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Uniqueness and Global Markov Property for Euclidean Fields: The Case of General Polynomial Interactions

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Abstract. We give a general method for proving uniqueness and global Markov property for Euclidean quantum fields. The method is based on uniform continuity of local specifications (proved by using potential theoretical tools) and exploitation of a suitable FKG-order structure. We apply this method to give a proof of uniqueness and global Markov property for the Gibbs states and to study extremality of Gibbs states also in the case of non-uniqueness. In particular we prove extremality for φ_2^4 (also in the case of non-uniqueness), and global Markov property for weak coupling φ_2^4 (which solves a long-standing problem). Uniqueness and extremality holds also at any point of differentiability of the pressure with respect to the external magnetic field.

1. Introduction

Among stochastic processes indexed by time t those with the Markov property, and in particular diffusion processes, play a fundamental role, see e.g. [DeMe, RogWi]. The search for a suitable extension of the Markov property and of Markov stochastic (diffusion) processes to the case where the one-dimensional time index set is replaced by a multidimensional indexing set has been of constant interest to probabilists. One direction in which such extensions has been looked for starts with work in 1945 by P. Lévy, and has been investigating fields with continuous realizations like homogeneous extensions of Brownian motion (Brownian sheet, Yeh-Wiener process), fields with independent increments, multiindices martingales, see e.g., for recent work and references [Ro 1–3, NuZ, Ru].

For application in physics, in particular quantum field theory, random fields which are homogeneous (stationary) with respect to symmetries of the indexing set (typically $\mathbb{R}^d$ with symmetry group the euclidean group) are particularly important. It turns out that to combine Markov property and homogeneity requirements generalized random fields (i.e. random fields with realizations which are generalized functions), rather than ordinary random fields, have to be considered, cf. [Mo, Nel2,4, Wo], (in which Gaussian Markov generalized fields are discussed). The usefulness of a Markov property for generalized random fields was briefly pointed out by K. Symanzik, but it was after fundamental work by Nelson [Nel 1–4] that the importance of Markov fields in the context of (constructive) quantum field theory became clear. In particular it was pointed out that essentially the Markov property together with homogeneity properties (Euclidean invariance) permit to construct relativistic fields from Euclidean fields. Guerra [Gu] gave the first striking convincing applications of such ideas to control the infinite volume limit of the pressure in $P(\varphi)_2$ models. Since then the basic work by Guerra, Rosen, Simon and others use the Markov property of free fields and interacting fields with a space time cut-off, see e.g. [Sim] and references therein.

Whereas the Markov property of free quantum fields was well understood and studied (see e.g. [Nel2–4, AHK2–4] and, especially for connections with potential theory and Brownian motion, [Dy, Röl,2, Kol]) as well as the one for space-time cut-off quantum fields [Nel 1–4], the Markov property for the non-Gaussian generalized random fields of Euclidean quantum field theory in the infinite volume limit remained unproven for many years until it was eventually proven in models with weak trigonometric interactions (Sine-Gordon model) in [AHK7] and with general exponential interactions [Gie 1, Ze 1]. Yet the case of polynomial interactions ($P(\varphi)_2$-model) remained open. The fact that the Markov property (in the global sense, made clear by work of [New1, AHK6–9,14, Fö1, Rö1,2] or at least with respect to half spaces) yields the cyclicity of time zero fields (in fact is equivalent with this) and hence a Schrödinger representation for quantum fields and a canonical formalism (in the original sense of [HePa, Ar]), was made clear by the work [AHK2–5,14,15, AHKR, AK, Her, Kl1,2]. In particular quantum fields with the global Markov property turn out to have generators described (at least on a dense domain) by infinite dimensional Dirichlet forms, and being connected then in this sense with infinite dimensional diffusion fields, for which there is presently a well developed mathematical theory, see [AHK2–5,14,15, Ku1, AR1,2] and references therein.

Despite the importance of these connections the difficulty of proving the Markov property made it necessary to find substitutes of it which, although, weaker, were sufficient to permit a passage from homogeneous (Euclidean) quantum fields to relativistic quantum fields. Such a substitute was found by Osterwalder and Schrader [OsSch 1,2], see also [Gl], and further discussed e.g. in [Heg, Kl1,2, Chal, Ac, Ok, Ku1], also in its relations with the Markov property.

The Osterwalder-Schrader property (also called T-property or reflection positivity property) have been verified in all constructed models. However, as mentioned, the Markov property is much stronger and it alone fully justifies e.g. the canonical Schrödinger representation. For this reason the problem of proving it remained a very important problem of quantum field theory and the theory of random fields. One of the purposes of the present paper is precisely to provide the first proof of the Markov property for the φ_2^4-model. By this we have also that the φ_2^4 models satisifies Nelson's axioms for quantum fields [Ne1–4, Sim]. Let us now discuss other topics of concern in our paper.

Whereas the global Markov property discussed above is difficult to prove, the local Markov property holds for all constructed fields and in fact it is at the basis of the construction of Gibbs states starting from quantum fields given first in a bounded region of space-time. This construction is a "multi-time" analogue of the Kolmogorov's construction of Markov processes from Markov kernels, the basic analogue of Markov (transition) kernels being the local specifications studied e. g. in [Do1, LaRu], (which handle in particular the case of statistical mechanics). See also, for a more general setting and the abstract study of the related Martin-Dynkin boundary [Fö2, Pr]. Gibbs states have been discussed in quantum field theory in [GRS1,2, DoMi, FrSi, AHK5,7], see also [Sim, GlJa] and references therein. Far reaching connections of the concept of specifications and Gibbs states with potential theory have been studied in [AHK3, Rö2,3, Ze2]. The construction of particular Gibbs states has been achieved in polynomial interactions, see [GlJa] and references therein, trigonometric [FrSe] and exponential interactions [AHK1, FrPa], see also [AHK15] and references therein. The question of the structure of the space of Gibbs states to a given interaction (specification), an extension of the one dimensional problem of constructing Martin-Dynkin-boundaries (cf. [Fö2, BlP, NgZ]) is of great interest.

Uniqueness results have been given in statistical mechanics [Do, Fö1, AHKO, Pr, Geo] (and references therein) and quantum field theory, for weak coupling trigonometric interactions [AHK7] and general exponential interactions [Ze1], see also [Gie1]. In the present paper we present a general condition for uniqueness for $P(\varphi)_2$ models and as an example we apply it to the φ^4 interaction. A related result is also discussed in [Gie3]. The case of weak $P(\varphi)_2$ models is solved in another paper of ours [AHKZ].

Weaker uniqueness results in sense of independence of classical boundary conditions for thermodynamic functions or Gibbs states are contained in [GRS1,2, GlJa, FrSi]. Structural results for the space of Gibbs states in the case of non-uniqueness have been obtained in classical statistical mechanics of lattice systems (following ideas by Dobrushin, Minlos-Sinai, Gercik) by Pirogov-Sinai [Sin] (and references therein). In specific models of classical lattice statistical mechanics, complete structural results are known, [Aiz, Hig], see also [Me]. The extension of Pirogov-Sinai results to the study of phase transitions in quantum field theory have been given in [Im2] (see also [GlJa, FrSi] and references therein). Structural results on the complete space of Gibbs states have been given in the case of free fields in [HoSt] (see also [Rö3]) and in the case of trigonometric and exponential interactions in [Ze4].

In the present paper we introduce an FKG-order in the set of Gibbs states for quantum fields, analogous to the well known FKG-order for lattice systems (cf. [Pr, BeHK, Sim]). In particular we consider FKG-maximal states and prove their extremality for some $P(\varphi)_2$ models.

As to the (global) Markov property the first proofs in classical statistical mechanics of lattice systems were obtained in the case where one has uniqueness of Gibbs states [AHK6,7, AHKO, Fö1, BePi, Go], extensions to the case of non-uniqueness for FKG-maximal states were given in [Fö1, Go, Ze5]. For other models of statistical mechanics see [Wi]. For a proof that all Pirogov-Sinai states in classical lattice statistical mechanics have the global Markov property see [Ze3]. For counterexamples to the conjecture that all extremal states have the global Markov property see [Ke1, Isr] (see also [Ke2, AFHKL] for further discussions of equivalence conditions of the global Markov property and criteria for it, also in relation to extremal Gibbs states).

We already mentioned above results on the Markov property of Gibbs states in quantum field theory (free fields, fields in a finite space-time volume with general interaction, fields over $\mathbb{R}^2$ for trigonometric and exponential interactions). In the present paper we give the first proof that also the weak coupling φ_2^4 interaction has the global Markov property. Our method is based on a combination of methods to establish uniform continuity of local specifications and the exploitation of the FKG-order we introduced. It should have applications also to other models. In a companion paper we prove the uniqueness of Gibbs measures for weak coupling $P(\varphi)_2$ models using cluster expansion [AHKZ].

The structure of the paper is as follows:
In Sect. 2 we introduce the basic space of regular probability measures, in which later on we will introduce our Gibbs structure. We also recall basic results on the Dirichlet boundary value problem with distributional data which will be used to represent the conditional expectations needed for discussing local specifications and the Markov property. A basic estimate on the solution of the above Dirichlet problem (of the "large deviation type") is given in Lemma 2.1.

In Sect. 3 we introduce the concept of local specifications and a concept of uniform continuity for them. Roughly speaking this expresses a weak dependence of the conditional expectations, associated with complements of bounded open sets, on boundary conditions near their boundary μ-a.s. with respect to a given regular probability measure, as well as a continuity property of sample paths of the field. In Theorem 3.1 we show that every local specification associated with quantum fields in two space-time dimensions is uniformly continuous a.s. with respect to Gibbs states of the specification.

In Sect. 4 we introduce an FKG-order for quantum fields on the lattice and in the continuum. For this we use the representation of specifications given by solutions of the Dirichlet problem for distributions mentioned above.

In Sect. 5 we discuss extremality in the set of Gibbs states. In particular we prove (Proposition 5.1) that uniform continuity of local specifications together with the convergence of the specification with boundary conditions "dominating at infinity" towards μ yields extremality of μ. This criterion is then applied to the φ_2^4-model to prove extremality of its Gibbs states, in particular the FKG-maximal Gibbs states are extremal (Propositions 5.3, 5.4). Extremality of Gibbs states for models with exponential respectively trigonometric interaction is also proven and uniqueness results are given (Theorem 5.6).

In Sect. 6 we give a general method for studying the global Markov property for quantum fields. This method is based on uniform continuity of local specifications for conditional Gibbs measures, together with control on the solutions of the Dirichlet boundary problem for distributions and the FKG-order for continuous fields which we introduced. We apply the method to prove the global Markov property for weak coupling symmetric φ_2^4 fields.

In Sect. 7 we give some remarks on the structure of the set of Gibbs measures. In Proposition 7.1 we show that certain Gibbs states $\mu^{\pm}$ are FKG-maximal and so extremal. In Proposition 7.2 we prove that at the points of differentiability of the pressure in $P(\varphi)_2$-models with respect to an extremal magnetic field there is a unique extremal Gibbs measure. We conclude the paper by formulating some expectations concerning the structure of Gibbs states for general $P(\varphi)_2$ models.

The main results of this paper have been announced at the Symposium of the Bernouilli Society in Rome (June 1988) and at the International IAMP Conference, Swansea (August 1988) (to appear in the Proceedings, eds. I. Davies, A. Truman, 1989).

2. Regular Probability Measures on $\mathscr{D}'(\mathbb{R}^2)$

Let $\mathscr{F}$ be a family of bounded open subsets Λ of $\mathbb{R}^2$ with piecewise $\mathscr{C}^1$-boundary $\partial\Lambda$. Let $\mathscr{F}_0 \equiv \{\Lambda_n \in \mathscr{F}, n \in \mathbb{N}\}$ be a countable base of $\mathscr{F}$, i.e. an increasing and absorbing (i.e. $\Lambda_n \subset \Lambda_{n+1}$ and for all $\Lambda \in \mathscr{F}$ there exists $n \in \mathbb{N}$ s.t. $\Lambda \subset \Lambda_n$) sequence of elements from $\mathscr{F}$. We shall always assume that $\mathscr{F}_0$ is a *Fisher sequence* in the sense of e.g. [Isr] i.e., as $n \to \infty$, $c_1 d(0, \partial\Lambda_n)^b \leqq |\partial\Lambda_n| \leqq c_2 d(0, \partial\Lambda_n)^b$ for some constants c_1, c_2, $b = d-1$, where $|\partial\Lambda|$ means the length of $\partial\Lambda$.

Let us introduce the Sobolev norm $\| \;\|_{-1}$ on the space of tempered distributions $f \in \mathscr{S}'(\mathbb{R}^2)$ with Fourier transforms which are functions $\hat{f}$ s.t. $\int\limits_{\mathbb{R}^2} \frac{|\hat{f}(q)|^2}{q^2 + m_0^2} dq < \infty$, by setting

$$\|f\|_{-1}^2 \equiv (2\pi)^{-2} \int\limits_{\mathbb{R}^2} \frac{|\hat{f}(q)|^2}{q^2 + m_0^2} dq , \tag{2.0}$$

with $m_0 > 0$ constant.

We call $H_{-1}(\mathbb{R}^d)$ the Sobolev space with this norm. Let $G \equiv (-\Delta + m_0^2)^{-1}$, with Δ the Laplacian on $\mathbb{R}^2$. Then

$$\|f\|_{-1}^2 = \int\limits_{\mathbb{R}^2} \int\limits_{\mathbb{R}^2} f(x) G(x-y) f(y) dx dy .$$

Let Σ denote the Borel σ-algebra in $\mathscr{D}' \equiv \mathscr{D}'(\mathbb{R}^2, \mathbb{R})$. Let $\varphi(\mathrm{f})$, $\mathrm{f} \in \mathscr{D}$ be the free Nelson Markov field, i.e. $\varphi(f)$ is the generalized Gaussian random field with mean zero and covariance

$$\mu_0(\varphi(f)\varphi(g)) = \langle f, g \rangle_{-1}$$

with $\langle f, g \rangle_{-1} \equiv \int\limits_{\mathbb{R}^2} \int\limits_{\mathbb{R}^2} f(x) G(x-y) g(y) dx dy$. $\varphi(f)$ is thus the coordinate map at f on $\mathscr{D}'$. We can look upon φ, cf. [AHK7], as a random field indexed by measures ϱ of finite energy i.e. such that $\varphi(\varrho)$ has mean zero and covariance $\langle \varrho, \varrho' \rangle_{-1} \equiv \int\limits_{\mathbb{R}^2} \int\limits_{\mathbb{R}^2} d\varrho(x) d\varrho'(y) G(x-y) < \infty$. Let $\Sigma(\Lambda)$ be the σ-algebra generated by $\varphi(\varrho)$ with $\operatorname{supp} \varrho \subset \Lambda$. Any function F on $(\mathscr{D}', \Sigma)$ which is $\Sigma(\Lambda)$-measurable for some $\Lambda \in \mathscr{F}$ is called a *local function*.

Definition 1. A probability measure μ on $(\mathscr{D}', \Sigma)$ is called *regular* if for some $p \geqq 2$, any $\Lambda_n \in \mathscr{F}$ there exists a constant $c_n > 0$ s.t.

$$\mu e^{\varphi(f)} \leqq e^{c_n(\|f\|_{-1} + \|f\|_{-1}^p)} \tag{2.1}$$

for any $f \in H_{-1}$ with supp $f \subset \Lambda_n$, and such that $d(\operatorname{supp} f, \partial\Lambda_n) \geqq 1$, where $d(A, B) \equiv \inf\{|x-y|, x \in A, y \in B\}$ is the distance of two sets A, B.

We call $\mathscr{M}_r$ the set of all regular probability measures. For any Σ-measurable function F and any probability measure μ on $(\mathscr{D}', \Sigma)$ we use the notation μF for the expectation $E(F)$ of F with respect to μ. We call p, c_n the *parameters of regularity* of μ.

Remark. The above definition of regularity is in fact independent of the chosen parameter m_0^2 (cf. [AHK 7]).

Remark. By the construction of 2-dimensional scalar quantum fields with exponential, trigonometric and polynomial interactions one obtains measures μ which are regular in the above sense. In fact all measures associated with such models satisfy the bound

$$\mu e^{\varphi(f)} \leqq \exp\left(a \|G * f\|_{L_1} + b \|f\|_{-1}^2 + c \|G * f\|_{L_p}^p\right), \tag{2.2}$$

for some constants $b > 0$, $a, c \geqq 0$ $p \geqq 4$ (where $\| \quad \|_{L_q}$ denotes the L_q-norm) (cf. [GlJa, AHK 1, 7, FrSi]).

It is shown in [AHKZ] that (2.2) implies the regularity bound (2.1) with c_n s.t. $c_n \leqq C|\Lambda_n|^{1/2}$ for some $C > 0$ (where $|\Lambda_n|$ denotes the volume of Λ_n).

Let $\Lambda \in \mathscr{F}$ and consider the harmonic measure (Poisson kernel) $\psi_{dz}^{\partial\Lambda}(x)$, for $z \in \partial\Lambda$, $x \in \Lambda$, i.e. the solution of $(-\Delta + m_0^2)\psi_{dz}^{\partial\Lambda}(x) = 0$ for $x \in \Lambda$ and $\psi_{dz}^{\partial\Lambda}(x) \to \delta_z(x')$ for $x \to x'$, $x' \in \partial\Lambda$.

It is possible to define $\psi_\eta^{\partial\Lambda}(x)$ for any $\eta \in \mathscr{D}'$, $x \in \Lambda$ in such a way that $x \to \psi_\eta^{\partial\Lambda}(x)$ is $-\Delta + m_0^2-$harmonic in Λ. Moreover there exists a μ-measure 1 subset $\Omega(\Lambda)$ of $\mathscr{D}'$ s.t. $\Omega(\Lambda) \in \Sigma(\partial\Lambda)$ and for $\eta \in \Omega(\Lambda)$, $\psi_\eta^{\partial\Lambda}(x)$ is the locally uniform limit in $x \in \Lambda$ of $\psi_{\eta_\kappa}^{\partial\Lambda}(x) \equiv \int_{\partial\Lambda} \psi_{dz}^{\partial\Lambda}(x) \langle \eta, h_\kappa(\cdot - z)\rangle$ for $h_\kappa \in \mathscr{C}_0^\infty(\mathbb{R}^2)$, $h_\kappa(x) \to \delta(x)$, $\kappa \to \infty$, $\eta_\kappa \equiv \eta * h_\kappa$. We set $\psi_\eta^{\partial\Lambda}(x) = 0$ for $\eta \in \mathscr{D}' - \Omega(\Lambda)$. We call $\psi_\eta^{\partial\Lambda}(x)$ the *solution of the Dirichlet boundary value problem in Λ with boundary condition η* (ref. [AHKZ, Rö 1, 2]).

It is useful to remark that $\psi_\eta^{\partial\Lambda}(\cdot)$ is also the $L^p(\mu \otimes dx, \mathscr{D}' \times \Lambda)$, $1 \leqq p < \infty$ limit of $\psi_{\eta_\kappa}^{\partial\Lambda}(\cdot)$ as $\kappa \to \infty$. Define for $\Lambda \in \mathscr{F}$, $|\Lambda| > 1$, $0 < \varepsilon < 1$:

$$\Lambda_\varepsilon \equiv \{x \in \mathbb{R}^d \mid d(x, \Lambda) \leqq \varepsilon\} .$$

For any $\omega \in C^\infty(\mathbb{R}^2)$, $\omega \geqq 0$ we shall set $\omega_{\partial\Lambda} \equiv \inf_{x \in \partial\Lambda} \omega(x)$.

We then have the basic estimate

Lemma 2.1. *Let $\omega \in C^\infty(\mathbb{R}^2)$. Then for any regular measure μ on $(\mathscr{D}', \Sigma)$ and any $\Lambda \in \mathscr{F}_0$ we have*

$$\mu\left\{\eta \in \mathscr{D}' \,\middle|\, \sup_{x \in \partial\Lambda} |\psi_\eta^{\partial\Lambda_\varepsilon}(x)| \geqq \omega_{\partial\Lambda}\right\} \leqq \exp\left\{-\frac{1}{a}\,\omega_{\partial\Lambda} + bc(\Lambda)|\partial\Lambda|^{p/2}\varepsilon^{-\frac{3}{2}p}\right\},$$

for some constants $a, b>0$ (independent of Λ, ε) and $c(\Lambda)$ with $c(\Lambda)=c_n$, c_n and $p\geqq 2$ being as in the Definition 1 of regularity.

Proof. Let $\chi\in C_0^\infty(\mathbb{R})$, $1\geqq\chi\geqq 0$, $\chi(x)=\begin{cases}1 & \text{for } |x|<\frac{1}{4}\\ 0 & \text{for } |x|>\frac{1}{2}\end{cases}$.
Define

$$\chi_{\varepsilon,\Lambda}(x)\equiv\chi\left(\frac{1}{\varepsilon}\,d(x,\partial\Lambda)\right). \tag{2.3}$$

Then we have [using Sobolev inequalities and the fact that $\chi_{\varepsilon,\Lambda}(x)=0$ for $d(x,\partial\Lambda)>\frac{1}{2}\varepsilon$]:

$$\begin{aligned}\sup_{x\in\partial\Lambda}|\psi_\eta^{\partial\Lambda_\varepsilon}(x)|&\leqq\sup_{x\in\mathbb{R}^2}|\chi_{\varepsilon,\Lambda}(x)\psi_\eta^{\partial\Lambda_\varepsilon}(x)|\\ &\leqq a\left\|(-\Delta+m_0^2)\chi_{\varepsilon,\Lambda}\psi_\eta^{\partial\Lambda_\varepsilon}\right\|_{L_2}\\ &\equiv a\left\|\chi_{\varepsilon,\Lambda}\psi_\eta^{\partial\Lambda_\varepsilon}\right\|_{+2}\end{aligned} \tag{2.4}$$

with a numerical constant $a>0$. Hence

$$\begin{aligned}P(\omega,\Lambda,\varepsilon)&\equiv\mu\left\{\eta\in\mathcal{D}':\sup_{x\in\partial\Lambda}|\psi_\eta^{\partial\Lambda_\varepsilon}(x)|\geqq\omega_{\partial\Lambda}\right\}\\ &\leqq\mu\left\{\eta\in\mathcal{D}':\left\|\chi_{\varepsilon,\Lambda}\psi_\eta^{\partial\Lambda_\varepsilon}\right\|_{+2}\geqq\frac{1}{a}\,\omega_{\partial\Lambda}\right\}\\ &=\mu\left\{\eta\in\mathcal{D}':\exp\left\|\chi_{\varepsilon,\Lambda}\psi_\eta^{\partial\Lambda_\varepsilon}\right\|_{+2}\geqq\exp\frac{\omega_{\partial\Lambda}}{a}\right\}.\end{aligned} \tag{2.5}$$

So using Tchebyshev's inequality we get

$$P(\omega,\Lambda,\varepsilon)\leqq\exp\left(-\frac{\omega_{\partial\Lambda}}{a}\right)\mu(\exp\left\|\chi_{\varepsilon,\Lambda}\psi_\eta^{\partial\Lambda_\varepsilon}\right\|_{+2}). \tag{2.6}$$

Let us introduce the notation:

$$\begin{aligned}H_{-1}\ni J_\cdot^{\varepsilon,\Lambda}(x)&\equiv(-\Delta+m_0^2)\chi_{\varepsilon,\Lambda}(x)\psi_\cdot^{\partial\Lambda_\varepsilon}(x)\\ &=(-\Delta\chi_{\varepsilon,\Lambda}(x))\psi_\cdot^{\partial\Lambda_\varepsilon}(x)-2\nabla\chi_{\varepsilon,\Lambda}(x)\cdot\nabla\psi_\cdot^{\partial\Lambda_\varepsilon}(x)\end{aligned} \tag{2.7}$$

(for all $x\in\mathbb{R}^2$, where we used harmonicity of $\psi_\cdot^{\partial\Lambda_\varepsilon}$) and denote by $J_\eta^{\varepsilon,\Lambda}(x)$ the corresponding random variable.

