p$ is a positive integer, and $M$ is a Fourier multiplier operator in the $x$ variable defined by

$$\widehat{Mv}(k) = m(k)\hat{v}(k) \tag{1.3}$$

for $k \in \mathbb{R}$, where a hat adorning a function of $x$ connotes that function's Fourier transform. Such equations have been derived as models for nonlinear dispersive waves in many different physical contexts, and in most situations where they arise $x$ is proportional to distance measured in the direction of wave propagation while $t$ is proportional to elapsed time. These connotations will be adopted as a descriptive device henceforth. Interest is often focused on the pure initial-value problem for (1.1) or (1.2) in which $\eta$ or $\zeta$ is specified for all real $x$ at some beginning value of $t$, say $t = 0$, and then the evolution equation is solved for $t \geq 0$ subject to the restriction that the solution respects the given initial condition. The thrust of our theory is that for suitably restricted initial conditions, the solutions $\eta$ and $\zeta$ emanating therefrom are nearly identical at least for values of $t$ in an interval $[0, T]$ where $T$ is quite large.

In the remainder of this introductory section we will first recall the original context in which this issue arose and the earlier work that spawned this study. A more detailed description of the present contribution is then provided. The section finishes with commentary on the import of our theory and a general appraisal of the relative merits of these two types of model equations.

The plan of subsequent sections is as follows. We begin in Section 2 with a brief review of the existence theory pertaining to the initial-value problems for equations (1.1) and (1.2). Section 3 is devoted to the statement and proof of our main comparison result, Theorem 3. The next section contains applications of Theorem 3 to well-known and often-used model equations. The relatively simple case where the symbol $m$ of the dispersion operator $M$ in (1.3) is homogeneous is treated first in Theorem 4 and the associated result Corollary 1. Then, as an example of the use of the main result in analyzing equations with nonhomogeneous symbols, the intermediate long-wave (ILW) equation is discussed in the statement and proof of Theorem 5. The last section provides some details pertaining to the existence theory described in Section 2.

### *Earlier Theory and Rationale*

To understand more precisely the results in view, and to grasp their importance, it is worthwhile to briefly review the theory developed earlier for the special case of the Korteweg–de Vries equation itself in Bona et al. (1983). Among the many assumptions that come to the fore in deriving models like those in (1.1) and (1.2) are that the wave motions in question have small amplitude and large wavelength. Letting $a$ and $\lambda$, respectively, denote typical, scaled, nondimensionalized values of these quantities, the assumption is that both $a$ and $\lambda^{-1}$ are small. However, in order that nonlinear and dispersive effects be balanced, these two small quantities must be related. In the case of the Korteweg–de Vries equation where $p = 1$ and $m(k) = k^2$, so that (1.1) and (1.2) take the forms

$$\eta_t + \eta_x + \eta\eta_x + \eta_{xxx} = 0 \tag{1.4}$$

and

$$\zeta_t + \zeta_x + \zeta\zeta_x - \zeta_{xxt} = 0, \tag{1.5}$$

this relationship is usually expressed by the stipulation that the Stokes number $S = a\lambda^2$ is of order one. The assumptions of small amplitude and large wavelength are embodied in requiring the initial data to possess the form

$$\eta(x,0) = \zeta(x,0) = ag(\lambda^{-1}x), \tag{1.6}$$

for $x \in \mathbb{R}$, where $g$ and $g'$ are order-one functions. Taking $S = 1$ and letting $\epsilon = a = \lambda^{-2}$, we may rescale the variables to obtain the initial-value problems

$$E_t + E_x + \epsilon E E_x + \epsilon E_{xxx} = 0, \tag{1.7}$$

and

$$Z_t + Z_x + \epsilon Z Z_x - \epsilon Z_{xxt} = 0, \tag{1.8}$$

with

$$E(x,0) = Z(x,0) = g(x).$$

It is typical that the formal error made in using these sort of approximate models is of higher order in the small parameter $\epsilon$, so that a more accurate rendition of the physical situations underlying (1.7) and (1.8) is to append a term of formal order $\epsilon^\alpha$ to the right-hand side, where $\alpha > 1$. In the case of plane waves on the surface of shallow water, $\alpha = 2$. A heuristic argument that applies to both (1.7) and (1.8), based on the solution $g(x - t) + \epsilon t$ to the simple initial-value problem $E_t + E_x = \epsilon, E(x,0) = g(x)$, leads one to the expectation that nonlinear and dispersive effects arising from the terms $\epsilon E E_x$ and $\epsilon E_{xxx}$, respectively, may contribute an order-one effect to the wave profile on a time scale of order $\epsilon^{-1}$. Similarly, one expects the error terms may accumulate in such a way that they have an order-one effect on the wave profile on a time scale of order $\epsilon^{-2}$. Thus interesting nonlinear and dispersive effects are expected to appear in times of order $\epsilon^{-1}$ while the model may be formally invalid by the time $t$ is of order $\epsilon^{-2}$. It was proved in Bona et al. (1983) that under quite reasonable smoothness hypotheses on $g$, there is a constant $C$ which depends only on $g$, and is therefore of order one, for which

$$\|E - Z\|_{L_\infty(\mathbb{R})} = \sup_{x\in\mathbb{R}} |E(x,t) - Z(x,t)| \leq C\epsilon^2 t \tag{1.9}$$

for $0 \leq t \leq \epsilon^{-1}$. The result (1.9), and others like it for different measures of the distance between two functions of $x$, are interpreted to mean that at least on the interesting time scale $\epsilon^{-1}$ the solutions of the initial-value problems for the models (1.7) and (1.8) agree to within the accuracy of either model.

The theoretical results just outlined laid to rest a controversy that had been actively debated for about a decade, the issue being to decide which of equations (1.7) and (1.8) is best suited for modeling the propagation of small-amplitude long waves. The point was raised implicitly by Peregrine (1966) in a paper on undular bores. For sufficiently gradual bores, Korteweg–de Vries-type equations provide a suitable model for this kind of wave motion, though dissipative effects might need incorporation should quantitatively accurate prediction be in view. In his work, Peregrine improved on the earlier theory due to Airy (cf., Stoker 1957) by keeping dispersion within the description. Taking as his starting point the Boussinesq system of equations, he made an assumption of unidirectional wave propagation and derived (1.8), noting in passing that this equation was like the Korteweg–de Vries equation. After a brief physical analysis, Peregrine used (1.8) as the basis for constructing a scheme to numerically follow the evolution of bore-like initial data

$$Z(x,0) = b(1 - \tanh(x/d)), \tag{1.10}$$

where $d$ and $b$ are positive constants. His finite-difference scheme for (1.8) was adequate to resolve the formation of undulations on the trailing side of the bore-like solution that emanated from (1.10). Although Peregrine's work appeared at about the same time that the inverse-scattering theory for the Korteweg–de Vries equation was being developed, the discussion of the relative merits of (1.7) and (1.8) began later.

Benjamin et al. (1972) made a careful study of the model equation (1.8), establishing a sound theoretical basis for the initial-value problem and also examining some of the qualitative properties of solutions. They recognized explicitly that the relation $E_t + E_x = O(\epsilon)$ implies that $E_{xxx} = -E_{xxt} + O(\epsilon)$, from which (1.7) and (1.8) are seen to be formally equivalent to within the order of the error terms. They also presented commentary on both the models (1.7) and (1.8), terming the latter the regularized long-wave equation. This appellation reflected their view that (1.8) was a more favorable prospect for simulating small-amplitude long waves than (1.7). At about the same time, Zabusky and Galvin (1971) presented some experimental results on surface water waves showing the Korteweg–de Vries equation to be a qualitatively reasonable model in certain circumstances, while Hammack (1973) obtained some experimental confirmation of the qualitative efficacy of the regularized long-wave equation. The line instigated by Zabusky and Galvin was continued and refined in the interesting study by Hammack and Segur (1974).

The issue of comparing (1.7) and (1.8) became sharply focused during the Clarkson meeting on nonlinear waves in 1972 (see Newell 1974), which included as participants almost all the just-named scientists and many other experts. A number of interesting points were made on both sides of the topic, and a rejoinder to the remarks of Benjamin et al. (1972) appears in Kruskal (1975).

The issue was greatly clarified by the results (1.9) which, as explained above, indicate that it does not matter which equation is chosen if it is intended to model genuinely small-amplitude long waves with the pure initial-value problem over time scales for which either model has a good chance of providing an accurate description. Adding to the work of Hammack, Bona et al. (1981) made an extensive, quantitative comparison between the predictions of (1.8) with suitable dissipative terms added and the outcome of laboratory experiments. In these comparisons, one may observe in some detail the gradual divergence of the predictions of the approximate theory and reality as the Stokes number increases.

***Description of New Results***

It is our purpose here to extend the arguments in Bona et al. (1983) so that they yield analogous results as regards the initial-value problems for (1.1) and (1.2). In terms of the variables pertaining to (1.4) and (1.5), and dropping the assumption that $S$ is exactly equal to one, the result (1.9) becomes

$$\|\eta - \zeta\|_{L_\infty(\mathbb{R})} \leq C a^3 \lambda^{-1} t, \tag{1.11}$$

provided $0 \leq t \leq a^{-1}\lambda$.

It is specifically this latter type of result that will be extended to the more general equations depicted in (1.1) and (1.2). The appropriate balance between nonlinearity and dispersion is struck in the regime where $a$ is small, $\lambda$ is large, and

$$\frac{a^p}{m(\lambda^{-1})} \quad \text{is of order one} \tag{1.12}$$

as $a,\ \lambda^{-1} \longrightarrow 0$. Indeed (1.12) guarantees that the nonlinear and dispersive terms of either equation (1.1) or (1.2) are balanced. Ignoring for the moment the questions of existence, uniqueness, and regularity of solutions, the results established in this article have the following general form. Let $\eta$ and $\zeta$ be solutions of (1.1) and (1.2), respectively, such that

$$\eta(x,0) = \zeta(x,0) = a g(\lambda^{-1}x) \tag{1.13}$$

for $x \in \mathbb{R}$. Consider values of $a$ and $\lambda^{-1}$ in the interval (0, 1) such that (1.12) is valid and let $\epsilon$ denote a value representing the common order of $a^p$ and $m(\lambda^{-1})$. Then for any integer $s \geq 0$, there is a constant $C = C_s > 0$ which is independent of $a$ and $\lambda$ such that

$$\|\partial_x^s(\eta - \zeta)\|_{L_2(\mathbb{R})} \leq C\epsilon^{(2p+1)/p}\lambda^{-(2s+1)/2}t \tag{1.14}$$

for all $t$ in the interval $0 \leq t \leq \epsilon^{-1}\lambda$. By interpolating the case $s = 0$ and $s = 1$ in (1.14), one finds that

$$\|\eta - \zeta\|_{L_\infty(\mathbb{R})} \leq C\epsilon^{1/p}(\epsilon^2\lambda^{-1}t) \tag{1.15}$$

for the same range of $a$, $\lambda$ and $t$.

