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Perturbation of Embedded Eigenvalues in the Generalized N-Body Problem

Shmuel Agmon[1,2], Ira Herbst[2]* and Erik Skibsted[3]

[1] Department of Mathematics, Hebrew University of Jerusalem, Israel
[2] Department of Mathematics, University of Virginia, Charlottesville, Virginia 22901, USA
[3] Mathematics Institute, Aarhus University, Denmark

Abstract. We discuss the perturbation of continuum eigenvalues without analyticity assumptions. Among our results, we show that generally a small perturbation removes these eigenvalues in accordance with Fermi's Golden Rule. Thus, generically (in a Baire category sense), the Schrödinger operator has no embedded non-threshold eigenvalues.

I. Introduction

It is well known [R–S1] that a one-body Schrödinger operator $-\Delta + V(x)$, where V is sufficiently well behaved at infinity, cannot have eigenvalues λ embedded in the continuous spectrum (except possibly at threshold, $\lambda = 0$). The situation is quite different in the N-body problem where continuum eigenvalues not only can exist, but do indeed exist in important physical situations: The operator $H_0 = -\Delta_1 - \Delta_2 - 2/|x_1| - 2/|x_2|$ in $L^2(\mathbb{R}^6)$ (describing the Helium atom without electronic repulsion) has eigenvalues embedded in the continuous spectrum. While this example has an obvious symmetry, such symmetry is not necessary for the existence of embedded eigenvalues. An example in [F–H–HO–HO] can be modified to produce an embedded eigenvalue where no symmetry is apparent.

In [How1, 2] and [S1], analyticity assumptions are made which allow the treatment of embedded eigenvalues using the perturbation theory developed for use with isolated eigenvalues. The major idea in this theory is that when a small perturbation βW is added to the Schrödinger operator H, the continuum eigenvalue E_0 turns into a "resonance," $E_0(\beta)$, which, while not necessarily an eigenvalue of $H + \beta W$, is a pole in the analytic continuation of certain matrix elements $(\varphi, (H + \beta W - z)^{-1}\varphi)$ of the resolvent. The function $E_0(\beta)$ is analytic in β for $|\beta|$ small. $E_0(\beta)$ has an imaginary part which appears first to second order in β:

$$\left|\operatorname{Im}\frac{d^2E_0(\beta)}{d\beta^2}\right|_{\beta=0} = 2\pi\frac{d}{dE}(W\psi_0, P(E)W\psi_0)|_{E=E_0}, \tag{1.1}$$

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where $(H-E_0)\psi_0=0$, $P(E)=E_H((-\infty,E)\backslash\{E_0\})$ and $H=\int\lambda dE_H(\lambda)$. Here we have assumed that ψ_0 is non-degenerate and normalized. The formula (1.1) is called Fermi's Golden Rule. The situation is reviewed more thoroughly in [R–S1].

The purpose of this paper is to examine the perturbation of embedded eigenvalues in the generalized N-body problem introduced by Agmon [Ag] (see also [F–H1, 2, 3]) without making any analyticity assumptions. While resonance poles have no meaning without some kind of analyticity, one would think that if Fermi's Golden Rule predicts the disappearance of an eigenvalue from the real axis (by producing a positive expression in (1.1)), then that eigenvalue should disappear (for small β). We show that this is indeed the case.

From what has just been said, it appears that embedded eigenvalues are very unstable, for one need only find W such that (1.1) is nonzero, and then for all small β, $H+\beta W$ will not have an eigenvalue near E_0. The major stumbling block to making this into a theorem is to show that the operator $(d/dE)P(E)|_{E=E_0}$ (which makes sense between certain weighted spaces) is not identically zero. This is accomplished only after a rather involved argument. But this argument produces as a bonus some information about the existence of generalized eigenfunctions and some new estimates for N-body Schrödinger operators between exponentially weighted Hilbert spaces.

In Sect. II, we introduce our notation and main assumptions and prove that embedded eigenvalues cannot suddenly appear under a small perturbation (Theorem 2.5). (See [K1] for general information about the perturbation of spectra.)

In Sect. III, certain estimates are proved for Schrödinger operators between exponentially weighted spaces. These estimates can be used to simplify the arguments of [F–H1, 2] considerably, although we do not do this here. We use the estimates in Sect. IV to show that for $P(E)$ as in (1.1), $(d/dE)P(E)|_{E=E_0}\neq 0$.

In Sect. V, it is shown that embedded eigenvalues are unstable and thus generically absent (Proposition 5.10 and Theorem 5.11), and in the last section we discuss open problems.

We remark that analogous results have been proved for perturbations of the hyperbolic Laplacian on a finite volume Riemann surface (see [V] and [Ph–Sa]).

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II. Semicontinuity of the Point Spectrum of *H*

Let us begin by setting our notation. We assume we are given a family $\{X_i\}_{i=1}^M$ of subspaces of $\mathbb{R}^n$ with associated orthogonal projection operators π_i, $\operatorname{Ran}\pi_i=X_i$. With each space X_i is associated a real-valued function v_i on X_i. Our generalized N-body Schrödinger operator is given by

$$H=-\Delta+V(x),$$

where

$$V(x)=\sum_{i=1}^{M}v_i(\pi_i x).$$

We will always assume that the potentials satisfy

$$\text{for all } l,\ v_l(-\Delta_l+1)^{-1} \quad \text{and} \quad (-\Delta_l+1)^{-1}x_l\cdot\nabla_l v_l(-\Delta_l+1)^{-1} \quad \text{are compact operators on } L^2(X_l), \tag{2.1}$$

and sometimes it will be necessary to also assume that for each l

$$(-\Delta_l+1)^{-1/2}x_l\cdot\nabla_l v_l(-\Delta_l+1)^{-1} \quad \text{is bounded}$$

and

$$(-\Delta_l+1)^{-1}(x_l\cdot\nabla_l)^2 v_l(-\Delta_l+1)^{-1} \quad \text{is bounded.} \tag{2.2}$$

It is convenient to introduce the family of subspaces, $\mathscr{L}$, consisting of $\{0\}$ and all subspaces of the form

$$\operatorname{span} X_{i_1}\cup\cdots\cup X_{i_l},$$

where $\{i_1,\ldots,i_l\}\subset\{1,\ldots,M\}$. Given $X\in\mathscr{L}$, let

$$V_X(x)=\sum_{i:X_i\subset X} v_i(\pi_i x),$$

$$\Delta_X = \text{Laplace operator for the subspace } X,$$

$$H_X = -\Delta_X + V_X, \quad \text{in } L^2(X).$$

By convention, $H_{\{0\}}=0$ on $\mathbb{C}$. Any $\mu\in\mathbb{R}$ which is an eigenvalue of H_X for some $X\in\mathscr{L}$, $X\neq\mathbb{R}^n$ is called a threshold of H. The set of all thresholds of H is denoted $\mathscr{T}(H)$.

The basic theorems about the spectrum of H were proved in [P–S–S]:

Theorem 2.1. *Suppose (2.1) holds. Then $\mathscr{T}(H)$ is a closed countable set. All eigenvalues, λ, of H which are not in $\mathscr{T}(H)$ have finite multiplicity. The only possible accumulation points of eigenvalues of H lie in $\mathscr{T}(H)$.*

The weighted spaces $L_s^2(\mathbb{R}^n)=\langle x\rangle^{-s}L^2(\mathbb{R}^n)$ will play a role in our discussion. Here $\langle x\rangle=\sqrt{1+|x|^2}$ and the norm in L_s^2 is $\|f\|_{L_s^2}=(\int|\langle x\rangle^{+s}f(x)|^2dx)^{1/2}$.

Theorem 2.2 [P–S–S]. *Suppose (2.1) and (2.2) hold. Then for any λ which is neither a threshold nor an eigenvalue of H, the strong limits*

$$\lim_{\varepsilon\downarrow 0}(H-\lambda\mp i\varepsilon)^{-1}=(H-\lambda\mp i0)^{-1}$$

exist as maps from L_s^2 to L_{-s}^2 for any $s>1/2$. The operators $(H-\lambda\mp i0)^{-1}$ are (norm) Hölder continuous in the variable λ.

The proofs of these two theorems are based on a crucial estimate first proved by Mourre [M1] and used by him to prove theorems similar to Theorems 2.1 and 2.2 for the 3-body problem. The "Mourre estimate" which follows was proved in general in [P–S–S]. See also [F–H1]. In the following, $A=(x\cdot D+D\cdot x)/2$, where D is the gradient operator.

Theorem 2.3. *Suppose (2.1) holds and $\lambda\notin\mathscr{T}(H)$. Then there is a compact operator K and an open interval I containing λ so that for some $c_0>0$,*

$$E_H(I)[H,A]E_H(I)\geqq c_0E_H(I)+K. \tag{2.3}$$

Note that in (2.3), the operator family $\{E_H(\beta):\beta$ a Borel set of $\mathbb{R}\}$ consists of the spectral projections of H.

We will need a slight generalization of this result which basically states that (2.3) is stable under the addition of a small perturbation to H.

Denote by $\mathscr{B}_1$ the set of all real-valued functions W such that

$$|W|_1 = \|W(-\Delta+1)^{-1}\| + \|(-\Delta+1)^{-1}[A,W](-\Delta+1)^{-1}\| < \infty.$$

Lemma 2.4. *Suppose (2.1) holds and that $\lambda \notin \mathscr{T}(H)$. Then*

(a) *there is an $\varepsilon_0 > 0$ and open interval J containing λ so that for any $W \in \mathscr{B}_1$ with $|W|_1 \leqq \varepsilon_0$, we have*

$$E_{H+W}(J)[H+W,A]E_{H+W}(J) \geqq \frac{c_0}{2} E_{H+W}(J) + E_{H+W}(J)K_1 E_{H+W}(J),$$

where c_0 is as in (2.3) and K_1 is a compact operator independent of W. If $K=0$ in (2.3), then $K_1 = 0$ in (2.4).

(b) *If W is a symmetric operator with $W(-\Delta+1)^{-1}$ and $(-\Delta+1)^{-1}[A,W]$ $(-\Delta+1)^{-1}$ compact, there is an open interval J containing λ and a compact operator K_W so that*

$$E_{H+W}(J)[H+W,A]E_{H+W}(J) \geqq c_0 E_{H+W}(J) + K_W. \tag{2.4}$$

Proof. Assuming that (2.3) holds, let ε_0 be small enough so that $H+W$ is self-adjoint. Let $H' = H+W$ and suppose $f \in C_0^\infty(\mathbb{R})$ is one in a neighbourhood of λ and zero outside I. Then by (2.3)

$$f(H)[H,A]f(H) \geqq c_0 f(H)^2 + K_1, \tag{2.5}$$

where $K_1 = f(H)Kf(H)$. Let $\mathcal{O}_H$ be the left side of (2.5). Then clearly

$$\mathcal{O}_{H'} \geqq c_0 f(H')^2 + K_1 + \mathscr{E},$$

where

$$\mathscr{E} = \mathcal{O}_{H'} - \mathcal{O}_H + c_0(f(H)^2 - f(H')^2). \tag{2.6}$$

We need to show that $\|\mathscr{E}\| \to 0$ as $|W|_1 \to 0$. Note that $\|(H+W+i)^{-1} - (H+i)^{-1}\| \to 0$, so that any polynomial in $(H+W+\Lambda)^{-1}$ converges to the same polynomial with argument $(H+\Lambda)^{-1}$ (if Λ is sufficiently large). This implies that $\|f(H) - f(H+W)\| \to 0$. All other terms in $\mathscr{E}$ can be controlled by this estimate alone. We omit the details. To prove (b), we need only show that $\mathscr{E}$ in (2.6) is compact. The fact that for any C_0^∞ function g, $g(H+W) - g(H)$ is compact, easily follows from the same result for polynomials. This in turn follows from the fact that the difference in resolvents is compact. All other terms in $\mathscr{E}$ are easily handled with this information. ■