Then we have by Hölder inequality

$$\begin{aligned}\mu\left\|\chi_{\varepsilon,\Lambda}\psi_\eta^{\partial\Lambda_\varepsilon}\right\|_{+2}^{2n}&=\mu\left\|J_\eta^{\varepsilon,\Lambda}(\cdot)\right\|_{L_2}^{2n}\\ &=\int_{(\operatorname{supp}\chi_{\varepsilon,\Lambda})^n}\prod_{i=1}^n d_2x_i\,\mu\left(\left(\prod_{i=1}^n J_\eta^{\varepsilon,\Lambda}(x_i)\right)^2\right)\\ &\leqq|\operatorname{supp}\chi_{\varepsilon,\Lambda}|^{n-1}\int_{\operatorname{supp}\chi_{\varepsilon,\Lambda}}d_2x\,\mu((J_\eta^{\varepsilon,\Lambda}(x))^{2n}).\end{aligned} \tag{2.8}$$

From this using regularity of μ we obtain

$$\begin{aligned}\mu(\exp \|\chi_{\varepsilon,\Lambda}\psi_\eta^{\partial\Lambda_\varepsilon}\|_{+2}) &\leqq 2\mu ch \|J_\eta^{\varepsilon,\Lambda}(\cdot)\|_{L_2} \\ &\leqq 2|\operatorname{supp}\chi_{\varepsilon,\Lambda}|^{-1} \int_{\operatorname{supp}\chi_{\varepsilon,\Lambda}} d_2x \exp(c(\Lambda)[|\operatorname{supp}\chi_{\varepsilon,\Lambda}|^{1/2}\|J_\cdot^{\varepsilon,\Lambda}(x)\|_{-1} \\ &\quad + |\operatorname{supp}\chi_{\varepsilon,\Lambda}|^{p/2}\|J_\cdot^{\varepsilon,\Lambda}(x)\|_{-1}^p]) \,. \end{aligned} \tag{2.9}$$

From this we get, estimating the volume of $\operatorname{supp}\chi_{\varepsilon,\Lambda}$:

$$\begin{aligned}\mu(\exp \|\chi_{\varepsilon,\Lambda}\psi_\eta^{\partial\Lambda_\varepsilon}\|_{+2}) \leqq 2\exp(c(\Lambda)(2|\partial\Lambda|)^{p/2} \sup_{d(x,\partial\Lambda)<\frac{\varepsilon}{2}} [\varepsilon^{1/2}\|J_\cdot^{\varepsilon,\Lambda}(x)\|_{-1} \\ + \varepsilon^{p/2}\|J_\cdot^{\varepsilon,\Lambda}(x)\|_{-1}^p]) \end{aligned} \tag{2.10}$$

if $|\partial\Lambda| \geqq 1$ (since we replaced $|\partial\Lambda|$ by $|\partial\Lambda|^{p/2}$. Now we have

$$\|J_\cdot^{\varepsilon,\Lambda}(x)_{-1} \leqq |\Delta\chi_{\varepsilon,\Lambda}(x)|\,\|\psi_\cdot^{\partial\Lambda_\varepsilon}(x)\|_{-1} + 4|\nabla_x\chi_{\varepsilon,\Lambda}(x)|\,|\|\nabla_x\psi_\cdot^{\partial\Lambda_\varepsilon}(x)\|_{-1}| \,. \tag{2.11}$$

By definition of $\chi_{\varepsilon,\Lambda}$ in (2.3) we have

$$|\Delta\chi_{\varepsilon,\Lambda}(x)| \leqq c\varepsilon^{-2}\,, \tag{2.12a}$$

$$|\nabla_i\chi_{\varepsilon,\Lambda}(x)| \leqq c\varepsilon^{-1}\,, \tag{2.12b}$$

with some constant $c>0$ (independent of $\Lambda\in\mathscr{F}_0$ and $\varepsilon>0$ $\left(\text{with } \nabla_i \equiv \frac{\partial}{\partial x_i} \equiv \partial_i\right)$.
We have also, using the fact that the singularity of $K^{\partial\Lambda_\varepsilon}(x,x)$ is logarithmic:

$$\sup_{d(x,\partial\Lambda)<\frac{\varepsilon}{2}} \|\psi_\cdot^{\partial\Lambda_\varepsilon}(x)\|_{-1} = \sup_{d(x,\partial\Lambda)<\frac{\varepsilon}{2}} (K^{\partial\Lambda_\varepsilon}(x,x))^{1/2} \leqq c|\ln_+\varepsilon|^{1/2} \tag{2.13a}$$

(where $K^{\partial\Lambda_\varepsilon} = G - G^{\partial\Lambda_\varepsilon}$ and $G^{\partial\Lambda_\varepsilon} = (-\Delta_{\partial\Lambda_\varepsilon} + m_0^2)^{-1}$, $\Delta_{\partial\Lambda_\varepsilon}$ being the Laplacian with Dirichlet boundary conditions on $\partial\Lambda_\varepsilon$, $\ln_+\varepsilon \equiv \max(|\ln\varepsilon|, 1)$), again with some $c>0$ (independent of $\Lambda\in\mathscr{F}_0$) and $0<\varepsilon<1$. Moreover

$$\begin{aligned}\sup_{d(x,\partial\Lambda)<\frac{\varepsilon}{2}} \|\nabla_{i,x}\psi_\cdot^{\partial\Lambda_\varepsilon}(x)\|_{-1} &= \sup_{d(x,\partial\Lambda)<\frac{\varepsilon}{2}} \lim_{y\to x} |\nabla_{i,x}\nabla_{i,y}K^{\partial\Lambda_\varepsilon}(x,y)|^{1/2} \\ &\leqq c\varepsilon^{-1} \end{aligned} \tag{2.13b}$$

with some $c>0$ (independent of $\Lambda\in\mathscr{F}_0$) and $0<\varepsilon<1$.

Combining (2.10) with the bounds (2.11)–(2.13) and using (2.6) we get the inequality in the lemma with some constants $a, b>0$ (independent of $\Lambda\in\mathscr{F}_0$) and $0<\varepsilon<1$. □

For an increasing function $\omega\in C^\infty(\mathbb{R}^2)$, $\omega\geqq 0$ and a decreasing function $\varepsilon:\mathscr{F}_0\to(0,1)$ we define, with $\Lambda_\varepsilon \equiv \Lambda_{\varepsilon(\Lambda)}$ for $\Lambda\in\mathscr{F}_0$, the following subset of $\mathscr{D}'$:

$$\begin{aligned}\Omega_{\varepsilon,\omega} &\equiv \bigcup_{n\in\mathbb{N}} \left\{\eta\in\mathscr{D}' : \forall\Lambda\in\mathscr{F}_0,\ \Lambda_n\subseteq\Lambda,\ \sup_{x\in\partial\Lambda} |\psi_\eta^{\partial\Lambda_\varepsilon}(x)| \leqq \omega_{\partial\Lambda}\right\} \\ &\equiv \bigcup_{n\in\mathbb{N}} \Omega_{\varepsilon,\omega,n}\,. \end{aligned} \tag{2.14}$$

Such a function $\omega_{\partial\Lambda}$ will be said to be *dominating at infinity* if the conclusion of the following lemma holds:

Lemma 2. *Let μ be a regular measure and $(c(\Lambda), p)$ be its corresponding parameters of regularity. Assume $c(\Lambda) \leqq |\Lambda|^{N_0}$ for some $N_0 \in \mathbb{N}$. Let*

$$\omega(x) \equiv e^{\alpha\|x\|}, \quad \alpha > 0 \tag{2.15}$$

$$\varepsilon(\Lambda) \equiv e^{-\frac{\alpha}{3p} d(0, \partial\Lambda)} . \tag{2.16}$$

Let the Fisher sequence Λ_n be s.t.

$$d(0, \partial\Lambda_n) \geqq \frac{4}{\alpha} \ln\left(\frac{1+\delta}{B(\alpha)} \ln n\right), \tag{2.17}$$

for any given $\delta > 0$ and some positive constant $B(\alpha)$. Then

$$\mu(\Omega_{\varepsilon,\omega}) = 1 . \tag{2.18}$$

Proof. For any $\Lambda_n \in \mathscr{F}_0$ we have by definition (2.14)

$$\mu(\mathscr{D}'\backslash\Omega_{\varepsilon,\omega}) \leqq \mu(\mathscr{D}'\backslash\Omega_{\varepsilon,\omega,n}) \tag{2.19}$$

and

$$\mu(\mathscr{D}'\backslash\Omega_{\varepsilon,\omega,n}) = \mu\left\{\eta \in \mathscr{D}' : \exists \Lambda \in \mathscr{F}_0,\ \Lambda_n \subseteqq \Lambda,\ \sup_{x \in \partial\Lambda} |\psi_\eta^{\partial\Lambda_\varepsilon}(x)| > \omega_{\partial\Lambda}\right\}$$

$$\leqq \sum_{\substack{\Lambda \in \mathscr{F}_0 \\ \Lambda_n \subseteqq \Lambda}} \mu\left\{\eta \in \mathscr{D}' : \sup_{x \in \partial\Lambda} |\psi_\eta^{\partial\Lambda_\varepsilon}(x)| > \omega_{\partial\Lambda}\right\} . \tag{2.20}$$

Now using Lemma 2.1 with ω, ε given by (2.15) and (2.16) and using the assumption on Λ_n respectively we get the bound

$$\text{rhs}\,(2.20) \leqq \sum_{\substack{\Lambda_m \in \mathscr{F}_0 \\ \Lambda_n \subset \Lambda_m}} \exp\left(-Be^{\frac{\alpha}{4} d(0, \partial\Lambda_m)}\right) \tag{2.21}$$

with some $B > 0$ independent of Λ_m (and α, if we take n sufficiently large).

The right-hand side of (2.21) can be made arbitrarily small if we use the assumption (2.17). This ends the proof. □

Remark. With the choice (2.17) we have

$$\omega_{\partial\Lambda} \geqq \left(\frac{1+\delta}{B} \ln n\right)^4 . \tag{2.22}$$

3. The Uniform Continuity of Local Specifications

Let μ be a regular probability measure on $(\mathscr{D}', \Sigma)$ (in the sense of Sect. 2).

We shall consider local specifications $\mathscr{E}$ in the sense of [Fö2, Pr], defined for $P(\varphi)_2$ models in [Rö3] (and references therein). $\mathscr{E}$ is by definition a family $E_{\Lambda^c}^\eta$, $\eta \in \Omega$ with Ω a μ-measure 1 Borel subset of $\mathscr{D}'$, $\Lambda \in \mathscr{F}$ and $E_{\Lambda^c}^\eta$ is given by

$$E_{\Lambda^c}^\eta(F) \equiv \mu_0^{\partial\Lambda}\left(e^{-U_\Lambda(\cdot + \psi_\eta^{\partial\Lambda})} F(\cdot + \psi_\eta^{\partial\Lambda})\right) \cdot \left[\mu_0^{\partial\Lambda}\left(e^{-U_\Lambda(\cdot + \psi_\eta^{\partial\Lambda})}\right)\right]^{-1} ,$$

with the interaction U_Λ given by

$$U_\Lambda(\varphi) = \int_\Lambda d_2 x : v(\varphi) :_0 (x) \ ,$$

with v of polynomial, exponential or trigonometric type and : : indicating normal ordering with respect to the Nelson's free field measure μ_0 [the Gaussian measure with mean zero and covariance $(-\Delta + m_0^2)^{-1}$].

$\mu_0^{\partial\Lambda}$ is correspondingly the Gaussian measure with mean zero and covariance $(-\Delta_{\partial\Lambda} + m_0^2)^{-1}$, with, as before, $\Delta_{\partial\Lambda}$ the Laplacian with Dirichlet boundary conditions on $\partial\Lambda$.

For any local specification $\mathscr{E}$ as above we define as Gibbs measure μ for $\mathscr{E}$ any probability measure on $(\Omega, \Sigma \cap \Omega)$ such that for any $\Lambda \in \mathscr{F}$

$$\mu E_{\Lambda^c}(F) = \mu F$$

for all bounded measurable F.

The set of all Gibbs measures for $\mathscr{E}$ will be denoted by $\mathscr{G}(\mathscr{E})$. By $\partial\mathscr{G}(\mathscr{E})$ we shall denote the subset of $\mathscr{G}(\mathscr{E})$ consisting of Gibbs measures which have no nontrivial convex linear representations in terms of other elements from $\mathscr{G}(\mathscr{E})$.

Remark. Gibbs measures and local specifications have been constructed for Euclidean fields with exponential, polynomial and trigonometric interaction in two dimensions see e.g. [GlJa, Sim, GRS1,2, FrSi, AHK1,15, Ze1] and references therein, see also [Rö2].

Definition 1. A local specification $\mathscr{E} \equiv \{E_{\Lambda^c}^\eta\}_{\Lambda \in \mathscr{F}}$ on $(\mathscr{D}', \Sigma)$ is called *uniformly continuous* μ-a.e. iff there is a function $\varepsilon : \mathscr{F}_0 \to (0,1)$ such that for any bounded measurable local function F (in the sense of Sect. 2)

$$\lim_{\mathscr{F}_0} |E_{\Lambda^c}^{\psi_\eta^{\partial\Lambda_\varepsilon}}(F) - E_{\Lambda^c}^\eta(F)| = 0 \ , \qquad \mu\text{-a.e.} \tag{3.1}$$

Theorem 3.1. *Let $\mathscr{E}$ be a local specification for any interaction U_Λ of the polynomial, trigonometric or exponential type and let μ be a regular Gibbs measure for $\mathscr{E}$. Then $\mathscr{E}$ is uniformly continuous μ-a.e.*

Proof. We prove first Lemma 3.2 for all interactions. The case of polynomial interactions is then handled by Lemma 3.3, the ones of trigonometric respectively exponential interactions by Lemma 3.4 respectively 3.5.

Let us denote for $\Lambda \in \mathscr{F}_0$, with $\varepsilon \equiv \varepsilon(\Lambda)$

$$\delta U_{\Lambda,\varepsilon}(\varphi) \equiv U_\Lambda(\varphi + \psi_\varphi^{\partial\Lambda_\varepsilon} - \psi_\varphi^{\partial\Lambda}) - U_\Lambda(\varphi) \ . \tag{3.2}$$

We have then

Lemma 3.2. *Let $\mu \in \mathscr{G}(\mathscr{E})$ be a regular measure. Suppose that*

$$\lim_{\mathscr{F}_0} \mu|\delta U_{\Lambda,\varepsilon}| = 0 \tag{3.3}$$

and with some constant $0 < C < \infty$ independent of $\Lambda \in \mathscr{F}_0$

$$\mu e^{-\delta U_{\Lambda,\varepsilon}} \leqq C \ . \tag{3.4}$$

Then $\mathscr{E}$ is uniformly continuous μ-a.e. *(if necessary by replacing $\mathscr{F}_0$ in Definition 1 by a subsequence $\mathscr{F}_0'$).*

Proof. For any cylinder function $F(\varphi) \equiv \tilde{F}(\varphi(f_1), \ldots, \varphi(f_n))$, $\tilde{F} \in \mathscr{C}_b^1(\mathbb{R}^n)$, $\tilde{F} \geqq 0$, $f_i \in \mathscr{D}$, $i=1, \ldots, n)$, we have, using the definition of the conditional expectation and calling φ the integration variable with respect to $\mu_0^{\partial\Lambda}$,

$$E_{\Lambda_\varepsilon^c}^{\psi_\eta^{\partial\Lambda_\varepsilon}} F = \mu_0^{\partial\Lambda}[e^{-U_\Lambda(\varphi+\psi_\eta^{\partial\Lambda})} e^{-\delta U_{\Lambda,\varepsilon}(\varphi+\psi_\eta^{\partial\Lambda})} F(\varphi+\psi_\eta^{\partial\Lambda}+\delta_\varepsilon\psi_\eta^{\partial\Lambda})]\,[\mu_0^{\partial\Lambda}(e^{-U_\Lambda(\varphi+\psi_\eta^{\partial\Lambda})} e^{-\delta U_{\Lambda,\varepsilon}(\varphi+\psi_\eta^{\partial\Lambda})})]^{-1}\,,$$

with

$$\delta_\varepsilon \psi_\eta^{\partial\Lambda} \equiv \psi_\eta^{\partial\Lambda_\varepsilon} - \psi_\eta^{\partial\Lambda}\,.$$

Dividing numerator and denominator by the factor $\mu_0^{\partial\Lambda}(e^{-U_\Lambda(\varphi+\psi_\eta^{\partial\Lambda})})$ and using the properties of conditional expectation we get that the above is equal to

$$E_{\Lambda^c}^{\eta}\left\{\frac{e^{-\delta U_{\Lambda,\varepsilon}(\varphi)}}{E_{\Lambda^c}^{\varphi}(e^{-\delta U_{\Lambda,\varepsilon}})} F(\varphi+\delta_\varepsilon\psi_\eta^{\partial\Lambda})\right\}\,.$$

Using this we get the equality which is the starting point of the following:

$$\begin{aligned}
&\mu|E_{\Lambda_\varepsilon^c}^{\psi_\eta^{\partial\Lambda_\varepsilon}} F(\varphi) - E_{\Lambda^c}^{\eta} F(\varphi)| \\
&= \mu\left|E_{\Lambda^c}^{\eta}\left[\left(\frac{e^{-\delta U_{\Lambda,\varepsilon}}}{E_{\Lambda^c}^{\eta}(e^{-\delta U_{\Lambda,\varepsilon}})} - 1\right) F(\varphi+\delta_\varepsilon\psi_\eta^{\partial\Lambda})\right]\right. \\
&\quad \left. + E_{\Lambda^c}^{\eta}(F(\varphi+\delta_\varepsilon\psi_\eta^{\partial\Lambda}) - F(\varphi))\right| \qquad (3.5)\\
&\leqq \mu\left(F(\varphi)\left|\frac{e^{-\delta U_{\Lambda,\varepsilon}}}{E_{\Lambda^c}^{\eta} e^{-\delta U_{\Lambda,\varepsilon}}} - 1\right|\right) \\
&\quad + 3\sum_{i=1}^{n} \|\nabla_i \tilde{F}\|_\infty \mu|\delta_\varepsilon\psi_\eta^{\partial\Lambda}(f_i)|\,,
\end{aligned}$$

where we used for the inequality simple properties of conditional expectations, measurability properties of the functions involved, a meanvalue theorem and the majorization

$$E_{\Lambda^c}^{\eta}\left|\frac{e^{-\delta U_{\Lambda,\varepsilon}}}{E_{\Lambda^c}^{\eta} e^{-\delta U_{\Lambda,\varepsilon}}} - 1\right| \leqq 2\,. \qquad (3.6)$$

The second sum from the right-hand side of (3.5) converges to zero as $\Lambda \uparrow \mathbb{R}^2$ through $\mathscr{F}_0$ (because of regularity, see [AHK7]).

Our assumption (3.3) implies that $\delta U_{\Lambda,\varepsilon}$ converges by subsequences to 0, hence

$$\lim_{\mathscr{F}_0'} e^{-\delta U_{\Lambda,\varepsilon}} = 1\,, \quad \mu\text{-a.e.} \qquad (3.7a)$$

for some $\mathscr{F}_0' \subseteq \mathscr{F}_0$.

We shall now use that the assumption (3.4) together with the property of conditional expectation implies

$$\mu(E_{\Lambda^c}^{\eta} e^{-\delta U_{\Lambda,\varepsilon}(\varphi)}) < C\,.$$

Consider, for any bounded measurable function F:

$$\mu(FE^{\eta}_{\Lambda^c}e^{-\delta U_{\Lambda,\varepsilon}})=(\mu(E^{\eta}_{\Lambda^c}F)e^{-\delta U_{\Lambda,\varepsilon}}) \ .$$

By martingale convergence theorem $E^{\eta}_{\Lambda^c}F$ converges in L^1 as $\Lambda\uparrow\mathbb{R}^2$, hence by subsequences a.e. On the other hand $\exp(-\delta U_{\Lambda,\varepsilon})$ converges to 1 by (3.7a) along $\mathscr{F}_0'$. By Fatou's lemma then for $\Lambda\in\mathscr{F}_0''\subset\mathscr{F}_0$,

$$\limsup \mu((E^{\eta}_{\Lambda^c}F)e^{-\delta U_{\Lambda,\varepsilon}})\leqq\mu F \ . \tag{3.7b}$$

On the other hand by conditional Jensen's inequality

$$E^{\eta}_{\Lambda^c}e^{-\delta U_{\Lambda,\varepsilon}}\geqq e^{-E^{\eta}_{\Lambda^c}(\delta U_{\Lambda,\varepsilon})} \ .$$

By (3.3) the right-hand side goes to one a.s. as $\Lambda\uparrow\mathbb{R}^2$ along $\mathscr{F}_0'$. Hence for $F\geqq 0$ bounded measurable, using again the properties of conditional expectation,

$$\liminf \mu(FE^{\eta}_{\Lambda^c}e^{-\delta U_{\Lambda,\varepsilon}})\geqq\mu F \ . \tag{3.7c}$$

From (3.7b), (3.7c) we get

$$\lim_{\mathscr{F}_0'} E^{\eta}_{\Lambda^c}e^{-\delta U_{\Lambda,\varepsilon}}=1 \qquad \mu\text{-a.e.} \tag{3.7d}$$

for some $\mathscr{F}_0'\subset\mathscr{F}_0$.