The inequality in (1.15) is interesting. For at the right-hand endpoint of its temporal range of proven validity one obtains that

$$\|\eta - \zeta\|_{L_\infty(\mathbb{R})} \leq C\epsilon^{1+1/p},$$

even though both $\eta$ and $\zeta$ separately have size of order $a = \epsilon^{1/p}$. Moreover, this latter estimate holds for all $t \leq \lambda\epsilon^{-1}$, and this interval of time is large enough that significant alteration of the wave profiles can occur due to the accumulation of nonlinear and dispersive effects. But, on the same time interval, the accumulation of the effects of the neglected error terms has formal order $\epsilon^{1+1/p}$. Thus it is concluded that over this time interval the outcome of predictions using either equation (1.1) or

(1.2) is the same to within the accuracy afforded by the approximation inherent in the use of either model.

***Relative Merits of the Two Models***

To complement the discussion of the preceding subsection, in which the similarities between the model equations (1.1) and (1.2) came to the fore, it may be useful to record here some of the more appreciable differences between the two equations. As seen from our current vantage point, the most obvious differences which present themselves are the following.

1. For certain, special equations of type (1.1), methods related to inverse-scattering theory have been used to generate exact solutions and various asymptotic expansions of solutions. In particular, for the Korteweg–de Vries equation (1.7), inverse-scattering theory has led to a deep understanding of the role of solitary waves in the evolution of general disturbances. The regularized long-wave equation (1.8), on the other hand, does not seem to possess an inverse-scattering theory, although its long-time asymptotics are very similar to those of the Korteweg–de Vries equation. Indeed, while one may learn a great deal about solutions of an equation solvable by an inverse-scattering transform, even among those of the form (1.1) such tools are rarely applicable. Thus a good deal of the value of this elegant theory is to teach us what to expect in more general situations where we lack methods, but where the intuition garnered from the detailed study of equations solvable by inverse-scattering transforms is still sensibly valid.
2. In a sense discussed by Kruskal (1975), equations of the form (1.1) are asymptotically pure, while those of the form (1.2) are not. In the case of the regularized long-wave equation (1.8), for example, this can be easily seen by consideration of the linearized equation

$$u_t + u_x - \epsilon u_{xxt} = 0. \tag{1.16}$$

Solving (1.16) by taking the Fourier transform in the $x$ variable, one obtains that

$$\hat{u}(k,t) = \exp(ikt/(1+\epsilon k^2))\hat{g}(k), \tag{1.17}$$

where $g$ is the order-one initial datum for $u$. If one removes from the right-hand side of (1.17) all effects that are of order higher than one in $\epsilon$, there appears $\exp(ik(1-\epsilon k^2))\hat{g}(k)$, which is the Fourier transform of the solution of the linearized Korteweg–de Vries equation

$$u_t + u_x + \epsilon u_{xxx} = 0. \tag{1.18}$$

Thus the Korteweg–de Vries equation is seen to be the lowest-order approximation to the regularized long-wave equation. This simple calculation has a precise analog for nonlinear equations with a general form of dispersion. For the case in which the operator $M$ has a homogeneous symbol, this may be seen by considering the initial-value problems which appear below in (4.5a). After performing on (4.5a) the change of variables (4.3), one obtains the "reduced" equations (3.1), in which the dependent variables have been scaled so that their size (and the sizes of their derivatives) is independent of the small parameter $\epsilon$. Of the two equations in (3.1), only the first (corresponding to an equation of the form (1.1)) is asymptotically pure, in the sense that its form is independent of $\epsilon$. For the case in which the symbol of $M$ is not homogeneous, an example of the asymptotic purity of equation (1.1) is given below in the reduction of the intermediate long-wave equation (4.9a) to the form (4.12a).

3. Global existence and well-posedness results for the pure initial-value problem for equations of type (1.2) are comparatively straightforward to establish, while such results may be problematic for equations of type (1.1) (compare Theorems 1 and 2 below).
4. In practical applications of either model equation, or in numerical simulations, one is often obliged to impose boundary conditions at finite values of $x$ rather than at infinity. Such initial boundary-value problems are considerably more tractable for equations of type (1.2). Consider, for example, the initial-value problem for equations (1.7) and (1.8) when posed on a semi-infinite interval (as would be appropriate for modeling a wavemaker at one end of a long channel). Although well-posedness results for both problems exist (see Bona and Bryant 1973; Bona et al. 1981; and Bona and Winther 1983, 1989), a comparison of these papers will quickly convince the reader of the difference in complexity between the two problems. As for two-point boundary-value problems, well-posedness results for the regularized long-wave equation are again seen to be straightforward in Bona and Dougalis (1980), whereas no such results are yet known for the Korteweg–de Vries equation.
5. Although the linearized dispersion relations for equations (1.1) and (1.2) are nearly equivalent for small values of the wavenumber $k$, they are widely disparate for large values of $k$. In this respect, the regularized long-wave equation (1.8) can be said to provide a slightly better model for water waves than the Korteweg–de Vries equation (1.7) because its linearized dispersion relation $\omega(k) = k/(1+k^2)$ more nearly mirrors that of the full Euler equations than does the relation $\omega(k) = k - k^3$ for the Korteweg–de Vries equation. Moreover, because the regularized long-wave equation has a linearized dispersion relation which tends to zero as the wavenumber becomes large, it is not prone to the dispersive blow-up that has been shown to hold for the Korteweg–de Vries equation (see Bona and Saut 1991).
6. Related to the last point is the fact that solutions of equations such as (1.8) are far easier to approximate numerically than solutions of the Korteweg–de Vries equation. Indeed, from the linearized dispersion relations given in the preceding paragraph, it follows that the linearized group and phase velocities for the Korteweg–de Vries equation are unbounded, whereas those of the regularized long-wave equation are not. Thus high-order accurate, unconditionally stable numerical schemes are not difficult to construct for the regularized long-wave equation (cf., Bona et al. 1981), whereas such schemes are very difficult to design for the Korteweg–de Vries equation (cf., Arnold and Winther 1982, Baker et al. 1983, Bona et al. 1986, 1990, 1991, Dougalis and Karakashian 1985, and Winther 1980).

## 2. Summary of the Existence Theory

In this section we recount briefly the general theory relating to the initial-value problems for (1.1) and (1.2). Some of the facts stated below are well known, while others have not appeared in the literature on these sort of nonlinear dispersive wave equations. The new theory that is stated here is established in Section 5, after the exposition of our main results in Sections 3 and 4. Some results for special realizations of equations (1.1) and (1.2) will also find their way into the exposition in Section 4, but these will be quoted in the context in which they arise.

Throughout our discussion, $p$ will denote a positive integer, the operator $M$ will be as formally defined in (1.3), and it will be assumed that the symbol $m$ of $M$ is such that there are positive constants $m_1$ and $m_2$ for which

$$\begin{aligned} m_1|k|^{\mu} &\leq m(k), && \text{for all } |k| \geq 1, \text{ and} \\ m(k) &\leq m_2(1+|k|)^{\nu}, && \text{for all } k \in \mathbb{R}, \end{aligned} \tag{2.1}$$

where $1 \leq \mu \leq \nu$.

The notational conventions will be those which are in current usage in the theory of partial differential equations. Thus $L_p = L_p(\mathbb{R})$ is the Banach space of measurable, $p$th-power integrable functions and the usual norm on this space is denoted $|\cdot|_p$. The Sobolev spaces of $L_2$-functions whose first $s$ derivatives lie in $L_2$ is written as $H^s$, and the usual norm on this space is denoted $\|\cdot\|_s$. A somewhat more general, and slightly less standard class of spaces will also be useful here. These are the Hilbert spaces $H = H_\alpha$ of real-valued functions $f$ defined on $\mathbb{R}$ for which

$$\|f\|_H^2 = \int_{-\infty}^{\infty} (1+\alpha(k))|\hat{f}(k)|^2 dk \tag{2.2}$$

is finite, equipped with the obvious inner product. It will always be assumed that the weight $\alpha$ satisfies the condition that

$$\alpha \geq 0 \text{ and } \alpha \text{ is an even, continuous function.} \tag{2.3$a$}$$

Since $\alpha \geq 0$, it follows that $H_\alpha$ is a linear subspace of $L_2$. It will also often be assumed that

$$\frac{1}{1+\alpha(k)} \in L_1. \tag{2.3$b$}$$

The assumption (2.3b) implies that $H_\alpha$ is embedded in $C_b$, the Banach space of bounded continuous functions defined on $\mathbb{R}$ equipped with the $L_\infty$-norm, and that $H_\alpha$ is a Banach algebra (cf., Hormander 1976, ch. 2). Thus there are constants $c_1$ and $c_2$ such that if $f, g \in H_\alpha$, then

$$|f|_\infty \leq c_1\|f\|_H \quad \text{and} \quad \|fg\|_H \leq c_2\|f\|_H\|g\|_H. \tag{2.4}$$

The most commonly encountered examples of such spaces $H$ are the previously mentioned spaces $H^s$. Indeed, if $s > \frac{1}{2}$, then $H^s = H_\alpha$ where $\alpha$ satisfies (2.3b), and consequently possesses the properties in (2.4).