Theorem 2.5. *Suppose $\lambda_0 \notin \mathscr{T}(H)$ and that (2.1) holds. Then there is an open interval J containing λ_0 and a number $\delta > 0$ so that*

(a) *if λ_0 is not an eigenvalue of H and $W \in \mathscr{B}_1$ with $|W|_1 < \delta$, then $H+W$ has no eigenvalues in J.*

(b) *If λ_0 is an eigenvalue of H with multiplicity m and $W \in \mathscr{B}_1$ with $|W|_1 < \delta$, then $H + W$ has, at most, m eigenvalues in J of total multiplicity m.*

Proof. Let $P_0 = E_H(\{\lambda_0\})$ and define

$$H_1 = H + P_0.$$

Then H_1 has no eigenvalues at λ_0 and $\mathscr{T}(H) = \mathscr{T}(H_1)$. Note that $AP_0 = cA(H+i)^{-1}P_0 = cA(H+i)^{-1}\langle x\rangle^{-1}\langle x\rangle P_0$. Since non-threshold eigenfunctions of H decay exponentially ([F−H2]), $\langle x\rangle P_0$ is compact. Thus AP_0 will be shown to be compact if we can show $\|A(H+i)^{-1}\langle x\rangle^{-1}\| < \infty$. Because of (2.1), $[A, (H+i)^{-1}]$ is bounded so that we must only show $\|(H+i)^{-1}A\langle x\rangle^{-1}\| < \infty$. This is obvious from the explicit form of A. It follows from these considerations that $[A, P_0]$ is compact, and thus by Lemma 2.4(b),

$$E_{H_1}(J)[H_1, A]E_{H_1}(J) \geqq c_0 E_{H_1}(J) + \tilde{K}$$

for some compact $\tilde{K}$ and some open interval J containing λ_0. By shrinking J we can assume $\|E_{H_1}(J)\tilde{K}E_{H_1}(J)\| \leqq c_0/2$ so that

$$E_{H_1}(J)[H_1, A]E_{H_1}(J) \geqq \frac{c_0}{2} E_{H_1}(J)$$

for some open interval J containing λ_0. As in the proof of Lemma 2.4(a), we can then find $\delta > 0$ and an open interval I centered at λ_0 so that for any operator $W \in \mathscr{B}_1$ with $|W|_1 \leqq \delta$ we have

$$E_{H_1+W}(I)[H_1 + W, A]E_{H_1+W}(I) \geqq \frac{c_0}{4} E_{H_1+W}(I). \tag{2.7}$$

Now suppose $H + W$ has one or more eigenvalues in $\tilde{I} \subset I$ with $\tilde{I} = (\lambda_0 - \gamma, \lambda_0 + \gamma)$ of total multiplicity $m_1 > m$. Choose an orthonormal set $\{\psi_1, \ldots, \psi_{m+1}\}$ such that

$$(H + W)\psi_l = \lambda_l \psi_l \quad l = 1, \ldots, m+1$$

and $\lambda_l \in \tilde{I}$. Choose a linear combination $\psi = \sum_{j=1}^{m+1} a_j \psi_j$ with norm 1 such that $P_0 \psi = 0$. (We treat the case where λ_0 is not an eigenvalue of H by taking $m = 0$.) Then

$$(H_1 + W - \lambda_0)\psi = (H + W - \lambda_0)\psi = \sum_j a_j(\lambda_j - \lambda_0)\psi_j,$$

so that $\|(H_1 + W - \lambda_0)\psi\| \leqq \gamma$. Then,

$$2\,\mathrm{Re}\,((H_1 + W - \lambda_0)\psi, A\psi) \leqq 2\|A\psi\|\gamma.$$

It thus follows from (2.7) that

$$2\|A\psi\|\gamma \geqq (\psi, [H_1 + W - \lambda_0, A]\psi) \geqq (\psi, (1-E)CE\psi)$$

$$+ (\psi, C(1-E)\psi) + \frac{c_0}{4}\|E\psi\|^2, \tag{2.8}$$

where

$$E = E_{H_1+W}(I) \quad \text{and} \quad C = [H_1 + W, A].$$

Since $\|(H_1 + W + i)^{-1} C(H_1 + W + i)^{-1}\|$ is bounded independently of W for $|W|_1$ small, and since

$$\|(1-E)\psi\| = \|(H_1 + W - \lambda_0)^{-1}(1-E)(H_1 + W - \lambda_0)\psi\| \leqq 2|I|^{-1}\gamma,$$

we conclude from (2.8) that

$$2\|A\psi\|\gamma \geqq \frac{c_0}{4} - k\gamma,$$

where k is a constant independent of W for $|W|_1$ small. Note that

$$\begin{aligned}\|A\psi\| &\leqq \sum_{j=1}^{m+1} \|A\psi_j\| \leqq \sum_{j=1}^{m+1} \|A(H+W-\lambda_j+i)^{-1}\langle x\rangle^{-1}\| \, \|\langle x\rangle\psi_j\| \\ &\leqq c\sum_{j=1}^{m+1} \|\langle x\rangle\psi_j\|.\end{aligned}$$

Thus assuming $\|\langle x\rangle\psi_j\|$ can be bounded independently of W for $|W|_1$ small enough, the theorem follows by taking $\gamma > 0$ and small enough. Thus our theorem can be deduced from the lemma which follows. ■

Lemma 2.6. *Suppose* (2.1) *is satisfied,* $H = -\Delta + V$, *and* $\lambda_0 \notin \mathscr{T}(H)$. *Then there is a* $\delta > 0$ *and* $\gamma > 0$ *such that if* W *is in* $\mathscr{B}_1$ *with* $|W|_1 < \delta$, *and* ψ *is an eigenfunction of* $H + W$ *with eigenvalue* $\lambda \in (\lambda_0 - \gamma, \lambda_0 + \gamma)$, *we have*

$$\|\langle x\rangle\psi\| \leqq k\|\psi\|,$$

where k *is independent of* W.

This lemma is proved in Appendix A. The proof uses certain uniform estimates to be given in the next section. These estimates will also be useful in showing that the expression in (1.1) is not always zero.

We give the following corollary of the proof of Theorem 2.5 which will be of later use:

Corollary 2.7. *In the situation of part* (b) *of Theorem 2.5, let* P_W *be the projection onto the span of all eigenvectors of* $H + W$ *with eigenvalue in* J. *Then, if* $\psi \in \operatorname{Ran} P_W$ *and* $E_H(\{\lambda_0\})\psi = 0$, *we have* $\psi = 0$.

The proof is essentially contained in the proof of Theorem 2.5.

III. Estimates

We assume at the outset that $H = -\Delta + V$ in $L^2(\mathbb{R}^n)$, where $V(x) = \sum_{i=1}^{M} v_i(\pi_i x)$ and (2.1) holds for the real potentials v_i. Define

$$\xi_1 = (\langle x\rangle(1 + \langle x\rangle/\mu)^{-1})^t$$
$$\xi_2 = (1 + \gamma\langle x\rangle/\mu)^\mu e^{\alpha\langle x\rangle}$$

for $\alpha, \gamma, t \geqq 0$ and $\mu > 0$. The purpose of this section is to prove estimates of the form

$$k\|\langle x\rangle^s \xi_j(H + W - \lambda)\varphi\| \geqq \|\xi_j\varphi\| - \|K\xi_j\varphi\| \tag{3.1}$$

under certain conditions. Here $W \in \mathscr{B}_1$ and K is compact. We will later take $\mu \to \infty$ so that $\xi_1 \uparrow \langle x \rangle^t$ and $\langle \xi_2 \rangle \uparrow e^{(\alpha+\gamma)\langle x \rangle}$. Thus our estimates need to be uniform in μ. We have the following result:

Theorem 3.1. *Fix t and α non-negative. Suppose $\lambda_0 + \alpha^2 \notin \mathscr{T}(H)$. Then there exist positive constants ε, δ, k and a compact operator K so that if $W \in \mathscr{B}_1$, $|W|_1 + |\lambda - \lambda_0| < \varepsilon, \gamma \leqq \delta$, and $\mu \geqq 1$, then (3.1) holds for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ if either*

(i) $s = 1$, $\alpha = 0$, *and* $j = 1$ *or* 2, *or*
(ii) $s = 1/2$, $\alpha > 0$, *and* $j = 2$.

Proof. Our proof of (3.1) is based on the ideas of [F–H2].

According to Lemma 2.4, there is an $\varepsilon_1 > 0$, and open interval I centered at $\lambda_0 + \alpha^2$, a function $f \in C_0^\infty(\mathbb{R})$ which is 1 on I, and a compact operator K_1 so that if W is a symmetric operator in $\mathscr{B}_1$ with $|W|_1 < \varepsilon_1$, we have

$$f(H+W)[H+W, A]f(H+W) \geqq c_0 f(H+W)^2 + f(H+W)K_1 f(H+W). \quad (3.2)$$

Denote ξ_1 or ξ_2 by ξ and define $F = \ln \xi$. In addition, let

$$\mathscr{C} = [H+W, A],$$
$$\nabla F = xg,$$
$$G = (x\cdot\nabla)^2 g - (x\cdot\nabla)|\nabla F|^2.$$

We use the following computations from [F–H2] for $\varphi \in C_0^\infty$:

$$2\,\mathrm{Re}(A\xi\varphi, \xi(H+W-\lambda)\varphi) = (\xi\varphi, \mathscr{C}\xi\varphi) + 4\|g^{1/2}A\xi\varphi\|^2 - (\xi\varphi, G\xi\varphi), \quad (3.3)$$

$$(H+W-\lambda-|\nabla F|^2)\xi\varphi = \xi(H+W-\lambda)\varphi - (D\cdot\nabla F + \nabla F\cdot D)\xi\varphi, \quad (3.4)$$

$$D\cdot\nabla F + \nabla F\cdot D = 2gA + x\cdot\nabla g. \quad (3.5)$$

We now insert (3.2) into (3.3) and find

$$2\,\mathrm{Re}(A\xi\varphi, \xi(H+W-\lambda)\varphi) \geqq c_0\|\xi\varphi\|^2 + 4\|g^{1/2}A\xi\varphi\|^2 - (\xi\varphi, G\xi\varphi) - \mathscr{E} + (\xi\varphi, K_1(W)\xi\varphi) \quad (3.6)$$

where

$$\mathscr{E} = c_0(\xi\varphi, (1-f(H+W)^2)\xi\varphi) - (\xi\varphi, (1-f(H+W))\mathscr{C}f(H+W)\xi\varphi) - (\xi\varphi, \mathscr{C}(1-f(H+W))\xi\varphi) \quad (3.7)$$

and $K_1(W) = f(H+W)K_1 f(H+W)$.

