Distance function and cut loci on a complete Riemannian manifold

By

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1. Introduction. The results of this note are a characterisation of cut loci (Theorem 1) and the fact that *a complete n-dimensional Riemannian manifold M is necessarily diffeomorphic to R^n if there is a point $p \in M$ for which the square of the distance function $d^2(p, .)$ is everywhere directional-differentiable* (*Theorem 2*).

By "directional-differentiable" at a point we mean that the directional derivative exists for all directions through that point. This is weaker than the existence of the gradient at that point. The proof of Theorem 2 makes use of considerations concerning the cut locus $\mathrm{Cu}(p)$ of an arbitrary point p on a complete Riemannian manifold M which we assume to be of C^∞ differentiability class for convenience throughout the paper though far less is necessary. By definition, a point q of M lies on $\mathrm{Cu}(p)$ if and only if there is a distance-minimal geodesic joining p to q whose every extension beyond q is no longer minimal. We prove here (Lemma 2) that $\mathrm{Cu}(p)$ is the closure of all points in M which have at least two minimal geodesic connections with p. Since we also show (Lemma 1) that $d^2(p, .)$ is not directional-differentiable at any point which lies on at least two minimal geodesics from p, it follows that there is a dense subset of points on the cut locus $\mathrm{Cu}(p)$ in which $d^2(p, .)$ is not differentiable. This explains Theorem 2 in view of the fact that $\exp_p$ is a diffeomorphism of T_pM onto M in case $\mathrm{Cu}(p)$ is empty.

The author wishes to thank D. Koutroufiotis for introducing him to the topic of this paper, as well as D. Ferus and H. Karcher for helpful discussions. Theorem 1 is essentially due to a telephon conversation between D. Ferus and the author. We also wish to thank the referee whose suggestions considerably shortened the original proof of Lemma 1.

Remark. Recently it has been pointed out to us by R. L. Bishop that our construction in the proof of Lemma 2 is very similar to the one used by H. Karcher in [1] for a quite different purpose. In [1] p. 116 is shown that an open geodesically convex set K does not meet its cut locus i.e. for all $p \in K$ is $\mathrm{Cu}(p) \cap K = \emptyset$. (Geodesically convex means here that any two points of K have within K a unique minimal geodesic connection. A priori it is here not excluded that outside of K exists another minimal geodesic connection for the given points.) Without much change in both proofs it certainly would be possible to give a combined proof of those results.

2. Statement of results. The main result follows immediately from the following two lemmas which we prove in section 3.

Lemma 1. *Suppose two minimal geodesics exist joining p to q on a Riemannian manifold M. Then $d^2(p, .)$ has no directional derivative at q for vectors in direction of those two minimal geodesics.*

Remark. The weaker claim that at the point q in Lemma 1 the gradient pf $d^2(p, .)$ is not continuous, was observed by R. L. Bishop [2] proposition.

Lemma 2. *Let $Cu(p)$ be the cut locus of an arbitrary point p on a complete Riemannian manifold M. The subset $Se(p)$ of $Cu(p)$ defined by those points where at least two minimal geodesics from p intersect (or meet so as to form a closed geodesic loop) is dense in $Cu(p)$.*

Remark. We show in the proof of Lemma 2 that the tangent cut-points corresponding to $\mathrm{Se}(p)$ are dense in the tangent cut locus of p. So our Lemma 2 constitutes the main theorem of [2]. Bishop uses strong results of Warner [3] in order to prove it. Our proof seems conceptionally and technically simpler, as well as being self-contained.

Because $\mathrm{Cu}(p)$ is a closed subset of M, we have from Lemma 2 the following characterisation of the cut locus.

Theorem 1. *The cut locus $Cu(p)$ on a complete Riemannian manifold is the closure of the set of all points in M which have at least two minimal geodesic connections to p.*

If $d^2(p, .)$ is directional-differentiable on all M the combination of both lemmas shows that the cut locus $\mathrm{Cu}(p)$ must be empty. Thus the exponential map $\exp_p \colon T_p \to M$ becomes a diffeomorphism between the tangent space $T_p M$ and M. Hence we have our main result.

Theorem 2. *Assume there exists on the complete n-dimensional Riemannian manifold M a point p with the property that $d^2(p, .)$ is directional-differentiable on all M; then M is diffeomorphic to R^n.*

Remark. An incomplete Riemannian manifold possessing an everywhere directional-differentiable (even real-analytic) $d^2(., .)$ need not be diffeomorphic to R^n; the punctered euclidean plane yields a counter example.