If $X$ is any Banach space and $T > 0$, then $C(0, T; X)$ is the space of continuous mappings of the interval $[0, T]$ into $X$ with the norm

$$\|u\|_{C(0,T;X)} = \sup_{0 \le t \le T} \|u(t)\|_X .$$

The value $T = +\infty$ is allowed in this definition and in those that follow, though in such cases it must be stipulated that $u$ is both continuous and bounded as a mapping from $[0,T]$ to $X$. If $k$ is a positive integer, $C^k(0, T; X)$ is the subspace of $C(0,T;X)$ of functions whose first $k$ derivatives with respect to $t$ also lie in $C(0,T;X)$, equipped with the obvious modification of the norm on $C(0,T;X)$; also, $C^\infty(0,T;X) = \cap_{k=1}^{\infty} C^k(0,T;X)$, but no topology will be needed for this function class. The inner product in $L_2$ will be denoted simply by $( \; , \; )$ while the pairing between $H^s$ and $H^{-s}$ is written $< \; , \; >_s$. Similarly, the pairing between one of the spaces $H_\alpha$ and its dual $H_\alpha^*$ is denoted $< \; , \; >_{H_\alpha}$. The space $H_\alpha^*$ may be identified with the class of tempered distributions $S$ whose Fourier transform is a measurable function $\hat{S}(k)$ such that

$$\int_{-\infty}^{\infty} \frac{1}{1 + \alpha(k)} |\hat{S}(k)|^2 dk$$

is finite.

First, attention is given to the pure initial-value problem for (1.1). For this, the following theorem applies.

**Theorem 1.** *Consider the initial-value problem*

$$\left.\begin{aligned} \eta_t + \eta_x + \eta^p \eta_x - M\eta_x &= 0, \\ \eta(x, 0) &= g(x), \end{aligned}\right\} \tag{2.5}$$

*where $p \ge 1$ is an integer and the symbol $m$ of $M$ satisfies (2.1). If $g \in H^s$ where $s > \frac{3}{2}$, then there exists a $T_0 > 0$ depending only upon $\|g\|_r$ such that (2.5) admits a unique solution which, for any $T < T_0$, lies in the class $C(0, T; H^s)$, where $r$ is any number between $\frac{3}{2}$ and $s$. Moreover, the correspondence that associates to initial data $g$ the unique solution $\eta$ is continuous from $H^s$ to $C(0, T; H^s)$ for any $T < T_0$. The existence time $T_0$ depends inversely upon $\|g\|_r$ with $\lim_{\|g\|_r \to 0} T_0(\|g\|_r) = +\infty$.*

*If it is known* a priori *that the solution $\eta$ remains bounded in $H^r$ on any bounded time interval, then it follows that $T_0$ may be taken to be $+\infty$ and the solution is therefore global in time. Thus if $p < 2\mu$, if $p = 2\mu$ and the data is not too large in $L_2$, or if $p > 2\mu$ and the data is small enough in $H^{\mu/2}$, then $T_0 = +\infty$.*

*Remarks.* It follows from the differential equation that the $\eta$ whose existence is asserted in Theorem 1 has $\eta_t \in C(0, T; H^{s-\nu-1})$. Further temporal differentiability is readily deduced provided $s$ is large enough. For commentary on this theorem, its proof, and some recent generalizations, please consult Section 5.

The theory for the initial-value problem for (1.2) is a little different as will be apparent from an inspection of the next theorem.

**Theorem 2.** *Consider the initial-value problem*

$$\left.\begin{aligned} \zeta_t + \zeta_x + \zeta^p \zeta_x + M\zeta_t &= 0, \\ \zeta(x,0) &= g(x), \end{aligned}\right\} \tag{2.6}$$

*where $p \geq 1$ is an integer and the symbol $m$ of $M$ satisfies (2.1). If $g \in H^s$ where $s > \frac{1}{2}$, then there exists a $T_0 > 0$ depending only upon $\|g\|_s$ such that (2.6) admits a unique solution which for any $T < T_0$, lies in $C^\infty(0, T; H^s)$. Moreover, the correspondence that associates to initial data $g$ the unique solution $\zeta$ of (2.6) is continuous from $H^s$ to $C^k(0, T; H^s)$ for any $T < T_0$ and any finite value of $k$. The existence time $T_0$ depends inversely on $\|g\|_s$ and $T_0 \to +\infty$ as $\|g\|_s \to 0$.*

*If it is known* a priori *that the solution $\zeta$ remains bounded in $H^s$ on any bounded time interval, then it follows that $T_0$ may be taken to be $+\infty$ and thus the solution is global. If $\mu > 1$ and $g \in H_m$, then for any $p$ the solution is global, while if $\mu = 1$, $p \leq 3$ and $g, g' \in H_m$, then the solution is global.*

## 3. The Main Theorem

The principal technical result that arises in our theory is enunciated and proved in this section.

Consider the equations

$$\left.\begin{aligned} &a) \quad u_t + u^p u_x - N u_x = 0 \\ &\text{and} \\ &b) \quad v_t + v^p v_x - N v_x + \epsilon N v_t = 0, \end{aligned}\right\} \tag{3.1}$$

subject to the initial conditions

$$u(x,0) = v(x,0) = g(x)$$

for $x \in \mathbb{R}$. As was the case before, $p$ is a positive integer while $N$ is defined by

$$\widehat{Nu}(k) = n(k)\hat{u}(k)$$

where the symbol $n$ of $N$ satisfies

$$\begin{aligned} n_1|k|^\mu \leq n(k), \qquad & \text{for } |k| \geq 1, \\ n(k) \leq n_2(1 + |k|)^\nu, \qquad & \text{for all } k \in \mathbb{R}, \end{aligned} \tag{3.2}$$

for some positive constants $n_1$ and $n_2$, where $1 \leq \mu \leq \nu$ as before.

**Theorem 3.** *Let $s \geq 0$ and suppose that $g \in H^{s+2\nu+1}$. Suppose also that both equations in (3.1) have solutions corresponding to initial data $g$ that lie in $C(0, T_0; H^{s+2\nu+1})$ for some $T_0 > 0$, and let $B = \|u\|_{C(0,T_0;H^{s+2\nu+1})}$. Then there exist positive constants $C$ and $T$, depending only on $s$, $n_1$, $n_2$ and $B$, such that the difference $u - v$ satisfies*

$$\|\partial_x^s(u - v)\|_0 \le C\epsilon t \tag{3.3}$$

*for all* $\epsilon$ *and* $t$ *for which* $0 < \epsilon \le 1$ *and* $0 \le t \le \min\{T, T_0\}$.

Before embarking on the proof of Theorem 3, it will be helpful to establish a couple of elementary lemmas.

**Lemma 1.** *Let* $k \ge 3$ *be an integer. Then there exists a constant* $C = C(k) > 0$ *such that for any* $A > 0$ *and* $\epsilon$ *in the interval (0,1], the inequality*

$$\epsilon A + A^2 + \epsilon^{-1/2}A^3 + \epsilon^{-1}A^4 + \cdots + \epsilon^{(2-k)/2}A^k \le C(\epsilon A + \epsilon^{2-k}A^k) \tag{3.4}$$

*is valid.*

*Proof.* An application of Young's inequality yields that

$$\epsilon^{(2-j)/2}A^j \le C(\epsilon^{(3-j)/2}A^{j-1} + \epsilon^{(2-k)/2}A^k) \tag{3.5}$$

provided that $3 \le j \le k$, where $C$ depends only on $j$. Applying (3.5) $k - 3$ times, the left-hand side of (3.4) is inferred to be less than

$$C(\epsilon A + A^2 + \epsilon^{(2-k)/2}A^k).$$

Another appeal to Young's inequality gives

$$A^2 \le C(\epsilon A + \epsilon^{2-k}A^k),$$

and (3.4) follows. □

**Lemma 2.** *Let* $\alpha, \beta > 0$ *and* $\rho \ge 1$ *be given. Define*

$$T = \begin{cases} \beta^{-1/\rho}\alpha^{(1-\rho)/\rho}\displaystyle\int_0^\infty (1 + z^\rho)^{-1}\,dz, & \text{if } \rho > 1, \\ 1, & \text{if } \rho = 1. \end{cases} \tag{3.6}$$

*Then there exists a constant* $C = C(\rho) > 0$ *which is independent of* $\alpha$ *and* $\beta$ *such that if* $y(t)$ *is any non-negative differentiable function defined on* $0 \le t \le T$ *satisfying*

$$\left.\begin{aligned} &\frac{d}{dt}(y^2(t)) \le \alpha y(t) + \beta y^{\rho+1}(t), \qquad \text{for } 0 \le t \le T, \\ &y(0) = 0, \end{aligned}\right\} \tag{3.7}$$

*then*

$$y(t) \le C\alpha t, \qquad \text{for } 0 \le t \le T.$$

*Proof.* From the inequality in (3.7), we know that for $t \in [0, T]$,

$$\frac{2y'(t)}{\alpha + \beta y^\rho(t)} \le 1.$$

Integrating this inequality over an interval $[0, t]$, where $0 < t \le T$, leads to

$$h((\beta/\alpha)^{1/p} y(t)) \leq \tfrac{1}{2}\alpha^{(p-1)/p}\beta^{1/p} t \tag{3.8}$$

where

$$h(z) = \int_0^z (1 + z^p)^{-1}\, dz.$$

The function $h$ is strictly monotone increasing on $\mathbb{R}^+$, and so $h^{-1}$ is well-defined on the interval $0 \leq x \leq \int_0^\infty (1 + z^p)^{-1}\, dz$. For $t \leq T$, the right-hand side of (3.8) lies in the domain of $h^{-1}$, and so it transpires that

$$(\beta/\alpha)^{1/p} y(t) \leq h^{-1}(\tfrac{1}{2}\alpha^{(p-1)/p}\beta^{1/p} t)$$

for $0 \leq t \leq T$. But if $0 \leq t \leq T$ where $T$ is specified in (3.6), then

$$\frac{1}{2}\alpha^{(p-1)/p}\beta^{1/p} t \leq \frac{1}{2}\int_0^\infty (1 + z^p)^{-1} dz,$$

and hence there exists a constant $C$ depending only upon $p$ such that $h^{-1}(w) \leq 2Cw$ provided that $0 \leq w \leq \frac{1}{2} t\alpha^{(p-1)/p}\beta^{1/p}$ and $t \in [0, T]$. It thus follows that

$$y(t) \leq C\alpha t$$

for $0 \leq t \leq T$, as advertised in the statement of the lemma. □

*Proof (of Theorem 3).* Define $w$ to be $v - u$ where $u$ and $v$ are the solutions of (3.1a) and (3.1b), respectively. Then the function $w$ satisfies the initial-value problem

$$w_t + w^p w_x + \sum_{j=1}^{p} \binom{p}{j} w^{p-j} u^j w_x + \sum_{j=0}^{p-1} \binom{p}{j} w^{p-j} u^j u_x - Nw_x + \epsilon N w_t + \epsilon N u_t = 0, \tag{3.9$a$}$$