Consider the quantity $\mathscr{E}$. Since for small ε_1, $\|(H+W+i)^{-1}\mathscr{C}(H+W+i)^{-1}\|$ is bounded uniformly in W, we have

$$|\mathscr{E}| \leqq c\|(H+W+i)\xi\varphi\|\cdot\|(H+W+i)(1-f(H+W))\xi\varphi\|.$$

We now demand that $|\lambda - \lambda_0| < \varepsilon_2$, where ε_2 is less than $\frac{1}{4}|I|$. This leads to the estimate

$$\begin{aligned}\|(1-f(H+W))\xi\varphi\| &= \|(H+W-\lambda-\alpha^2)^{-1}(1-f(H+W))(H+W-\lambda-\alpha^2)\xi\varphi\| \\ &\leqq 4|I|^{-1}\|(1-f(H+W))(H+W-\lambda-\alpha^2)\xi\varphi\| \\ &\leqq c\|(H+W-\lambda-\alpha^2)\xi\varphi\|.\end{aligned}$$

This kind of analysis leads to the estimate,

$$\begin{aligned}|\mathscr{E}| &\leqq c(\|(H+W-\lambda-\alpha^2)\xi\varphi\|+\|\xi\varphi\|)\cdot\|(H+W-\lambda-\alpha^2)\xi\varphi\|\\ &\leqq \tfrac{1}{4}c_0\|\xi\varphi\|^2+c\|(H+W-\lambda-\alpha^2)\xi\varphi\|^2.\end{aligned} \tag{3.8}$$

Inserting (3.8) into (3.6) gives

$$\begin{aligned}2\,\mathrm{Re}(A\xi\varphi,\xi(H+W-\lambda)\varphi) &\geqq \tfrac{3}{4}c_0\|\xi\varphi\|^2+4\|g^{1/2}A\xi\varphi\|^2\\ &-c\|(H+W-\lambda-\alpha^2)\xi\varphi\|^2-(\xi\varphi,G\xi\varphi)+(\xi\varphi,K_1(W)\xi\varphi).\end{aligned} \tag{3.9}$$

If $\xi=\xi_1$, we have

$$|\nabla F|^2=t^2(1-\langle x\rangle^{-2})\langle x\rangle^{-2}(1+\mu^{-1}\langle x\rangle)^{-2}\leqq t^2\langle x\rangle^{-2}. \tag{3.10}$$

$$G\leqq c\langle x\rangle^{-2},\quad |x\cdot\nabla g|\leqq c\langle x\rangle^{-2},\quad g\leqq c\langle x\rangle^{-2}, \tag{3.11}$$

while if $\xi=\xi_2$,

$$\nabla F=x\langle x\rangle^{-1}(\alpha+\gamma(1+\gamma\langle x\rangle/\mu)^{-1})$$

so that

$$\alpha|x|\langle x\rangle^{-1}\leqq|\nabla F|\leqq\alpha+\gamma. \tag{3.12}$$

In addition, for $\gamma\leqq 1$,

$$G\leqq c\langle x\rangle^{-1}+\gamma(\alpha+\gamma)/2, \tag{3.13}$$

$$\alpha\langle x\rangle^{-1}\leqq g\leqq(\alpha+\gamma)\langle x\rangle^{-1}, \tag{3.14}$$

$$|x\cdot\nabla g|\leqq c\langle x\rangle^{-1}. \tag{3.15}$$

We now use (3.4) and (3.5) to bound $\|(H+W-\lambda-\alpha^2)\xi\varphi\|$:

$$\begin{aligned}\|(H+W-\lambda-\alpha^2)\xi\varphi\| &\leqq \|\xi(H+W-\lambda)\varphi\|+2\|gA\xi\varphi\|\\ &\quad+\|(x\cdot\nabla g)\xi\varphi\|+\|(\alpha^2-(\nabla F)^2)\xi\varphi\|.\end{aligned}$$

For the sake of efficiency, we treat the cases $\xi=\xi_1$ and $\xi=\xi_2$ together. Let χ_N be the characteristic function of the ball $\langle x\rangle<N$. We have

$$|(\alpha^2-|\nabla F|^2)|\leqq c\gamma+c\langle x\rangle^{-2},\quad |x\cdot\nabla g|\leqq c\langle x\rangle^{-1}.$$

In addition

$$g=\chi_N g+(1-\chi_N)g\leqq c(\chi_N\langle x\rangle^{-1}+N^{-1/2}g^{1/2}).$$

We thus have

$$\begin{aligned}\|(H+W-\lambda-\alpha^2)\xi\varphi\| &\leqq \|\xi(H+W-\lambda)\varphi\|+c\|\chi_N\nabla(\xi\varphi)\|\\ &\quad+cN^{-1/2}\|g^{1/2}A\xi\varphi\|+C\|\langle x\rangle^{-1}\xi\varphi\|+\gamma c\|\xi\varphi\|.\end{aligned} \tag{3.16}$$

If h is a bounded function with $h(x)\to 0$ as $|x|\to\infty$, then

$$\|h\xi\varphi\|\leqq\|h\tilde{f}(H+W)\xi\varphi\|+\|h(1-\tilde{f}(H+W))\xi\varphi\|.$$

Choose $\tilde{f}\in C_0^\infty(\mathbb{R})$ so that $\tilde{f}=1$ on an interval of length $2L+1$ centered at $\lambda_0+\alpha^2$.

Then, if ε_2 is small enough

$$\|(1-\tilde{f}(H+W))\xi\varphi\| = \|(H+W-\lambda-\alpha^2)^{-1}(1-\tilde{f}(H+W))(H+W-\lambda-\alpha^2)\xi\varphi\|$$
$$\leqq L^{-1}\|(H+W-\lambda-\alpha^2)\xi\varphi\|,$$

so that

$$\|h\xi\varphi\| \leqq \|h\tilde{f}(H+W)\xi\varphi\| + cL^{-1}\|(H+W-\lambda-\alpha^2)\xi\varphi\|. \tag{3.17}$$

Note that the map $W \mapsto \tilde{f}(H+W)$ is continuous in the sense that for $|W_0|_1$ small, as $|W-W_0|_1 \to 0$, we have $\|\tilde{f}(H+W)-\tilde{f}(H+W_0)\| \to 0$. Thus the compact operator $h\tilde{f}(H+W)$ is also continuous in the variable W.

Let us bound the term $\|\chi_N(\nabla(\xi\varphi))\|$ in (3.16): If $\tilde{\chi}_N$ is a smooth function with $\chi_N \leqq \tilde{\chi}_N \leqq \chi_{N+1}$, we have

$$\|\chi_N\nabla(\xi\varphi)\|^2 \leqq \sum_j (D_j(\xi\varphi), \tilde{\chi}_N^2 D_j(\xi\varphi))$$
$$= \mathrm{Re}(\tilde{\chi}_N^2\xi\varphi, -\Delta(\xi\varphi)) + \|h\xi\varphi\|^2$$

where $h \in C_0^\infty(\mathbb{R}^n)$. Thus,

$$\|\chi_N\nabla(\xi\varphi)\|^2 \leqq c(\|\chi_{N+1}\xi\varphi\|\ \|\Delta\xi\varphi\| + \|h\xi\varphi\|^2)$$
$$\leqq c(\|\chi_{N+1}\xi\varphi\|\cdot(\|(H+W-\lambda-\alpha^2)\xi\varphi\| + \|\xi\varphi\|) + \|h\xi\varphi\|^2)$$
$$\leqq \tfrac{1}{2}L^{-1}\|\xi\varphi\|^2 + c_L\|\chi_{N+1}\xi\varphi\|^2 + \tfrac{1}{2}L^{-1}\|(H+W-\lambda-\alpha^2)\xi\varphi\|^2 + \|h\xi\varphi\|^2$$
$$\leqq L^{-1}\|\xi\varphi\|^2 + L^{-1}\|(H+W-\lambda-\alpha^2)\xi\varphi\|^2 + \|K_2(W)\xi\varphi\|^2, \tag{3.18}$$

where we have used (3.17). The operator $K_2(W)$ is compact and continuous in W.

Putting together (3.16), (3.17), and (3.18), we have

$$\|(H+W-\lambda-\alpha^2)\xi\varphi\| \leqq 2\|\xi(H+W-\lambda)\varphi\| + cN^{-1/2}\|g^{1/2}A\xi\varphi\|$$
$$+ c\gamma\|\xi\varphi\| + \|K_3(W)\xi\varphi\|, \tag{3.19}$$

where $K_3(\cdot)$ is continuous and compact. Using (3.17) again, we find from (3.9), (3.11), and (3.13),

$$2\,\mathrm{Re}(A\xi\varphi, \xi(H+W-\lambda)\varphi) \geqq \tfrac{3}{4}c_0\|\varphi\|^2 + 4\|g^{1/2}A\xi\varphi\|^2$$
$$- c\gamma\|\xi\varphi\|^2 - c\|(H+W-\lambda-\alpha^2)\xi\varphi\|^2 + (\xi\varphi, K_4(W)\xi\varphi),$$

where $K_4(\cdot)$ is continuous and compact. Using (3.19) with N large enough, we have

$$2\,\mathrm{Re}(A\xi\varphi, \xi(H+W-\lambda)\varphi) \geqq (\tfrac{3}{4}c_0 - c\gamma)\|\xi\varphi\|^2 + 2\|g^{1/2}A\xi\varphi\|^2$$
$$- c\|\xi(H+W-\lambda)\varphi\|^2 + (\xi\varphi, K_5(W)\xi\varphi), \tag{3.20}$$

where again $K_5(\cdot)$ is continuous and compact.

We now consider two cases. If $\alpha = 0$, we have

$$2\,\mathrm{Re}(A\xi\varphi, \xi(H+W-\lambda)\varphi) \leqq 2\|\langle x\rangle^{-1}A\xi\varphi\|\cdot\|\langle x\rangle\xi(H+W-\lambda)\varphi\|$$
$$\leqq c(\|\nabla(\xi\varphi)\| + \|\xi\varphi\|)\cdot\|\langle x\rangle\xi(H+W-\lambda)\varphi\|.$$

An easy estimate gives

$$\|\nabla(\xi\varphi)\|^2 \leqq c(\|\xi\varphi\|^2 + \|\xi(H+W-\lambda)\varphi\|^2),$$

so that

$$2\,\mathrm{Re}(A\xi\varphi, \xi(H+W-\lambda)\varphi) \leqq c(\|(H+W-\lambda)\varphi\| + \|\xi\varphi\|)\cdot\|\langle x\rangle\xi(H+W-\lambda)\varphi\|$$
$$\leqq c\|\langle x\rangle\xi(H+W-\lambda)\varphi\|^2 + \tfrac{1}{4}c_0\|\xi\varphi\|^2.$$

Combining this with (3.20) gives the result that for some $c>0$,

$$c\|\langle x\rangle\xi(H+W-\lambda)\varphi\|^2 \geqq (\tfrac{1}{2}c_0 - c\gamma)\|\xi\varphi\|^2 + (\xi\varphi, K_5(W)\xi\varphi). \tag{3.21}$$

If $\alpha>0$, we estimate differently:

$$2\,\mathrm{Re}(A\xi\varphi, \xi(H+W-\lambda)\varphi) \leqq 2\|g^{1/2}A\xi\varphi\|\cdot\|g^{-1/2}\xi(H+W-\lambda)\varphi\|$$
$$\leqq \|g^{1/2}A\xi\varphi\|^2 + \|g^{-1/2}\xi(H+W-\lambda)\varphi\|^2.$$

But from (3.14), $g^{-1/2} \leqq \alpha^{-1/2}\langle x\rangle^{1/2}$, so that

$$2\,\mathrm{Re}(A\xi\varphi, \xi(H+W-\lambda)\varphi) \leqq \|g^{1/2}A\xi\varphi\|^2 + \alpha^{-1}\|\langle x\rangle^{1/2}\xi(H+W-\lambda)\varphi\|^2.$$

Combining this with (3.20) gives

$$c\|\langle x\rangle^{1/2}\xi(H+W-\lambda)\varphi\|^2 \geqq (\tfrac{1}{2}c_0 - c\gamma)\|\xi\varphi\|^2 + (\xi\varphi, K_5(W)\xi\varphi). \tag{3.22}$$