Let us now consider

$$\mu\left(F(\varphi)\left|\frac{e^{-\delta U_{\Lambda,\varepsilon}(\varphi)}}{E^{\varphi}_{\Lambda^c}e^{-\delta U_{\Lambda,\varepsilon}}}-1\right|\right) \ . \tag{3.8}$$

By (3.7a) and (3.7d) the integrand goes pointwise, as $\Lambda\uparrow\mathbb{R}^2$ in $\mathscr{F}_0'$, to zero. On the other hand F being bounded we can bound the integral as

$$\leqq\|F\|_{\infty}\mu\left(\left|\frac{e^{-\delta U_{\Lambda,\varepsilon}(\varphi)}}{E^{\varphi}_{\Lambda^c}(e^{-\delta U_{\Lambda,\varepsilon}})}-1\right|\right) \ .$$

Using that μ is a Gibbs measure together with the bound (3.6) the right-hand side is bounded by $2\|F\|_{\infty}$. Hence by weak compactness we can choose a subsequence $\mathscr{F}_0''$ s.t.

$$\mu\left(F(\varphi)\left|\frac{e^{-\delta U_{\Lambda,\varepsilon}}}{E^{\varphi}_{\Lambda^c}(e^{-\delta U_{\Lambda,\varepsilon}})}-1\right|\right)\to 0 \tag{3.8}$$

as $\Lambda\uparrow\mathbb{R}^2$ in $\mathscr{F}_0''$. This gives the stated uniform continuity of $\mathscr{E}$. □

Let now

$$U_{\Lambda}(\varphi)\equiv\int_{\Lambda} d_2x : P(\varphi) :_0 (x) \tag{3.9}$$

be the function describing a $P(\varphi)_2$ interaction in the region Λ, with P a semibounded polynomial. The Wick ordering is, as before, with respect to the free field measure. Let μ be a Gibbs state for the local specification given by the above interaction.

We have the following

Lemma 3.3. *Let*

$$\varepsilon(\Lambda)\equiv e^{-\gamma d(0,\partial\Lambda)} \ , \qquad \gamma>0 \ . \tag{3.10}$$

Then, with $\delta U_{\Lambda,\varepsilon}$ *given by* (3.2), (3.9):

$$\lim_{\mathscr{F}_0} \mu|\delta U_{\Lambda,\varepsilon}|=0 \ , \tag{3.11}$$

and there is a constant $0<c<\infty$ *independent of* $\Lambda \in \mathscr{F}_0$ *such that*

$$\mu e^{-\delta U_{\Lambda,\varepsilon}}<c \ . \tag{3.12}$$

Proof. To get (3.11) it is sufficient to show, by the definition (3.2), (3.9) of $\delta U_{\Lambda,\varepsilon}$, that with

$$F_{\Lambda,\varepsilon}\equiv\int_\Lambda d_2x : \varphi^k :_0(x)\,(\psi_\varphi^{\partial\Lambda_\varepsilon}(x)-\psi_\varphi^{\partial\Lambda}(x)) \tag{3.13}$$

we have $\lim_{\mathscr{F}_0} \mu F^2_{\Lambda,\varepsilon}=0$, for any $0\leqq k\leqq \deg P-1$ and $1\leqq n\leqq \deg P$, $n+k\leqq \deg P$, where $\deg P$ is the degree of P.

Let us set

$$\delta_\varepsilon\psi_\varphi^{\partial\Lambda}(x)\equiv\varphi(\delta_\varepsilon\psi_{\cdot}^{\partial\Lambda}(x))\equiv\varphi(\psi_{\cdot}^{\partial\Lambda_\varepsilon}(x)-\psi_{\cdot}^{\partial\Lambda}(x)) \ , \qquad x\in\Lambda \ . \tag{3.14}$$

We can perform the integration by parts in the expectation $\mu F^2_{\Lambda,\varepsilon}$ to eliminate $\delta\psi_\varphi^{\Lambda,\varepsilon}(x)$, using the local equivalence of μ with the free measure μ_0, and we get:

$$\begin{aligned}
&\mu[(\textstyle\iint d_2z d_ix \delta_\varepsilon\psi_z^{\partial\Lambda}(x)G(z-x)k:\varphi^{k-1}:_0(x)\,(\delta_\varepsilon\psi_\varphi^{\partial\Lambda}(x))\,(n-1)\chi_\Lambda(x))F_{\Lambda,\varepsilon}]\\
&+\mu[(\textstyle\int d_2x:\varphi^k:_0(x)\,(\delta_\varepsilon\psi_\varphi^{\partial\Lambda}(x))^{n-2}(n-1)\|\delta_\varepsilon\psi_{\cdot}^{\partial\Lambda}(x)\|_{-1}^2\chi_\Lambda(x))F_{\Lambda,\varepsilon}]\\
&+\mu[\textstyle\int d_2x:\varphi^k:_0(x)\,(\delta_\varepsilon\psi_\varphi^{\partial\Lambda}(x))^{n-1}\chi_\Lambda(x)\,\{\iint d_2z d_2y \delta_\varepsilon\psi_z^{\partial\Lambda}(x)G(z-y)]\\
&\quad k:\varphi^{k-1}:_0(y)\,(\delta_\varepsilon\psi_\varphi^{\partial\Lambda}(y))\chi_\Lambda(y)\\
&+\textstyle\int d_2y\chi_\Lambda(y)\,(\iint d_2z d_2z' \delta_\varepsilon\psi_z^{\partial\Lambda}(x)G(z-z')\delta_\varepsilon\psi_{z'}^{\partial\Lambda}(y):\varphi^k:_0(y)\\
&\quad(\delta_\varepsilon\psi_\varphi^{\partial\Lambda}(y))^{n-1}n\}]\\
&-\mu[F\textstyle\int d_2x\chi_\Lambda(x):\varphi^k:_0(x)\,(\delta_\varepsilon\psi_\varphi^{\partial\Lambda}(x))^{n-1}\int d_2z d_2y\delta_\varepsilon\psi_z^{\partial\Lambda}(x)\\
&\quad G(z-y):P'(\varphi):_0(y)]
\end{aligned} \tag{3.15}$$

(for such computations see the "integration by parts formula" in [GlJa], following [DiGl]). We note that by definition of $\delta_\varepsilon\psi_{\cdot}^{\partial\Lambda}$ we have

$$\begin{aligned}
v_1(x)&\equiv\textstyle\int d_2z\delta_\varepsilon\psi_z^{\partial\Lambda}(x)\chi_\Lambda(x)=[K^{\partial\Lambda_\varepsilon}(x,y)-K^{\partial\Lambda}(x,x)]\chi_\Lambda(x)\\
&=\|\delta_\varepsilon\psi_{\cdot}^{\partial\Lambda}(x)\|_{-1}^2\chi_\Lambda(x) \ ,
\end{aligned} \tag{3.16}$$

$$\begin{aligned}
v_2(x,y)&\equiv\textstyle\int d_2z\delta_\varepsilon\psi_z^{\partial\Lambda}(x)G(z-y)\\
&=(K^{\partial\Lambda_\varepsilon}(x,y)-K^{\partial\Lambda}(x,y))\chi_\Lambda(x)\chi_\Lambda(y)\\
&\quad+(K^{\partial\Lambda_\varepsilon}(x,y)-G(x,y))\chi_\Lambda(x)\chi_{\Lambda_\varepsilon\setminus\Lambda}(y) \ ,
\end{aligned} \tag{3.17}$$

where

$$K^{\partial\Lambda}(x,y)\equiv G(x,y)-G^{\partial\Lambda}(x,y) \ , \tag{3.18a}$$

$$K^{\partial\Lambda}(x,x)\equiv\lim_{y\to x} K^{\partial\Lambda}(x,y) \tag{3.18b}$$

and analogously for $\partial\Lambda_\varepsilon$.

By successive integration by parts analogous to (3.15) we get the expectation of a sum of (at most 5^{2n}) monomials of the form

$$\int \prod_{i=1}^{\ell} :\varphi^{k_i}:_0(x_i)\, w(x_1,\dots,x_\ell)\, dx_1 \dots dx_\ell \tag{3.19}$$

($k_i < \deg P - 1$, $l \leqq 2n+2$) with the kernels $w(x_1,\dots,x_l)$ defined as the products of the kernels of type (3.16) and (3.17).

The expectation of (3.19) can be estimated by $L_s(dx_1 \dots dx_l)$-norm of $w(x_1,\dots,x_l)$ for some $s \in \mathbb{N}$, see e.g. [GlJa]. From the definition of these kernels this norm can be estimated by a finite product of L_r norms (for some $r \in \mathbb{N}$) of the kernels $v_1(\cdot)$ and $v_2(\cdot,\cdot)$. Since as is shown in Appendix 3.A, for any $r \in \mathbb{N}$:

$$\|v_1\|_{L_r(\mathbb{R}^2)} \leqq c\,|\partial\Lambda|^q \varepsilon^b \ , \tag{3.20a}$$

$$\|v_2\|_{L_r(\mathbb{R}^2 \times \mathbb{R}^2)} \leqq c\,|\partial\Lambda|^q \varepsilon^b \ , \tag{3.20b}$$

with some $a, b > 0$ and a constant $c > 0$ independent of Λ, ε, so by our assumption about $\varepsilon(\Lambda)$ we get (3.13). This finishes the proof of (3.11).

Let us assume from now on, for simplicity of notation, that μ is of the following form:

$$\mu(\cdot) = \lim_{\mathscr{F}_0} \mu_{\tilde{\Lambda}}(\cdot) \ ,$$

with

$$\mu_{\tilde{\Lambda}}(\cdot) \equiv \mu_0(e^{-U_{\tilde{\Lambda}}} \cdot)/Z_{\tilde{\Lambda}}$$

with $Z_{\tilde{\Lambda}} \equiv \mu_0(e^{-U_{\tilde{\Lambda}}})$.
(The general case can be handled similarly.)
We have then, using the local Markov property of $\mu_{\tilde{\Lambda}}$:

$$\begin{aligned} &Z_{\tilde{\Lambda}} \mu_{\tilde{\Lambda}} e^{-\delta U_{\Lambda,\varepsilon}} \\ &= \mu_0[\exp(-U_{\tilde{\Lambda}\setminus\Lambda}(\varphi) - U_\Lambda(\varphi + \psi_\varphi^{\partial\Lambda_\varepsilon} - \psi_\varphi^{\partial\Lambda}))] \\ &= \mu_0[e^{-U_{\tilde{\Lambda}\setminus\Lambda}(\varphi)} \mu_0^{\partial\Lambda}(e^{-U_\Lambda(\varphi' + \psi_\varphi^{\partial\Lambda_\varepsilon})})] \\ &= \mu_0[e^{-U_{\tilde{\Lambda}\setminus\Lambda_\varepsilon}(\varphi)} (E_0^{\partial\Lambda_\varepsilon} e^{-U_{\Lambda_\varepsilon\setminus\Lambda}}) \mu_0^{\partial\Lambda}(e^{-U_\Lambda(\varphi' + \psi_\varphi^{\partial\Lambda_\varepsilon})})] \ , \end{aligned} \tag{3.21}$$

with $E_0^{\partial\Lambda_\varepsilon}$ the free conditional expectation, i.e. the conditional expectation with respect to μ_0, given the σ-algebra $\Sigma(\partial\Lambda_\varepsilon)$ [we used here again the local Markov property, and the $\Sigma(\partial\Lambda_\varepsilon)$-measurability of $\psi_\varphi^{\partial\Lambda_\varepsilon}(x)$].

By conditioning inequality [GRS1, Sim] we have:

$$\begin{aligned} \mu_0^{\partial\Lambda} e^{-U_\Lambda(\cdot + \psi_\varphi^{\partial\Lambda_\varepsilon})} &\leqq \mu_0^{\partial\Lambda_\varepsilon} e^{-U_\Lambda(\cdot + \psi_\varphi^{\partial\Lambda_\varepsilon})} \\ &= E_0^{\partial\Lambda_\varepsilon}(e^{-U_\Lambda})(\varphi) \end{aligned} \tag{3.22}$$

(where we used the definition of $\psi_\varphi^{\partial\Lambda_\varepsilon}$ and conditional expectation). From (3.21), (3.22) we get then

$$\begin{aligned} Z_{\tilde{\Lambda}} \mu_{\tilde{\Lambda}} e^{-\delta U_{\Lambda,\varepsilon}} &\leqq \mu_0[e^{-U_{\tilde{\Lambda}\setminus\Lambda_\varepsilon}(\varphi)} E_0^{\partial\Lambda_\varepsilon}(e^{-U_{\Lambda_\varepsilon\setminus\Lambda}}) \\ &\quad (E_0^{\partial\Lambda_\varepsilon} e^{-U_\Lambda})] = \mu_0[e^{-U_{\tilde{\Lambda}\setminus\Lambda_\varepsilon}(\varphi) - U_\Lambda(\varphi)} E_0^{\partial\Lambda_\varepsilon}(e^{-U_{\Lambda_\varepsilon\setminus\Lambda}})(\varphi)] \\ &\equiv \mu_{\tilde{\Lambda},\varepsilon}(E_0^{\partial\Lambda_\varepsilon}(e^{-U_{\Lambda_\varepsilon\setminus\Lambda}})) \cdot \mu_0(e^{-U_{\tilde{\Lambda}\setminus\Lambda_\varepsilon} + U_\Lambda}) \\ &\equiv \mu_{\tilde{\Lambda},\varepsilon} \otimes \mu_0^{\partial\Lambda_\varepsilon}(e^{-U_{\Lambda_\varepsilon\setminus\Lambda}(\varphi' + \psi_\varphi^{\partial\Lambda_\varepsilon})}) Z_{(\tilde{\Lambda}\setminus\Lambda_\varepsilon)\cup\Lambda} \ , \end{aligned} \tag{3.23}$$

(with φ, φ' integration variables).

Using (3.23) and Jensen inequality to estimate $Z_{(\tilde{\Lambda}\setminus\Lambda_\varepsilon)\cup\Lambda}\backslash Z_{\tilde{\Lambda}}$ we get

$$\mu_{\tilde{\Lambda}} e^{-\delta U_{\Lambda,\varepsilon}} \leqq \mu_{\tilde{\Lambda},\varepsilon} \otimes \mu_0^{\partial\Lambda_\varepsilon}(e^{-U_{\Lambda_\varepsilon\setminus\Lambda}(\varphi'+\psi_\varphi^{\partial\Lambda_\varepsilon})}) \exp(\mu_{\tilde{\Lambda},\varepsilon}(U_{\Lambda_\varepsilon\setminus\Lambda}(\varphi))) . \tag{3.24}$$

By standard arguments [GlJa] we get the bound uniform in the volume $\tilde{\Lambda}$

$$|\mu_{\tilde{\Lambda},\varepsilon}(U_{\Lambda_\varepsilon\setminus\Lambda}(\varphi))| \leqq c|\Lambda_\varepsilon\backslash\Lambda| \tag{3.25}$$

with some $c>0$ independent of Λ, $\tilde{\Lambda}$ and ε.

Using the Duhamel expansion (cf. [GlDi, GRS2]) we get also

$$\mu_{\tilde{\Lambda},\varepsilon} \otimes \mu_0^{\partial\Lambda_\varepsilon}(e^{-U_{\Lambda_\varepsilon\setminus\Lambda}(\varphi'+\psi_\varphi^{\partial\Lambda_\varepsilon})}) \leqq e^{c|\Lambda_\varepsilon\setminus\Lambda|} , \tag{3.26}$$

with $c>0$ independent of Λ, $\tilde{\Lambda}$ and ε (see Appendix 3.2).

This together with the definition of μ and our assumption about $\varepsilon(\Lambda)$ finishes the proof of (3.12), since $|\Lambda_\varepsilon\backslash\Lambda|$ can then be bounded by a constant. □

By combining Lemmas 3.2 and 3.3 we get the proof of the theorem in the case of polynomial interactions.

We shall now complete the proof for the case of trigonometric interactions. For this we use the following

Lemma 3.4. *Let* $U_\Lambda(\varphi) \equiv \int_\Lambda d_2x \int d\nu(\alpha) : \cos\alpha\varphi :_0(x)$ *with* $\operatorname{supp}\nu \subset (-2\sqrt{\pi}, 2\sqrt{\pi})$, $\int d\nu < \infty$.

Let μ *be a Gibbs state for the local specification corresponding to the interaction* U_Λ. *Let* $\varepsilon(\Lambda)$ *be as in Lemma 3.3, then the same conclusions as in Lemma 3.3 hold.*

Proof. Similarly as in the case of polynomial interaction we only treat explicitly the case of Gibbs measures constructed with free boundary conditions, the other cases can be handled analogously. To show (3.11) in the present case it is convenient to prove that

$$\lim_{\mathscr{F}_0} \mu(\delta U_{\Lambda,\varepsilon})^2 = 0 . \tag{3.27}$$

We find an estimation as $\Lambda \uparrow \mathbb{R}^2$ on $\mu(\delta U_{\Lambda,\varepsilon})^2$, namely that it goes to zero, using the trigonometric identities, integration by parts formula (to remove $\delta_\varepsilon \psi_\varphi^{\partial\Lambda}$) and standard bounds for the measure μ [FrSe, FrPa].

To show (3.12) we use Jensen inequality and analogous arguments as in the polynomial case to get the bounds

$$\begin{aligned} &\exp(-\mu(\delta U_{\Lambda,\varepsilon})) \leqq \mu e^{-\delta U_{\Lambda,\varepsilon}} \leqq \mu_{\Lambda,\varepsilon} \otimes \mu_0^{\partial\Lambda_\varepsilon}(e^{-U_{\Lambda_\varepsilon\setminus\Lambda}(\varphi'+\psi_\varphi^{\partial\Lambda_\varepsilon})} \\ &\exp \mu_{\Lambda,\varepsilon}(U_{\Lambda_\varepsilon\setminus\Lambda}(\varphi)) , \end{aligned} \tag{3.28}$$

with

$$\mu_{\Lambda,\varepsilon}(\cdot) \equiv \lim_{\mathscr{F}_0} \mu_0(e^{-U_{\tilde{\Lambda}\setminus\Lambda_\varepsilon}(\cdot)-U_\Lambda(\cdot)} \cdot)[\mu_0(e^{-U_{\tilde{\Lambda}\setminus\Lambda_\varepsilon}(\cdot)-U_\Lambda(\cdot)})]^{-1} ,$$

and φ respectively φ' is the integration variable of $\mu_{\Lambda,\varepsilon}$ respectively $\mu_0^{\partial\Lambda_\varepsilon}$. The left-hand side of (3.28) converges to one [from (3.27)] and by analogous arguments we have also

$$\lim_{\mathscr{F}_0} \mu_{\Lambda,\varepsilon} U_{\Lambda_\varepsilon\setminus\Lambda}(\varphi) = 0 . \tag{3.29}$$

For estimation of the first factor from right-hand side of (3.28) we make the expansion of the exponential into power series and then estimate each term separately, using the bounds for expectations with respect to $\mu_{\Lambda,\varepsilon}$ and $\mu_0^{\partial\Lambda_\varepsilon}$ (if $\operatorname{supp} dv(\alpha) \subset (-2\sqrt{\pi}, 2\sqrt{\pi})$ then the gaussian integrations with $\mu_0^{\partial\Lambda_\varepsilon}$ are sufficient to get the estimation yielding convergence to one). □

Lemma 3.4 together with Lemma 3.2 yields the proof of Theorem 3.2 for trigonometric interactions. Similarly we get the proof of the theorem for exponential interactions using the following:

Lemma 3.5. *Let* $U_\Lambda \equiv \lambda \int_\Lambda d_2x \int dv(\alpha) : e^{\alpha\varphi} :_0(x)$ *with* $\lambda > 0$ *and* v *a probability measure on* $(-2\sqrt{\pi}, 2\sqrt{\pi})$, $: \, :_0$ *being normal ordering with respect to free field measure* μ_0.

Let μ *be a Gibbs state for the local specification corresponding to the interaction* U_Λ. *Let* $\varepsilon(\Lambda)$ *be as in Lemma 3.3, then the same conclusions as in Lemma 3.3 hold.*

Proof. As before we consider explicitly only the case where μ corresponds to half-Dirichlet boundary conditions [AHK 7, Sim], the other cases being similar. Then we have $\mu = \lim_{\mathscr{F}_0} \mu_{\tilde\Lambda}$, with

$$\mu_{\tilde\Lambda}(\cdot) \equiv \mu_0^{\partial\tilde\Lambda}(e^{-U_{\tilde\Lambda}}\cdot) Z_{\tilde\Lambda}^{-1} \; ,$$

with $Z_{\tilde\Lambda} \equiv \mu_0^{\partial\tilde\Lambda}(e^{-U_{\tilde\Lambda}})$.

Moreover for $\tilde\Lambda \in \mathscr{F}_0$, $\Lambda_\varepsilon \subset \tilde\Lambda$ (as in Lemma 3.3), we have (using the definition of $\mu_{\tilde\Lambda}$ and conditional expectation)

$$\begin{aligned} Z_{\tilde\Lambda}\mu_{\tilde\Lambda} e^{-\delta U_{\Lambda,\varepsilon}} &= \mu_0^{\partial\tilde\Lambda} e^{-U_{\tilde\Lambda\setminus\Lambda}(\varphi)} e^{-U_\Lambda(\varphi+\delta_\varepsilon\psi_\varphi^{\partial\Lambda})} \\ &= \mu_0^{\partial\tilde\Lambda} e^{-U_{\tilde\Lambda\setminus\Lambda}(\varphi)} \mu_0^{\partial\Lambda} e^{-U_\Lambda(\varphi+\psi_\varphi^{\partial\Lambda_\varepsilon})} \\ &\leqq \mu_0^{\partial\tilde\Lambda} e^{-U_{\tilde\Lambda\setminus\Lambda}(\varphi)} \mu_0^{\partial\Lambda_\varepsilon} e^{-U_\Lambda(\varphi+\psi_\varphi^{\partial\Lambda_\varepsilon})} \; , \end{aligned} \tag{3.30}$$

where we used the conditioning inequalities (proven similarly as for the polynomial interactions, by expansion of the interaction term in a power series).