with

$$w(x, 0) = 0. \tag{3.9$b$}$$

Consider first the case $s = 0$. Multiplying equation (3.9a) by $w$, integrating over the spatial domain, and integrating by parts several times (this is justified under the smoothness assumed about $u$ and $v$) leads to the relation

$$\frac{d}{dt}\int_{-\infty}^{\infty} [w^2 + \epsilon w N w] dx = \sum_{j=2}^{p+1} \gamma_j \int_{-\infty}^{\infty} w^j u^{p+1-j} u_x dx - 2\epsilon \int_{-\infty}^{\infty} w N u_t dx \tag{3.10}$$

where the $\gamma_j$ are particular constants. Let $A_0(t)$ denote the square root of the integral on the left-hand side of (3.10). Then $\|w(\cdot, t)\|_0 \leq A_0(t)$ and, since $0 < \epsilon \leq 1$,

$$\|w(\cdot, t)\|_{\frac{1}{2}} \leq \frac{C}{n_1}\left\{ \|w(\cdot, t)\|_0 + \left[\int_{-\infty}^{\infty} (wNw) dx\right]^{\frac{1}{2}} \right\} \leq C\epsilon^{-\frac{1}{2}} A_0(t)$$

for all $t \in [0, T_0]$, where $n_1$ is the constant appearing in (3.2) and $C$ connotes an absolute constant depending only upon the particular norm chosen for the spaces $H^s$. These observations may be used to bound the terms on the right-hand side of (3.9) as follows:

$$\begin{aligned}
\left|\int_{-\infty}^{\infty} wNu_t dx\right| &= \left|\int_{-\infty}^{\infty} wN(Nu_x - u^p u_x)dx\right| \\
&\leq \|w\|_0(\|N^2 u_x\|_0 + \|N(u^p u_x)\|_0) \\
&\leq Cn_2^2\|w\|_0(\|u\|_{2\nu+1} + \|u\|_\nu^p\|u\|_{\nu+1}) \\
&\leq C(B + B^{p+1})\|w\|_0 \\
&\leq CA_0(t);
\end{aligned} \tag{3.11}$$

$$\begin{aligned}
\left|\int_{-\infty}^{\infty} w^j u^{p+1-j} u_x dx\right| &\leq C|w|_j^j|u|_\infty^{p+1-j}|u_x|_\infty \\
&\leq C|w|_j^j \\
&\leq C\|w\|^j_{\frac{1}{2}-\frac{1}{j}} \\
&\leq C(\|w\|_0^2\|w\|_{\frac{1}{2}}^{j-2}) \\
&\leq CA_0(t)^j\epsilon^{(2-j)/2};
\end{aligned} \tag{3.12}$$

where use has been made of standard Sobolev inequalities, interpolation results and the fact that $H^s$ is an algebra for $s > \frac{1}{2}$. Combining (3.10), (3.11), (3.12), and Lemma 1 yields the differential inequality

$$\frac{d}{dt}[A_0^2(t)] \leq C[\epsilon A_0(t) + \epsilon^{(1-p)}A_0^{p+1}(t)].$$

Apply Lemma 2 with $\alpha = C\epsilon, \beta = C\epsilon^{(1-p)}$ and $\rho = p$ to conclude that

$$A_0(t) \leq C\epsilon t$$

for $0 \leq t \leq T(0)$ for some $T(0) > 0$ which is independent of $\epsilon$. Because

$$\|u(\cdot, t) - v(\cdot, t)\|_0 = \|w(\cdot, t)\|_0 \leq A_0(t),$$

this completes the proof of the Theorem in the case $s = 0$.

Next consider the case $s = 1$. Multiplying equation (3.9) by $w_{xx}$, integrating over $\mathbb{R}$ and integrating by parts leads to the equation

$$\frac{d}{dt}[A_1(t)^2] = 2\int_{-\infty}^{\infty} w^p w_x w_{xx} dx - 2\epsilon\int_{-\infty}^{\infty}(w_x N u_{xt})dx$$

$$+\sum_{k=1}^{p-1}\gamma_k\int_{-\infty}^{\infty} w^{p-k-1}w_x^3u^k\,dx+\sum_{k=1}^{p}\mu_k\int_{-\infty}^{\infty} w^{p-k}w_x^2u^{k-1}u_x\,dx \tag{3.13}$$

$$+\sum_{k=1}^{p=1}\nu_k\int_{-\infty}^{\infty} w^{p-k}w_xu^ku_x\,dx$$

for some constants $\gamma_k$, $\mu_k$, $\nu_k$, where

$$A_1(t)=\left\{\int_{-\infty}^{\infty}[w_x^2+\epsilon w_xNw_x]dx\right\}^{\frac{1}{2}}.$$

As before, we have that $\|w_x\|_0\le A_1(t)$ and

$$\|w_x\|_{\frac{1}{2}}\le\frac{C}{\epsilon^{\frac{1}{2}}n_1}A_1(t).$$

The second term on the right-hand side of (3.13) may be estimated as follows:

$$\begin{aligned}\left|\int_{-\infty}^{\infty}(w_xNu_{xt})dx\right| &\le \|w_x\|_0\|Nu_{xt}\|_0=\|w_x\|_0\|N(\partial_x(Nu_x-u^pu_x))\|_0\\ &\le C\|w_x\|_0\big[n_2^2\|u\|_{2\nu+2}+n_2\|u^pu_x\|_{\nu+1}\big] \qquad (3.14)\\ &\le C\|w_x\|_0\big[\|u\|_{2\nu+2}+\|u\|_{\nu+1}^p\|u\|_{\nu+2}\big]\\ &\le CA_1(t).\end{aligned}$$

To obtain an effective upper bound on the first term, we proceed to write the inequalities

$$\begin{aligned}\left|\int_{-\infty}^{\infty}w^pw_xw_{xx}\,dx\right| &\le \frac{p}{2}\int_{-\infty}^{\infty}|w|^{p-1}|w_x|^3dx\\ &\le\frac{p}{2}|w|_{\infty}^{p-1}|w_x|_3^3\\ &\le C\|w\|_0^{(p-1)/2}\|w_x\|_0^{(p-1)/2}|w_x|_3^3.\end{aligned}$$

Since the case $s=0$ is already in hand, $\|w\|_0\le C$ for all $t$ with $0\le t\le T(0)$. Hence the right-hand side of the last inequality may be majorized by

$$\begin{aligned}C\|w_x\|_0^{(p-1)/2}|w_x|_3^3 &\le CA_1(t)^{(p-1)/2}\|w_x\|_{\frac{1}{6}}^3\\ &\le CA_1(t)^{(p-1)/2}\|w_x\|_0^2\|w_x\|_{\frac{1}{2}}\\ &\le CA_1(t)^{(p+3)/2}\left[\epsilon^{-\frac{1}{2}}A_1(t)\right].\end{aligned}$$

Indeed, all the terms under a summation sign in (3.13) may be estimated in a manner similar to that used for the term that was just bounded. These estimates may be combined with (3.13), (3.14) and the fact that $\epsilon \leq 1$ to yield

$$\frac{d}{dt}[A_1(t)^2] \leq C(\epsilon A_1(t) + \epsilon^{-p} A_1(t)^{p+2})$$

and it then follows from Lemma 2 that

$$A_1(t) \leq C\epsilon t$$

for $0 \leq t \leq T(1)$ for some positive value $T(1)$ which is independent of $\epsilon$. This proves the theorem in case $s = 1$.

The cases $s \geq 2$ may be deduced in exactly the same way as the case $s = 1$, though the details are naturally a little more complicated. Multiplying equation (3.9a) by $\partial_x^{2s} w$ and integrating over the spatial domain results in the identity

$$\begin{aligned}\frac{d}{dt}[A_s(t)]^2 = &-2\epsilon \int_{-\infty}^{\infty} (\partial_x^s w)(\partial_x^s N u_t)dx + \int_{-\infty}^{\infty} \partial_x^s (w^p w_x)\partial_x^s w \\ &+ \sum_{k=1}^{p} \binom{p}{k} \int_{-\infty}^{\infty} \partial_x^s (w^{p-k} w_x u^k)\partial_x^s w\, dx \\ &+ \sum_{k=0}^{p-1} \binom{p}{k} \int_{-\infty}^{\infty} \partial_x^s (w^{p-k} u_x u^k)\partial_x^s w\, dx,\end{aligned} \tag{3.15}$$

where

$$A_s(t) = \left\{ \int_{-\infty}^{\infty} [(\partial_x^s w)^2 + \epsilon \partial_x^s w N(\partial_x^s w)]dx \right\}^{\frac{1}{2}},$$

and the formal calculations leading to this relation are easily justified.

We now set about bounding the terms on the right-hand side of equation (3.15). The first integral on the right-hand side of (3.15) is majorized by $C\epsilon \|\partial_x^s w\|_0 \leq C\epsilon A_s(t)$ where $C$ depends only on $B$. Upon using Leibnitz' rule to expand the differentiated terms in the remaining integrals, one obtains expressions which are easily estimated using the assumed bound on $\|u\|_{s+2\nu+1}$ and the already established bound on $\|w\|_s$ for $s = 0$. For example, one of the terms arising from the second integral on the right-hand side of (3.15) is bounded above as follows:

$$\begin{aligned}\left| \int_{-\infty}^{\infty} w^p (\partial_x^{s+1} w)(\partial_x^s w)dx \right| &\leq C \int_{-\infty}^{\infty} |w|^{p-1} |w_x| |\partial_x^s w|^2 dx \\ &\leq C|w|_\infty^{p-1} |w_x|_\infty \|\partial_x^s w\|_0^2 \\ &\leq C\|w\|_s^p \|\partial_x^s w\|_0^2 \\ &\leq C(\|w\|_0^p + \|\partial_x^s w\|_0^p)\|\partial_x^s w\|_0^2\end{aligned}$$

$$\leq C(1 + A_s(t)^p)A_s(t)^2$$

$$\leq C(A_s(t)^2 + \epsilon^{-p}A_s(t)^{p+2}).$$

Continuing with such estimates, one derives from (3.15) an inequality of the form

$$\frac{d}{dt}[A_s(t)]^2 \leq C(\epsilon A_s(t) + A_s(t)^2 + \cdots + \epsilon^{-p}A_s(t)^{p+2})$$

$$\leq C(\epsilon A_s(t) + \epsilon^{-p}A_s(t)^{p+2}).$$

As before, Lemma 2 then implies that

$$A_s(t) \leq C\epsilon t$$

for $0 \leq t \leq T(s)$ where $C$ and $T(s) > 0$ are independent of $\epsilon$.