Choose ε_1 small enough so that $\|K_5(W) - K_5(0)\| < \frac{1}{4}c_0$, and note that from the inequality

$$2x \geqq -\beta^2 x^2 - \beta^{-2}$$

we have

$$K_5(0) \geqq -(K_5(0))^2\beta^2 - \beta^{-2} = -K^2 - \beta^{-2}.$$

We choose $\gamma < \delta$. Then

$$(\tfrac{1}{2}c_0 - c\gamma)\|\xi\varphi\|^2 + (\xi\varphi, K_5(W)\xi\varphi) \geqq (\tfrac{1}{4}c_0 - c\delta)\|\xi\varphi\|^2 - \beta^{-2}\|\xi\varphi\|^2 - \|K\xi\varphi\|^2$$
$$\geqq \tfrac{1}{8}c_0\|\xi\varphi\|^2 - \|K\xi\varphi\|^2,$$

if $\delta>0$ is small enough and β is large enough. Combining this with (3.21) and (3.22) gives the desired estimates. ■

We will need the following corollary of Theorem 3.1 in the next section:

Corollary 3.2. *Let Q be a finite-dimensional orthogonal projection and $\beta\in\mathbb{R}$. Suppose $\|e^{\alpha_0|x|}Q\| < \infty$ for some $\alpha_0 > 0$. Fix t and α non-negative with $\lambda_0 + \alpha^2 \notin \mathscr{T}(H)$ and $\alpha < \alpha_0$. Then there exist positive constants ε and δ so that for each $W \in \mathscr{B}_1$ with $|W|_1 < \varepsilon$ and λ_0 not an eigenvalue of $H + W + \beta Q$, the estimate*

$$k\|\langle x\rangle^s\xi_j(H+W+\beta Q-\lambda_0)\varphi\| \geqq \|\xi_j\varphi\| \tag{3.23}$$

holds for all $\varphi \in C_0^\infty(\mathbb{R}^n)$ if either

(i) $s=1, \alpha=0$, *and* $j=1$ *or* 2, *or*
(ii) $s=1/2$, $\alpha>0$, *and* $j=2$.

Here $0 \leqq \gamma \leqq \delta$, $\mu \geqq 1$, and k is a constant depending on W and β.

Proof. We first show that given $\varepsilon>0$ there is a compact operator K_ε such that

$$\|\langle x\rangle^s\xi_j Q\varphi\| \leqq \varepsilon\|\xi_j\varphi\| + \|K_\varepsilon\xi_j\varphi\|, \tag{3.24}$$

where K_ε is independent of μ and γ. It is enough to assume Q is one dimensional so that $Q\varphi = (\psi, \varphi)\psi$. Then

$$\|\langle x\rangle^s \xi_j Q\varphi\| = \|\langle x\rangle^s \xi_j \psi\| \, |(\psi,\varphi)| \leqq c|(\psi,\varphi)|,$$

if δ is small enough so that $\alpha+\gamma<\alpha_0$. Let $H_0 = -\Delta$, and suppose $f \in C_0^\infty(\mathbb{R})$. Then

$$\|\langle x\rangle^s \xi_j Q\varphi\| \leqq c\|(1-f(H_0))\psi\| \, \|\varphi\| + c|(f(H_0)\psi,\varphi)|.$$

Now

$$(f(H_0)\psi,\varphi) = ((H_0+1)\langle x\rangle \xi_j^{-1} f(H_0)\psi, (H_0+1)^{-1}\langle x\rangle^{-1}\xi_j\varphi).$$

It is easy to see that $\|(H_0+1)\langle x\rangle \xi_j^{-1} f(H_0)\psi\|$ is bounded independent of $\mu \geqq 1$ and $\gamma \leqq \delta$ (say by M) so that

$$\begin{aligned}\|\langle x\rangle^s \xi_j Q\varphi\| &\leqq c\|(1-f(H_0))\psi\|\cdot\|\varphi\| + cM\|K\xi_j\varphi\| \\ &\leqq c'\|(1-f(H_0))\psi\|\cdot\|\xi_j\varphi\| + cM\|K\xi_j\varphi\|.\end{aligned}$$

We obtain (3.24) by choosing f so that $c'\|(1-f(H_0))\psi\| < \varepsilon$.

From (3.1), we now obtain

$$k\|\langle x\rangle^s \xi_j (H+W+\beta Q-\lambda_0)\varphi\| \geqq \|\xi_j\varphi\| - \|K\xi_j\varphi\| \tag{3.25}$$

under the stated conditions. Now suppose that for some sequence $\varphi_m \in C_0^\infty$, $0 \leqq \gamma_m \leqq \delta$, and $\mu_m \geqq 1$, we have $\|\xi_j^m \varphi_m\| = 1$ and

$$\|\langle x\rangle^s \xi_j^m (H+W+\beta Q-\lambda_0)\varphi_m\| \to 0. \tag{3.26}$$

Here ξ_j^m corresponds to μ_m and γ_m. We can assume that $\gamma_m \to \gamma$ and $\mu_m \to \mu$, where possibly $\mu = \infty$, in which case $\xi_j^m \to \xi_j$ uniformly on compact sets. Since λ_0 is not an eigenvalue of $H+W+\beta Q$, and since (3.26) implies $(H+W+\beta Q-\lambda_0)\varphi_m \to 0$, we learn that $\varphi_m \to 0$ weakly. Since for each $\varphi \in C_0^\infty$,

$$(\varphi, \xi_j^m \varphi_m) - (\xi_j \varphi, \varphi_m) \to 0,$$

we see that $(\varphi, \xi_j^m\varphi_m) \to 0$, and since $\|\xi_j^m \varphi_m\| = 1$, we learn that $\xi_j^m \varphi_m \to 0$ weakly. But, since K is compact, $K\xi_j^m\varphi_m \to 0$ in norm which contradicts (3.25) and (3.26). This implies the result. ■

IV. $\delta(H-\lambda) \neq 0$

In this section we will always assume that (2.1) and (2.2) are satisfied and that $H = -\Delta + V$ in $L^2(\mathbb{R}^n)$. We also suppose $\lambda_0 \notin \mathcal{T}(H)$ but that $\lambda_0 \in \sigma_{\mathrm{ess}}(H)$.

If λ_0 is an eigenvalue of H, we will want to study the continuous spectrum in which λ_0 is embedded. Let $P_0 = E_H(\{\lambda_0\})$. The operator $\bar{H} = H + P_0$ has only continuous spectrum in a neighbourhood of λ_0 and is therefore a convenient object to analyze.

Lemma 4.1. *For some $\alpha_0 > 0$, $\exp(\alpha_0|x|)P_0$ is bounded and the operators $[A, P_0]$ and $[A,[A,P_0]]$ are compact.*

Proof. The first statement follows from [F-H2] which gives the additional result that α_0^2 can be any number less than the distance from λ_0 to the next highest threshold.

If $\psi \in \operatorname{Ran} P_0$, we claim that ψ is in the domain of A^2. Note that

$$(A+1)^2\psi = B\langle x\rangle^2(H+i)\psi, \quad B=(A+1)^2(H+i)^{-1}\langle x\rangle^{-2},$$

so we need only show that B is bounded. We can compute

$$B=[A,[A,(H+i)^{-1}]]\langle x\rangle^{-2}+2[A,(H+i)^{-1}](A+1)\langle x\rangle^{-2} + (H+i)^{-1}(A+1)^2\langle x\rangle^{-2}.$$

Using (2.1) and (2.2), it is not difficult to show that the first two terms above are bounded. The third term can be handled by elementary means. This implies that $AP_0, P_0A, A^2P_0, AP_0A$, and P_0A^2 are all bounded. These operators are compact because they are finite rank. ■

From the estimate (2.7) and the proof of Theorem 2.2 ([P–S–S]) we learn that the strong limits

$$\lim_{\varepsilon\downarrow 0}(\bar{H}-\lambda_0\mp i\varepsilon)^{-1}=(\bar{H}-\lambda_0\mp i0)^{-1}$$

exist as maps from L_s^2 to L_{-s}^2 for any $s>1/2$.

Let

$$\delta(\bar{H}-\lambda_0)\equiv\frac{1}{2\pi i}\{(\bar{H}-\lambda_0-i0)^{-1}-(\bar{H}-\lambda_0+i0)^{-1}\}.$$

The operator $\delta(\bar{H}-\lambda_0)$ is a bounded operator from L_s^2 to L_{-s}^2 for any $s>1/2$. In order to understand the perturbation of the eigenvalue λ_0, it is important to know that this operator is not identically zero. For this purpose, we introduce the following condition:

For each $i=1,\dots,M, v_i$ has the decomposition

$$v_i=v_i^s+v_i^L, \quad \langle y\rangle^{1/2}v_i^s(y)(-\Delta_i+1)^{-1} \quad \text{is compact, and}$$
$$v_i^L\in C^1(X_i) \quad \text{with} \quad \lim_{|y|\to\infty}(|v_i^L(y)|+|y|\cdot|\nabla v_i^L(y)|)=0. \tag{4.1}$$

Theorem 4.2. *Suppose, in addition to the assumptions at the beginning of this section, that (4.1) holds. Then* $\delta(\bar{H}-\lambda_0)\neq 0$.

Remark. This result will be needed in Sect. V. As an aside, we note here the fact that $\delta(\bar{H}-\lambda_0)\neq 0$ implies the existence of nonzero solutions u of $(-\Delta+V-\lambda_0)u=0$ with $u\in L_{-s}^2(\mathbb{R}^n)$, $s>1/2$. Just set $u=\delta(\bar{H}-\lambda_0)\varphi$ for suitable φ. (From Eq. (5.18), we see that $P_0\delta(\bar{H}-\lambda_0)\varphi=0$.) For the existence of generalized eigenfunctions for a.e. value of the spectral parameter, see [S2, Ki, J–Ki].

Proof. According to [M2] and [P–S–S], the operators

$$P_{\mp}(\bar{H}-\lambda_0\mp i0)^{-1}\langle x\rangle^{-s}$$

are bounded. Here P_+ is the spectral projection, $\chi_{[0,\infty)}(-iA)$, $P_-=1-P_+$, and $s>1$. Suppose $\delta(\bar{H}-\lambda_0)=0$. Then

$$R=(\bar{H}-\lambda_0-i0)^{-1}=(\bar{H}-\lambda_0+i0)^{-1}.$$

Formally $R\langle x\rangle^{-s}=(P_-+P_+)R\langle x\rangle^{-s}$ is also bounded if $s>1$. We prove this in Appendix B. Suppose $\varphi\in L^2$. Then, if $\psi\in C_0^\infty$,

$$((\bar{H}-\lambda_0)\psi, R\langle x\rangle^{-s}\varphi)=\lim_{\varepsilon\downarrow 0}((\bar{H}-\lambda_0)\psi,(\bar{H}-\lambda_0-i\varepsilon)^{-1}\langle x\rangle^{-s}\varphi)=(\psi,\langle x\rangle^{-s}\varphi).$$

since C_0^∞ is a core for $\bar{H}$, $R\langle x\rangle^{-s}\varphi$ is in the domain of $\bar{H}-\lambda_0$ and $(\bar{H}-\lambda_0)R\langle x\rangle^{-s}\varphi=\langle x\rangle^{-s}\varphi$. Thus $\langle x\rangle^{-s}\varphi$ is in the domain of $(\bar{H}-\lambda_0)^{-1}$ and

$$R\langle x\rangle^{-s}\varphi=(\bar{H}-\lambda_0)^{-1}\langle x\rangle^{-s}\varphi.$$

We thus learn that $(\bar{H}-\lambda_0)^{-1}\langle x\rangle^{-s}$ is a bounded operator if $s>1$.