3. Proof of the lemmas. In the proofs which follows all geodesics are parametrized by arc length.

Proof of Lemma 1. Let $g_i \colon [0, \hat{t}] \to M$ $i = 1, 2$ be two distinct geodesics with $g_1(0) = g_2(0) = p$, $g_1(l) = g_2(l) = q$, $l = d(p, q)$ and $0 < l < \hat{t}$. At $t = l$ the derivative of $d(p, g_i(t))$ in direction toward p is clearly 1. (We consider here the left hand derivative i.e. the limit is taken with $t - l \leqq 0$.) Now if $\omega \in (0, \pi]$ is the angle between g_1 and g_2 at q then for $t \leqq l$ one has an upper bound for $d(p, g_i(t))$. The upper bound equals at $t = l$ and its right hand derivative is bounded away from 1 at $t = l$. Therefore $d^2(p, .)$ does not have derivatives along g_i at q. Namely: Using the triangle inequality we can define an upper bound

$$u(t) := d(p, g_1(l-\varepsilon)) + d(g_1(l-\varepsilon), g_2(t))$$

for $d(p, g_i(t))$ here $i = 2$ (analogues for $i = 1$). Using Fermi-coordinates one has with $\tau = t - l \geqq 0$

$$d(g_1(l-\varepsilon), g_2(l+\tau)) = \sqrt{\varepsilon^2 + \tau^2 + 2\varepsilon\tau\cos\omega}\,(1 + O(\tau^2)).$$

Hence we get for $u(t)$ at $t = l$ the right hand derivative $u'_+(l) = \cos\omega < 1$.

The proof of Lemma 2 will use the following two well-known facts.

(A) Let $\{g_j\}$ be a sequence of minimal geodesics in a complete Riemannian manifold and suppose each g_j joins p to a point q_j. If $\{q_j\}$ converges to q, then a subsequence of $\{g_j\}$ converges to a minimal geodesic from p to q. (See [4], p. 24.)

(B) Let $S^{n-1}(p)$ be the unit sphere in the tangent space T_pM of a complete n-dimensional Riemannian manifold M. For x in $S^{n-1}(p)$ let $s(x)$ denote the supremum of those t-values for which the geodesic g defined by $g(t) = \exp_p(tx)$ is minimal. Then the function $s\colon S^{n-1}(p) \to [0, \infty]$ is continuous, where $[0, \infty]$ denotes the Alexandrov one-point compactification of $[0, \infty)$. (See [5], p. 169.)

Proof of Lemma 2. Let $\bar{K}(x_0, \delta)$ denote the closed ball with center $x_0 \in S^{n-1}(p)$ and radius $\delta > 0$ in the tangent space T_pM. If $s(x_0) = r < \infty$, then for small enough δ-values, (B) easily implies that s restricted to $\bar{K}(x_0, \delta) \cap S^{n-1}(p)$ is a continous real-valued function. Consider the cone

$$\mathrm{Co}(x_0, \delta) = \{tx \mid 0 \leqq t \leqq 1,\ x \in \bar{K}(x_0, \delta) \cap S^{n-1}(p)\}$$

in T_pM. For small enough δ, it follows from the continuity of s that both $\mathrm{Co}(x_0, \delta)$ and

$$\mathrm{Co}^*(x_0, \delta) = \{x' \mid x' = s(x/\|x\|)\,x,\ \theta \neq x \in \mathrm{Co}(x_0, \delta)\} \cup \{\theta\} \qquad \theta = 0$$

are homeomorphic to a closed euclidean n-ball.