The proof of the theorem is thus completed. □

*Remark.* A little more care with the above energy estimates allow one to establish the conclusion (3.3) of Theorem 3 based on the possibly weaker assumptions that $g \in H_\alpha$ where $\alpha(k) = k^{s+1}n^2(k)$ and that, for some $T_0 > 0$, $\|u\|_{C(0,T_0;H_\alpha)} = B$ is finite.

## 4. Applications to Model Equations

In this section the technical results of the preceding section will be used to effect a comparison between solutions of the initial-value problems (1.1) and (1.2), repeated here for convenience,

$$\left.\begin{aligned} &a)\ \eta_t + \eta_x + \eta^p\eta_x - M\eta_x = 0,\\ &b)\ \zeta_t + \zeta_x + \zeta^p\zeta_x + M\zeta_t = 0,\\ &c)\ \eta(x,0) = \zeta(x,0) = \epsilon^{1/p}g(\lambda^{-1}x), \end{aligned}\right\} \tag{4.1}$$

where the use of $\epsilon$ to stand for $a^p$ introduced below (1.12) is continued and

$$\widehat{Mv}(k) = m(k)\hat{v}(k)$$

for all $k \in \mathbb{R}$. If we define

$$n(k) = \epsilon^{-1}m(\lambda^{-1}k), \tag{4.2}$$

then solutions $u$ and $v$ of (3.1) are related to solutions $\eta$ and $\zeta$ of (4.1) via the changes of variables

$$\left.\begin{aligned} \eta(x,t) &= \epsilon^{1/p}u(\lambda^{-1}(x-t), \epsilon\lambda^{-1}t),\\ \zeta(x,t) &= \epsilon^{1/p}v(\lambda^{-1}(x-t), \epsilon\lambda^{-1}t). \end{aligned}\right\} \tag{4.3}$$

Consideration is first given to operators $M = M_\mu$ whose symbol $m$ is homogeneous of degree $\mu$ where $\mu \geq 1$, so that $m(k) = |k|^\mu$. As explained in the introduction, the regime in which the models (4.1) may be expected to apply is where the amplitude parameter $\epsilon$ is small and the combination $\epsilon^{1/\mu}\lambda$ has order one. The following theorem shows that in the just-mentioned regime, the solutions of (4.1a) and (4.1b) corresponding to the initial data (4.1c) are close to each other over appropriately long time scales.

**Theorem 4.** *Let $p$ be a non-negative integer, let $s \geq 0$, and suppose $g \in H^{s+2\mu+1}$. Suppose also that the initial-value problems*

$$\left.\begin{aligned} &a)\ u_t + u^p u_x - M_\mu u_x = 0,\\ &b)\ v_t + v^p v_x - M_\mu v_x + \epsilon M_\mu v_t = 0,\\ &c)\ u(x,0) = v(x,0) = g(x) \end{aligned}\right\} \tag{4.4}$$

*have solutions $u, v \in C(0, T_0; H^{s+2\mu+1})$ for some $T_0 > 0$. Let $B = \|u\|_{C(0,T_0;H^{s+2\mu+1})}$. Then there exist positive constants $C$ and $T$ which depend only on $s$ and $B$ such that the solutions $\eta$ and $\zeta$ of the initial-value problems*

$$\left.\begin{aligned} &a)\ \eta_t + \eta_x + \eta^p\eta_x - M_\mu\eta_x = 0,\\ &b)\ \zeta_t + \zeta_x + \zeta^p\zeta_x + M_\mu\eta_t = 0,\\ &c)\ \eta(x,0) = \zeta(x,0) = \epsilon^{1/p} g(\epsilon^{1/\mu}x) \end{aligned}\right\} \tag{4.5a}$$

*satisfy the inequality*

$$\|\partial_x^s\eta - \partial_x^s\zeta\|_0 \leq C\epsilon^{(2+\frac{1}{p}+\frac{2s+1}{2\mu})}t \tag{4.5b}$$

*for $0 < \epsilon \leq 1$ and $0 \leq t \leq \epsilon^{-(1+\frac{1}{\mu})} min(T, T_0)$.*

*Proof.* For any $\epsilon$ in $(0,1]$, define $\lambda$ to be $\epsilon^{-1/\mu}$. Then the solutions $\eta$ and $\zeta$ of (4.5) are related to solutions $u$ and $v$ of (3.1) by the changes of variables (4.3), where the operator $N$ appearing in (3.1) is exactly equal to $M_\mu$ because of (4.2) and the homogeneity of the symbol of $M_\mu$. (In particular, the constants $n_1$ and $n_2$ appearing in (3.2) may be taken equal to 1.) In view of the stated existence assumptions about solutions of (4.4), we may apply Theorem 3 to conclude that there exist constants $C, T > 0$ which depend only on $s$ and $B$ such that

$$\|\partial_x^s u - \partial_x^s v\|_0 \leq C\epsilon t \tag{4.6}$$

for all $0 < \epsilon \leq 1$ and $0 \leq t \leq \min(T, T_0)$. The assertion contained in Theorem 4 now follows from (4.6) by inverting the changes of variables (4.3). □

Combining Theorem 4 with the existence theory reviewed in Section 2 gives the following set of results.

**Corollary 1.** *Let $p$, $s$, and $M_\mu$ be as in Theorem 4 where $\mu \geq 1$. Then the hypotheses concerning the initial-value problems (4.4) are valid and the constants $T_0$ and $B$ are both of order one if $g$ is. Consequently, if $g$ is order one, then so are the constants $C$ and $T$ appearing in the inequalities (4.5).*

As examples of the application of Corollary 1 to concrete problems of interest, consider the generalized Benjamin-Ono equation (Benjamin 1967)

$$\eta_t + \eta_x + \eta^p \eta_x - H\eta_{xx} = 0, \tag{4.7a}$$

and its regularized version

$$\zeta_t + \zeta_x + \zeta^p \zeta_x + H\zeta_{xt} = 0, \tag{4.7b}$$

where $H$ here connotes the Hilbert transform. Taking $g \in H^{3+s}$ and assigning initial data

$$\eta(x,0) = \zeta(x,0) = ag(\lambda^{-1}x)$$

where both $a$ and $\lambda^{-1}$ are small and $a^p\lambda$ has order one leads to the conclusion that

$$\|\partial_x^s(\eta - \zeta)\|_0 \le C_s a^{2p+1} \lambda^{-(2s+1)/2} t$$

for $0 \le t \le T\lambda a^{-p}$, where $C$ and $T$ are order-one constants. Interpolation leads to the conclusion that

$$|\partial_x^j(\eta - \zeta)|_\infty \le C_j a^{2p+1} \lambda^{-(j+1)} t$$

provided $0 \le j < s$, for the same range of $t$. In particular, for the Benjamin-Ono equation itself, the case $p = 1$ in (4.7), we have that

$$|\eta - \zeta|_\infty \le C a^3 \lambda^{-1} t$$

provided that $0 \le t \le T\lambda a^{-1}$, $a$ and $\lambda^{-1}$ are small and $a\lambda$ has order one (see Benjamin 1967, p. 559).

In the remainder of this section attention is given to showing how to use our theory in cases where the symbol $m$ of the dispersive operator $M$ is not homogeneous. In such situations, it is often useful to keep in mind the underlying physical situation being modeled to guide the formulation of precise theorems. We shall show how the issues are resolved for the particular case of the intermediate long-wave (ILW) equation, though the reader will readily appreciate how the analysis can be adapted to encompass other such equations. The ILW equations have the forms in (4.1) with $p = 1$ and $M = M_H$, where the symbol $m_H$ of $M_H$ is given by

$$m_H(k) = k \coth(kH) - \frac{1}{H}$$

and $H > 0$ is a parameter having a definite physical significance to be explained presently. Rewriting the symbol $m_H(k)$ as $\frac{1}{H}\beta(kH)$ where $\beta(z) = z\coth(z) - 1$, and noting that $\beta(z) \sim z$ as $z \to \infty$ while $\beta(z) \sim z^2/3$ as $z \to 0$, one understands that $m_H(k) \sim k$ if $kH \gg 1$, but that $m_H(k) \sim k^2 H$ for $kH \ll 1$. Thus $m_H$ is far from being homogeneous, and in fact the relation (1.11) which guarantees that a balance is struck between nonlinear and dispersive effects depends strongly on $H$. It is therefore useful to recall how the parameter $H$ is related to the waves being modeled.

The role of the ILW equation in modeling small-amplitude long waves in a two-layer fluid system has been explained by Kubota, Ko and Dobbs (1978). This paper considers a system consisting of a homogeneous layer of finite depth resting on a thin pycnocline which in turn lies over another, heavier, homogeneous layer. The whole fluid body is bounded above and below by horizontal, impermeable planes. The layers are supposed to extend indefinitely in the horizontal plane and motions are considered that are uniform in one of the horizontal directions, say along the $y$-axis of a standard Cartesian coordinate system, and which propagate in the perpendicular direction, so along the $x$-axis. The top and bottom layers have depths $H_1$ and $H_2$, respectively. The ILW equation itself arises in two different circumstances related to the aforementioned fluid system. In one instance the depths of the two layers are essentially the same, so $H_1 = H_2$ and up to a constant of proportionality these quantities both equal $H$. In the other, one of the layers is very much smaller than the other, so that the pycnocline is located quite close to a solid boundary. In this case, $H$ essentially corresponds to $H_1 + H_2$, the total depth of the two-fluid system. In either of the above cases, $H$ may take any value in $(0, \infty)$. Both of the parameters $a$ and $\lambda$ are scaled by the pycnocline thickness, which in turn must be small compared to $H$ in either of the above cases.