Let $\xi=(\langle x\rangle(1+\mu^{-1}\langle x\rangle)^{-1})^t$ with $t\geqq 0$ and μ positive. We will estimate

$$\|\xi(\bar{H}-\lambda_0)^{-1}\langle x\rangle^{-1}\xi^{-1}(1+\mu^{-1}\langle x\rangle)^{-1}\|\equiv N(\mu).$$

Note that $N(\mu)<\infty$ because ξ and ξ^{-1} are bounded. Suppose $\psi\in C_0^\infty(\mathbb{R}^n)$, and $\varphi=(\bar{H}-\lambda_0)^{-1}\langle x\rangle^{-1}\xi^{-1}(1+\mu^{-1}\langle x\rangle)^{-1}\psi$. We have

$$\begin{aligned}&\|\xi(\bar{H}-\lambda_0)^{-1}\langle x\rangle^{-1}\xi^{-1}(1+\mu^{-1}\langle x\rangle)^{-1}\psi\|/\|\psi\|\\ &=\|\xi\varphi\|/\|(1+\mu^{-1}\langle x\rangle)\xi\langle x\rangle(\bar{H}-\lambda_0)\varphi\|\\ &\leqq\|\xi\varphi\|/\|\xi\langle x\rangle(\bar{H}-\lambda_0)\varphi\|.\end{aligned}\tag{4.2}$$

According to Corollary 3.2, we have

$$k\|\xi\langle x\rangle(\bar{H}-\lambda_0)\tilde{\varphi}\|\geqq\|\xi\tilde{\varphi}\|\tag{4.3}$$

for all $\tilde{\varphi}\in C_0^\infty(\mathbb{R}^n)$. Equation (4.3) easily extends to $\tilde{\varphi}$ in $\mathscr{D}(\bar{H})$ with compact support, so if $\eta\in C_0^\infty(\mathbb{R}^n)$ with $\eta(x)=1$ for $|x|<1$, define $\eta_m(x)=\eta(x/m)$ and let $\varphi_m=\eta_m\varphi$. Then

$$k\|\xi\langle x\rangle(\bar{H}-\lambda_0)\varphi_m\|\geqq\|\xi\varphi_m\|.\tag{4.4}$$

We have

$$\langle x\rangle(\bar{H}-\lambda_0)\varphi_m=\eta_m\langle x\rangle(\bar{H}-\lambda_0)\varphi-\langle x\rangle[\Delta,\eta_m]\varphi+\beta\langle x\rangle[P_0,\eta_m]\varphi.\tag{4.5}$$

Now, $[\Delta,\eta_m]=2\nabla\eta_m\cdot D+\Delta\eta_m$ so that the middle term is given by

$$-(2/m)\langle x\rangle\nabla\eta(x/m)\cdot\nabla\varphi-\frac{\langle x\rangle}{m^2}\Delta\eta(x/m)\varphi.$$

Clearly this is bounded uniformly in m by an L^2 function and converges pointwise to zero. Thus we get L^2 convergence by Lebesgue's dominated convergence theorem. The last term in (4.5) is easily seen to converge to zero. Since $\psi\in C_0^\infty$, $\langle x\rangle(\bar{H}-\lambda_0)\varphi$ has compact support, and thus the first term in (4.5) converges. We conclude that

$$\xi\langle x\rangle(\bar{H}-\lambda_0)\varphi_m\to\xi\langle x\rangle(\bar{H}-\lambda_0)\varphi,$$

and thus by (4.4),

$$k\|\xi\langle x\rangle(\bar{H}-\lambda_0)\varphi\|\geqq\|\xi\varphi\|.$$

From (4.2), it follows that $N(\mu)\leqq k$. For $\psi_1,\psi_2\in C_0^\infty$, we thus have

$$|(\xi\psi_2,(\bar{H}-\lambda_0)^{-1}\langle x\rangle^{-1}\xi^{-1}(1+\mu^{-1}\langle x\rangle)^{-1}\psi_1)|\leqq k\|\psi_1\|\cdot\|\psi_2\|.$$

Taking the limit $\mu\uparrow\infty$, we thus have

$$|(\langle x\rangle^t\psi_2,(\bar{H}-\lambda_0)^{-1}\langle x\rangle^{-1-t}\psi_1)|\leqq k\|\psi_1\|\cdot\|\psi_2\|.$$

This implies

$$\langle x\rangle^t(\bar{H}-\lambda_0)^{-1}\langle x\rangle^{-1-t} \tag{4.6}$$

is a bounded operator for $t\geqq 0$. We now use Corollary 3.2 again to show that for some small $\gamma_0>0$,

$$e^{\gamma_0\langle x\rangle}(\bar{H}-\lambda_0)^{-1}\langle x\rangle^{-1}e^{-\gamma_0\langle x\rangle} \tag{4.7}$$

is bounded by repeating the argument above with $\xi=\xi_2, \alpha=0$.

Let

$$F(z)=\langle x\rangle^{zt}e^{(1-z)\gamma_0\langle x\rangle}(\bar{H}-\lambda_0)^{-1}e^{-(1-z)\gamma_0\langle x\rangle}\langle x\rangle^{-1}\langle x\rangle^{-zt}$$

for $\mathrm{Re}\,z\in[0,1]$. Matrix elements of F between vectors in C_0^∞ are analytic in a neighborhood of $\{z:\mathrm{Re}\,z\in[0,1]\}$. Thus, by interpolation using the boundedness of (4.6) and (4.7), we find $F(1/2)$ is bounded. Thus for all $t\geqq 0$ and $\alpha=\gamma_0/2$,

$$\langle x\rangle^t e^{\alpha\langle x\rangle}(\bar{H}-\lambda_0)^{-1}e^{-\alpha\langle x\rangle}\langle x\rangle^{-1-t} \tag{4.8}$$

is bounded. We now improve (4.8) by using Corollary 3.2 with $\xi_2=e^{\alpha\langle x\rangle}(1+\mu^{-1}\gamma\langle x\rangle)^\mu$ for small enough α and γ. We find that for some $\gamma_1>0$,

$$e^{\gamma_1\langle x\rangle}(\bar{H}-\lambda_0)^{-1}e^{-\gamma_1\langle x\rangle}\langle x\rangle^{-1/2} \tag{4.9}$$

is bounded. Let

$$G(z)=\langle x\rangle^{-1/2(1-z)}e^{2\gamma_1(z-1/2)\langle x\rangle}(\bar{H}-\lambda_0)^{-1}e^{-2\gamma_1(z-1/2)\langle x\rangle}\langle x\rangle^{-z/2}.$$

Again, by interpolation, we find $G(1/2)$ is bounded so that

$$\|\langle x\rangle^{-1/4}(\bar{H}-\lambda_0)^{-1}\langle x\rangle^{-1/4}\|<\infty. \tag{4.10}$$

We claim that if $\lambda_0<0$, then there is an $X\in\mathscr{L}$ with $X\neq\mathbb{R}^n$ such that H_X has an eigenvalue $\mu_0<\lambda_0$ with $\mu_0<\inf\sigma_{\mathrm{ess}}(H_X)$. Assume the contrary. Since $\inf\sigma_{\mathrm{ess}}(H)\in\mathscr{T}(H)$, there is a subspace $Y_1\in\mathscr{L}$, $Y_1\neq\mathbb{R}^n$, so that $\inf\sigma_{\mathrm{ess}}(H)$ is an eigenvalue of H_{Y_1}. Since $\inf\sigma_{\mathrm{ess}}(H_{Y_1})\in\mathscr{T}(H_{Y_1})$, there is a subspace $Y_2\in\mathscr{L}$, $Y_2\subset Y_1$, $Y_2\neq Y_1$ such that $\inf\sigma_{\mathrm{ess}}(H_{Y_1})$ is an eigenvalue of H_{Y_2}. Continuing in this way, we have a chain of subspaces

$$Y_1\supset Y_2\supset\cdots,$$

no two of which are equal. This chain can only terminate if $Y_j=\{0\}$ for some j, but this is impossible, for then $\lambda_0>\inf\sigma_{\mathrm{ess}}(H_{Y_{j-1}})=0$.

Assuming $\lambda_0<0$, we will show that (4.10) is false, thereby obtaining a contradiction.

Applying the operator $\langle x\rangle^{-1/4}(\bar{H}-\lambda_0)^{-1}\langle x\rangle^{-1/4}$ to $f=\langle x\rangle^{1/4}(\bar{H}-\lambda_0)u$ with

$u \in C_0^\infty(\mathbb{R}^n)$ and using (4.10), we find the estimate

$$\| \langle x \rangle^{-1/4} u \| \leqq c \| \langle x \rangle^{1/4} (H - \lambda_0) u \| + \| K \langle x \rangle^{-1/4} u \| \tag{4.11}$$

for some compact operator K and all $u \in C_0^\infty(\mathbb{R}^n)$. Let X' be a subspace in $\mathscr{L}$ with $X' \neq \mathbb{R}^n$ such that $H_{X'}$ has an eigenvalue $\mu_0 < \lambda_0$ with $\mu_0 < \inf \sigma_{\text{ess}}(H_{X'})$. Fixing X' we set $Y = (X')^\perp$. We denote generic points in X' and Y by x' and y respectively. Note that $X_i \not\subset X'$ means that $Y \cap X_i^\perp$ is a proper subspace of Y. It follows that there is a point $y_0 \in Y$ with $|y_0| = 1$ such that

$$\pi_i y_0 \neq 0 \quad \text{for all } i \text{ with} \quad X_i \not\subset X'.$$

Thus there is an open cone Γ containing y_0 and a $\delta > 0$ so that for all i with $X_i \not\subset X'$,

$$|\pi_i x| \geqq \delta |x|, \quad x \in \Gamma. \tag{4.12}$$

Define

$$W^s(x) = \sum_{X_i \not\subset X'} v_i^s(\pi_i x)$$

and

$$\Gamma_R = \{ x \in \Gamma : |x| > R \}.$$

Let χ_B be the characteristic function of the set B. Using (4.11) we obtain

$$\begin{aligned} \| \langle x \rangle^{-1/4} u \| \leqq c \| \langle x \rangle^{-1/4} (H - W^s - \lambda_0) u \| + \| K \langle x \rangle^{-1/4} u \| \\ + c \| K_1 \langle x \rangle^{-1/4} (H - W^s - \lambda_0 + i) u \| \end{aligned} \tag{4.13}$$

for all $u \in C_0^\infty(\Gamma)$, where

$$K_1 = \langle x \rangle^{1/4} W^s \chi_\Gamma (H - W^s - \lambda_0 + i)^{-1} \langle x \rangle^{1/4}.$$

It is easily seen from (4.1) and (4.12) that K_1 is compact. For any compact operator $\tilde{K}$ we have

$$\lim_{R \to \infty} \| \tilde{K} \chi_{\Gamma_R} \| = 0,$$

and thus we obtain for R sufficiently large

$$\| \langle x \rangle^{-1/4} u \| \leqq D \| \langle x \rangle^{1/4} (H - W^s - \lambda_0) u \| \tag{4.14}$$

for all $u \in C_0^\infty(\Gamma_R)$.