Using the definition of conditional expectation value we rewrite the right-hand side of (3.30) in the form

$$\begin{aligned} &\mu_0^{\partial\tilde\Lambda} e^{-U_{\tilde\Lambda\setminus\Lambda_\varepsilon}(\varphi)} (E_0^{\partial\Lambda_\varepsilon} e^{-U_{\Lambda_\varepsilon\setminus\Lambda}(\varphi)}) E_0^{\partial\Lambda_\varepsilon} e^{-U_\Lambda(\varphi)} \\ &= \mu_0^{\partial\tilde\Lambda} e^{-U_{(\tilde\Lambda\setminus\Lambda_\varepsilon)\cup\Lambda}(\varphi)} (E_0^{\partial\Lambda_\varepsilon} e^{-U_{\Lambda_\varepsilon\setminus\Lambda}(\varphi)}) \\ &\leqq \mu_0^{\partial\tilde\Lambda} (e^{-U_{(\tilde\Lambda\setminus\Lambda_\varepsilon)\cup\Lambda}(\varphi)}) \; , \end{aligned} \tag{3.31}$$

where in the first equality we used the properties of conditional expectation and in the last inequality we used the positivity of the considered interaction. Now using the Jensen inequality we get

$$\begin{aligned} \mu_{\tilde\Lambda} e^{-\delta U_{\Lambda,\varepsilon}} &\leqq (\mu_0^{\partial\tilde\Lambda}(e^{-U_{\tilde\Lambda}}))^{-1} \mu_0^{\partial\tilde\Lambda} e^{-U_{(\tilde\Lambda\setminus\Lambda_\varepsilon)\cup\Lambda}} \\ &\leqq \exp \mu_{\tilde\Lambda,\varepsilon}(U_{\Lambda_\varepsilon\setminus\Lambda}(\varphi)) \; , \end{aligned} \tag{3.32}$$

with

$$\begin{aligned} \mu_{\tilde\Lambda,\varepsilon}(\cdot) \equiv {} & [\mu_0^{\partial\tilde\Lambda}(e^{-U_{(\tilde\Lambda\setminus\Lambda_\varepsilon)\cup\Lambda}})]^{-1} \\ & \cdot \mu_0^{\partial\tilde\Lambda}(e^{-U_{(\tilde\Lambda\setminus\Lambda_\varepsilon)\cup\Lambda}}\cdot) \; . \end{aligned} \tag{3.33}$$

On the other hand, again from the Jensen inequality, we get

$$\exp(-\mu_{\tilde{A}}\delta U_{A,\varepsilon}) \leqq \mu_{\tilde{A}} e^{-\delta U_{A,\varepsilon}} . \tag{3.34}$$

From (3.32) and (3.34) using the exponential bound for the measures with exponential interactions we get as in [Ze1] (Lemma 1.4.2) that

$$\lim_{\mathscr{F}_0} \mu_{\tilde{A},\varepsilon} U_{A_\varepsilon \setminus A}(\varphi) = 0 \tag{3.35}$$

and (using also Lemma 1.5.1 of [Ze1])

$$\lim_{\mathscr{F}_0} \mu_{\tilde{A}} |\delta U_{A,\varepsilon}(\varphi)| = 0 . \tag{3.36}$$

This ends the proof of Lemma 3.5 and Theorem 3.1. □

4. The FKG-Structure on a Lattice and in the Continuum

Let $\mathscr{F}$ be as in Sect. 2. For $\Lambda \in \mathscr{F}$ and $\omega \in \mathscr{C}(\mathbb{R}^2)$ let $\psi_\omega^{\partial\Lambda}$ be the solution of the following free Dirichlet problem

$$\begin{aligned} (-\Delta + m_0^2)\psi_\omega^{\partial\Lambda}(x) &= 0 \qquad \text{in } \Lambda \\ \psi_\omega^{\partial\Lambda}(x) &= \omega(x) \qquad \text{in } \Lambda^c \end{aligned} \tag{4.1}$$

(this correspondends to the quantity $\psi_\eta^{\partial\Lambda_\varepsilon}$ introduced in Sect. 2, in the case where $\eta = \omega$). Let μ_0 respectively $\mu_0^{\partial\Lambda}$ be the Gaussian measure with mean zero and the covariance G respectively $G^{\partial\Lambda}$ (as in Sect. 2). For any polynomial, exponential or trigonometric v, as in Sect. 2, let

$$U_\Lambda(\varphi) \equiv \int_\Lambda :v(\varphi):_0(x) d_2x \tag{4.2}$$

with the normal ordering $: :_0$ with respect to μ_0.

Let us define correspondingly $E_{\Lambda^c}^\omega$ as the local specification to the interaction U_Λ, as in Sect. 2 (with the continuous function ω instead of the distribution η).

Let $\delta > 0$ and let $\mathbb{Z}_\delta^2 \equiv \{n\delta \equiv (n^1\delta, n^2\delta),\ n \in \mathbb{Z}^2\}$.

Let $\mathscr{F}_\delta \equiv \mathscr{F} \cap \mathbb{Z}_\delta^2$. For $\Lambda_\delta \in \mathscr{F}_\delta$ we define the energy functional on $(\Omega_\delta \equiv \mathbb{R}^{\mathbb{Z}_\delta^2}, \Sigma_\delta)$ (Σ_δ the Borel σ-algebra in $\mathbb{R}^{\mathbb{Z}_\delta^2}$):

$$H_{\Lambda_\delta}(q) \equiv \frac{1}{2}\delta^2 \sum_{i \in \Lambda_\delta} q_i((-\Delta_\delta + m_0^2)q)_i , \tag{4.3}$$

with Δ_δ the Laplacian on $\mathbb{Z}_\delta^2$ (see Appendix A4).

We also define the free lattice measure $\mu_{0,\delta}$ by

$$\begin{aligned} &\mu_{0,\delta} \equiv \lim_{\mathscr{F}_0} \mu_{0,\delta}^\Lambda , \qquad \text{with} \\ &\mu_{0,\delta}^\Lambda \equiv \delta_0\{(\textstyle\int dq_{\Lambda_\delta}(e^{-H_{\Lambda_\delta}(q)} \cdot))/(\textstyle\int dq_{\Lambda_\delta} e^{-H_{\Lambda_\delta}(q)})\} , \end{aligned} \tag{4.4}$$

with δ_0 a point measure concentrated on $\{q_i \equiv 0\}$ and $\mathscr{F}_0$ a filter of finite subsets of $\mathbb{Z}_\delta^2$ invading all the lattice. (We remark that the above limit is unique in the set of all probability measures supported on tempered sequences.)

Define the lattice interaction by

$$U_{\Lambda_\delta}(q) \equiv \delta^2 \sum_{i \in \Lambda_\delta} :v(q_i):_{0,\delta} , \tag{4.5}$$

with normal ordering with respect to $\mu_{0,\delta}$. Then we define the measures $E^{\omega}_{\Lambda^c,\delta}$ by

$$E^{\omega}_{\Lambda^c,\delta}(F) \equiv \delta_\omega \{ (\int dq_{\Lambda_\delta} e^{-H_{\Lambda_\delta}(q) - U_{\Lambda_\delta}(q)} F(q)) \cdot (\int dq_{\Lambda_\delta} e^{-H_{\Lambda_\delta}(q) - U_{\Lambda_\delta}(q)})^{-1} \} , \tag{4.6}$$

with δ_ω the point measure on Ω_δ concentrated on $\{\omega(i\delta), i \in \mathbb{Z}^d\}$ for a function $\omega \in \mathscr{C}(\mathbb{R}^2)$.

Lemma 4.1. *For any rectangle $\Lambda \in \mathscr{F}$, in the sense of weak convergence of measures:*

$$\lim_{\delta \to 0} E^{\omega}_{\Lambda^c,\delta} = E^{\omega}_{\Lambda^c} . \tag{4.7}$$

Remark. The lemma can be extended to hold for any sufficiently regular set $\Lambda \in \mathscr{F}$, see e.g. [GRS1].

In the following we shall call for simplicity *regular sets* the sets in $\mathscr{F}$ for which (4.7) holds.

Proof. First we note that changing the integration variables

$$q_i \to q_i = q'_i + \psi^{\partial\Lambda}_{\omega,\delta} \tag{4.8}$$

with $\psi^{\partial\Lambda}_{\omega,\delta}$ a solution of Dirichlet problem (4.1) but on the lattice, we get

$$E^{\omega}_{\Lambda^c,\delta} F(q) = (\mu^{\Lambda}_{0,\delta} e^{-U_{\Lambda_\delta}(q+\psi^{\partial\Lambda}_{\omega,\delta})})^{-1} \cdot \mu^{\Lambda}_{0,\delta}(e^{-U_{\Lambda_\delta}(q+\psi^{\partial\Lambda}_{\omega,\delta})} F(q+\psi^{\delta\Lambda}_{\omega,\delta})) . \tag{4.9}$$

Now it is known [GRS1, Sim] that the measure $\mu^{\Lambda}_{0,\delta}$ can be represented as the restriction of μ_0 to the σ-algebra generated by $\{\varphi(f_{n\delta}) : n\delta \in \Lambda_\delta\}$, for suitable test functions $f_{n\delta}$ (defined in Appendix A.4). Using our Lemma A.4.4 together with the Theorem VIII.5 [Sim] (see also [GRS1]) for polynomial interactions we get (using also the definition of U_Λ)

$$\lim_{\delta \to 0} (\mu^{\Lambda}_{0,\delta}(e^{-U_{\Lambda_\delta}(q+\psi^{\partial\Lambda}_{\omega,\delta})})^{-1} \mu^{\Lambda}_{0,\delta}(e^{-U_{\Lambda_\delta}(q+\psi^{\partial\Lambda}_{\omega,\delta})} F(q+\psi^{\partial\Lambda}_{\omega,\delta})) = E^{\omega}_{\Lambda^c}(F) , \tag{4.10}$$

for any cylinder function $F(\varphi) \equiv \tilde{F}(\varphi(f_1), \ldots, \varphi(f_n))$, $\tilde{F} \in \mathscr{C}(\mathbb{R}^n)$, $f_i \in \mathscr{D}$, $\operatorname{supp} f_i \subset \Lambda$. This ends the proof of our lemma, for the case of polynomial interactions. In the case of trigonometric respectively exponential interactions one proceedes in a similar way. □

It is known (see [FKG, GRS1, Sim, GlJa]...) that on lattice we have an FKG structure which we formulate as follows:

For any increasing measurable function F, $i\delta \in \mathbb{Z}^2_\delta$:

$$\omega(i\delta) \leqq \tilde{\omega}(i\delta) \Rightarrow E^{\omega}_{\Lambda^c,\delta}(F) \leqq E^{\tilde{\omega}}_{\Lambda^c,\delta}(F) . \tag{4.11}$$

This is expressed by the writing $E^{\omega}_{\Lambda^c,\delta} \underset{\text{FKG}}{\leqq} E^{\tilde{\omega}}_{\Lambda^c,\delta}$.

We extend the definition of the symbol $\underset{\text{FKG}}{\leqq}$ also to the continuum case, writing

$$E^{\omega}_{A^c} \underset{\text{FKG}}{\leqq} E^{\tilde{\omega}}_{A^c} \quad \text{iff} \quad \omega \leqq \tilde{\omega}\ , \qquad \omega, \tilde{\omega} \in \mathscr{C}(\mathbb{R}^2) \Rightarrow E^{\omega}_{A^c} \leqq E^{\tilde{\omega}}_{A^c}\ .$$

Our Lemma 4.1 and (4.11) imply immediately:

Proposition 4.2. (FKG-structure in continuum). *For any regular set $\Lambda \in \mathscr{F}$ and any $\omega, \tilde{\omega} \in \mathscr{C}(\mathbb{R}^2)$:*

$$\omega \underset{\text{FKG}}{\leqq} \tilde{\omega} \Rightarrow E^{\omega}_{A^c} \underset{\text{FKG}}{\leqq} E^{\tilde{\omega}}_{A^c}\ . \tag{4.12}$$

Remark. FKG-structures have various consequences see e.g. [Pr, Sim, BeHK, Go, Ze 5]. In the next sections we shall exploit the above FKG-structure for the study of Gibbs measures of Euclidean fields.

In the following the concept of FKG-maximal Gibbs measure will be useful.

We call a Gibbs state $\mu \in \mathscr{G}(\mathscr{E})$ FKG-*maximal* if it is maximal with respect to the partial order $\underset{\text{FKG}}{\leqq}$ defined in $\mathscr{G}(\mathscr{E})$ by $\mu \underset{\text{FKG}}{\leqq} \mu'$ iff for any bounded measurable local increasing function F we have $\mu F \leqq \mu' F$.

Remark. In the case of compact specifications (defined in [Pr, BeHK]) the existence of FKG-maximal Gibbs states have been shown and properties of them have been studied (see [Pr, BeHK, Ze5, Go, Fö1]).

5. The Extremality

Let μ be a regular Gibbs measure for a local specification $\mathscr{E} \equiv \{E_{A^c}\}_{A \in \mathscr{F}}$ on $(\mathscr{D}', \Sigma)$ (as in Sect. 2). Let $\mathscr{F}_0$ be a countable increasing absorbing family of open sets as in Sect. 2. Let $\mathscr{G}(\mathscr{E})$ be the set of Gibbs states for $\mathscr{E}$ and $\partial\mathscr{G}(\mathscr{E})$ be the set of extremal points of $\mathscr{G}(\mathscr{E})$. We have the following

Proposition 5.1. *Suppose there is $\omega \in \mathscr{C}(\mathbb{R}^d)$ satisfying $\omega_{|\partial A} = e^{\alpha d(0, \partial A)}$ for some $\alpha > 0$ and any $\Lambda \in \mathscr{F}_0$, such that*

$$\mu = \lim_{\mathscr{F}_0} E^{\omega}_{A^c}\ . \tag{5.1}$$

If $\mathscr{E}$ is uniformly continuous μ-a.e. (*in the sense of Sect. 3*) *then*

$$\mu \in \partial\mathscr{G}(\mathscr{E})\ .$$

Proof. Let F be an increasing bounded measurable local cylinder function. Let Λ_n, $\Lambda \in \mathscr{F}_0$, $\Lambda_n \subset \Lambda$. Then for $\eta \in \Omega_{\varepsilon, \omega, n}$ [where $\Omega_{\varepsilon, \omega, n}$ is defined in (2.14)] we have

$$\begin{aligned} E^{\eta}_{A^c}(F) &= (E^{\eta}_{A^c}(F) - E^{\psi_n^{\eta}\,\partial A_\varepsilon}_{A^c}(F)) + E^{\psi_n^{\eta}\,\partial A_\varepsilon}_{A^c}(F) \\ &\leqq (E^{\eta}_{A^c}(F) - E^{\psi_n^{\eta}\,\partial A_\varepsilon}_{A^c}(F)) + E^{\omega}_{A^c}(F) \end{aligned} \tag{5.2}$$

[where we used FKG-order (Sect. 4) and the definition (2.14)].

Since $\mathscr{E}$ is uniformly continuous by our assumption and $\bigcup_{n \in \mathbb{N}} \Omega_{\varepsilon, \omega, n}$ is from Lemma 2.2 of μ-measure one (if ε, ω are suitably chosen) so the first term from the right-hand side of (5.2) converges to zero as $\Lambda \uparrow \mathbb{R}^2$. The second by our assumption

(5.1) converges to $\mu(F)$. Hence we get μ-a.e.

$$\lim_{\mathscr{F}_0'} E^{\eta}_{\Lambda^c}(F) \leqq \mu(F) = \lim_{\mathscr{F}_0'} E^{\omega}_{\Lambda^c}(F) \tag{5.3}$$

($\mathscr{F}_0' \subseteq \mathscr{F}_0$, the limit by subsequences exist by martingale convergence theorem). On the other hand by the definition of Gibbs measure

$$\mu \lim_{\mathscr{F}_0'} E^{\eta}_{\Lambda^c}(F) = \mu F \ .$$

This then, together with (5.3) implies

$$\lim E^{\eta}_{\Lambda^c}(F) = \mu(F) \ , \qquad \mu\text{-a.e.} \tag{5.4}$$

This in turn implies $\mu \in \partial\mathscr{G}(\mathscr{E})$ (see e.g. [Fö2, Pr]). □

Remark. a) The proof is similar to the case of lattice fields [Ze5].

b) The same result holds if we replace $\omega^+ \equiv \omega$ by $\omega^- \equiv -\omega$.

c) In the set of Gibbs states supported on $\mathscr{S}'$ this implies μ is FKG maximal, as shown by using similar methods as in [BeHK, Ze5], as we shall discuss later on.

The above proposition implies also a uniqueness result if we have $\lim\limits_{\mathscr{F}_0} E^{\pm\omega}_{\Lambda^c} = \mu$. Namely if $\mathscr{E}$ is $\tilde{\mu}$-a.e. uniformly continuous for ω as in Proposition 5.1 for some other probability measure $\tilde{\mu}$ and $\tilde{\mu}\ (\bigcup \Omega_{\varepsilon,\omega,n}) = 1$ we get for $\tilde{\mu}$ a.e. $\eta \in \mathscr{S}'$,

$$\mu = \lim_{\mathscr{F}_0} E^{-\omega}_{\Lambda^c} \underset{\text{FKG}}{\leqq} \lim_{\mathscr{F}_0} E^{\eta}_{\Lambda^c} \underset{\text{FKG}}{\leqq} \lim_{\mathscr{F}_0} E^{+\omega}_{\Lambda^c} = \mu \ , \tag{5.5}$$

so we have

$$\lim_{\mathscr{F}_0} E^{\eta}_{\Lambda^c} = \mu \ , \qquad \tilde{\mu} \text{ a.e.} \tag{5.6}$$

This is the uniqueness result we alluded to.

Now we will verify (5.1) in some particular models of euclidean field theory in $\mathbb{R}^2$. We will discuss separately the case of $P(\varphi)_2$ interactions (starting with the $:\varphi^4:_2$ case), exponential and trigonometric interactions.

The Extremality of the $:\varphi^4:_2$ *Model.* Let for $\Lambda \in \mathscr{F}$,

$$U_\Lambda(\varphi) \equiv \int_\Lambda :\lambda\varphi^4 + b\varphi^2 + h\varphi:_0(x)\,d_2x \tag{5.7}$$

with $\lambda > 0$, $h \geqq 0$, $b \in \mathbb{R}$ and if $h = 0$ we take $\lambda > 0$ and $b \in \mathbb{R}$ both small or $\lambda > 0$, $b \in \mathbb{R}_+$ sufficiently big (as specified below).

Let $\mathscr{E} = \{E^{\eta}_{\Lambda^c}\}_{\Lambda \in \mathscr{F}}$ be a local specification corresponding to the interaction (5.7) (cf. Sect. 3).

Let

$$\mu \equiv \lim_{\mathscr{F}_0} E^{0}_{\Lambda^c} \ . \tag{5.8}$$

This measure is known to exist (e.g. [Nel3, Sim]).

Lemma 5.2. *If* $\omega \in \mathscr{C}(\mathbb{R}^2)$, $\omega \geqq 0$ *then*

$$E^{0}_{\Lambda^c} \underset{\text{FKG}}{\leqq} E^{\omega}_{\Lambda^c} \tag{5.9}$$

and for any $f \in \mathscr{D}, f \geqq 0$

$$E^{\omega}_{\Lambda^c}\varphi(f) \leqq E^0_{\Lambda^c}\varphi(f) + \psi^{\partial\Lambda}_{\omega}(f)$$
$$- E^0_{\Lambda^c}(\varphi(f), U'(\varphi)(\psi^{\partial\Lambda}_{\omega}\chi_\Lambda)) \,, \tag{5.10}$$

with $E(F,G) \equiv E(FG) - E(F)E(G)$ *and* $\psi^{\partial\Lambda}_{\omega}(x)$ *the solution of the boundary value problem* $\psi^{\partial\Lambda}_{\omega}(x)$ *of Sect. 3, as a distribution tested with the test function f.*

Proof. The first statement has been proven in Proposition 4.2. To prove (5.10) we use the lattice approximation and the GHS inequality (e.g. [Sim]). We have

$$E^{\omega}_{\Lambda^c,\delta}(\cdot) = [\mu^{\partial\Lambda}_{0,\delta}(e^{-U_{\Lambda,\delta}(\varphi_\delta)}e^{\varphi_\delta(h^{\partial\Lambda}_{\omega,\delta})})]^{-1}$$
$$\cdot\, \mu^{\partial\Lambda}_{0,\delta}(e^{-U_{\Lambda,\delta}(\varphi_\delta)}e^{\varphi_\delta(h^{\partial\Lambda}_{\omega,\delta}}\cdot) \,, \tag{5.11}$$

with

$$h^{\partial\Lambda}_{\omega,\delta}(i\delta) \equiv \sum_{\substack{|j-i|=1\\ j\in\partial\Lambda^c_\delta}} \omega(j\delta) \quad \text{for} \quad i\delta \in \partial\Lambda_\delta \,, \tag{5.12}$$

and zero otherwise.

For $f \in \mathscr{D}, f \geqq 0$ we have

$$E^{\omega}_{\Lambda^c,\delta}\varphi(f) = E^0_{\Lambda^c,\delta}\varphi(f) + \int\limits_0^1 ds \frac{d}{ds} E^{s\omega}_{\Lambda^c,\delta}\varphi(f)$$
$$= E^0_{\Lambda^c,\delta}\varphi(f) + \int\limits_0^1 ds E^{s\omega}_{\Lambda^c,\delta}(\varphi(f), \varphi(h^{\partial\Lambda}_{\omega,\delta}))$$
$$\leqq E^0_{\Lambda^c,\delta}\varphi(f) + E^0_{\Lambda^c,\delta}(\varphi(f), \varphi(h^{\partial\Lambda}_{\omega,\delta})) \,, \tag{5.12}$$

where in the inequality we used GHS inequality.

From integration by parts we get

$$E^0_{\Lambda^c,\delta}(\varphi(f), \varphi(h^{\partial\Lambda}_{\omega,\delta}))$$
$$= \psi^{\partial\Lambda}_{\omega,\delta}(f) - E^0_{\Lambda^c,\delta}(\varphi(f), U'_\delta(\psi^{\partial\Lambda}_{\omega,\delta}(\cdot))) \,, \tag{5.14}$$

where

$$U'_\delta(\psi^{\Lambda}_{\omega,\delta}(\cdot)) \equiv \delta^2 \sum_{i\delta\in\Lambda_\delta} (4\lambda : \varphi^3_\delta :_0 (i\delta)$$
$$+ 2b\varphi_\delta(i\delta) + h)\psi^{\partial\Lambda}_{\omega,\delta}(i\delta) \,, \tag{5.15}$$

and we used that

$$\psi^{\partial\Lambda}_{\omega,\delta}(i\delta) = \sum_{j\delta\in\partial\Lambda_\delta} G(i\delta, j\delta) \sum_{\substack{j'\delta\in\partial\Lambda^c_\delta\\ |j-j'|=1}} \varrho(j'\delta) \tag{5.16}$$

is the solution of the free Dirichlet problem for $(-\Delta_\delta + m_0^2)$ on the lattice (see Lemma A4.2).

Now using (5.13) and the fact that from Lemma 4.1 respectively A4.3 $E^0_{\Lambda^c,\delta}$ and $\psi^{\partial\Lambda}_{\omega,\delta}$ converge as $\delta\to 0$ we get (5.10). □

From the above lemma we get that if $E^0_{\Lambda^c}$ has the cluster property uniformly in the volume (which we have by the above mentioned choice of parameters λ, h, b)

then the right-hand side of (5.10) converges to zero as $\Lambda \uparrow \mathbb{R}^2$ if $\omega(x)=e^{\alpha\|x\|}$ for some $0<\alpha<m$, where m is the physical mass for the model [Sim]. Using the fact that equality of first moments and FKG inequality imply equality [FrSi] and Proposition 5.1 we obtain

Proposition 5.3. *If the interaction U_Λ is given by (5.7) with λ, h, b as specified there, then*

$$\mu \equiv \lim_{\mathscr{F}_0} E^0_{\Lambda^c} = \lim_{\mathscr{F}} E^\omega_{\Lambda^c} \tag{5.17}$$

for $\omega(x)=e^{\alpha\|x\|}$ with some α such that $0<\alpha<m$ (m the physical mass). Moreover

$$\mu \in \partial\mathscr{G}(\mathscr{E}) \; .$$

Proof. a) By analogous arguments we can get the same result for $h<0$.
b) A similar result has been obtained before in the lattice case [Ze5].
c) The parameter α in the above Proposition can be chosen to be an increasing function of m_0, since the physical mass is increasing with m_0 (cf. [GlJa]).