**Theorem 5.** *Let $s \geq 0$ and suppose $g \in H^{s+5}$. Fix a positive depth $H$ and consider values of the amplitude $a$ and wavelength $\lambda$ such that $a, \lambda^{-1} \in (0, 1]$ and for which*

$$0 < C_0 \leq \frac{a}{\lambda^{-1}\coth(H\lambda^{-1}) - H^{-1}} \leq C_1. \tag{4.8}$$

*Then there exist constants $C, T > 0$ depending only on $s, C_0, C_1$ and $\|g\|_{s+5}$, but not on $H$, such that the solutions $\eta$ and $\zeta$ of*

$$\left.\begin{aligned} &a)\ \eta_t + \eta_x + \eta\eta_x - M_H\eta_x = 0,\\ &b)\ \zeta_t + \zeta_x + \zeta\zeta_x + M_H\zeta_t = 0,\\ &c)\ \eta(x,0) = \zeta(x,0) = ag(\lambda^{-1}x) \end{aligned}\right\} \tag{4.9}$$

*satisfy the relation*

$$\|\partial_x^s\eta - \partial_x^s\zeta\|_0 \leq Cta^3\lambda^{-(s+\frac{1}{2})}$$

*for all $t$ such that $0 \leq t \leq \lambda a^{-1}T$.*

*Proof.* Both the initial-value problems in (4.9) are globally well posed in $H^{s+5}$ as assured by the results quoted in Section 2, so that both $\eta$ and $\zeta$ lie in $C(0, T; H^{s+5})$ for every $T > 0$. Moreover, because of the bounds that follow from the countably-infinite string of conservation laws appertaining to (4.9a) (cf., Abdelouhab et al. 1989), $\|\eta(\cdot, t)\|_r$ is bounded in terms of $\|g\|_r$ for all $t \geq 0$, for $r = \frac{1}{2}k$, $k = 0, 1, 2, \ldots$.

Rewriting (4.8) as

$$C_0 \leq aH/\beta(H/\lambda) \leq C_1,$$

and using the properties mentioned above of the function $\beta(z) = z\coth z - 1$, one easily obtains that there exist positive constants $B_0$ and $B_1$, depending only on $C_0$ and $C_1$, such that

$$B_0 \leq \lambda/\lambda_0 \leq B_1 \tag{4.10}$$

where

$$\lambda_0 = \lambda_0(a,H) = \begin{cases} 1/a, & \text{if } aH \geq 1, \\ \sqrt{H/a}, & \text{if } aH \leq 1. \end{cases}$$

If one transforms the equations (4.9) via the changes of variables

$$\eta(x,t) = au(\lambda_0^{-1}(x-t),\ a\lambda_0^{-1}t)$$

and

$$\zeta(x,t) = av(\lambda_0^{-1}(x-t),\ a\lambda_0^{-1}t), \tag{4.11}$$

then the initial-value problems which result may be written in the form

$$\begin{aligned} &a)\ u_t + uu_x - L_K u_x = 0, \\ &b)\ v_t + vv_x - L_K v_x + aL_K v_t = 0, \\ &c)\ u(x,0) = v(x,0) = g\left(\frac{\lambda_0}{\lambda}x\right), \end{aligned} \tag{4.12}$$

where $K = aH$, and the symbol $n_K$ of $L_K$ is given by

$$n_K(k) = \begin{cases} k\coth(kK) - \dfrac{1}{K}, & \text{if } K \geq 1, \\ \dfrac{k}{\sqrt{K}}\coth(k\sqrt{K}) - \dfrac{1}{K}, & \text{if } K \leq 1. \end{cases}$$

An application of Theorem 3, with $\mu = 1$ and $\nu = 2$, to the problems (4.12) then yields the estimate

$$\|\partial_x^s(u-v)\|_0 \leq Cat \tag{4.13}$$

for $0 \leq t \leq T$. Here the values of the constants $C$ and $T$ are determined by the quantities $s$, $n_1(K)$, $n_2(K)$, and $B$, where $B = B(u) = \|u\|_{C(0,\infty;H^{s+5})}$ and $n_1(K)$, $n_2(K)$ are such that

$$\begin{aligned} n_1(K)|k| &\leq n_K(k), && \text{for } |k| \geq 1, \\ n_K(k) &\leq n_2(K)(1+|k|)^2, && \text{for all } k \in \mathbb{R}. \end{aligned}$$

The quantity $B(u)$ varies with $a$, $\lambda$, and $H$ through the dependence of the solution $u$ of (4.12) (a),(c) on these parameters. However, there exists a constant $B_0$, depending only on $C_0$, $C_1$, and $\|g\|_{s+5}$ which majorizes the value of $B(u)$ for all values of the parameters $a$, $\lambda$, and $H$. To see this technical fact, first apply Theorems 7.1.1 and Lemma 7.2.2 of Abdelouhab et al. (1989) to obtain the estimate

$$\|u\|_{C(0,\infty;H^{s+5})} \leq A_1\|u(x,0)\|_{s+5},$$

where $A_1$ is an absolute constant which is independent of the value of $K$ in the range $\{0 < K < \infty\}$. Then note that (4.10) implies

$$\|u(x,0)\|_{s+5} = \|g(\frac{\lambda_0}{\lambda}x)\|_{s+5} \le A_2\|g\|_{s+5},$$

where $A_2$ depends only on $C_0$ and $C_1$. Hence the desired constant $B_0$ is given by $B_0 = A_1A_2\|g\|_{s+5}$.

Next, it will be shown that the constants $n_1(K)$ and $n_2(K)$ may be chosen independently of $K$. To prove this, define the function $\gamma(\theta)$ for $\theta \in \mathbb{R}$ by $\gamma(\theta) = |(\theta\coth(\theta) - 1)/\theta|$, and note that $\gamma$ is an even function which is monotone increasing and bounded on the interval $0 \le \theta < \infty$, with $\sup_{\theta\ge 0}(\gamma(\theta)/\theta) = Q < \infty$ and $\inf_{0\le\theta\le 1}(\gamma(\theta)/\theta) = R > 0$. For $K \ge 1$ one has $n_K(k) = |k|\gamma(kK)$, so that

$$n_K(k) \le |\gamma|_\infty|k| \le |\gamma|_\infty(1 + |k|)^2, \qquad \text{for all } k,$$

and

$$n_K(k) \ge |k|\gamma(1), \qquad \text{for } |k| \ge 1.$$

On the other hand, for $K \le 1$ one has

$$n_K(k) = k^2\left(\frac{\gamma(k\sqrt{K})}{|k|\sqrt{K}}\right),$$

so that

$$n_K(k) \le Q\,k^2 \le Q(1 + |k|)^2, \qquad \text{for all } k \in \mathbb{R},$$

and

$$n_K(k) = |k|\left(\frac{\gamma(|k|\sqrt{K})}{\sqrt{K}}\right) \ge |k|\frac{\gamma(\sqrt{K})}{\sqrt{K}} \ge R|k|, \qquad \text{for all } |k| \ge 1.$$

Therefore, for any $K \in (0,\infty)$, one may choose $n_1(K) = \min\{\gamma(1), R\}$ and $n_2(K) = \max\{|\gamma|_\infty, Q\}$.

It follows from the assertions proved in the preceding two paragraphs that (4.13) holds for constants $C$ and $T$ which depend only on $s, C_0, C_1$, and $\|g\|_{s+5}$. Inverting the change of variables (4.11) now gives

$$\|\partial_x^s(\eta - \zeta)\|_0 \le Ca^3\lambda_0^{-(s+\frac{1}{2})}t$$

for $0 \le t \le \lambda_0 a^{-1}T$. A final application of (4.10) then yields the result. □

## 5. The Existence Theory

The existence theory enunciated in Section 2 is discussed here. We begin with the theory for the initial-value problem (2.6). As before, $p$ is a positive integer and the symbol $m$ of the operator $M$ is taken to satisfy (2.1). The first result is a theory of local existence following the lines put forth in Benjamin et al. (1972).

**Lemma 3.** *Suppose $\mu \geq 1$ and $p \geq 1$ and that the initial data lies in a space $H = H_\alpha$ where $\alpha$ satisfies the conditions (2.3). Then there exists a maximal time $T_0$ which depends only upon $g$ such that for each $T < T_0$, the initial-value problem (2.6) has a unique distributional solution $\zeta$ lying in $C(0, T; H)$. The maximal time interval has a lower bound which is related to $\|g\|_\alpha^{-1}$, and which approaches $+\infty$ as $\|g\|_\alpha$ approaches 0. This solution is infinitely differentiable in its temporal variable, and its temporal derivatives of all orders lie in $C(0, T; H_\beta)$ where $\beta(k) = k\alpha(k)/(1+m(k))$. The mapping that associates to initial data $g \in H_\alpha$ the solution $\zeta$ of (2.6) is, for any $R > 0$, continuous from the ball of radius $R$ about zero in $H$ into $C(0, T; H)$ where $T = T(R)$. If $\mu > 1$ and $g \in H_\gamma$ where $\gamma(k) \geq \alpha(k)$ and $\gamma(k)$ has polynomial growth at infinity, then it follows that the solution $\zeta$ of (2.6) lies in $C(0, T; H_\gamma)$.*

*Proof.* Consider the integral equation

$$\zeta(x, t) = g(x) - \int_0^t Q[\zeta(\cdot, \tau) + \frac{1}{p+1}\zeta^{p+1}(\cdot, \tau)]d\tau \tag{5.1}$$

where $Q$ is the operator defined by

$$\widehat{Q\psi}(k) = \frac{ik}{1+m(k)}\hat{\psi}(k). \tag{5.2}$$

The assumption (2.1) on $m$ coupled with the fact that $\mu \geq 1$ implies that $Q$ defines a bounded linear operator on any space $H = H_\alpha$. Let $R = 2\|g\|_H$ and let $B_R$ be the ball of radius $R$ centered at the origin in $C(0, T; H)$, where the value of $T$ will be chosen presently. For any function $Z \in C(0, T; H)$, define $A(Z)$ to be that function of $(x, t) \in \mathbb{R} \times [0, T]$ obtained upon replacing $\zeta$ by $Z$ in the right-hand side of (5.1). It will be shown that the correspondence $Z \mapsto A(Z)$ maps $C(0, T; H)$ into itself and that if $T$ is well chosen, then in fact $A : B_R \to B_R$ and $A$ is contractive on this latter set.

To see that $A$ maps $C(0, T; H)$ to itself, simply note that $A$ is the composition of three operators, all of which have the desired property, namely

$$Z \mapsto Z + (p+1)^{-1}Z^{p+1}, \; W \mapsto QW, \text{and } Y \mapsto \int_0^t Y$$

(In analyzing the first mapping listed above, use is made of the fact that $H$ is an algebra.)