Let $\psi \in L^2(X')$ be a normalized eigenfunction of $H_{X'}$ with eigenvalue μ_0. Define

$$W^L(x) = \sum_{X_i \not\subset X'} v_i^L(\pi_i x),$$

and fix $t_0 > 0$ so that

$$|W^L(x)| < \tfrac{1}{2}(\lambda_0 - \mu_0)$$

for $x \in \Gamma_{t_0}$. For each $t \geqq t_0$ choose $\eta_t \in Y$ such that

$$|\eta_t|^2 = \lambda_0 - \mu_0 - W^L(t y_0),$$

and define

$$\varphi_t(x) = \psi(x')\exp(i\eta_t \cdot y), \quad x = (x', y).$$

It is readily checked that φ_t satisfies the equation

$$(-\Delta + V(x) - W^s(x) - \lambda_0)\varphi_t(x) = (W^L(x) - W^L(ty_0))\varphi_t(x). \tag{4.15}$$

Choose $\zeta \in C_0^\infty(\mathbb{R}^n)$ so that $\zeta(x) = 1$ if $|x| \leqq 1/2$ and $\zeta(x) = 0$ if $|x| \geqq 1$. Set

$$\tilde{\chi}_t(x) = \zeta(r_t^{-1}(x - ty_0)), \quad u_t = \varphi_t \tilde{\chi}_t,$$

where r_t will be chosen later. For now we only specify that

(i) $\lim_{t\to\infty} r_t/t = 0$,

(ii) $\lim_{t\to\infty} r_t = \infty$.

From (4.15) we obtain

$$\begin{aligned}(-\Delta + V(x) - W^s(x) - \lambda_0)u_t(x) = &(W^L(x) - W^L(ty_0))u_t(x)\\ &- 2\nabla\varphi_t(x)\cdot\nabla\tilde{\chi}_t(x) - \varphi_t\Delta\tilde{\chi}_t,\end{aligned}$$

so that with $m = \dim Y$

$$\begin{aligned}&\|\langle x\rangle^{1/4}(-\Delta + V - W^s - \lambda_0)u_t\|\\ &\leqq \left(\int_0^1\int \langle x\rangle^{1/2}|\nabla W^L(\lambda x + (1-\lambda)ty_0)\cdot(x - ty_0)u_t(x)|^2 dx\, d\lambda\right)^{1/2}\\ &\quad + c(r_t^{-1}r_t^{m/2} + r_t^{-2}r_t^{m/2})t^{1/4}\\ &\leqq ct^{1/4}(r_t/(t-r_t))\sup\{|x|\cdot|\nabla W^L(x)| : |x| \geqq t - r_t\}r_t^{m/2} + cr_t^{-1}r_t^{m/2}t^{1/4}.\end{aligned}$$

Let

$$\varepsilon_t = \sup\{|x|\cdot|\nabla W^L(x)| : |x| \geqq \tfrac{1}{2}t\}.$$

Then for large t,

$$\|\langle x\rangle^{1/4}(H - W^s - \lambda_0)u_t\| \leqq cr_t^{m/2}t^{1/4}(\varepsilon_t r_t t^{-1} + r_t^{-1}).$$

On the other hand, given any R, for large enough t we have $\operatorname{supp} u_t \subset \Gamma_R$. It is thus easy to see that we can use (4.14) with $u = u_t$ for sufficiently large t. We have

$$\begin{aligned}c\|\langle x\rangle^{-1/4}u_t\|^2 &\geqq t^{-1/2}\left(\int_{|y|<r_t/4} |\psi(x')|^2 dx'dy - \int_{|y|<r_t/4, |x'|>r_t/4} |\psi(x')|^2 dx'dy\right)\\ &\geqq t^{-1/2}(c_0 r_t^m - c' e^{-\beta r_t} r_t^m)\end{aligned}$$

for some $c_0 > 0$ and $\beta > 0$. Thus for large t

$$c\|\langle x\rangle^{-1/4}u_t\| \geqq t^{-1/4}r_t^{m/2}.$$

From (4.14)

$$t^{-1/4}r_t^{m/2} \leqq cr_t^{m/2}t^{1/4}(\varepsilon_t r_t t^{-1} + r_t^{-1})$$

or

$$1 \leqq c(\varepsilon_t(r_t/\sqrt{t}) + \sqrt{t}/r_t). \tag{4.16}$$

Set

$$\beta_t = \mathrm{Max}(\varepsilon_t, t^{-1/2}), \quad r_t = (t/\beta_t)^{1/2}.$$

Then (i) and (ii) above are satisfied while (4.16) implies

$$1 \leqq 2c\sqrt{\beta_t},$$

which is a contradiction for large t.

The proof in case $\lambda_0 > 0$ is even simpler. Here we get a contradiction to the estimate (4.10) by using $u = u_t$ with

$$u_t(x) = \exp(i\eta_t \cdot x)\tilde{\chi}_t(x)$$

in (4.14), where $\eta_t \in \mathbb{R}^n$, $|\eta_t|^2 = \lambda_0 - W^L(ty_0)$. We omit the details. ■

V. Instability of Embedded Eigenvalues

In this section we assume (2.1) and (2.2) are in force and $H = -\Delta + V$ in $L^2(\mathbb{R}^n)$. We also assume that $\lambda_0 \notin \mathscr{T}(H)$ but that λ_0 is an eigenvalue of H in $\sigma_{\mathrm{ess}}(H)$.

Let $\mathscr{B}_2$ be the space of all real-valued functions, W, such that

$$|W|_2 = \|W(-\Delta + 1)^{-1}\| + \|(-\Delta + 1)^{-1/2}[A, W](-\Delta + 1)^{-1}\| \\ + \|(-\Delta + 1)^{-1}[A, [A, W]](-\Delta + 1)^{-1}\| < \infty.$$

Lemma 5.1. *Let $P_0 = E_H(\{\lambda_0\})$, and $\bar{H} = H + P_0$. There is a $\delta > 0$ so that if $|\lambda - \lambda_0| + |W|_2 < \delta$, then the strong limits*

$$\lim_{\varepsilon \downarrow 0} (\bar{H} + W - \lambda \mp i\varepsilon)^{-1} = (\bar{H} + W - \lambda \mp i0)^{-1} \tag{5.1}$$

exist as maps from L^2_s to L^2_{-s} for any $s > 1/2$. The operators $(\bar{H} + W - \lambda \pm i0)^{-1}$ are norm Hölder continuous in the variables (λ, W) for $|\lambda - \lambda_0| + |W|_2 < \delta$. If $|\lambda - \lambda_0| + |W|_2 < \delta$, and λ is not an eigenvalue of $H + W$, (5.1) also holds with $\bar{H}$ replaced by H. The Hölder continuity is also valid for the operators $(H + W - \lambda \pm i0)^{-1}$.

Proof. As in the proof of Theorem 2.5, we find for some $c_0 > 0$,

$$E_{\bar{H}+W}(I)[\bar{H} + W, A]E_{\bar{H}+W}(I) \geqq c_0 E_{\bar{H}+W}(I) \tag{5.2}$$

for some open interval I containing λ_0 and all W with $|W|_1 < \delta_1$ if δ_1 is sufficiently small. The proof of [P–S–S] then shows that the limits (5.1) exist and are Hölder continuous in λ if $W \in \mathscr{B}_2$. The Hölder continuity in W (in the norm $|\cdot|_2$) is proved by exactly the same technique. We do not repeat the argument here. The Mourre estimate also holds for $H + W$ (see Lemma 2.4) so that if λ is not an eigenvalue of $H + W$, the proof of [P–S–S] again shows that boundary values exist as maps from L^2_s into $L^2_{-s}(s > 1/2)$ and that they are Hölder continuous in (λ, W). ■

We now present a result using a formalism which has proved very useful in the study of eigenvalues [K2, Howl, 2]:

Proposition 5.2. *There is an open interval J containing λ_0 and a $\delta > 0$ so that the* *following holds: Define* $P_0 = E_H(\{\lambda_0\})$ *and* $\bar{H} = H + P_0$. *For* $\lambda \in J$ *and* $|W|_2 < \delta$, *(5.1) holds. Define*

$$Q_+(\lambda, W) = P_0(\bar{H} + W - \lambda - i0)^{-1} P_0.$$

Then $\lambda \in J$ *is an eigenvalue of* $H + W$ *if and only if the operator* $1 - Q_+(\lambda, W)$ *is not invertible. (Note that* Q_+ *is well defined since all functions in* $\operatorname{Ran} P_0$ *decay exponentially.)*

The proof of this result is very similar to that of similar results found in the literature. We sketch it mainly to establish notation: For $\operatorname{Im} z > 0$, we have

$$(H + W - z)^{-1} = (\bar{H} + W - z)^{-1} + (H + W - z)^{-1} P_0 (\bar{H} + W - z)^{-1}. \tag{5.3}$$

Multiplying by P_0, we find

$$(H + W - z)^{-1} P_0 (1 - P_0 (\bar{H} + W - z)^{-1} P_0) = (\bar{H} + W - z)^{-1} P_0. \tag{5.4}$$

Letting $z = \lambda + i\varepsilon$ and taking $\varepsilon \downarrow 0$ gives (for λ not an eigenvalue of $H + W$)

$$(H + W - \lambda - i0)^{-1} P_0 (1 - Q_+(\lambda, W)) = (\bar{H} + W - \lambda - i0)^{-1} P_0. \tag{5.5}$$

Suppose $\psi \in \operatorname{Ran} P_0$ and $(1 - Q_+(\lambda, W))\psi = 0$. Then from (5.5), $(\bar{H} + W - \lambda - i0)^{-1}\psi = 0$, which implies $\psi = 0$, so that since on $\operatorname{Ran} P_0$, $1 - Q_+(\lambda, W)$ is just a finite dimensional matrix, $1 - Q_+(\lambda, W)$ is invertible. Conversely, suppose $1 - Q_+(\lambda, W)$ is invertible. Let

$$Q(z, W) = P_0 (\bar{H} + W - z)^{-1} P_0$$

for $\operatorname{Im} z > 0$. It is easy to see from (5.4) that $1 - Q(z, W)$ is invertible, and thus from (5.4),

$$(H + W - z)^{-1} P_0 = (\bar{H} + W - z)^{-1} P_0 (1 - Q(z, W))^{-1}. \tag{5.6}$$

Substituting (5.6) into (5.3) gives

$$(H + W - z)^{-1} = (\bar{H} + W - z)^{-1} + (\bar{H} + W - z)^{-1} P_0 (1 - Q(z, W))^{-1} P_0 (\bar{H} + W - z)^{-1}. \tag{5.7}$$

From (5.7), it follows that the limit

$$\lim_{\varepsilon \downarrow 0} (H + W - \lambda - i\varepsilon)^{-1}$$

exists strongly as maps from L_s^2 into L_{-s}^2 for $s > 1/2$. Thus the projection

$$E_{H+W}(\{\lambda\}) = s - \lim_{\varepsilon \downarrow 0} - i\varepsilon (H + W - \lambda - i\varepsilon)^{-1} = 0,$$

and λ is not an eigenvalue of $H + W$. ■

Lemma 5.3. *There is an open interval* J *containing* λ_0, *an* $\eta > 0$, *and a* $\delta > 0$ *such that if* W *is in* $\mathscr{B}_2$ *with* $|W|_2 < \delta$, *and* $\lambda \in J$, *then with* $\gamma(\lambda) = \lambda_0 + 1 - \lambda$

$$Q_+(\lambda, W) = \gamma^{-1} P_0 - \gamma^{-2} P_0 W P_0 - \gamma^{-2} P_0 W (\bar{H} - \lambda - i0)^{-1} W P_0 + O(|W|_2^{2+\eta}). \tag{5.8}$$

Proof. We use the resolvent formulae

$$(\bar{H}+W-z)^{-1}=(\bar{H}-z)^{-1}-(\bar{H}-z)^{-1}W(\bar{H}+W-z)^{-1}$$
$$=(\bar{H}-z)^{-1}-(\bar{H}+W-z)^{-1}W(\bar{H}-z)^{-1}. \quad (5.9)$$

Thus

$$Q(z,W)=\gamma(z)^{-1}P_0-\gamma(z)^{-2}P_0WP_0$$
$$+\gamma(z)^{-2}P_0W(\bar{H}+W-z)^{-1}WP_0. \quad (5.10)$$