Since to get Proposition 5.3 we used only (the convergence of the lattice approximation and) the fact that the measure $E^{\omega_0}_{\Lambda^c}$ for $0<\omega_0 \leqq \omega$ has a cluster property uniform in the volume, so by analogous arguments (taking ω_0 to be a constant boundary condition) we get in the multiphase region (when $h=0$, b sufficiently smaller than -1) the extremality for the FKG-maximal measures.

We note that the existence of $E^{\omega_0}_{\Lambda^c}$ in this case follows from [Im2, Gid].

Let μ_h be the measure defined by (5.8) with $h \neq 0$ in (5.7). We define the Gibbs measure μ_+ as

$$\mu_+(\cdot) \equiv \lim_{\Lambda \uparrow \mathbb{R}^2} \mu_h(e^{h\varphi(\chi_\Lambda)} \cdot)/\int \mu_h(e^{h\varphi(\chi_\Lambda)}) \;, \tag{5.18}$$

for $h>0$ large enough [FrSi]. Similarly we define μ_- by the same formula with $h<0$, $|h|$ sufficiently large. By [FrSi] we have $\mu_\pm$ are FKG-maximal.

We can then formulate the following

Proposition 5.4. *For the interaction (5.7) with $h=0$ and $b<-1$, $|b|$ sufficiently big, the* FKG-*maximal measures $\mu_\pm$ satisfy*

$$\mu_\pm = \lim_{\mathscr{F}_0} E^{\pm\omega}_{\Lambda^c} \tag{5.19}$$

for any ω of the form $\omega(x)=e^{\alpha\|x\|}$ for some $\alpha>0$. Moreover they are extremal.

Proof. Let $\mathscr{E}=\{E_{\Lambda^c}\}_{\Lambda\in\mathscr{F}}$ be the local specification for the interaction (5.2) with h, b as assumed. Then, by Theorem 3.1, $\mathscr{E}$ is μ-a.e. uniformly continuous for any $h\in\mathbb{R}$. Hence we get, as for (5.2)

$$\mu_+ \underset{\text{FKG}}{\leqq} \lim_{\mathscr{F}_0} E^\omega_{\Lambda^c} \tag{5.20}$$

(and this holds also, with the reverse inequality, for $h<0$ and $-\omega$ instead of ω). On the other hand for any Gibbs measure μ for $\mathscr{E}$ we have

$$\mu \leqq \mu_+ \; . \tag{5.21}$$

So if $\lim_{\mathscr{F}_0} E^{+\omega}_{\Lambda^c}$ exists we get (5.19) (and analogously for μ_-), [using $\mu_\pm$ are maximal and (5.18)].

Since one can show [Im2, Gid] that

$$\mu^+ = \lim_{\mathscr{F}_0} E^{\omega_0^+}_{\Lambda^c}$$

with $\omega_0^+ > 0$ being a translationally invariant configuration of minimal energy, so using Lemma 5.2 (with ω_0^+ instead of zero) and the fact that $E^{\omega_0^+}_{\Lambda^c}$ has a uniform in $\Lambda \in \mathscr{F}_0$ cluster property [Im2, Gid] we get

$$\mu^+ = \lim_{\mathscr{F}_0} E^{\omega_0^+}_{\Lambda^c} = \lim_{\mathscr{F}_0} E^{\omega^+}_{\Lambda^c} \tag{5.22}$$

(here we used the GHS argument of Lemma 5.2.)

(Analogous arguments work for μ_-.) □

The Extremality for Exponential Models. Let for $\Lambda \in \mathscr{F}$

$$U_\Lambda(\varphi) \equiv \lambda \int_\Lambda d_2x \int d\varrho(\alpha) : e^{\alpha\varphi} :_0 (x) \tag{5.23}$$

with $\lambda > 0$ and $d\varrho(\alpha)$ a probability measure on $(-2\sqrt{\pi}, 2\sqrt{\pi})$.

Define $E^\eta_{\Lambda^c}$ for above interaction $U_\Lambda(\varphi)$ as in Sect. 3.

Lemma 5.5. *Let* $\omega \in \mathscr{C}(\mathbb{R}^2)$, $\omega \geqq 0$, *then*

$$E^0_{\Lambda^c} \leqq E^\omega_{\Lambda^c}\ , \tag{5.24}$$

and for any $f \in \mathscr{D}, f \geqq 0$,

$$E^\omega_{\Lambda^c}\varphi(f) \leqq E^0_{\Lambda^c}\varphi(f) + \psi^{\partial\Lambda}_\omega(f)\ . \tag{5.25}$$

Proof. (5.24) follows from the convergence of lattice approximation. To show (5.25) we take first the lattice approximation $E^\omega_{\Lambda^c,\delta}$ of $E^\omega_{\Lambda^c}$. Then we have

$$\begin{aligned} E^\omega_{\Lambda^c,\delta}(\varphi_\delta(f)) &= E^0_{\Lambda^c,\delta}(\varphi_\delta(f)) + \int_0^1 ds \frac{d}{ds} E^{s\omega}_{\Lambda^c,\delta}(\varphi_\delta(f)) \\ &= E^0_{\Lambda^c,\delta}(\varphi_\delta(f)) + \int_0^1 ds\, E^{s\omega}_{\Lambda^c,\delta}(\varphi_\delta(f), \varphi_\delta(h^{\partial\Lambda}_{\omega,\delta}))\ , \end{aligned} \tag{5.26}$$

where $h^{\partial\Lambda}_{\omega,\delta}$ was defined in (5.12).

Using the arguments from the proof of Lemma 1.5.1 of [Ze1] we get for $\omega \geqq 0$,

$$\begin{aligned} 0 &\leqq E^{s\omega}_{\Lambda^c,\delta}(\varphi_\delta(f), \varphi_\delta(h^{\partial\Lambda}_{\omega,\delta})) \leqq \mu^{\partial\Lambda}_{0,\delta}(\varphi_\delta(f)\varphi_\delta(h^{\partial\Lambda}_{\omega,\delta})) \\ &= \psi^{\partial\Lambda}_{\omega,\delta}(f) \end{aligned} \tag{5.27}$$

(where we used the definition of $\psi^{\partial\Lambda}_{\omega,\delta}$).

From (5.26) and (5.27) and convergence of lattice approximation (Sect. 4) we get

$$E^\omega_{\Lambda^c}\varphi(f) \leqq E^0_{\Lambda^c}\varphi(f) + \psi^{\partial\Lambda}_\omega(f)\ . \quad \square \tag{5.28}$$

Remark. The same holds if one adds the term $h\varphi(\chi_\Lambda)$ with $h \in \mathbb{R}$ to the interaction (5.23) [since (5.27) holds in the same way).

As a consequence of Lemma 5.5 and Proposition 5.1 we get

Proposition 6.5. *The measure* $\mu \equiv \lim_{\mathcal{F}_0} E^0_{\Lambda^c}$ *satisfies*

$$\mu = \lim_{\mathcal{F}_0} E^{\omega}_{\Lambda^c} \tag{5.29}$$

for $\omega = e^{\alpha \|x\|}$ *and* $0 < \alpha < m_0$, *and so*

$$\mu \in \partial \mathcal{G}(\mathcal{E}) \ .$$

Remark. The parameter α can be chosen to be an increasing function of m_0 (cf. [Ze1]). We can extend the above proof to the case of trigonometric interactions

$$U_\Lambda(\varphi) \equiv \lambda \int_\Lambda d_2x : \sin(\alpha\varphi(x) + \beta): \ , \tag{5.30}$$

λ small, $|\alpha| < \sqrt{2\pi}$, $0 \leqq \beta < 2\pi$, by using an integration by parts in (5.27) and using cluster expansion arguments of [AHK7, FrSe].

We summarize the results of this section in the following

Theorem 5.6. *Let* U_Λ *be of the* φ^4 *respectively exponential respectively trigonometric form given by* (5.7), *respectively* (5.23) *respectively* (5.30). *Then the Gibbs measures* $\mu^{\pm} = \lim_{\mathcal{F}_0} E^{\pm\omega}_{\Lambda^c}$ *with* $\omega(x) = e^{\alpha\|x\|}$, $0 < \alpha < m$, *(with m the physical mass) are extremal Gibbs states. In the case of weak coupling* φ^4 *or trigonometric or exponential interactions we have uniqueness of Gibbs states in the sense that* $\mu_+ = \mu_-$ *(hence the set of tempered Gibbs states has only 1 point).*

Remark. The parameter α can be chosen to be an increasing function of the free mass m_0.

6. The Global Markov Property

Let $Q \subset \mathbb{R}^2$ be an unbounded open set with a piecewise $\mathcal{C}^1$-boundary ∂Q and such that $\mathbb{R}^2 - Q$ is also unbounded. Let $\Lambda \in \mathcal{F}_0$ (with $\mathcal{F}_0$ as in Sect. 2), $\Lambda \cap Q \neq \emptyset$ and $\Lambda \cap Q^c \neq \emptyset$. We assume that $\partial\Lambda \cap \partial Q$ consists of a finite number of points. We say that a probability measure μ on $(\mathcal{D}', \Sigma)$ has the *global Markov property* (GMP) if for any $F \in \Sigma(Q)$, $G \in \Sigma(Q^c)$ bounded measurable we have

$$E(FG|\Sigma(\partial Q)) = E(F|\Sigma(\partial Q))E(G|\Sigma(\partial Q)) \ ,$$

where $E(\cdot|\Sigma(\partial Q))$ means conditional expectation with respect to μ and $\Sigma(\partial Q)$ (cf. [AHK7, Fö1]).

We write then for simplicity $\mu \in$ GMP. We say μ has the local Markov property if the above relations only holds with Q replaced by $Q \cap \Lambda$ with Λ bounded and open.

Let $\omega \in \mathcal{C}(\mathbb{R}^2)$ and μ an extremal Gibbs measure as in Proposition 5.1 such that

$$\mu \equiv \lim_{\mathcal{F}_0} E^0_{\Lambda^c} = \lim_{\mathcal{F}_0} E^{\omega}_{\Lambda^c} \ , \tag{6.1}$$

with $E^{\omega}_{\Lambda^c}$ belonging to a specification for an interaction of the type considered in Sect. 5. Let $\psi_z^{\partial(\Lambda \cap Q)}(x)$ be the Poisson kernel considered in Sect. 2 and consider

$$z \to f(z) \equiv \chi_{z \in \partial Q} \psi_z^{\partial(\Lambda \cap Q)}(x) \ .$$

This is an element of $H_{-1}(\mathbb{R}^2)$. Consider $\varphi(f)$ with $\varphi\in\operatorname{supp}\mu$. We denote this random variable also by

$$\eta(\chi_{z\in\partial Q}\psi_z^{\partial(\Lambda\cap Q)}(x))\equiv\psi_{\eta|\partial Q}^{\partial(\Lambda\cap Q)}(x) .$$

Let us define

$$\mu_{Q,\Lambda}(\cdot)\equiv\mu(E_{(\Lambda\cap Q)^c}^{\eta|\partial Q}(\cdot)) . \tag{6.2}$$

Here $E_{(\Lambda\cap Q)^c}^{\eta/\partial Q}$ is defined as $E_{(\Lambda\cap Q)^c}^{\eta}$ but with $\eta(\chi(z\in\partial Q)\psi_z^{\partial(\Lambda\cap Q)}(x))$ instead of $\psi_\eta^{\partial(\Lambda\cap Q)}(x)$.

Remark. If more general interactions than those of Sect. 5 are considered then we should replace $E_{(\Lambda\cap Q)^c}^{\eta|\partial Q}$ by $E_{(\Lambda\cap Q)^c}^{\eta|\partial Q+\omega_0|\partial\Lambda}$ for some bounded configuration $\omega_0\in\mathscr{C}(\mathbb{R}^2)$.

Suppose that the interaction which gives the specification is symmetric (i.e. invariant under $\varphi\to-\varphi$) and we have using this symmetry and the definition of $E_{(\Lambda\cap Q)^c}^{\eta|\partial Q}$ for any $f\in\mathscr{D}, f\geqq 0$

$$\mu\varphi(f)=0=\mu_{Q,\Lambda}\varphi(f) \tag{6.3}$$

(for any $\Lambda\in\mathscr{F}_0$).

Our assumption (6.3) implies that if

$$\lim_{\mathscr{F}_0}\mu_{Q,\Lambda}\underset{\text{FGK}}{\leqq}\mu \tag{6.4}$$

then by [FrSi],

$$\lim_{\mathscr{F}_0}\mu_{Q,\Lambda}=\mu . \tag{6.5}$$

By an argument in [Go] this implies that μ has the global Markov property i.e.

$$\mu\in\text{GMP} \tag{6.6}$$

To prove (6.4) let us take, for a fixed $\Lambda_0\in\mathscr{F}_0$, $G\in\Sigma_{Q^c\cap\Lambda_0}$, $F\in\Sigma_{Q\cap\Lambda_0}$ to be some bounded (measurable) cylindric and non-negative functions. Then for $\Lambda_0\subset\Lambda\in\mathscr{F}_0$ we have with $\varepsilon>0$, using the definition of $\mu_{Q,\Lambda}$ and the properties of specifications

$$\begin{aligned}\mu_{Q,\Lambda}GF&\equiv\mu(GE_{(\Lambda\cap Q)^c}^{\eta|\partial Q}F)=\mu[E_{\Lambda^c}^{\eta}(GE_{(\Lambda\cap Q)^c}^{\tilde\eta|\partial Q}F)]\\&=\mu[(E_{\Lambda^c}^{\eta}-E_{\Lambda^c}^{\psi_\eta^{\partial\Lambda_\varepsilon}})(GE_{(\Lambda\cap Q)^c}^{\tilde\eta|\partial Q}F)]\\&\quad+\mu[E_{\Lambda^c}^{\psi_\eta^{\partial\Lambda_\varepsilon}}(GE_{(\Lambda\cap Q)^c}^{\tilde\eta|\partial Q}F)] ,\end{aligned} \tag{6.7}$$

with $\tilde\eta$ the integration variable with respect to the measure $E_{\Lambda^c}^{\eta}$.

Let us consider $E_{(\Lambda\cap Q)^c}^{\tilde\eta|\partial Q}F$, for F bounded increasing.

By first replacing $\tilde\eta$ by a regularized version $\tilde\eta_\kappa$, taking a lattice approximation $E_{(\Lambda\cap Q)^c,\delta}^{\tilde\eta_\kappa}(F)$ of $E_{(\Lambda\cap Q)^c}^{\tilde\eta_\kappa}$, using FKG order [since $E_{(\Lambda\cap Q)^c,\delta}^{\tilde\eta_\kappa}(F)$ is an increasing cylinder function] we get for G bounded increasing

$$E_{\Lambda^c}^{\psi_\eta^{\partial\Lambda_\varepsilon}}(GE_{(\Lambda\cap Q)^c,\delta}^{\tilde\eta_\kappa}F)\leqq E_{\Lambda^c}^{\omega}(GE_{(\Lambda\cap Q)^c,\delta}^{\tilde\eta_\kappa}F) , \tag{6.8}$$

if $\eta\in\Omega_{\varepsilon,\omega,n}$ [this set is defined in (2.14)] so that $\psi_\eta^{\partial\Lambda_\varepsilon}\leqq\omega$.

Now taking the continuum limit $\delta\downarrow 0$ and afterwards removing the regularization κ we get

$$E_{\Lambda^c}^{\psi_\eta^{\partial\Lambda_\varepsilon}}(GE_{(\Lambda\cap Q^c)}^{\tilde\eta}F)\leqq E_{\Lambda^c}^{\omega}(GE_{(\Lambda\cap Q)^c}^{\tilde\eta}F) . \tag{6.9}$$

Let $\chi_{\Lambda_n,\omega}$ be the characteristic function of $\Omega_{\varepsilon,\omega,n}$. Then we get from (6.8), integrating with respect to μ, inserting $(1-\chi_{\Lambda,\omega})+\chi_{\Lambda,\omega}$ on the right-hand side and bounding $\chi_{\Lambda,\omega}$ by 1:

$$\mu E_{\Lambda^c}^{\psi_\eta^{\partial\Lambda_\varepsilon}}(GE_{(\Lambda\cap Q)^c}^{\tilde\eta|\partial Q}F)\leqq E_{\Lambda^c}^{\omega}(GE_{(\Lambda\cap Q)^c}^{\tilde\eta|\partial Q}F)+\mu(1-\chi_{\Lambda,\omega})\|F\|_\infty\|G\|_\infty \ . \tag{6.10}$$

Using

$$E_{(\Lambda\cap Q)^c}^{\tilde\eta|\partial Q}(F)\leqq E_{(\Lambda\cap Q)^c}^{\tilde\eta|\partial Q+\omega_{\partial\Lambda}}(F) \tag{6.11}$$

[which is proven again by going to a lattice as in (6.8) and using FKG order], and the compatibility conditions of a specification, we get from (6.11)

$$E_{\Lambda^c}^{\omega}(GE_{(\Lambda\cap Q)^c}^{\tilde\eta|\partial Q}(F))\leqq E_{\Lambda^c}^{\omega}(GF) \ . \tag{6.12}$$

Inserting (6.12) into (6.10) we get

$$\mu E_{\Lambda^c}^{\psi_\eta^{\partial\Lambda_\varepsilon}}(GE_{(\Lambda\cap Q)^c}^{\tilde\eta|\partial Q}F)\leqq E_{\Lambda^c}^{\omega}(GF)+\mu(1-\chi_{\Lambda,\omega})\|F\|_\infty\|G\|_\infty \ . \tag{6.13}$$

Recalling $\chi_{\Lambda,\omega}=\chi_{\Omega_\varepsilon,\omega,n}$ for some n (since $\Lambda\in\mathscr{F}$) we can choose ε as a function of Λ s.t.

$$\mu(1-\chi_{\Lambda,\omega})\to 0 \quad \text{as} \quad \Lambda\uparrow\mathbb{R}^2 \ , \tag{6.14}$$

as in Lemma 2.2.

Inserting this into (6.13) and by going to subsequences we get

$$\lim_{\mathscr{F}_0}\ \mu E_{\Lambda^c}^{\psi_\eta^{\partial\Lambda_\varepsilon}}(GE_{(\Lambda\cap Q)^c}^{\tilde\eta|\partial Q}F)\leqq\lim_{\mathscr{F}_0}\ E_{\Lambda^c}^{\omega}(GF) \ , \tag{6.15}$$

and the limit on the right-hand side, by (6.1), is equal to

$$\mu(GF) \ .$$

Hence, using (6.7) we see that if

$$\lim_{\mathscr{F}_0}\ |\mu(E_{\Lambda^c}^{\eta}-E_{\Lambda^c}^{\psi_\eta^{\partial\Lambda_\varepsilon}})(GE_{(\Lambda\cap Q)^c}^{\tilde\eta|\partial Q}F)|=0 \ , \tag{6.16}$$

then

$$\lim_{\mathscr{F}_0}\ \mu_{Q,\Lambda}GF\leqq\mu GF \ . \tag{6.17}$$

This implies, by an approximation argument and the definition of FKG-order

$$\lim_{\mathscr{F}_0}\ \mu_{Q,\Lambda}\underset{\text{FKG}}{\leqq}\mu \ , \tag{6.18}$$

which together with our assumption (6.3) gives (6.5) and so the global Markov property for the measure μ.

Let us now consider (6.16). We can and do assume that

$$F(\varphi)=\tilde F(\varphi(f_1),\dots,\varphi(f_n)) \tag{6.19}$$

with $\tilde F\in\mathscr{C}^1(\mathbb{R}^n)$, $f_i\in\mathscr{D}$, $\operatorname{supp} f_i\subset Q\cap\Lambda$, $f_i\geqq 0$ and $\|\tilde F\|_\infty$, $\|\partial_i\tilde F\|<\infty$ (with ∂_i the derivation in the i-th argument).

We also assume corresponding properties for G.

Let

$$\delta U_{\Lambda,\varepsilon}(\varphi)\equiv U_\Lambda(\varphi+\psi_\varphi^{\partial\Lambda_\varepsilon}-\psi_\varphi^{\partial\Lambda})-U_\Lambda(\varphi) \ . \tag{6.20}$$

Using the definition of $E_{A^c}^{\psi_\eta^{\partial A_\varepsilon}}$ given at the beginning of Sect. 3 we have, calling φ the integration parameter with respect to $E_{A^c}^{\psi_\eta^{\partial A_\varepsilon}}$, setting $\delta_\varepsilon \psi_\eta^{\partial A} \equiv \psi_\eta^{\partial A_\varepsilon} - \psi_\eta^{\partial A}$ and proceeding as in the proof of (3.5),

$$\mu(E_{A^c}^\eta - E_{A^c}^{\psi_\eta^{\partial A_\varepsilon}})(GE_{(A\cap Q)^c}^{\varphi|\partial Q}F)$$
$$= \mu E_{A^c}^\eta \left\{ [G(\varphi)E_{(A\cap Q)^c}^{\varphi|\partial Q}(F)] - \left[\frac{e^{-\delta U_{A,\varepsilon}(\varphi)}}{E_{A^c}^\varphi(e^{-\delta U_{A,\varepsilon}(\cdot)})} G(\varphi + \delta_\varepsilon \psi_\eta^{\partial A}) E_{(A\cap Q)^c}^{(\varphi+\delta_\varepsilon \psi_\eta^{\partial A})|\partial Q}(F) \right] \right\} . \tag{6.21}$$

Using the definition of conditional expectation we can omit $E_{A^c}^\eta$ and replace everywhere η by φ. By adding and subtracting the term

$$\mu[G(\varphi+\delta_\varepsilon \psi_\varphi^{\partial A}) E_{(A\cap Q)^c}^{(\varphi+\delta_\varepsilon \psi_\varphi^{\partial A})|\partial Q}(F)] , \tag{6.22}$$

we can rewrite (6.21) in the form $A+B$, with

$$A \equiv \mu\{G(\varphi)E_{(A\cap Q)^c}^{\varphi|\partial Q}(F) - G(\varphi+\delta_\varepsilon \psi_\varphi^{\partial A}) E_{(A\cap Q)^c}^{(\varphi+\delta_\varepsilon \psi_\varphi^{\partial A})|\partial Q}(F)\} , \tag{6.23}$$

$$B \equiv \mu \left\{ \left(\left[1 - \frac{e^{-\delta U_{A,\varepsilon}(\varphi)}}{E_{A^c}^\varphi(e^{-\delta U_{A,\varepsilon}(\cdot)})} \right] G(\varphi+\delta_\varepsilon \psi_\varphi^{\partial A}) \right) E_{(A\cap Q)^c}^{[\varphi+\delta_\varepsilon \psi_\varphi^{\partial A}]|\partial Q}(F) \right\} . \tag{6.24}$$

Since by our assumption $G \in \Sigma_{Q^c \cap A_0}$ for some $A_0 \in \mathscr{F}_0$, so

$$G(\varphi+\delta_\varepsilon \psi_\varphi^{\partial A}) \underset{\mathscr{F}_0}{\longrightarrow} G(\varphi)\, \mu\text{-a.e. } \varphi . \tag{6.25}$$

This is seen using the fact that G is a cylinder function with the assumed properties (6.19), so that

$$|G(\varphi+\delta_\varepsilon \psi_\varphi^{\partial A}) - G(\varphi)| \leq \sum_{i=1}^n \|\partial_i \tilde{G}\|_\infty |\delta_\varepsilon \psi_\varphi^{\partial A}(g_i)|$$

and the right-hand side goes to zero by the properties of $\psi_\varphi^{\partial A}$ as $A \uparrow \mathbb{R}^2$.