To prove the other two assertions, consider $\zeta_1$ and $\zeta_2$ in $B_R$ and estimate $A\zeta_1 - A\zeta_2$ as follows:

$$\begin{aligned} \|A\zeta_1 - A\zeta_2\|_{C(0,T;H)} &\leq \sup_{0\leq t\leq T} \int_0^t \left\|Q\left[\zeta_1 - \zeta_2 + \frac{1}{p+1}(\zeta_1^{p+1} - \zeta_2^{p+1})\right]\right\|_H d\tau \\ &\leq \sup_{0\leq t\leq T} C\int_0^t \|\zeta_1 - \zeta_2\|_H(1 + \|\zeta_1^p + \zeta_1^{p-1}\zeta_2 + \cdots + \zeta_2^p\|_H)d\tau \end{aligned}$$

$$\leq \sup_{0\leq t\leq T} C\int_0^t \|\zeta_1-\zeta_2\|_H(1+C(\|\zeta_1\|_H^p+\|\zeta_2\|_H^p))d\tau$$
$$\leq CT(1+R^p)\|\zeta_1-\zeta_2\|_{C(0,T;H)}, \tag{5.3}$$

where essential use was again made of the fact that $H$ is an algebra. As the radius $R$ is fixed, we may choose $T$ so that $CT(1+R^p)=\frac{1}{2}$, say. It then follows that

$$\|A\zeta_1-A\zeta_2\|_{C(0,T;H)}\leq \tfrac{1}{2}\|\zeta_1-\zeta_2\|_{C(0,T;H)}$$

provided that $\zeta_1,\zeta_2\in B_R$. Furthermore, if $\zeta\in B_R$, then

$$\|A\zeta\|_{C(0,T;H)}\leq \|g\|_H+\|A\zeta-A(0)\|_{C(0,T;H)}$$
$$\leq \|g\|_H+\tfrac{1}{2}\|\zeta\|_{C(0,T;H)}$$
$$\leq \tfrac{1}{2}R+\tfrac{1}{2}R=R,$$

so that $A$ maps $B_R$ to itself.

It follows from the contraction-mapping theorem that (5.1) has a unique solution $\zeta\in B_R$. Moreover, if (5.1) is differentiated with respect to $t$, there appears the relation

$$\zeta_t=-Q\left(\zeta+\frac{1}{p+1}\zeta^{p+1}\right), \tag{5.4}$$

and since $(I+M)Q=\partial_x$ (as operators from $H_\alpha$ to $H_{\alpha/(1+|k|)}$), it follows from (5.4) that

$$(I+M)\zeta_t=-\zeta_x-\zeta^p\zeta_x, \tag{5.5}$$

at least as an equation relating distributions in $H_\beta$, where $\beta$ is defined above. Further examination of (5.4) reveals that $\zeta_t$ lies in $C(0,T;H_\beta)$. The right-hand side of (5.4) may therefore be differentiated with respect to $t$, and so $\zeta_{tt}$ is seen to exist and is given by

$$\zeta_{tt}=-Q(\zeta_t+\zeta^p\zeta_t).$$

Continuing this argument inductively leads to the conclusion that $\zeta\in C^\infty(0,T;H_\beta)$. Plainly (5.1) implies that $\zeta(\cdot,t)\to g$ as $t\to 0$.

Because the solution $\zeta$ above was obtained by use of the contraction-mapping principle, there is automatically implied uniqueness of the solution, at least within the ball $B_R$. It is worth note that uniqueness can be established in the large, and not just locally. Thus one may assert that for any value of $T$, there is at most one solution in $C(0,T;H^s)$ of the integral equation (5.1), and so there is at most one solution of the initial-value problem (2.6) corresponding to $g$. To see this, note that for small values of $t$, $\zeta$ is unique by virtue of the uniqueness aspect of the contraction-mapping theorem. Let $\zeta_1$ and $\zeta_2$ be two solutions and suppose them to be equal for $0\leq t\leq t_0$, but on no interval $[0,t_0+\epsilon]$ where $\epsilon>0$ is sufficiently small do they agree identically.

Rewrite the equation (5.1) as

$$\begin{aligned}\zeta(x,t) &= g(x) - \int_0^{t_0} Q[\zeta(\cdot,\tau) + \frac{1}{p+1}\zeta^{p+1}(\cdot,\tau)]d\tau \\ &\quad - \int_{t_0}^{t} Q[\zeta(\cdot,\tau) + \frac{1}{p+1}\zeta^{p+1}(\cdot,\tau)]d\tau \\ &= G(x) - \int_{t_0}^{t} Q[\zeta(\cdot,\tau) + \frac{1}{p+1}\zeta^{p+1}(\cdot,\tau)]d\tau \\ &= \tilde{A}(\zeta),\end{aligned}$$

say. Our assumption implies that the integral equation $\zeta = \tilde{A}(\zeta)$ has two distinct solutions on any time interval $[t_0, t_0+\epsilon]$ for small enough $\epsilon$. However, by choosing $R$ large enough and $\epsilon$ small enough, we can assure that the mapping $\zeta \mapsto \tilde{A}(\zeta)$ maps the ball $B_R$ of radius $R$ centered at the origin in $C(t_0, t_0+\epsilon; H)$ into itself and is contractive there, where $H$ is as in the statement and proof of the lemma. Indeed, the argument is the same as that exposed above. However, by choosing $R$ yet larger, both $\zeta_1$ and $\zeta_2$ restricted to the time interval $[t_0, t_0+\epsilon]$ will lie in $B_R$, and thus a contradiction is reached.

Having established that the initial-value problem (2.6) possesses a solution corresponding to any $g \in H_\alpha$, at least for some time interval $[0, T_1]$, say, attention is now given to extending the interval of existence. The contraction-mapping argument may be used again starting with $\zeta(\cdot, T_1)$ as initial data. This will yield a solution of (2.6) on a time interval $[0, T_2]$ where $T_2 > T_1$. Continuing this line of argument inductively leads to an increasing sequence $\{T_k\}_{k=1}^{\infty}$ of times such that a solution of (2.6) exists on the time interval $[0, T_k]$ for all $k = 1, 2, \ldots$. Moreover, on each temporal interval $[T_k, T_{k+1}]$, $k = 1, 2, \ldots$, the solution $\zeta$ is given as the fixed point of an integral equation like (5.1) by virtue of the contraction-mapping theorem. Two possibilities now arise, either $T_0 = \lim_{k\to\infty} T_k$ is finite or it is $+\infty$. If $T_0 = +\infty$, then the solution of (2.6) is global in time, whereas if $T_0 < +\infty$, then

$$\limsup_{t\to T_0} \|\zeta(\cdot, t)\|_H = +\infty. \tag{5.6}$$

Otherwise, if $K$ is a finite upper bound for $\|\zeta(\cdot, t)\|_H$ on $[0, T_0)$, then the local existence theory obtained by the contraction-mapping theorem as above, when applied with initial data $\zeta(\cdot, t_0)$ where $t_0 \in [0, T_0)$, will always extend the solution by at least $\frac{1}{2}(1 + (2K)^p)^{-1}C^{-1}$ where $C = C_p$ is the constant appearing on the right-hand side of (5.3). In consequence of this lower bound, it follows that in a finite number of iterations of the contraction-mapping argument, the solution will have been defined on a temporal interval $[0, T_0+\epsilon]$ where $\epsilon > 0$. This contradiction forces the validity of (5.6). It follows from these arguments that

$$T_0 = \sup\{T : \exists \text{ a solution } \zeta \text{ of (2.6) in } C(0, T; H)\},$$

and that the solution $\zeta$ can certainly be extended over any time interval $[0, T]$ for which one has an *a priori* deduced bound on the norm of $\zeta$ in $H$.

Use will be made of the last-mentioned conclusion to verify the statement concerning the maximal interval of existence. To this end, consider again equation (5.4) and take its inner product with $\zeta$ in the Hilbert space $H$. Because $Q$ is a bounded operator on $H$ and because $H$ is an algebra, it is readily deduced from the result of following the above prescription that

$$\frac{d}{dt}\|\zeta(\cdot,t)\|_H^2 \leq C_1\|\zeta(\cdot,t)\|_H^2 + C_2\|\zeta(\cdot,t)\|_H^{p+2}.$$

Integrating the differential equation obtained by demanding equality in the last inequality leads to the upper bound

$$\|\zeta(\cdot,t)\|_H^2 \leq \frac{e^{C_1 t}\|g\|_H^2}{\left[1 - \|g\|_H^p \frac{C_2}{C_1}(e^{C_1 pt/2} - 1)\right]^{2/p}},$$

and therefore it is concluded that

$$T_0 \geq \frac{2}{pC_1}\log\left(1 + \frac{C_1}{C_2}\|g\|_H^{-p}\right).$$

Combining this bound with the result from the last paragraph leads to the stated conclusion about the maximal time of existence.

The continuous-dependence results follow from the fact that a solution $\zeta$ is obtained at least locally in time by iterating the operator $A$ on any function in $B_R$. More precisely, let $R > 0$ be given and let $T_1$ be determined by the relation $CT_1(1 + R^p) = \frac{1}{2}$, where $C$ is the constant appearing in the last line of (5.3). Let $g_1,\ g_2 \in H$ and suppose $\|g_1\|,\ \|g_2\| \leq \frac{1}{2}R$. Define $A_1$ and $A_2$ to be the operators given by the right-hand side of (5.1) with $g$ replaced by $g_1$ and $g_2$, respectively. Then if we view $g_1$ and $g_2$ as elements of $C(0, T_1; H)$ which are constant in time, the contraction-mapping principle assures that as $n$ tends to infinity, $A_i^n g_i$ converges in $C(0, T_1; H)$ to the solution $\zeta_i$ of (5.1) with $g$ replaced by $g_i$, $i = 1, 2$, where $A_i^n$ is the $n$th iterate of $A_i$. The inequality (5.3) with $CT_1(1 + R^p) = \frac{1}{2}$, and the triangle inequality quickly lead to the estimate

$$\|A_1^n g_1 - A_2^n g_2\|_{C(0,T_1;H)} \leq (1 + \tfrac{1}{2} + \cdots + \tfrac{1}{2^n})\|g_1 - g_2\|_H,$$

which holds for all $n$. Thus if $\zeta_1$ and $\zeta_2$ are the two solutions of (2.6) corresponding to $g_1$ and $q_2$, respectively, then

$$\|\zeta_1 - \zeta_2\|_{C(0,T_1,H)} \leq 2\|g_1 - g_2\|_H.$$

Thus the solution depends continuously on the data at least on $[0, T_1]$. In particular, $\|\zeta_1(\cdot, T_1) - \zeta_2(\cdot, T_1)\|_H \leq 2\|g_1 - g_2\|$. Making the same argument starting with data $\zeta_1(\cdot, T_1)$ and $\zeta_2(\cdot, T_2)$ rather than $g_1$ and $g_2$ leads to the conclusion that the solution depends continuously on the data on the time interval $[0, T_2]$. Continuing in this manner leads to the desired conclusion.