We have

$$P_0W[(\bar{H}+W-\lambda-i0)^{-1}-(\bar{H}-\lambda-i0)^{-1}]WP_0$$
$$=P_0W\langle x\rangle^s\{\langle x\rangle^{-s}[(\bar{H}+W-\lambda-i0)^{-1}-(\bar{H}-\lambda-i0)^{-1}]\langle x\rangle^{-s}\}\langle x\rangle^sWP_0.$$

If $1>s>1/2$, the expression in curly brackets is bounded in norm by $c|W|_2^\eta$ for some $\eta>0$, while

$$\|\langle x\rangle^sWP_0\|\leqq\|W(H+i)^{-1}\|\cdot\|(H+i)\langle x\rangle^sP_0\|\leqq c|W|_2.$$

Taking $z=\lambda+i\varepsilon$ and $\varepsilon\downarrow 0$ in (5.10) thus gives (5.8). ■

Lemma 5.4. *Let J be as in Lemma 5.3. Suppose $W\in\mathscr{B}_2$ and $\{\mu_j: j=1,2,\dots\}$ is the set of eigenvalues of P_0WP_0. Then, $\lambda\in J$ is an eigenvalue of $H+W$,*

$$\lambda=\lambda_0+\mu_j+O(|W|_2^2)\quad \text{for some } j.$$

Proof. Let $\xi=\gamma(\lambda)(\lambda-\lambda_0)$. Then we calculate using (5.8),

$$Q_+(\lambda,W)-P_0=\gamma(\lambda)^{-2}P_0\{\xi-P_0WP_0+O(|W|_2^2)\}. \quad (5.11)$$

Now $\|(\xi-P_0WP_0)^{-1}\|=\left(\operatorname*{Min}_j|\xi-\mu_j|\right)^{-1}$ so that

$$\xi-P_0WP_0+O(|W|_2^2)=(\xi-P_0WP_0)(1+(\xi-P_0WP_0)^{-1}O(|W|_2^2))$$

is invertible if

$$\left(\operatorname*{Min}_j|\xi-\mu_j|\right)^{-1}\|O(|W|_2^2)\|<1/2.$$

or, in other words, for some $c>0$

$$|\xi-\mu_j|\geqq c|W|_2^2,\quad \text{all } j. \quad (5.12)$$

There exists a constant $c_1>0$ so that if

$$|\lambda-\lambda_0-\mu_j|\geqq c_1|W|_2^2\quad \text{for all } j, \quad (5.13)$$

then (5.12) is satisfied. Thus, in view of Proposition 5.2, if λ is an eigenvalue of $H+W$, we must have $|\lambda-\lambda_0-\mu_j|<c_1|W|_2^2$ for some j. ■

Lemma 5.5. *Suppose* dim Ran $P_0=m$. *Then there is a real-valued function $W\in C_0^\infty(\mathbb{R}^n)$ such that P_0WP_0 has m distinct eigenvalues as an operator on* Ran P_0.

Proof. We first show that if $m>1$, we can find $W_1\in C_0^\infty$ such that $P_0W_1P_0$ is not a multiple of P_0. Let $\{\psi_i\}_{i=1}^m$ be an orthonormal basis for Ran P_0. If $(\psi_1,\varphi\psi_2)=0$

for all $\varphi \in C_0^\infty$, clearly $\psi_1\psi_2 = 0$ a.e. If $(\psi_1, \varphi\psi_1) = (\psi_2, \varphi\psi_2)$ for all $\varphi \in C_0^\infty$, then $|\psi_1|^2 = |\psi_2|^2$ a.e. These two statements imply $\psi_1 = \psi_2 = 0$ a.e. and we have proved our claim.

Suppose we have found a real $W_2 \in C_0^\infty$ such that $P_0 W_2 P_0$ has $m_2 < m$ distinct eigenvalues. We will show how to construct a real $W_3 \in C_0^\infty$ with at least $m_2 + 1$ distinct eigenvalues. This will complete the proof.

Suppose $\{\psi_i\}_{i=1}^m$ is an orthonormal basis for Ran P_0 so that

$$(\psi_i, W_2\psi_j) = \mu\delta_{ij}, \quad 1 \leqq i, j \leqq l,$$

where $l \geqq 2$. Find a real $\varphi \in C_0^\infty$ such that $\{(\psi_i, \varphi\psi_j)\}_{1 \leqq i,j \leqq l}$ is not a multiple of the identity. By making a unitary change of basis we can assume

$$(\psi_i, W_2\psi_j) = \mu\delta_{ij}, \quad 1 \leqq i, \quad j \leqq l,$$
$$(\psi_i, \varphi\psi_j) = \mu_i\delta_{ij}, \quad 1 \leqq i, \quad j \leqq l,$$

where $\mu_1 \neq \mu_2$. Let

$$\bar{W}_2 = \{(\psi_i, W_2\psi_j)\}_{1 \leqq i,j \leqq m};$$
$$\bar{\varphi} = \{(\psi_i, \varphi\psi_j)\}_{1 \leqq i,j \leqq m},$$

and let $\bar{W}(\varepsilon) = \bar{W}_2 + \varepsilon\bar{\varphi}$. The projection onto the eigenspace of eigenvalues for $\bar{W}(\varepsilon)$ near μ is for small ε

$$P(\varepsilon) = \frac{1}{2\pi i} \int_{|z-\mu|=\delta} (z - \bar{W}(\varepsilon))^{-1} dz.$$

If the eigenvalues near μ were all equal, we would have ($\{e_i\}_{i=1}^m$ is the standard basis in $\mathbb{R}^m$)

$$(e_1, \bar{W}(\varepsilon)P(\varepsilon)e_1)/(e_1, P(\varepsilon)e_1) = (e_2, \bar{W}(\varepsilon)P(\varepsilon)e_2)/(e_2, P(\varepsilon)e_2).$$

But a simple calculation gives for $j = 1, 2$,

$$(e_j, \bar{W}(\varepsilon)P(\varepsilon)e_j)/(e_j, P(\varepsilon)e_j) = \mu + \varepsilon\mu_j + O(\varepsilon^2).$$

Thus the eigenvalue μ splits into at least 2 eigenvalues. If ε is small enough, the number of other distinct eigenvalues cannot decrease. Thus, for small enough $\varepsilon > 0$, the number of distinct eigenvalues of $\bar{W}(\varepsilon)$ is at least $m_2 + 1$. ∎

Lemma 5.6. *Let P_0, J, and δ be as in Proposition 5.2 and suppose $m = \operatorname{rank}(P_0)$. Then there exists a real $W \in C_0^\infty(\mathbb{R}^n)$ and a $t_0 > 0$ such that*

$$\operatorname{rank}(1 - Q_+(\lambda, tW)) \geqq m - 1$$

for all $\lambda \in J$ and all t with $0 < |t| \leqq t_0$.

Proof. Let W be as in Lemma 5.5. Then according to Lemma 5.3,

$$Q_+(\lambda, tW) = \gamma^{-1}P_0 - \gamma^{-2}tP_0WP_0 + O(t^2),$$

and with $\xi = \gamma(\lambda)(\lambda - \lambda_0)$

$$1 - Q_+(\lambda, tW) = -\gamma(\lambda)^{-2}P_0(\xi - t\bar{W} + O(t^2)),$$

where $\bar{W} = P_0WP_0$. Choose a basis so that we can write (with some abuse of notation)

$$\bar{W} = D = \begin{pmatrix} \mu_1 & & 0 \\ & \ddots & \\ 0 & & \mu_m \end{pmatrix}.$$

If $\xi = t\mu_1 - t\zeta$, where $|\zeta| \leq c_1 |t|$, then

$$(\xi - t\bar{W} + O(t^2)) = -t\left[\begin{pmatrix} \zeta & & & \\ & \mu_2 - \mu_1 & & 0 \\ & 0 & \ddots & \\ & & & \mu_m - \mu_1 \end{pmatrix} + O(t)\right].$$

Since the μ_i are distinct, the matrix in brackets clearly has rank $\geq m-1$ for small t. This conclusion also holds if $\xi = t\mu_j - t\zeta$ for any j if $|\zeta| \leq c_1 |t|$. According to Lemma 5.4, unless $\xi = t\mu_j + O(t^2)$, the operator $1 - Q_+(\lambda, tW)$ is invertible for $\lambda \in J$ and thus has rank m. This proves the result. ■

Lemma 5.7. *With W as in Lemma 5.6, any eigenvalue of $H + tW$ in J has multiplicity one for $0 < |t| \leq t_0$.*

Proof. Multiplying (5.4) by P_0 and defining

$$Q_t(z) = P_0(H + tW - z)^{-1} P_0,$$

we obtain

$$Q_t(z)(1 - Q(z, tW)) = Q(z, tW). \tag{5.14}$$

Suppose $\lambda \in J$ is an eigenvalue of $H + tW$. Set $z = \lambda + i\varepsilon$ in (5.14) and note that

$$s - \lim_{\varepsilon \downarrow 0} (H + tW - \lambda - i\varepsilon)^{-1}(-i\varepsilon) = E_{H+tW}(\{\lambda\}) \equiv P_1.$$

We obtain from (5.14),

$$P_0 P_1 P_0 (1 - Q_+(\lambda, tW)) = 0. \tag{5.15}$$

Since $\operatorname{rank}(1 - Q_+(\lambda, tW)) \geq m - 1$, clearly (5.15) implies

$$\operatorname{rank}(P_0 P_1 P_0) \leq 1. \tag{5.16}$$

From (5.16), it follows that $P_1(P_0P_1P_0)P_1 = (P_1P_0P_1)^2$ has at most rank 1, and thus

$$\operatorname{rank}(P_1 P_0 P_1) \leq 1. \tag{5.17}$$

Suppose φ_1 and φ_2 are two linearly independent vectors in $\operatorname{Ran} P_1$. We can find a non-zero vector $\varphi = \alpha_1\varphi_1 + \alpha_2\varphi_2$ with $P_1P_0P_1\varphi = 0$. But this implies $\|P_0P_1\varphi\|^2 = (\varphi, P_1P_0P_1\varphi) = 0$, or $P_0\varphi = 0$. This contradicts Corollary 2.7 so that $\operatorname{rank} P_1 \leq 1$. ■

We are, of course, heading toward a result which says that not only can one split a degenerate eigenvalue, but remove it completely. Before we prove this, we need to know a bit more about $\delta(\bar{H} - \lambda_0)$ in addition to the fact that it is not the zero operator.

Lemma 5.8.

$$P_0 \delta(\bar{H} - \lambda_0) = 0. \tag{5.18}$$

Proof. With convergence in norm as maps from L_s^2 to L_{-s}^2, $s > 1/2$, we have

$$\delta(\bar{H}-\lambda_0) = \lim_{\varepsilon\downarrow 0}\left(\frac{1}{2\pi i}\right)[(\bar{H}-\lambda_0-i\varepsilon)^{-1}-(\bar{H}-\lambda_0+i\varepsilon)^{-1}]$$
$$= \lim_{\varepsilon\downarrow 0}\delta_\varepsilon(\bar{H}-\lambda_0),$$

where

$$\delta_\varepsilon(\bar{H}-\lambda_0) = \frac{\varepsilon}{\pi}\frac{1}{(\bar{H}-\lambda_0)^2+\varepsilon^2}. \tag{5.19}$$

But $\bar{H} = H + P_0$ so that

$$P_0\delta_\varepsilon(\bar{H}-\lambda_0) = \frac{\varepsilon}{\pi}\frac{1}{1+\varepsilon^2}P_0.$$

This proves (5.18). ∎

At this point, in order to learn more about the eigenfunctions of H we need to make further regularity assumptions about the potentials v_j, in addition to (2.1) and (2.2).