From the proof of the uniform continuity of local specifications we know by (3.8) that for a subsequence $A \uparrow \mathbb{R}^2$

$$\left| 1 - \frac{e^{-\delta U_{A,\varepsilon}(\varphi)}}{E_{A^c}^\varphi(e^{-\delta U_{A,\varepsilon}(\cdot)})} \right| \underset{\mathscr{F}_0}{\to} 0 , \quad \mu\text{-a.e. } \varphi . \tag{6.26}$$

(6.25), (6.26), together with properties of conditional expectations, imply that $B \to 0$ as $A \uparrow \mathbb{R}^2$, by subsequences. Hence to prove (6.6) using (6.25) and the uniform bounds on F, G, we need only show that μ-a.e.

$$[E_{(A\cap Q)^c}^{\varphi|\partial Q}(F) - E_{(A\cap Q)^c}^{(\varphi+\delta_\varepsilon \psi_\varphi^{\partial A})|\partial Q}(F)] \underset{\mathscr{F}_0}{\longrightarrow} 0 . \tag{6.27}$$

This will be shown in Lemmas 6.1–6.3, in which we assume that our interaction U_A is such that the measure μ is constructed by the cluster expansion [as for weak $P(\varphi)_2$ and weak trigonometric interactions].

Lemma 6.1. *For any F like in* (6.19) *we have for $\Lambda \uparrow \mathbb{R}^2$ along some subsequence $\mathscr{F}_0' \subset \mathscr{F}_0$*

$$[E^{\varphi|\partial Q}_{(\Lambda\cap Q)^c}(F) - E^{(\varphi+\delta_\varepsilon\psi^{\partial\Lambda}_\varphi)_{|\partial Q}}_{(\Lambda\cap Q)^c}(F)] \to 0 \ , \qquad \mu\text{-a.e. .}$$

Proof. Let us introduce the notation, for any $s \in [0,1]$:

$$[F]_{s\varepsilon} \equiv [\mu_0^{\partial\Lambda\cap Q} e^{-U_{\Lambda\cap Q}(\varphi'+\psi_\varphi^{\partial\Lambda\cap Q})}]^{-1}$$
$$\mu_0^{\partial\Lambda\cap Q}(e^{-U_{\Lambda\cap Q}(\varphi'+\Psi^{\partial\Lambda\cap Q}_{\varepsilon,s}(\varphi))} F(\varphi'+\psi^{\partial\Lambda\cap Q}_{\varphi|\partial Q})) \tag{6.28}$$

with φ' an integration variable with respect to the measure $\mu_0^{\partial\Lambda\cap Q}$ and

$$\Psi^{\partial\Lambda\cap Q}_{\varepsilon,s}(\varphi) \equiv \psi^{\partial\Lambda\cap Q}_{(\varphi+s\delta_\varepsilon\psi^{\partial\Lambda}_\varphi)_{|\partial Q}} \ . \tag{6.29}$$

Then we have for F as in (6.19), by adding and subtracting $F(\varphi'+\psi^{\partial\Lambda\cap Q}_{\varphi_{|Q}})$ and using the mean value theorem:

$$|E^{(\varphi+\delta_\varepsilon\psi^{\partial\Lambda}_\varphi)_{|\partial Q}}_{(\Lambda\cap Q)^c}(F) - E^{\varphi|\partial Q}_{(\Lambda\cap Q)^c}(F)|$$
$$\leqq \sum_{i=1}^{n} \|\partial_i\tilde{F}\|_\infty |\psi^{\partial\Lambda\cap Q}_{\delta_\varepsilon\psi^{\partial\Lambda}_\varphi|\partial Q}(f_i)|$$
$$+\left|\frac{[F]_\varepsilon}{[\mathbb{1}]_\varepsilon} - \frac{[F]_0}{[\mathbb{1}]_0}\right| \ . \tag{6.30}$$

The first term in the right-hand side goes to 0 as $\Lambda \uparrow \mathbb{R}^2$ by the construction of the Dirichlet solution $\psi^{\partial\Lambda}_\eta$ and the regularity of μ (this is similar to [AHK7]).

The second term goes to zero as a consequence of $[F]_\varepsilon - [F] \to 0$ and $[\mathbb{1}]_\varepsilon \to [\mathbb{1}]_0$ pointwise for a subsequence of $\mathscr{F}_0$, on a subset of μ-measure one, as shown in the next lemma. □

Lemma 6.2. *Let $\varepsilon(\Lambda) \equiv e^{-\gamma d(0,\partial\Lambda)}$ for some $\gamma > 0$, and let $[F]_\varepsilon$ be as in* (6.28). *Then there exists a $\gamma_0 > 0$* s.t. *for all $\gamma \geqq \gamma_0$ and some $\mathscr{F}_0' \subset \mathscr{F}_0$, F any bounded measurable function*

$$\lim_{\mathscr{F}_0'} \mu|[F]_{\varepsilon(\Lambda)} - [F]_0| = 0 \ .$$

Here $\mu = \lim_{\Lambda\in\mathscr{F}_0} \mu_\Lambda$ is a limit of finite volume measures

$$\mu_\Lambda \equiv \mu_0(e^{-U_\Lambda}\cdot)/Z_\Lambda \ , \qquad Z_\Lambda \equiv \mu_0(e^{-U_\Lambda}) \ .$$

U_Λ is an interaction as in Proposition 5.1 $[F]_{\varepsilon(\Lambda)}$ is defined by (6.28).

Proof. Let Λ, $\tilde{\Lambda} \in \mathscr{F}_0$, $\Lambda \cap Q \subset \tilde{\Lambda}$. Then by definition of $\mu_{\tilde{\Lambda}}$ and $[F]_{s\varepsilon}$ we have:

$$\mu_{\tilde{\Lambda}}(|[F]_\varepsilon - [F]_0) = Z_{\tilde{\Lambda}}^{-1} \ \cdot$$
$$\mu_0 e^{U_{\tilde{\Lambda}-(\Lambda\cap Q)}} |\mu_0^{\partial\Lambda\cap Q}(e^{-U_{\Lambda\cap Q}(\varphi'+\Psi^{\partial\Lambda\cap Q}_{\varepsilon,1}(\varphi))}$$
$$F(\varphi'+\psi^{\partial\Lambda\cap Q}_{\varphi|\partial Q})) - \mu_0^{\partial\Lambda\cap Q}(e^{-U_{\Lambda\cap Q}(\varphi'+\psi^{\partial\Lambda\cap Q}_{\varphi|\partial Q})} \tag{6.31}$$
$$F(\varphi'+\psi^{\partial\Lambda\cap Q}_{\varphi|\partial Q}))| \ ,$$

with $\Psi^{\partial\Lambda\cap Q}_{\varepsilon,1}$ given by (6.29).

Using the definition of conditional expectation with respect to μ_0, (6.31) can be bounded as follows (using also the definition of $\mu_{\tilde{\Lambda}}$):

$$\begin{aligned}&\mu_{\tilde{\Lambda}}(|[F]_\varepsilon-[F]_0|)\\&\leqq \mu_{\tilde{\Lambda}}\{|\exp(-[U_{\Lambda\cap Q}(\varphi-\psi^{\partial\Lambda\cap Q}_{\varphi|\partial\Lambda}+\psi^{\partial\Lambda\cap Q}_{\delta_\varepsilon\psi^{\partial\Lambda}_\varphi|\partial Q})\\&\quad -U_{\Lambda\cap Q}(\varphi)])F(\varphi-\psi^{\partial\Lambda\cap Q}_{\varphi|\partial\Lambda})\\&\quad -\exp(-[U_{\Lambda\cap Q}(\varphi-\psi^{\partial\Lambda\cap Q}_{\varphi|\partial\Lambda})-U_{\Lambda\cap Q}(\varphi)])\\&\quad \cdot F(\varphi-\psi^{\partial\Lambda\cap Q}_{\varphi|\partial\Lambda})|\}\ .\end{aligned}\tag{6.32}$$

Since $\tilde{\Lambda}\in\mathscr{F}_0$ was arbitrary we get, using the definition of μ, the same inequality with μ replacing $\mu_{\tilde{\Lambda}}$.

We shall now prove that from Lemma 6.3 we can finish the proof of Lemma 6.2 (and hence also of Lemma 6.1).

It is easy to see that what is needed is an estimate of the right-hand side of (6.32) with F replaced by 1 (this is so as seen by the fact that $\|F\|_\infty<\infty$). This latter estimate is a consequence of the following Lemma 6.3.

Lemma 6.3. *Let $\Lambda\in\mathscr{F}_0$ be a rectangle with $\Lambda\cap Q\neq\emptyset$, $\Lambda\cap Q^c\neq\emptyset$. We assume also that $|\Lambda\cap\partial Q|<c'|\partial\Lambda|$ with a constant $c'>0$ independent of $\Lambda\in\mathscr{F}_0$. Then there exists* $\gamma_0>0$ s.t. *for $\gamma\geqq\gamma_0$ and $\varepsilon=e^{-\gamma d(0,\partial\Lambda)}$:*

$$\begin{aligned}&\mu|e^{-[U_{\Lambda\cap Q}(\varphi-\psi^{\partial\Lambda\cap Q}_{\varphi|\partial\Lambda}+\psi^{\partial\Lambda\cap Q}_{\delta_\varepsilon\psi^{\partial\Lambda}_\varphi|\partial Q})-U_{\Lambda\cap Q}(\varphi)]}\\&\quad -e^{-[U_{\Lambda\cap Q}(\varphi-\psi^{\partial\Lambda\cap Q}_{\varphi|\partial\Lambda})-U_{\Lambda\cap Q}(\varphi)]}|\leqq\varepsilon^a\end{aligned}\tag{6.33}$$

for some $1>a>0$ independent of $\Lambda\in\mathscr{F}_0$, γ and μ.

Proof. Let $\tilde{\Lambda}_k\equiv\{x\in\Lambda\cap Q: d(x,\partial Q\cup\partial\Lambda)>3k\}$ for $1\leqq k\leqq\bar{n}$, with $\bar{n}=\dfrac{d(0,\partial\Lambda)}{3}+1$. Let $\chi_{\tilde{\Lambda}_k}$ be the characteristic function of $\tilde{\Lambda}_k$. First adding and subtracting a $\tilde{\Lambda}_1$-depending term, we get by the triangle inequality the estimate:

$$\begin{aligned}\text{l.h.s. (6.33)}&\leqq\mu|\exp(-[U_{\Lambda\cap Q}(\varphi-\psi^{\partial\Lambda\cap Q}_{\varphi|\partial\Lambda}+\psi^{\partial\Lambda\cap Q}_{\delta_\varepsilon\psi^{\partial\Lambda}_\varphi|\partial Q})\\&\quad -U_{\Lambda\cap Q}(\varphi)])-\exp(-[U_{\Lambda\cap Q}(\varphi-\psi^{\partial\Lambda\cap Q}_{\varphi|\partial\Lambda}+\chi_{\tilde{\Lambda}_1}\psi^{\partial\Lambda\cap Q}_{\delta_\varepsilon\psi^{\partial\Lambda}_\varphi|\partial Q})\\&\quad -U_{\Lambda\cap Q}(\varphi)])|\\&\quad +\mu|\exp(-[U_{\Lambda\cap Q}(\varphi-\psi^{\partial\Lambda\cap Q}_{\varphi|\partial\Lambda}+\chi_{\tilde{\Lambda}_1}\psi^{\partial\Lambda\cap Q}_{\delta_\varepsilon\psi^{\partial\Lambda}_\varphi|\partial Q})-U_{\Lambda\cap Q}(\varphi)])\\&\quad -\exp(-[U_{\Lambda\cap Q}(\varphi-\psi^{\partial\Lambda\cap Q}_{\varphi|\partial\Lambda})-U_{\Lambda\cap Q}(\varphi)])|\ .\end{aligned}\tag{6.34}$$

In the same way, adding and subtracting a $\tilde{\Lambda}_2$-depending term in the second term of (6.34), estimating as before and then going in this way by $\bar{n}-1-$fold iteration we get:

$$\begin{aligned}\text{l.h.s. (6.33)}&\leqq\mu|\exp(-[U_{\Lambda\cap Q}(\varphi-\psi^{\partial\Lambda\cap Q}_{\varphi|\partial\Lambda}+\psi^{\partial\Lambda\cap Q}_{\delta_\varepsilon\psi^{\partial\Lambda}_\varphi|\partial Q})\\&\quad -U_{\Lambda\cap Q}(\varphi)])-\exp(-[U_{\Lambda\cap Q}(\varphi-\psi^{\partial\Lambda\cap Q}_{\varphi|\partial\Lambda}+\chi_{\tilde{\Lambda}_1}\psi^{\partial\Lambda\cap Q}_{(\delta_\varepsilon\psi^{\partial\Lambda}_\varphi)|\partial Q})\\&\quad -U_{\Lambda\cap Q}(\varphi)])|\end{aligned}$$

$$+\sum_{k=1}^{\bar{n}-1} \mu|\exp(-[U_{\Lambda\cap Q}(\varphi-\psi_{\varphi|\partial\Lambda}^{\partial\Lambda\cap Q}+\chi_{\tilde{\Lambda}_k}\psi_{(\delta_\varepsilon\psi_\varphi^{\partial\Lambda})|\partial Q}^{\partial\Lambda\cap Q})$$
$$-U_{\Lambda\cap Q}(\varphi)])-\exp(-[U_{\Lambda\cap Q}(\varphi-\psi_{\varphi|\partial\Lambda}^{\partial\Lambda\cap Q}+\chi_{\tilde{\Lambda}_{k+1}}\psi_{(\delta_\varepsilon\psi_\varphi^{\partial\Lambda})|\partial Q}^{\partial\Lambda\cap Q})$$
$$-U_{\Lambda\cap Q}(\varphi)])|+\mu|\exp(-[U_{\Lambda\cap Q}(\varphi-\psi_{\varphi|\partial\Lambda}^{\partial\Lambda\cap Q}+\chi_{\tilde{\Lambda}_{\bar{n}}}\psi_{(\delta_\varepsilon\psi_\varphi^{\partial\Lambda})|\partial Q}^{\partial\Lambda\cap Q})$$
$$-U_{\Lambda\cap Q}(\varphi)])-\exp(-[U_{\Lambda\cap Q}(\varphi-\psi_{\varphi|\partial\Lambda}^{\partial\Lambda\cap Q})-U_{\Lambda\cap Q}(\varphi)])| \ . \qquad (6.35)$$

Since on the right-hand side of (6.35) there are less than $[d(0,\partial\Lambda)]$ number of terms (by our choice of $\bar{n}$), (6.33) will be proven if one will get estimations for each term from the right-hand side of (6.35) analogous to the one for (6.33) (since ε is decaying exponentially).

Let us thus consider a single term from right-hand side (6.35) with some $0\leqq k\leqq\bar{n}$ (with obvious definitions for $k=0$ and $k=\bar{n}$). Using the representation of μ given in [FrSi] (recalling our assumption on the interaction!) we get, with

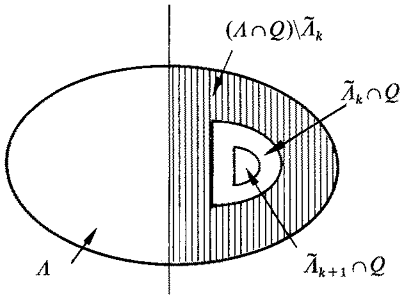
$$F_{k,k+1}(\varphi)\equiv\exp(-[U_{\Lambda\cap Q}(\varphi-\psi_{\varphi|\partial\Lambda}^{\partial\Lambda\cap Q}+\chi_{\tilde{\Lambda}_k}\psi_{(\delta_\varepsilon\psi_\varphi^{\partial\Lambda})|\partial Q}^{\partial\Lambda\cap Q})$$
$$-U_{\Lambda\cap Q}(\varphi)])-\exp(-[U_{\Lambda\cap Q}(\varphi-\psi_{\varphi|\partial\Lambda}^{\partial\Lambda\cap Q}+\chi_{\tilde{\Lambda}_{k+1}}\psi_{(\delta_\varepsilon\psi_\varphi^{\partial\Lambda})|\partial Q}^{\partial\Lambda\cap Q})$$
$$-U_{\Lambda\cap Q}(\varphi)]) \qquad (6.36)$$

that

$$\mu(|F_{k,k+1}|)=\mu_0(\varrho^{\partial\Lambda_\varepsilon}e^{-\alpha_\infty|\Lambda_\varepsilon|}e^{-U_{\Lambda_\varepsilon}(\varphi)},\ |F_{k,k+1}(\varphi)|)$$
$$=\mu_0(\varrho^{\partial\Lambda_\varepsilon}e^{-\alpha_\infty|\Lambda_\varepsilon|}\exp(-[U_{\Lambda_\varepsilon\setminus(\Lambda\cap Q)}(\varphi)$$
$$+U_{((\Lambda\cap Q)\setminus\tilde{\Lambda}_k)\cup\tilde{\Lambda}_{k+1}}(\varphi-\psi_{\varphi|\partial\Lambda}^{\partial\Lambda\cap Q}+\chi_{\tilde{\Lambda}_{k+1}}\psi_{(\delta_\varepsilon\psi_\varphi^{\partial\Lambda})|\partial Q}^{\partial\Lambda\cap Q})])$$
$$\cdot|\exp(-[U_{\tilde{\Lambda}_k\setminus\tilde{\Lambda}_{k+1}}(\varphi-\psi_{\varphi|\partial\Lambda}^{\partial\Lambda\cap Q}+\chi_{\tilde{\Lambda}_k}\psi_{(\delta_\varepsilon\psi_\varphi^{\partial\Lambda})|\partial Q}^{\partial\Lambda\cap Q})])$$
$$-\exp(-U_{\tilde{\Lambda}_k\setminus\tilde{\Lambda}_{k+1}}(\varphi-\psi_{\varphi|\partial\Lambda}^{\partial\Lambda\cap Q}))| \ , \qquad (6.37)$$

with $\varrho^{\partial\Lambda_\varepsilon}\geqq 0$ (the boundary density), $\alpha>0$ (the infinite volume pressure) defined as in [FrSi] (we also used the definition of $F_{k,k+1}$).

Now we take the free conditional expectations first with respect to $\Sigma(S_k^c)$ with $S_k^c\equiv\Lambda^c\cup(\tilde{\Lambda}_k\setminus\tilde{\Lambda}_{k+1})$. We shall denote by Π_S the conditional expectation with respect to μ_0 and to the σ-algebra $\Sigma(S)$, for any measurable set $S\subset\mathbb{R}^2$. Afterwards we will apply the Hölder inequalities together with conditioning inequalities (cf. [GRS1, Sim]) to remove the volume factor.



The right-hand side of (6.37) can be rewritten by splitting

$$\alpha_\infty |\Lambda_\varepsilon| = \alpha_\infty |\Lambda_\varepsilon \backslash \Lambda| + \alpha_\infty |S_k| + \alpha_\infty |(\tilde{\Lambda}_k \backslash \tilde{\Lambda}_{k+1})|$$

and inserting a conditional expectation (using that in expectations conditional expectations drop out). So we get:

$$\begin{aligned}\mu(|F_{k,k+1}|) = \mu_0 (\varrho^{\partial \Lambda_\varepsilon} e^{-\alpha_\infty |\Lambda_\varepsilon \backslash \Lambda|} e^{-U_{\Lambda_\varepsilon \backslash \Lambda}(\varphi)} A \\ e^{-\alpha_\infty |\tilde{\Lambda}_k \backslash \tilde{\Lambda}_{k+1}|} |\exp(-U_{\tilde{\Lambda}_k \backslash \tilde{\Lambda}_{k+1}} (\varphi - \psi_{\varphi|\partial\Lambda}^{\partial(\Lambda\cap Q)} \\ + \chi_{\tilde{\Lambda}_k} \psi^{\partial\Lambda \cap Q}_{(\delta_\varepsilon \psi_\varphi^{\partial\Lambda})|\partial Q}) - \exp(-(U_{\tilde{\Lambda}_k \backslash \tilde{\Lambda}_{k+1}} (\varphi - \psi_{\varphi|\partial\Lambda}^{\partial(\Lambda\cap Q)}))| \;, \end{aligned} \tag{6.38}$$

where

$$\begin{aligned} A \equiv \Pi_{S_k^c} [e^{-\alpha_\infty |S_k|} e^{-U_{S_k \backslash Q}(\varphi)} \\ e^{-U_{S_k \cap Q} (\varphi - \psi_{\varphi|\partial\Lambda}^{\partial(\Lambda\cap Q)} + \chi_{\tilde{\Lambda}_{k+1}} \psi^{\partial(\Lambda\cap Q)}_{(\delta_\varepsilon \psi_\varphi)|\partial Q})}] \;. \end{aligned} \tag{6.39}$$

Now using Hölder inequality with exponents $\frac{1}{q} + \frac{1}{p} + \frac{1}{r} + \frac{1}{s} = 1$, $1 < q < \frac{4}{3}$ we get that the right-hand side of (6.38) is less or equal

$$\begin{aligned} (\mu_0 ((\varrho^{\partial\Lambda_\varepsilon})^q))^{1/q} [\mu_0 (e^{-\alpha_\infty |\Lambda_\varepsilon \backslash \Lambda| - U_{\Lambda_\varepsilon \backslash \Lambda}(\varphi)})^p)]^{1/p} \\ (\mu_0 (A^r))^{1/r} e^{-\alpha_\infty |\tilde{\Lambda}_k \backslash \tilde{\Lambda}_{k+1}|} \\ [\mu_0 |\exp(-[U_{\tilde{\Lambda}_k \backslash \tilde{\Lambda}_{k+1}} (\varphi - \psi^{\partial\Lambda\cap Q}_{\varphi|\partial\Lambda} + \chi_{\tilde{\Lambda}_k} \psi^{\partial\Lambda \cap Q}_{(\delta_\varepsilon \psi_\varphi^{\partial\Lambda})|\partial Q})] \\ - \exp(-[U_{\tilde{\Lambda}_k \backslash \tilde{\Lambda}_{k+1}} (\varphi - \psi^{\partial\Lambda\cap Q}_{\varphi|\partial\Lambda})])|^s]^{1/s} \;. \end{aligned} \tag{6.40}$$

The first factor in (6.40) is less or equal to one [FrSi]. The second is uniformly bounded in $\Lambda \in \mathscr{F}$ if $\varepsilon = e^{-\gamma d(0, \partial\Lambda)}$. We prove below that

$$(\mu_0 A^r)^{1/r} \leqq e^{c|\partial\Lambda|} \tag{6.41}$$

with a constant $c > 0$ independent of $\Lambda \in \mathscr{F}_0$ and decreasing to zero with $\frac{1}{m_0^2}$ (m_0 being the free mass in μ_0) and that

$$\begin{aligned} \{\mu_0 (|\exp(-[U_{\tilde{\Lambda}_k \backslash \tilde{\Lambda}_{k+1}} (\varphi - \psi^{\partial\Lambda\cap Q}_{\varphi|\partial\Lambda} + \chi_{\tilde{\Lambda}_k} \psi^{\partial\Lambda \cap Q}_{(\delta_\varepsilon \psi_\varphi^{\partial\Lambda})|\partial Q})]) \\ - \exp(-[U_{\tilde{\Lambda}_k \backslash \tilde{\Lambda}_{k+1}} (\varphi - \psi^{\partial\Lambda\cap Q}_{\varphi|\partial\Lambda})])|^s)\}^{1/s} \\ \leqq \varepsilon^{a'} \;, \end{aligned} \tag{6.42}$$

with some constant $0 < a' < 1$ independent of Λ and ε. Since the constant γ from definition of ε can be taken arbitrary big if m_0 is sufficiently big, (cf. Remark after Theorem 5.6) so the estimates (6.41), (6.42) give us (6.33).