Suppose now that $\exp_p(s(x_0)x_0) = q$ is a point with only one minimal geodesic connection g to p, so that q is in $\mathrm{Cu}(p)$ but not in $\mathrm{Se}(p)$. We wish to show that in each neighborhood of q there is a point of $\mathrm{Se}(p)$. Suppose otherwise. Then for some fixed small $\delta > 0$, the restriction of $\exp_p$ to $\mathrm{Co}^*(x_0, \delta)$ is a homeomorphism onto its image (being a continuous univalent mapping on a compact set). Now if $\mathscr{K}(q, \varepsilon)$ is the open ball in M centered at q with arbitrary radius $\varepsilon > 0$, then $\mathscr{K}(q, \varepsilon)$ contains points lying in the complement of $\exp_p(\mathrm{Co}^*(x_0, \delta))$ in M, because $s(x_0)x_0$ has in $\mathrm{Co}^*(x_0, \delta)$ a neighborhood homeomorphic to a closed n-dimensional euclidean half space. Thus we can take a sequence $\{q_j\}$ of points with q_j in $\mathscr{K}(q, 1/j)$ but *not* in $\exp_p(\mathrm{Co}^*(x_0, \delta))$. Each q_j has a minimal geodesic connection g_j to p (perhaps several, but one can be choosen) since M is complete. By (A) a subsequence of $\{g_j\}$ converges to a minimal geodesic $\tilde{g}$ from p to q, so that the unit tangents of the g_j at p converge to the unit tangent $\tilde{x}$ to $\tilde{g}$ at p. Note that any sequence from $S^{n-1}(p)$ which converges to x_0 must lie in $\mathrm{Co}(x_0, \delta)$ from some point on, because x_0 is an interior point of the set $\bar{K}(x_0, \delta) \cap S^{n-1}(p)$. But the unit tangents to the g_j at p all lie outside of $\mathrm{Co}(x_0, \delta)$ since q_j is not in $\exp_p(\mathrm{Co}^*(x_0, \delta))$. Thus $\tilde{x} \neq x_0$ and $\tilde{g} \neq g$ contradicting the assumption that q is not in $\mathrm{Se}(p)$.

Remark. We have just proved that for any given neighborhood $U(s(x_0)x_0)$ in T_pM, $\exp_p$ cannot be injektive on

$$U(s(x_0)x_0) \cap \{s(x)x/x \in S^{n-1}(p)\}$$

if $\exp_p(s(x_0)x_0)$ is not in $\mathrm{Se}(p)$.

This means that the points corresponding to $\mathrm{Se}(p)$ are dense in the *tangent cut locus* $\{s(x)x/x \in S^{n-1}(p)\}$ of p.

4. Further considerations.

Remark on Lemma 2. If we denote by $Se(p)$ the points with several minimal geodesic connections to a point p in a complete Riemannian manifold M it is in general certainly not true that $\mathrm{Se}(p)$ is open in the cut locus $\mathrm{Cu}(p)$ of p. (Several here means more than one.) At least if the dimension of M is two it is true that $\mathrm{Se}(p)$ is open in $\mathrm{Cu}(p)$ if M is real analytic. If M is only of C^∞ differentiability class it can however be shown that $\mathrm{Cu}(p)\setminus\mathrm{Se}(p)$ is nowhere dense in $\mathrm{Cu}(p)$. This means $\mathrm{Se}(p)$ is a residual subset of $\mathrm{Cu}(p)$.

Examples. On a complete Riemannian manifold differentiability of $d^2(.,.)$ on all M is equivalent to the condition that any two points of M have a unique minimal geodesic connection and this is by Lemma 2 equivalent to the condition "points have a unique geodesic connection". Examples for manifolds of this type are simply connected complete Riemannian manifolds with non positive sectional curvature, the so called Hadamard manifolds. But it is not true that all manifolds with unique geodesic connection for any two given points are Hadamard manifolds. There exist examples of complete manifolds with unique geodesic connection for any two given points where the sectional curvature changes sign. All universal coverings of Riemannian manifolds without conjugate points and small regions of positive sectional curvature provide examples by Hadamard's theorem that the exponential map for a point p is a covering map if the conjugate locus of p is empty. R. Gulliver in [6] gives examples of the requested type.

Generalisations of Theorem 2. A Riemannian manifold is called nonextendable if it cannot be isometrically embedded in a larger Riemannian manifold of equal dimension. The nonextendable Riemannian manifolds include the complete ones, however Theorem 2 cannot be generalised to this class. The two sheeted Riemannian covering of the punctered Euclidean plane is nonextendable and its squared distance function is everywhere real analytic but this manifold is not diffeomorphic to R^2. It is however true that certain forms of generalisation of Theorem 2 do hold for manifolds with boundary which are complete as a metric space.

This problem in relation to other questions will be discussed in a further note.

Remark. After submitting our manuskript to the redaction we learned by T. Sakai that this note could be applied in a joint article of M. Buchner, K. Fritzsche und T. Sakai about cohomology and cut-loci in submanifolds of complex projective spaces. This joint article of M. Buchner, K. Fritzsche and T. Sakai is still in preparation.

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