As for the final statement in the Lemma, suppose $\mu > 1$, $g \in H_\gamma$ and that there is a solution $\zeta$ of (5.1) in $C(0,T;H_\alpha)$. It should follow that $\zeta \in C(0,T;H_\gamma)$. Consider the set $\Gamma$ of all weights $\beta$ such that $\alpha(k) \leq \beta(k) \leq \gamma(k)$ for all $k$ and which are such that $\zeta \in C(0,T;H_\beta)$. The set $\Gamma$ is not empty since $\alpha \in \Gamma$. If $\beta \in \Gamma$, and $H_\beta \not\supseteq H_\gamma$, then direct consideration of the integral equation (5.1) and the operator $Q$ assures that the right-hand side of (5.1) lies in $C(0,T;H_\omega)$ where $\omega(k) = \min\{\gamma(k), \beta(k)(1+|k|^{\mu-1})\}$. Hence, $\zeta$ itself belongs to $C(0,T;H_\omega)$. Iterating this argument a finite number of times leads to the stated conclusion since $\gamma$ has polynomial growth at infinity. □

*Remarks.* The continuous dependence of the solution on the initial data extends to the temporal derivatives. Thus for any positive integer $j$, the mapping that associates to data $g$ the $j$th temporal derivative $\partial_t^j \zeta$ is continuous from $H_\alpha$ to $C(0,T;H_\beta)$, where $\beta$ is defined above. Indeed, the solution $\zeta$ of the initial-value problem (2.6) can easily be shown to depend analytically on its temporal variable $t$.

Consider the special case of the spaces $H_\alpha$ in which $\alpha = m$, the symbol of the operator $M$ appearing in (1.3). It is a simple consequence of Plancherel's theory that, on account of (2.1), if $f \in H^\infty$, then

$$\int_{-\infty}^{\infty} [f^2(x) + f(x)Mf(x)]dx = c\|f\|_{H_m}^2 \tag{5.7}$$

for some constant $c$ depending only upon the particular normalization given to the Fourier transform. It follows by taking limits that the relation (5.7) holds for $f \in H_m$, provided the second term on the left-hand side is interpreted as $< f, Mf >_m$ where the brackets connote the pairing between $H_m$ and its dual space $H_m^*$. The space $H_m$ intervenes naturally in the following energy-type estimates.

**Lemma 4.** *Suppose $\mu \geq 1$ and that $\zeta$ is a solution that lies in $C(0,T;H_m \cap H^s)$ of the initial-value problem (2.6) with initial data $g$, where $s > \frac{1}{2}$. Then for each $t$ in* $[0,T]$,

$$\|\zeta(\cdot,t)\|_{H_m} = \|g\|_{H_m}. \tag{5.8}$$

*If, in addition, $\zeta_x \in C(0,T;H_m)$, then for each $t$ in $[0,T]$ we also have*

$$\|\zeta_x(\cdot,t)\|_{H_m}^2 = \|g'\|_{H_m}^2 - p\int_0^t \int_{-\infty}^{\infty} \zeta^{p-1}\zeta_x^3\,dx\,ds. \tag{5.9}$$

*Remark.* If $\mu > 1$, then $H_m \subset H^{\mu/2}$, so the restriction that $\zeta$ lie in $C(0,T;H_m \cap H^s)$ for some $s > \frac{1}{2}$ only amounts to the requirement that $\zeta$ lie in $C(0,T;H_m)$.

*Proof.* Both of these relations follow readily for smooth solutions. Formula (5.8) follows from multiplying the evolution equation in (2.6) by $\zeta$, integrating with respect to the spatial variable over $\mathbb{R}$ and with respect to the temporal variable over $[0,t]$, and noting that

$$\int_{-\infty}^{\infty} \zeta(\zeta_x + \zeta^p \zeta_x)dx = 0.$$

Formula (5.9) follows via the same route after multiplying by $\zeta_{xx}$ and integrating by parts suitably. Having established these formulas for smooth solutions, the stated results then follow from a standard limiting procedure and the aforementioned continuous-dependence results. □

We are now prepared to prove Theorem 2.

*Proof.* Let $g$ be fixed data satisfying the hypotheses of the theorem, and as before, let

$$T_0 = \sup\{T : \exists \text{ a solution } \zeta \text{ of (2.6) in } C(0, T; H)\}$$

where $H = H_m$ if $\mu > 1$ and $H = H_m \cap H^s$ if $\mu = 1$. In either case, we certainly know that $T_0 > 0$ by virtue of Lemma 3. Moreover, as remarked earlier, to establish the theorem it suffices to deduce that $T_0 = +\infty$, and for this it is sufficient to show *a priori* that $\|\zeta(\cdot, t)\|_H$ is bounded on bounded time intervals.

First consider the case wherein $\mu > 1$. In this case, $H = H_m \subset H^{\mu/2}$ satisfies the conditions for the direct applicability of the local existence theory. Moreover, by Lemma 4, $\|\zeta(\cdot, t)\|_H$ does not depend on $t$. Hence in this case the desired result follows easily (and is independent of the exponent $p$).

The case $\mu = 1$ is less obvious because the local existence theory does not apply to $H_m$. Instead, we let $H = H_\alpha$ where $\alpha(k) = k^2 m(k)$. The local existence theory certainly applies in this space, and we are left to find bounds on the norm of $\zeta$ in this space. Because of (5.7), a bound on $f$ in the space $H_\alpha$ is equivalent to bounding $\|\zeta_x(\cdot, t)\|_{H_m}$, and because of (5.9), the latter will be provided as soon as the last term on the right-hand side of (5.9) is seen to be under control.

Attention is now turned to this latter task. First, since (5.8) holds, it is inferred that $\|\zeta(\cdot, t)\|_{1/2}$ is bounded, independently of $t$ since $\mu \geq 1$. With this remark in hand, proceed as follows:

$$\begin{aligned}
\left|\int_{-\infty}^{\infty} \zeta^{p-1}\zeta_x^3 dx\right| &\leq |\zeta(\cdot, t)|_\infty^{p-1} |\zeta_x(\cdot, t)|_3^3 \\
&\leq C|\zeta|_\infty^{p-1}\|\zeta_x\|_{1/6}^3 \\
&\leq C|\zeta|_\infty^{p-1}\|\zeta\|_{7/6}^3 \\
&\leq C|\zeta|_\infty^{p-1}\|\zeta\|_{3/2}^2\|\zeta\|_{1/2} \\
&\leq C\left[\|\zeta\|_{1/2}(1 + \log(1 + \|\zeta\|_{3/2}))^{1/2}\right]^{p-1} \|\zeta\|_{3/2}^2\|\zeta\|_{1/2},
\end{aligned}$$

where in the second step use was made of a standard Sobolev imbedding theorem, the fourth step is a standard interpolation, while the last step makes use of an inequality of Brezis and Wainger (1980). Since $\|\zeta(\cdot, t)\|_{1/2}$ is bounded, independently of $t$, this quantity may be absorbed into the constant $c$, thus leaving the upper bound

$$\|\zeta_x(\cdot,t)\|_{1/2}^2 \le \|\zeta_x(\cdot,t)\|_{H_m}^2$$
$$\le C_1 + C_2\int_0^t \|\zeta(\cdot,s)\|_{3/2}^2 + \|\zeta(\cdot,s)\|_{3/2}^2 \log(1+\|\zeta(\cdot,s)\|_{3/2})^{(p-1)/2} ds.$$

The left-hand side of the last inequality is equivalent to the $H^{3/2}$-seminorm, and because the $L_2$-norm is known to be bounded independently of $t$, it is concluded that this inequality amounts to

$$y^2(t) \le C_1 + C_2\int_0^t \{y^2(s) + y^2(s)[\log(y(s))]^{\frac{p-1}{2}}\}ds,$$

where $y(t) = 1 + \|\zeta(\cdot,t)\|_{3/2}$. A Gronwall-type argument then assures that $y(t)$ is finite on bounded time intervals provided that $(p-1)/2 \le 1$, which amounts to the restriction $p \le 3$.

With this bound in hand, the desired result follows from our previous remarks. □

As for Theorem 1, these results are essentially all contained in the work of Abdelouhab et al. (1989) (see also Saut 1979 and Felland 1991). The local theory of existence and the continuous dependence can be obtained directly from the semigroup theory of Kato (1975, 1983). The global theory depends on *a priori* bounds. As just explained with reference to the initial-value problem (2.6), the length of the time interval over which the local theory guarantees existence depends upon $\|g\|_s$, where $s > \frac{3}{2}$. Thus, arguing as before, if it can be shown that solutions of (2.5) remain bounded in $H^s$ at least on any time interval of finite extent, it will follow that such solutions can be globally extended.

We content ourselves with an indication of how the aforementioned bounds are obtained. The details may be carried out exactly as indicated above, and we may suitably pass over them here. First, because $M$ is a self-adjoint operator, it follows readily that if $\eta$ is a smooth solution of (2.5), then

$$\int_{-\infty}^{\infty} \eta^2(x,t)dx = \int_{-\infty}^{\infty} g^2(x)dx, \tag{5.11}$$

so the $L_2$-norm is seen to be an invariant of the motion associated with the evolution equation. A somewhat more elaborate calculation reveals another invariant, namely

$$\int_{-\infty}^{\infty} (\tfrac{1}{2}\eta M\eta - \tfrac{1}{p+2}\eta^{p+2})dx = \int_{-\infty}^{\infty} (\tfrac{1}{2}gMg - \tfrac{1}{p+2}g^{p+2})dx. \tag{5.12}$$

(Naturally, it must be assumed that $g \in H_m$ in order that the right-hand side of (5.12) be finite). It then follows from (5.11), (5.12), and standard Sobolev-space estimates that if $p < 2\mu$, if $p = 2\mu$, and $|g|_2$ is not too large or if $p > 2\mu$ and $\|g\|_{\mu/2}$ is small enough, then $\eta$ remains bounded in $H^{2\mu}$ on bounded time intervals with a bound that depends only on $\|g\|_{2\mu}$.

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