It remains therefore only to prove (6.41), (6.42). For this it is useful to recall the relation between the conditional expectations Π_S with respect to μ_0 and $\Sigma(S)$ and the solution $\psi_\eta^{\partial S}$ of the Dirichlet problem for $-\Delta + m_0^2$ in S discussed in Sect. 2:

$$(\Pi_S F(\cdot))(\eta) = \mu_0^{\partial S} F(\cdot + \psi_\eta^{\partial S}) \;, \tag{6.43}$$

$\eta \in \mathscr{D}'(\mathbb{R}^2)$, F bounded measurable on $\mathscr{D}'(\mathbb{R}^2)$ (or positive measurable). For this see e.g. [AHK7, DoMi, Rö2,3].

Proof of (6.41): Using (6.43) and the definition (6.39) of A, we can rewrite A in the form

$$A=\mu_0^{\partial S_k}[e^{-\alpha_\infty|S_k|}\\ e^{-U_{S_k}(\varphi'+\Phi_\varphi)}] , \tag{6.44}$$

where φ' are the variables of integration with respect to $\mu_0^{\partial S_k}$.

We have set

$$\Phi_\varphi\equiv\psi_\varphi^{\partial S_k}+[-\psi_{\varphi|\partial\Lambda}^{\partial(\Lambda\cap Q)}+\chi_{\tilde{\Lambda}_{k+1}}\psi_{(\delta_\varepsilon\psi_\varphi^{\partial\Lambda})|\partial Q}^{\partial(\Lambda\cap Q)}]\cdot\chi_{S_k\cap Q} \tag{6.45}$$

and we used the properties of conditional expectations [with respect to μ_0 and $\Sigma(\partial S_k)$], to simplify the expressions for Φ_φ entering (6.44).

Let $\tilde{\chi}_k\in\mathscr{C}^\infty(\mathbb{R}^2)$ with $0\leqq\tilde{\chi}_k\leqq 1$ and

$$\tilde{\chi}_k(x)\equiv\begin{cases}1 & \text{for } d(x,\partial S_k\cup\partial Q)>3\\ 0 & \text{for } d(x,\partial S_k\cup\partial Q)<2 .\end{cases}$$

Let us change the integration variable $\varphi\to\varphi+\tilde{\chi}_k\,\Phi_\varphi$ in the $\mu_0^{\partial S_k}$-integration. Then we get

$$\begin{aligned}&\mu_0^{\partial S_k}(e^{-\alpha_\infty|S_k|}e^{-U_{S_k}(\varphi'+\Phi_\varphi)})\\ &=\mu_0^{\partial S_k}[e^{-\alpha_\infty|S_k|}e^{-U_{S_k}(\varphi'+(1-\tilde{\chi}_k)\Phi_\varphi)}\\ &\quad e^{-\varphi'(h_k)}e^{-1/2\|h_k\|_{-1}^2}] ,\end{aligned} \tag{6.46}$$

with

$$\begin{aligned}h_k(x)&\equiv(-\Delta+m_0^2)\tilde{\chi}_k\Phi_\varphi(x)\\ &=-2\nabla\tilde{\chi}_k\cdot\nabla\Phi_\varphi(x)-(\Delta\tilde{\chi}_k)\Phi_\varphi(x) .\end{aligned}$$

We remark that $h_k(x)$ is localized close to $\partial S_k\cup\partial Q$, since it is built with the special smooth function $\tilde{\chi}_k$ and the harmonic function Φ_φ (observe also the support properties of $\tilde{\chi}_k$ and $\chi_{\tilde{\Lambda}_{k+1}}$).

In order to bound (6.46) we proceed as in the proof of extremality for the weak coupling $P(\varphi)_2$ model, see [AHKZ], cancelling the volume factor necessary to bound the φ'-expectations of $\exp[-U_{S_k}(\varphi'+\Phi_\varphi)]$ by introducing a Neumann condition on the boundary $\partial\tilde{S}_k$ of the set $\tilde{S}_k\equiv\{x\in\Lambda\cap Q|d(x,\partial S_k\cup\partial Q)\geqq 4\}$ and using a conditioning inequality (cf. [GRS1, Sim]).

We shall also use from now on the notation μ_0^X for the field measure with X being a Dirichlet boundary condition on ∂S_k and Neumann boundary condition on $\partial\tilde{S}_k$. Then we get that (6.46) is less or equal

$$\mu_0^X(e^{-\alpha_\infty|\tilde{S}_k|}e^{-U_{\tilde{S}_k}})\cdot B(\varphi) \tag{6.47a}$$

with

$$\begin{aligned}B(\varphi)\equiv\mu_0^X[(e^{-\alpha_\infty|S_k\backslash\tilde{S}_k|}e^{-U_{S_k\backslash\tilde{S}_k}(\varphi'+(1-\tilde{\chi}_k)\Phi_\varphi)}\\ e^{-\varphi'(h_k)}])e^{-1/2\|h_k\|_{-1}^2} .\end{aligned} \tag{6.47b}$$

We observe that we used the fact that $\operatorname{supp}h_k\subset(S_k\backslash\tilde{S}_k)$ [to drop the factor $\varphi'(h_k)$ in the first expectation]. We have the bound for the first factor in (6.47):

$$\mu_0^X(e^{-\alpha_\infty|\tilde{S}_k|}e^{-U_{\tilde{S}_k}})\leqq e^{c|\partial\Lambda|} \tag{6.48}$$

with $0<c\to 0$ as $\dfrac{\lambda}{m_0^2}\to 0$, see [AHKZ].

Here we assumed that

$$|\Lambda \cap \partial Q| \leqq C|\partial \Lambda| \tag{6.49}$$

for some constant $C>0$, independent of λ, m_0^2 (and $\partial\Lambda$).

The quantity $B(\varphi)$ in (6.47) is localized in a set which has volume bounded by $(2C+3)4|\partial\Lambda|$ (this is due to our definition of S_k, $\tilde{S}_k$). Using this and (6.48) we get (6.41) by using in (6.46) repeated Hölder inequalities to control separately the term containing the interaction U_{S_k} and the term $\exp[-\varphi'(h_k)-\frac{1}{2}\|h_k\|_{-1}^2]$. Here we also use the fact that $\|h_k\|_{-1}^2$ contains exponentially decaying factors in the distance from the boundary $\partial S_k \cup \partial Q$, as seen from (6.45), which gives us the estimate

$$\mu_0(e^{\frac{a}{2}\|h_k\|_{-1}^2}) \leqq e^{b|\partial\Lambda|} \tag{6.50}$$

for any fixed constant $a>0$ and a constant b decreasing in m_0.

Moreover we also use a Gaussian integration to control the $\exp(\varphi'(h_k))$-term, followed again by an estimate of the type (6.50).

This completes the proof of (6.41).

There remains the

Proof of (6.42). We have with Hölder exponents t, t', such that $\frac{1}{t}+\frac{1}{t'}=\frac{1}{s}$, that the left-hand side of (6.42) is bounded as follows

$$\begin{aligned}
&(\mu_0|e^{-U_{\tilde{\Lambda}_k\setminus\tilde{\Lambda}_{k+1}}(\varphi-\psi_{\varphi|\partial\Lambda}^{\partial(\Lambda\cap Q)}+\chi_{\tilde{\Lambda}_k}\psi^{\partial(\Lambda\cap Q)}_{(\delta_\varepsilon\psi_\varphi^{\partial\Lambda})|\partial Q})} \\
&\quad -e^{-U_{\tilde{\Lambda}_k\setminus\tilde{\Lambda}_{k+1}}(\varphi-\psi_{\varphi|\partial\Lambda}^{\partial(\Lambda\cap Q)})}|^s)^{1/s} \\
&\leqq \int_0^1 dz\,(\mu_0 e^{-tU_{\tilde{\Lambda}_k\setminus\tilde{\Lambda}_{k+1}}(\varphi-\psi_{\varphi|\partial\Lambda}^{\partial(\Lambda\cap Q)}+z\chi_{\tilde{\Lambda}}\psi^{\partial(\Lambda\cap Q)}_{(\delta_\varepsilon\psi_\varphi^{\partial\Lambda})|\partial Q})}) \\
&\quad \left(\mu_0\left|\frac{d}{dz}U_{\tilde{\Lambda}_k\setminus\tilde{\Lambda}_{k+1}}\left((\varphi-\psi_{\varphi|\partial\Lambda}^{\partial(\Lambda\cap Q)})+z\chi_{\tilde{\Lambda}_k}\psi^{\partial(\Lambda\cap Q)}_{(\delta_\varepsilon\psi_\varphi^{\partial\Lambda})|\partial Q}\right)\right|^{t'}\right)^{1/t'}
\end{aligned} \tag{6.51}$$

(where we used the fundamental theorem of calculus and Hölder's inequality).

The first factor from the right-hand side of (6.51) has (from the Duhamel expansion as e.g. in [GRS2, GlJa]) the following estimation

$$\begin{aligned}
&[\mu_0 e^{-tU_{\tilde{\Lambda}_k\setminus\tilde{\Lambda}_{k+1}}}(\varphi-\psi_{\varphi|\partial\Lambda}^{\partial(\Lambda\cap Q)}+z\chi_{\tilde{\Lambda}}\psi^{\partial\Lambda\cap Q}_{(\delta_\varepsilon\psi_\varphi^{\partial\Lambda})|\partial Q})]^{1/t} \\
&\leqq e^{c|\partial\Lambda|},
\end{aligned} \tag{6.52}$$

with a constant $c>0$ independent of $\Lambda\in\mathscr{F}$, ε and z, and decreasing to zero as $\frac{\lambda}{m_0^2}$. (We used here that $|\tilde{\Lambda}_k\setminus\tilde{\Lambda}_{k+1}|<c'|\partial\Lambda|$ with some numerical constant $c'>0$.)

The second factor from the right-hand side of (6.51) has the simple estimation

$$\begin{aligned}
&\left[\mu_0\left|\frac{d}{dz}U_{\tilde{\Lambda}_k\setminus\tilde{\Lambda}_{k+1}}(\varphi-\psi_{\varphi|\partial\Lambda}^{\partial(\Lambda\cap Q)}+z\chi_{\tilde{\Lambda}_k}\psi^{\partial\Lambda\cap Q}_{(\delta_\varepsilon\psi_\varphi^{\partial\Lambda})|\partial Q})\right|^{t'}\right]^{1/t'} \\
&\leqq \varepsilon^{b'}
\end{aligned} \tag{6.53}$$

with some $0<b'<1$ independent of $z\in[0,1]$, ε and $\Lambda\in\mathscr{F}_0$.

Here we use the definition of $\delta_\varepsilon \psi_\varphi^{\partial\Lambda}|_{\partial Q}$, in the proof of Lemma 3.3.

By combining (6.51) with (6.52), (6.53) we get (6.42). By what we said before engaging in the proof of (6.41), (6.42) this proves (6.33), hence Lemma 6.3. By arguments preceding Lemma 6.3, Lemma 6.1 and Lemma 6.2 are also proven, which in turn prove (6.27), hence (6.16). But this then yields (6.17), (6.18), which, as remarked after (6.18) completes the proof of the global Markov property of μ, with respect to any unbounded open set Q with a piecewise $\mathscr{C}^1$-boundary ∂Q such that $\mathbb{R}^2 - Q$ is also unbounded and $\Lambda \cap Q \neq \emptyset$, $\Lambda \cap Q^c \neq \emptyset$. Moreover $\partial\Lambda \cap \partial Q$ consists of a finite number of points and $|\Lambda \cap \partial Q| \leqq C|\partial\Lambda|$ for some constant $C>0$ independent of $0 \leqq \lambda$, a free mass $m_0^2 > 0$ and $\Lambda \in \mathscr{F}_0$. In the proof we used both λ small and m_0^2 large, but this model is equivalent with the model with $\frac{\lambda}{m_0^2}$ small (cf. [GlJa, GRS1]).

Hence we have proven the following:

Theorem 6.4. *Let $\Lambda \in \mathscr{F}_0$, with $\mathscr{F}_0$ as in Sect. 2 and let U_Λ be the φ_2^4-interaction $U_\Lambda(\varphi) = \lambda(\int :\varphi^4:(x) + b:\varphi^2:(x))$, $b \in \mathbb{R}$, $\lambda \geqq 0$. Let μ be a Gibbs measure to this interaction. Then there exist $K>0$ s.t. for all $\frac{\lambda}{m_0^2} \leqq K$ the measure μ is extremal and is the unique point in the set of regular Gibbs measures. Moreover μ has the global Markov property.*

Remark. In particular μ in Theorem 6.4 has the Markov property with respect to halfplanes (just take Q to be such). As well known this implies in particular that the cyclicity of the time zero fields hold and moreover that $t \to \varphi(t,x)$, $t \in \mathbb{R}$, $x \in \mathbb{R}$ is a symmetric Markov process (cf. [AHK15]). The global Markov property, together with the known results on the quantum fields associated with μ, imply that μ yields models satisfying all Nelson's axioms for quantum fields (cf. [Nel1–4, Sim]).

7. On the Structure of the Set of Gibbs Measures

Let $P(\varphi)$ be a fixed semibounded polynomial. Let h_∞ be a positive constant sufficiently large so that the measures

$$\mu_{P(\varphi)\mp h_\infty \varphi}(\cdot) \equiv \lim_{\mathscr{F}_0} \frac{\mu_0(e^{-(P(\varphi)_\Lambda \mp h_\infty \varphi_\Lambda)} \cdot)}{\mu_0(e^{-(P(\varphi)_\Lambda \mp h_\infty \varphi_\Lambda)})} \tag{7.1}$$

constructed by the cluster expansion [Sp] are unique [AHKZ].

We will assume from now on that the following property holds:

Assumption. There is a function $\omega \in \mathscr{C}(\mathbb{R}^2)2$ $\omega \geqq 0$ dominating at infinity as in Lemma 2.2 such that

$$\mu_{P(\varphi)\mp h_\infty \varphi} = \lim_{\mathscr{F}_0} E^{\pm\omega}_{\Lambda^c, P(\varphi)\mp h_\infty \varphi}\,. \tag{7.2}$$

Remark. This assumption is satisfied for the $:\varphi^4:_2$ model, as discussed in Sect. 5. Moreover the corresponding statement is satisfied for general interactions on a lattice [BeHK].

For a fixed $h \in \mathbb{R}$, $h^{(\pm)}$ be defined by $\pm(h_\infty - h^{(\pm)}) \equiv h$. Let

$$\mu^{\pm}_{P(\varphi)-h\varphi}(\cdot) \equiv \lim_{\mathscr{F}_0} \frac{\mu_{P(\varphi)\mp h_\infty \varphi}(e^{\mp h^{(\pm)}\varphi_\Lambda} \cdot)}{\mu_{P(\varphi)\mp h_\infty \varphi}(e^{\mp h^{(\pm)}\varphi_\Lambda})} \tag{7.3}$$

be the ultraregular states constructed in [FrSi]. If $h \in \mathbb{R}$ is fixed we will write $\mu^{\pm} \equiv \mu^{\pm}_{P(\varphi)-h\varphi}$.

Proposition 7.1 *Suppose that* (7.2) *is fulfilled then* $\mu^{\pm}$ *are* FKG-*maximal Gibbs measures (in the sense of Sect. 4) for the interaction* $P(\varphi) - h\varphi$, i.e.

$$\begin{gathered} \forall \mu \in \mathscr{G}(\mathscr{E}_{P(\varphi)-h\varphi}) \cap \mathscr{M}_r : \\ \mu^- \underset{\text{FKG}}{\leqq} \mu \underset{\text{FKG}}{\leqq} \mu^+ \end{gathered} \tag{7.4}$$

(where, as in Sect. 1, $\mathscr{M}_r$ *denotes the set of regular probability measures, as in Sect. 2* $\mathscr{E}_{P(\varphi)-h\varphi}$ *denotes the specification of the interaction* $P(\varphi) - h\varphi$ *and* $\mathscr{G}(\mathscr{E})$ *the set of Gibbs states to the specification* $\mathscr{E}$*).*

Proof. Let $\omega \in \mathscr{C}(\mathbb{R}^2)$, $\omega \geqq 0$ be dominating at infinity for μ (in the sense of Lemma 2.2) and $\mu \in \mathscr{G}(\mathscr{E}_{P(\varphi)-h\varphi}) \cap \mathscr{M}_r$. Using the uniform continuity property for the local specification, proven in Sect. 3, we get using the FKG order

$$\lim_{\mathscr{F}_0} E^{-\omega}_{\Lambda^c} \underset{\text{FKG}}{\leqq} \mu \underset{\text{FKG}}{\leqq} \lim_{\mathscr{F}_0} E^{+\omega}_{\Lambda^c}, \tag{7.5}$$

with $E_{\Lambda^c} \in \mathscr{E}_{P(\varphi)-h\varphi}$. By the definition (7.3) we have that the measures $\mu^{\pm}$ are in $\mathscr{G}(\mathscr{E}_{P(\varphi)-h\varphi})$. On the other hand we have for any $\tilde{\Lambda}$, $\Lambda \in \mathscr{F}_0$, $\tilde{\Lambda} \subset \Lambda$,

$$\begin{aligned} &\frac{E^{-\omega}_{\Lambda^c, P(\varphi)+h_\infty\varphi}(e^{h^{(-)}\varphi_{\tilde{\Lambda}}} \cdot)}{E^{-\omega}_{\Lambda^c, P(\varphi)+h_\infty\varphi}(e^{h^{(-)}\varphi_{\tilde{\Lambda}}})} \underset{\text{FKG}}{\leqq} E^{-\omega}_{\Lambda^c, P(\varphi)-h\varphi} \\ &\underset{\text{FKG}}{\leqq} E^{+\omega}_{\Lambda^c, P(\varphi)-h\varphi} \underset{\text{FKG}}{\leqq} \frac{E^{+\omega}_{\Lambda^c, P(\varphi)-h_\infty\varphi}(e^{-h^{(+)}\varphi_{\tilde{\Lambda}}} \cdot)}{E^{+\omega}_{\Lambda^c, P(\varphi)-h_\infty\varphi}(e^{-h^{(+)}\varphi_{\tilde{\Lambda}}})}, \end{aligned} \tag{7.6}$$

where we have explicitly denoted to which specification a given measure belongs. We note that (7.6) gives us the compactness of the sequences

$$\{E^{-\omega}_{\Lambda^c, P(\varphi)-h\varphi}\}_{\Lambda \in \mathscr{F}_0}; \quad \{E^{+\omega}_{\Lambda^c, P(\varphi)-h\varphi}\}_{\Lambda \in \mathscr{F}_0}.$$

Passing to the limit with $\Lambda \uparrow \mathbb{R}^2$ through a subsequence $\mathscr{F}_0'$ using (7.2) together with the definition (7.3) we get

$$\mu^- \underset{\text{FKG}}{\leqq} \lim_{\mathscr{F}_0'} E^{-\omega}_{\Lambda^c} \underset{\text{FKG}}{\leqq} \lim_{\mathscr{F}_0'} E^{+\omega}_{\Lambda^c} \underset{\text{FKG}}{\leqq} \mu^+. \tag{7.7}$$

This together with (7.5) gives us (7.4). □

The above proof implies the identification

$$\mu^{\pm} = \lim_{\mathscr{F}_0'} E^{\pm\omega}_{\Lambda^c} \tag{7.8}$$

(which follows from (7.4) and the choice of ω as dominating at infinity).

By using the uniform continuity of $\mathscr{E}_{P(\varphi)-h\varphi}$, we have by Proposition 5.1 the extremality of the considered measures.

Let

$$\alpha_\infty(P(\varphi)-h\varphi)\equiv\lim_{\mathscr{F}_0}\frac{1}{|\Lambda|}\ln\,\mu_0(e^{-[P(\varphi)_\Lambda-h\varphi_\Lambda]})\ , \tag{7.9}$$

be the infinite volume pressure.

We recall that $\alpha_\infty(P(\varphi)-h\varphi)$ as a convex continuous function of h is a.e. differentiable with respect to h. Combining that with the result of [FrSi, Corollary 4.3] we get the following result

Proposition 7.2. *At the points h of differentiability of the pressure* $\alpha_\infty(P(\varphi)-h\varphi)$ *with respect to* $h\in\mathbb{R}$, *there is a unique (in the space of regular measures) extremal Gibbs measure*

$$\mu\equiv\mu^+=\mu^-=\lim_{\mathscr{F}_0'} E_{\Lambda^c}^{\pm\omega}\ . \tag{7.10}$$

Remark. By the method used in Sect. 6 it should be then possible, using (7.8), to extend the proof of the global Markov property to the case of the FKG maximal measures for general $P(\varphi)_2$ models.

Let us close with some expectations concerning the structure of Gibbs states for general $P(\varphi)_2$ models:

1. $\mathscr{G}(\mathscr{E}(P(\varphi)_2))\subset\mathscr{M}_r$.
Moreover we expect that (if P is non-identically zero) $\mathscr{G}(\mathscr{E}(P(\varphi)_2))\subset\mathscr{M}_{\mathscr{S}'(\mathbb{R}^2)}$, where $\mathscr{M}_{\mathscr{S}'(\mathbb{R}^2)}$ is the set of probability measures with support on $\mathscr{S}'(\mathbb{R}^2)$. Let us note however that for free specifications [HoSt], for specifications corresponding to trigonometric interactions and some exponential interactions [Ze4] there are Gibbs measures which are regular but not in $\mathscr{M}_{\mathscr{S}'(\mathbb{R}^2)}$.

2. We expect that there are only finitely many extremal Gibbs measures for any $P(\varphi)_2$-interaction. For $\lambda{:}\varphi^4{:}_2+b{:}\varphi^2{:}_2$, $b\ll-1$, one can expect on the basis of [Aiz, Hig] and approximations arguments of the continuum model by Ising-like models (cf. [Sim]) that $\partial\mathscr{G}(\mathscr{E}_{\lambda:\varphi^4:_2+b:\varphi^2:})\equiv\{\mu^-,\mu^+\}$.

It is expected that in multiphase region $\partial\mathscr{G}(P(\varphi)_2)$ consists of measures constructed by [Im2] (using an extension of Pirogov-Sinai theory [PiSi2]).

3. In two dimensions there is no breaking of translational symmetry, i.e. $\forall x\in\mathbb{R}^2$, $\forall\mu\in\mathscr{G}(\mathscr{E}_{P(\varphi)_2})$ satisfy $T_x\mu=\mu$ (with T_x being translation by x) and every Gibbs state is invariant under time-reflections and is reflection positive.

4. For any $\mu\in\partial\mathscr{G}(\mathscr{E}_{P(\varphi)_2})$ we have $\mu\in$ GMP.
The work [Ze3] suggests $\forall\mu\in\mathscr{G}(\mathscr{E}_{P(\varphi)_2})$, $\mu\in$ GMP.
Furthermore we expect $\mu\in\mathscr{G}(\mathscr{E}_{P(\varphi)_2})\Leftrightarrow\mu\in$ GMP.

5. The set $\partial\mathscr{G}(\mathscr{E}_{P(\varphi)_2})$ of extremal Gibbs measures should be totally FKG-ordered (i.e. $\forall\mu,\mu'\in\partial\mathscr{G}(\mathscr{E}_{P(\varphi)_2})$, $\mu^-\underset{\text{FKG}}{\leqq}\mu\underset{\text{FKG}}{\leqq}\mu'$ or $\mu'\underset{\text{FKG}}{\leqq}\mu\underset{\text{FKG}}{\leqq}\mu^+$) and so any Gibbs measure for $\mathscr{E}_{P(\varphi)_2}$ fulfills FKG inequality.

Appendix 3.A

In this Appendix we prove lemmas yielding the estimates (3.20) [recalling the definitions (3.16), (3.17)].

Lemma A3.1. *For* $0<\varepsilon<1$, $p\in\mathbb{N}$ *and* $\Lambda\in\mathscr{F}_0$,

$$\|\chi_\Lambda(K^{\partial\Lambda_\varepsilon}-K^{\partial\Lambda})\|_p\leqq cp(\varepsilon^{1/4}|\partial\Lambda|)^{1/p} \tag{A3.1}$$

with a constant $c>0$ *independent of* p, ε *and* Λ.

Proof. Since for $d(x,\partial\Lambda)<\frac{1}{2}$ we have

$$K^{\partial\Lambda}(x,x)\leqq a|\ln d(x,\partial\Lambda)| \tag{A3.2}$$

(see e.g. [GlJa, AHK 7]) and analogously for $\partial\Lambda_\varepsilon$, so we have

$$\int\limits_{d(x,\partial\Lambda)<\varepsilon^{1/2}} d_2x|K^{\partial\Lambda_\varepsilon}(x,x)-K^{\partial\Lambda}(x,x)|^p$$
$$\leqq c_1^p|\partial\Lambda| \int\limits_{0<\varepsilon<\varepsilon^{1/2}} ds|\ln s|^p\leqq c_2^p p!\,|\partial\Lambda|\varepsilon^{1/4}$$

(the last estimate coming from integration by parts and estimating a logarithmic term against $\varepsilon^{1/4}$).

For $d(x,\partial\Lambda)>\varepsilon^{1/2}$ we have

$$|K^{\partial\Lambda_\varepsilon}(x,x)-K^{\partial\Lambda}(x,x)|\leqq c_3\varepsilon^{1/4}e^{-bd(x,\partial\Lambda)} \tag{A3.4}$$

which implies, using the exponential decay in (A.3.4)

$$\int\limits_{d(x,\partial\Lambda)>\varepsilon^{1/2}} d_2x|K^{\partial\Lambda_\varepsilon}(x,x)-K^{\partial\Lambda}(x,x)|^p\leqq c_4^p|\partial\Lambda|\varepsilon^{p/4}\ . \tag{A.3.5}$$

All constants a, b, $c_i>0$ are independent of Λ, ε, p. From (A.3.3) and (A.3.5) the bound (A.3.1) follows. □

Lemma A.3.2. *With a constant* $c>0$ *independent of* $0<\varepsilon<1$, Λ *and* $p\in\mathbb{N}$ *we have*

a) $$\|(K^{\partial\Lambda_\varepsilon}(x,y)-K^{\partial\Lambda}(x,y))\chi_\Lambda(x)\chi_\Lambda(y)\|_p$$
$$\leqq cp(\varepsilon^{1/4}|\partial\Lambda|)^{1/p}\ , \tag{A.3.6}$$

b) $$\|(K^{\partial\Lambda_\varepsilon}(x,y)-G(x,y))\chi_\Lambda(x)\chi_{\Lambda_\varepsilon\setminus\Lambda}(y)\|_p$$
$$\leqq Cp(\varepsilon^{1/4}|\partial\Lambda|)^{1/p}\ . \tag{A.3.7}$$

Proof. We have, for $x,y\in\Lambda$,

$$|K^{\partial\Lambda_\varepsilon}(x,y)-K^{\partial\Lambda}(x,y)|=|G^{\partial\Lambda}(x,y)-G^{\partial\Lambda_\varepsilon}(x,y)|\ . \tag{A.3.8}$$

For $d(x.\partial\Lambda)<\varepsilon^{1/2}$ using the exponential decay of covariances and arguments similar to those leading to (A.3.3) we get

$$\int\limits_{d(x,\partial\Lambda)<\varepsilon^{1/2}} d_2x\int\limits_\Lambda d_2y|K^{\partial\Lambda_\varepsilon}(x,y)-K^{\partial\Lambda}(x,y)|^p\leqq c^p p!\,\varepsilon^{1/4}|\partial\Lambda|\ . \tag{A.3.9}$$

For $d(x,\partial\Lambda)>\varepsilon^{1/2}$ we use continuity of the difference (A.3.8) in ε and the exponential decay in $d(x,y)$. We then obtain

$$\int\limits_{d(x,\partial\Lambda)>\varepsilon^{1/2}} d_2x\int\limits_\Lambda d_2y|K^{\partial\Lambda_\varepsilon}(x,y)-K^{\partial\Lambda}(x,y)|^p$$
$$\leqq c^p\varepsilon^{p/4}|\partial\Lambda|\ . \tag{A.3.10}$$

From (A.3.9) and (A.3.10) we get (A.3.6) and (A.3.7). □

Appendix 3.B

The exponential bound (3.26) follows from Duhamel expansion, for which we need the following two lemmas below:

Lemma B.3.1.

$$U_{\Lambda_\varepsilon\setminus\Lambda}(\varphi'_\kappa+\psi^{\partial\Lambda_\varepsilon}_{\varphi_\kappa})\geqq -c|\Lambda_\varepsilon\setminus\Lambda|c_\kappa^{\deg P/2}\ , \tag{B.3.1}$$

with $c_\kappa\equiv a\ln(\kappa+1)$ *and the constants* $c, a>0$ *independent of* $\Lambda, \varepsilon, \kappa$.

The proof of (B.3.1) is standard and follows from the definition of normal ordered semibounded polynomials in the field with a cutoff.

Lemma B.3.2.

$$\begin{aligned}&\mu_{\tilde\Lambda,\varepsilon}\otimes\mu_0^{\partial\Lambda_\varepsilon}|U_{\Lambda_\varepsilon\setminus\Lambda}(\varphi'_\kappa+\psi^{\partial\Lambda_\varepsilon}_{\varphi_\kappa})-U_{\Lambda_\varepsilon\setminus\Lambda}(\varphi'+\psi^{\partial\Lambda_\varepsilon}_{\varphi})|^p\\ &\leqq\kappa^{-\delta p}(p\deg P)!\ c^p|\Lambda_\varepsilon\setminus\Lambda|^{ap}\ ,\end{aligned} \tag{B.3.2}$$

with $a, c, \delta>0$ *independent of* $\tilde\Lambda, \Lambda, \varepsilon, p$, *and* κ.

Proof. It is sufficient to consider p even (using Hölder inequality).

Then we start with Gaussian integration of the field φ'. We integrate the $\psi^{\partial\Lambda_\varepsilon}_{\varphi_\kappa}$, $\psi^{\partial\Lambda_\varepsilon}_{\varphi}$ variables using the integration by parts formula [GlDi, GlJa] for the measure $\mu_{\tilde\Lambda,\varepsilon}$.

From that we get the representation of the left-hand side of (B.3.2) as the sum of two terms

$$\mu_0|U_{\Lambda_\varepsilon\setminus\Lambda}(\varphi_\kappa)-U_{\Lambda_\varepsilon\setminus\Lambda}(\varphi)|^p+A\ .$$

The first term has just the estimation (B.3.2) (see [GlJa, Sim]). The estimation for the second term A follows from the standard bounds with measures $\mu_{\tilde\Lambda,\varepsilon}$ [GlJa] and is uniform in $\tilde\Lambda$. □

Appendix 4.A. The Potential Theory on a Lattice

Let $\mathscr{F}$ be the family of open bounded sets $\Lambda\subset\mathbb{R}^d$ with piecewise $\mathscr{C}^1$-boundaries $\partial\Lambda$. For $\delta>0$ let $\mathbb{Z}^d_\delta:\equiv\{n\delta\equiv(n^1\delta,\dots,n^d\delta):n\in\mathbb{Z}^d\}$. If $\Lambda\in\mathscr{F}$ then $\Lambda_\delta\equiv\Lambda\cap\mathbb{Z}^d_\delta$ and $\Lambda^c_\delta\equiv\mathbb{Z}^d_\delta\setminus\Lambda_\delta$. The boundary $\partial\Lambda_\delta$ of Λ_δ is defined by

$$\partial\Lambda_\delta\equiv\{n\delta\in\Lambda_\delta:d(n\delta,\Lambda^c_\delta)=\delta\} \tag{A.4.1}$$

with $d(\cdot,\cdot)$ the usual euclidean distance in $\mathbb{R}^d$.

The lattice distance is given by

$$|n-m|\equiv\min_{i=1,\dots,d}|n^i-m^i|\ . \tag{A.4.2}$$

Let

$$L_{2,\delta}\equiv L_2(\mathbb{Z}^d_\delta)\equiv\{f:\mathbb{Z}^d_\delta\to\mathbb{R}\,|\,\delta^d\sum_{n\in\mathbb{Z}^d}|f(n\delta)|^2<\infty\}\ , \tag{A.4.3}$$

and let

$$e_{n\delta}(n'\delta)\equiv\left\{\begin{matrix}\delta^{-d} & \text{if } n=n'\\ 0 & \text{otherwise}\end{matrix}\ ,\quad n\in\mathbb{Z}^d\right\} \tag{A.4.4}$$

be its base. For $f \in L_{2,\delta}$ the lattice Laplacian is defined by

$$(-\Delta_\delta f)(n\delta) \equiv \delta^{-2} \sum_{|n'-n|=1} (f(n\delta)-f(n'\delta)) \ . \tag{A.4.5}$$

The standard result from the potential theory on the lattice is given by the following lemma (see [GRS1, Roy, Sim]):

Lemma A.4.1. a) *There is a unique function $G_\delta(n\delta, \cdot)$, $n\delta \in \mathbb{Z}_\delta^d$ which tends to zero at infinity such that*

$$(-\Delta_\delta + m_0^2)\, G_\delta(n\delta, \cdot) = e_{n\delta} \ . \tag{A.4.6}$$

The matrix $G_\delta(n\delta, m\delta)$ is symmetric and positive definite.

b) *For any $\Lambda_\delta = \mathbb{Z}_\delta^d \cap \Lambda$, $\Lambda \in \mathscr{F}$ there is a unique symmetric and positive definite matrix $G_\delta^{\partial\Lambda}(x_\delta, y_\delta)$, x_δ, $y_\delta \in \mathbb{Z}_\delta^d$ such that the function $G_\delta^{\partial\Lambda}(x_\delta, \cdot)$ fulfills*

$$\begin{aligned} (-\Delta_\delta + m_0^2)\, G_\delta^{\partial\Lambda}(x_\delta, \cdot) &= e_{x_\delta} \quad \textit{for} \ \ x_\delta \in \Lambda_\delta \cup \operatorname{int} \Lambda_\delta^c \\ G_\delta^{\partial\Lambda}(x_\delta, \cdot) &= 0 \ , \quad \textit{for} \ \ x_\delta \in \partial\Lambda_\delta^c \ . \end{aligned} \tag{A.4.7}$$

Moreover

$$0 \leqq G_\delta^{\partial\Lambda}(x_\delta, y_\delta) \leqq G_\delta(x_\delta, y_\delta) \ . \tag{A.4.8}$$

Let G and $G^{\partial\Lambda}$ for $\Lambda \in \mathscr{F}$ be the counterparts of G_δ and $G_\delta^{\partial\Lambda}$ (respectively) in the present continuum case. Let $\hat{G}(k)$, $k \in \mathbb{R}^d$ respectively $G_\delta(k)$, $k \in \left(-\frac{\pi}{\delta}, \frac{\pi}{\delta}\right)^d$ be the Fourier transforms of G respectively G_δ.

Denote by $f_{n\delta}(x) \in H_{-1}$ the function defined by

$$\hat{f}_{n\delta}(k) \equiv \begin{cases} \dfrac{e^{-ikn\delta}}{(2\pi)^{d/2}} (\hat{G}_\delta(k) | \hat{G}(k))^{1/2} & \text{for } |k_i| < \pi/\delta \\ 0 & \text{otherwise} \ . \end{cases} \tag{A.4.9}$$

From the definition (A.4.9) we have [GRS1, Sim]

$$G_\delta(n\delta, m\delta) = (f_{n\delta}, G * f_{m\delta})_{L_2(\mathbb{R}^d)} \ . \tag{A.4.10}$$

For Λ_δ being a product of $[-l^i\delta, l^i\delta] \cap \mathbb{Z}_\delta$, $i = 1, \ldots, d$ the lattice Green function $G_\delta^{\partial\Lambda}$ with Dirichlet boundary conditions on $\partial\Lambda_\delta^c$ can be represented using the method of images as follows [GRS2, Sect. III.3; GlJa]:

$$G_\delta^{\partial\Lambda}(x_\delta, y_\delta) = \sum_{n \in \mathbb{Z}^d} (-1)^{\varepsilon_n} G_\delta(x_\delta, r_n y_\delta) \ , \tag{A.4.11}$$

where

$$(r_n y_\delta)^i \equiv (-1)^{n_i}(y_\delta^i - n^i 2 l^i \delta) \ , \tag{A.4.12}$$

and $\varepsilon_n \in \mathbb{Z}$ suitably chosen so that $G_\delta^{\partial\Lambda}(x_\delta, y_\delta)$ vanishes if $y_\delta \in \partial\Lambda_\delta^c$.

Lemma A.4.2. *Let $d = 2$ and $\omega \in \mathscr{C}(\mathbb{R}^2)$. For any $\Lambda_\delta \in \mathscr{F}_\delta$ the solution $\psi_{\omega,\delta}^{\partial\Lambda}(\cdot)$ of the following Dirichlet problem:*

$$\begin{aligned} (-\Delta_\delta + m_0^2)\, \psi_{\omega,\delta}^{\partial\Lambda}(x_\delta) &= 0 \qquad \textit{for} \ \ x_\delta \in \Lambda_\delta \ , \\ \psi_{\omega,\delta}^{\partial\Lambda}(x_\delta) &= \omega(x_\delta) \quad \textit{for} \ \ x_\delta \in \Lambda_\delta^c \end{aligned} \tag{A.4.13}$$

is given in Λ_δ by the lattice Poisson formula

$$\psi^{\partial\Lambda}_{\omega,\delta}(x_\delta)=\sum_{n\delta\in\partial\Lambda_\delta} G^{\partial\Lambda}_\delta(x_\delta,n\delta)\sum_{\substack{|n'-n|=1\\ n'\delta\in\partial\Lambda^c_\delta}}\omega(n'\delta)\ . \tag{A.4.14}$$

Proof. The proof of this lemma follows simply from the definition of $G^{\partial\Lambda}_\delta$ in Lemma A.4.1b. □

Now we will consider the limit of the solution $\psi^{\partial\Lambda}_{\omega,\delta}$ as $\delta\to 0$. For that we assume Λ of the special form s.t. $\bar\Lambda=\prod_{i=1,2}[-n_i,m_i]$, $n_i,m_i\in\mathbb{Z}^+$, and take $\delta\equiv 2^{-l}$, $l\in\mathbb{Z}^+$.

(The general case can be treated analogously.) By our assumptions we have

$$\partial\Lambda^c_\delta=\partial\Lambda\cap\mathbb{Z}^2_\delta\ . \tag{A.4.15}$$

Lemma A.4.3. *For any* $x\in\Lambda\cap\bigcap_{\delta\equiv 2^{-\ell}}\mathbb{Z}^2_\delta$,

$$\lim_{\delta\equiv 2^{-\ell}\to 0}\psi^{\partial\Lambda}_{\omega,\delta}(x)=\psi^{\partial\Lambda}_\omega(x)\ , \tag{A.4.16}$$

where $\psi^{\partial\Lambda}_\omega(x)$ *is the solution of the Dirichlet problem in the continuum which corresponds to* (A.4.13).

Proof. The proof follows from the fact that for any $x\in\Lambda\cap\bigcap_\delta\mathbb{Z}^2_\delta$ and $n\delta\in\partial\Lambda_\delta$.

$$\lim_{\substack{\delta\to 0\\ n\delta\to y\in\partial\Lambda}}\frac{1}{\delta}G^{\partial\Lambda}_\delta(x,n\delta)=\psi^{\partial\Lambda}_y(x)\ . \tag{A.4.17}$$

This can be seen using the formula (A.4.11) for $G^{\partial\Lambda}_\delta$ (see e.g. [BrFrSp]). Hence using (A.4.14) we get

$$\lim_{\delta\to 0}\psi^{\partial\Lambda}_{\omega,\delta}(x)=\int_{\partial\Lambda}d_2y\,\psi^{\partial\Lambda}_y(x)\omega(x)\equiv\psi^{\partial\Lambda}_\omega(x)\ .\ \square \tag{A.4.18}$$

We need a stronger result to get the convergence of the lattice approximation of euclidean field theory.

Define

$$\hat\psi^{\partial\Lambda}_{\omega,\delta}(k)\equiv\left(\frac{\delta^2}{2\pi}\sum_{n\delta\in\Lambda_\delta}\psi^{\partial\Lambda}_{\omega,\delta}(n\delta)e^{-ikn\delta}\right)\chi_\delta(k)\ , \tag{A.4.19}$$

with $\chi_\delta(k)$ the characteristic function of the set $\left\{k:|k_i|<\frac{\pi}{\delta}\right\}$.

Lemma A.4.4. *For any* $2\leqq p<\infty$,

$$\lim_{\delta\to 0}\hat\psi^{\partial\Lambda}_{\omega,\delta}=\hat\psi^{\partial\Lambda}_\omega\chi_\Lambda \tag{A.4.20}$$

in $L_p(\mathbb{R}^2)$.

Proof. From definition (A.4.19) of $\hat\psi^{\Lambda}_{\omega,\delta}$ we have

$$\|\hat\psi^{\partial\Lambda}_{\omega,\delta}\|_\infty\leqq\|\psi^{\partial\Lambda}_{\omega,\delta}\|_\infty\frac{\delta^2}{2\pi}\frac{|\Lambda|}{\delta^2}\ , \tag{A.4.21}$$

and using the maximum principle on the lattice (see e.g. [BJS]) we get, uniformly in δ,

$$\|\hat{\psi}^{\partial\Lambda}_{\omega,\delta}\|_\infty \leqq \frac{1}{2\pi} \|\omega_{|\partial\Lambda}\|_\infty |\Lambda| \ . \tag{A.4.22}$$

From Plancherel theorem we have

$$\begin{aligned}\|\hat{\psi}^{\partial\Lambda}_{\omega,\delta}\|_{L_2} &= \sum_{n\delta\in\Lambda_\delta} |\psi^{\partial\Lambda}_{\omega,\delta}(n\delta)|^2\delta^2 \\ &= \Bigg(\sum_{\substack{n\delta\in\Lambda_\delta \\ d(n\delta,\partial\Lambda)<\varepsilon}} + \sum_{\substack{n\delta\in\Lambda_\delta \\ d(n\delta,\partial\Lambda)\geqq\varepsilon}}\Bigg)|\psi^{\partial\Lambda}_{\omega,\delta}(n\delta)|^2\delta^2\end{aligned} \tag{A.4.23}$$

for any $\varepsilon>0$. Since from Lemma A.4.3

$$\psi^{\partial\Lambda}_{\omega,\delta}(x) \underset{\delta\to 0}{\longrightarrow} \psi^{\partial\Lambda}_{\omega}(x) \ ,$$

uniformly on compact subsets of Λ, the second sum on the right-hand side of (A.4.23) converges to

$$\int_{\Lambda\cap\{d(x,\partial\Lambda)\geqq\varepsilon\}} |\psi^{\partial\Lambda}_{\omega}(x)|^2 d_2x \ .$$

Since from the lattice maximum principle

$$|\psi^{\partial\Lambda}_{\omega,\delta}| \leqq \|\omega_{|\partial\Lambda}\|_\infty \tag{A.4.24}$$

the first sum from the right-hand side of (A.4.23) is bounded by

$$\|\omega_{|\partial\Lambda}\|_\infty |\Lambda\cap\{d(x.\partial\Lambda)<\varepsilon\}| \ .$$

Due to the fact that $\varepsilon>0$ is arbitrary we get

$$\|\hat{\psi}^{\partial\Lambda}_{\omega,\delta}\|_{L_2} \underset{\delta\to 0}{\longrightarrow} \|\psi^{\partial\Lambda}_{\omega}\chi_\Lambda\|_{L_2} \ . \tag{A.4.25}$$

For any $\hat{f}\in\mathscr{S}(\mathbb{R}^2)$, we have also

$$\begin{aligned}\int \hat{f}(k)\hat{\psi}^{\partial\Lambda}_{\omega,\delta}(k)d_2k &\to \int f(x)\psi^{\partial\Lambda}_{\omega}(x)\chi_\Lambda(x)d_2x \\ &= \int \hat{f}(k)\hat{\psi}^{\partial\Lambda}_{\omega}(k)\chi_\Lambda(k)d_2k \ ,\end{aligned}$$

that is $\hat{\psi}^{\partial\Lambda}_{\omega,\delta} \underset{\delta\to 0}{\to} \hat{\psi}^{\partial\Lambda}_{\omega}\chi_\Lambda$ weakly in $L_2(\mathbb{R}^2)$ and by (A.4.25) we have also strong convergence in L_2. By interpolation using also (A.4.22) we have L_p-convergence. This ends the proof. □

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