Assumption R:

(a) The potentials v_j belong to the Kato class $K_{d_j}^{\mathrm{loc}}$ [A–S, S2], where $d_j = \dim X_j$.

(b) If $\langle x\rangle^{-s}\psi\in\mathscr{D}(\Delta)$ for some s and if $(-\Delta + V + W - \lambda)\psi = 0$, where W is a real function in $C_0^\infty(\mathbb{R}^n)$, and ψ vanishes in an open set, then $\psi = 0$.

It has been conjectured [S2] and proved for low dimension [Saw], that (a)⇒(b). At this point, however, theorems guaranteeing (b) are not optimal for N-body type potentials (see, for example, [J–K] and [G]).

Lemma 5.9. *Suppose, in addition to the assumptions (2.1) and (2.2) in force in this section, that assumptions R and (4.1) hold. Then if Ω is an open set in $\mathbb{R}^n$ and ψ is a (non-zero) eigenfunction of H with eigenvalue λ_0, there is a real function $W\in C_0^\infty(\Omega)$ such that*

$$(W\psi, \delta(\bar{H}-\lambda_0)W\psi) \neq 0.$$

Proof. If $(W\psi, \delta(\bar{H}-\lambda_0)W\psi) = 0$ for all real $W\in C_0^\infty(\Omega)$, the Schwarz inequality for non-negative quadratic forms implies that

$$(W_1\psi, \delta(\bar{H}-\lambda_0)W_2\psi) = 0 \tag{5.20}$$

for all W_1 and $W_2\in C_0^\infty(\Omega)$. Let

$$\psi' = \delta(\bar{H}-\lambda_0)W_2\psi.$$

According to (5.18),

$$(-\Delta + V - \lambda_0)\psi' = 0. \tag{5.21}$$

From assumption $R(a)$ and [A–S], we can assume that ψ and ψ' are continuous. Thus, from (5.20) we obtain

$$\bar{\psi}(x)\psi'(x) = 0, \quad x\in\Omega.$$

From $R(b)$, it follows that $\{x\in\Omega : \psi(x)\neq 0\}$ is dense in Ω so that $\psi'(x) = 0$ for $x\in\Omega$. From (5.21) and assumption $R(b)$, it then follows that $\psi' \equiv 0$.

Choosing $\eta \in C_0^\infty(\mathbb{R}^n)$. We have

$$(\eta, \delta(\bar{H}-\lambda_0)W_2\psi)=0$$

for all $W_2 \in C_0^\infty(\Omega)$. Thus, repeating the arguments above, we obtain that

$$\psi''=\delta(\bar{H}-\lambda_0)\eta \equiv 0.$$

Since this holds for all $\eta \in C_0^\infty(\mathbb{R}^n)$, we have a contradiction to Theorem 4.2. ∎

Remark. The unique continuation property $R(b)$ does not guarantee that $\{x:\psi(x)=0\}$ has measure zero. Thus $\{W\psi: W\in C_0^\infty\}$ is not known to be dense in L_s^2 (for any s). For this reason the proof of Lemma 5.9 is somewhat involved.

Proposition 5.10. *Suppose the assumptions of Lemma 5.9 are satisfied. There is an open interval J containing λ_0 so that, given any $\varepsilon>0$, we can find a real C_0^∞ function W with $|||W|||<\varepsilon$ such that $H+W$ has no eigenvalues in J. Here $|||\cdot|||$ is any norm on C_0^∞.*

Proof. Choose W_1 as in Lemma 5.7 so that any eigenvalue of $H+tW_1$ in $\bar{J}$($\bar{J}$ is the closure of J, an open interval containing λ_0) has multiplicity one for $0<|t|\leqq t_0$. Choose $t_1\in(0,t_0)$ so that $|||t_1W_1|||<\varepsilon/2$. We can assume (by shrinking J if necessary) that $\bar{J}\cap\mathscr{T}(H)=\varnothing$. We will now remove the eigenvalues of $H_1=H+t_1W_1$ which are in $\bar{J}$, one at a time. For simplicity of exposition, suppose there are just two such eigenvalues, λ_1 and λ_2. Suppose then that $(H_1-\lambda_1)\psi_1=0$ where $\|\psi_1\|=1$. Note that the results of this section apply equally well to H_1($\mathscr{T}(H_1)=\mathscr{T}(H)$ so $\lambda_1\notin\mathscr{T}(H_1)$). Choose a real function $W_2\in C_0^\infty$ so that

$$(W_2\psi_1,\delta(\bar{H}_1-\lambda_1)W_2\psi_1)=\alpha_1>0. \tag{5.22}$$

Here $\bar{H}_1=H_1+P_1, P_1=(\psi_1,\cdot)\psi_1$. If

$$Q_+^1(\lambda,W)\equiv P_1(\bar{H}_1+W-\lambda-i0)^{-1}P_1,$$

it follows from Lemma 5.3 and Proposition 5.2 that for λ in some open interval J_1 containing λ_1,

$$\operatorname{Im} Q_+^1(\lambda,tW_2)=\pi\gamma_1(\lambda)^{-2}P_1W_2\delta(\bar{H}_1-\lambda)W_2P_1t^2+O(t^{2+\eta}). \tag{5.23}$$

and for small enough $|t|$, H_1+tW_2 has no eigenvalues in J_1 if and only if $1-Q_+^1(\lambda,tW_2)$ is invertible. Here $\gamma_1(\lambda)=\lambda+1-\lambda_1$. Since $P_1W_2\delta(\bar{H}_1-\lambda)W_2P_1$ is continuous in λ, we can assume (by virtue of (5.22) and (5.23) that for small $|t|>0$, $1-Q_+^1(\lambda,tW_2)$ is invertible for all $\lambda\in J_1$ (we may have to shrink J_1 again). Thus H_1+tW_2 has no eigenvalues in J_1 for small non-zero t, say $0<|t|\leqq t'$. For each $\lambda\in\bar{J}\backslash J_1$ there is an open interval $J(\lambda)$ containing λ such that (by Theorem 2.5) if $|t|\leqq t(\lambda)$ ($t(\lambda)>0$), the following is true: If $\lambda_2\in J(\lambda)$, there is at most one eigenvalue of H_1+tW_2 in $J(\lambda)$ and this eigenvalue has multiplicity one, while if $\lambda_2\notin J(\lambda)$, then H_1+tW_2 has no eigenvalues in $J(\lambda)$. A finite family $\{J(\lambda^i):i=1,\dots,N\}$ covers $\bar{J}\backslash J_1$. We can assume that only $J(\lambda^1)$ contains λ_2. Let

$$t_2'=\operatorname{Min}(t',t(\lambda^1),\dots,t(\lambda^N)).$$

Then if $0<|t|\leqq t_2'$, H_1+tW_2 has no eigenvalues in $\bar{J}$, except perhaps one, of multiplicity one, in $J(\lambda^1)$. Choose t_2 in $(0, t_2')$ so that $\| t_2 W_2 \| < \varepsilon/4$. Then the operator $H_2 = H + t_1 W_1 + t_2 W_2$ has at most one eigenvalue in $\bar{J}$, and this eigenvalue has multiplicity one. If this eigenvalue indeed exists, denote it by μ_2.

Choose $W_3 \in C_0^\infty$ such that

$$P_2 W_3 \delta(\bar{H}_2 - \mu_2) W_3 P_2 = \alpha_2 P_2, \quad \alpha_2 > 0,$$

where P_2 is the orthogonal projection on the eigenfunction associated with μ_2. We proceed to remove this eigenvalue in the same way we removed the previous eigenvalue of H_1. If t is non-zero and small enough, $H_2 + tW_3$ will have no eigenvalues in $\bar{J}$. We choose such a non-zero t, t_3, such that $\| t_3 W_3 \| < \varepsilon/4$. Then the proposition holds with

$$W = t_1 W_1 + t_2 W_2 + t_3 W_3. \quad \blacksquare$$

This proposition and Theorem 2.5 are the main ingredients in the genericity result to follow.

Let $\mathscr{B}$ be the closure of the set of all real $W \in C_0^\infty(\mathbb{R}^n)$ in the nom $|\cdot|_1$.

Theorem 5.11. *Suppose, in addition to the assumptions (2.1) and (2.2) in force in this section, that assumptions R and (4.1) hold. Then the set of all $W \in \mathscr{B}$ such that $H + W$ has no eigenvalues in $\sigma_{\text{ess}}(H) \backslash \mathscr{T}(H)$ is a dense G_δ.*

Proof. Let Λ be a compact subset of $\sigma_{\text{ess}}(H) \backslash \mathscr{T}(H)$. If $W \in \mathscr{B}$ and $H + W$ has no eigenvalues in Λ, then by Theorem 2.5 (and a compactness argument) there is an open ball B (in $\mathscr{B}$) with center at 0 such that if $\tilde{W} \in B, H + W + \tilde{W}$ has no eigenvalues in Λ. Hence

$$D_\Lambda \equiv \{W \in \mathscr{B} : H + W \text{ has no eigenvalues in } \Lambda\}$$

is open. If $W \in C_0^\infty$, according to Proposition 5.10, we can find $W_m \in C_0^\infty$ with $|W_m|_1 \to 0$ so that $H + W + W_m$ has no eigenvalues in Λ. Since C_0^∞ is dense in $\mathscr{B}$ in the norm $|\cdot|_1$, it follows that D_Λ is also dense. Choose a sequence of compact sets Λ_m with $\Lambda_m \uparrow \sigma_{\text{ess}}(H) \backslash \mathscr{T}(H)$. We see that if

$$W \in \bigcap_m D_{\Lambda_m} \equiv G,$$

then $H + W$ has no eigenvalues in $\sigma_{\text{ess}}(H) \backslash \mathscr{T}(H)$. G is a G_δ and is dense by the Baire category theorem. $\blacksquare$

Remark. Suppose that the potentials v_i satisfy (2.1) and (2.2). Then our discussion shows that the following weak form of Theorem 5.11 holds.

Theorem 5.12. *The set of all $W \in \mathscr{B}$ such that $H + W$ has only simple eigenvalues in $\sigma_{\text{ess}}(H) \backslash \mathscr{T}(H)$ is a dense G_δ.*

This theorem should be compared with results for the Dirichlet problem in [U].

To emphasize the local nature of our results, we will state another theorem.

For a compact set $K \subset \mathbb{R}^n$ with non-empty interior, we denote by $\mathscr{D}_K$ the set of all real functions in $C_0^\infty(\mathbb{R}^n)$ with support in K. Equipped with the family of seminorms

$$\sup \{ |D^\alpha \varphi(x)| : x \in \mathbb{R}^n, |\alpha| \leq m \}; \quad m = 0, 1, \ldots,$$

$\mathscr{D}_K$ is a Frechet space. We then have

Theorem 5.11′. *With the same assumptions as in Theorem 5.11 and with $\mathscr{D}_K$ as above, the set of all $W \in \mathscr{D}_k$ such that $H + W$ has no eigenvalues in $\sigma_{ess}(H) \setminus \mathscr{T}(H)$ is a dense G_δ.*

Proof. The proof of Theorem 5.11′ is almost exactly the same as that of Theorem 5.11 if one observes that under the additional assumption R, Lemma 5.5 through 5.7 and Proposition 5.10 hold for some $W \in C_0^\infty(\Omega)$, where $\Omega \subset$ interior (K). ■

We end this section by giving a simple set of potentials for which all of our results are valid:

Suppose that for each $i = 1, 2, \ldots, M$,

$$v_i \in L^{p_i}_{loc}(X_i) \quad \text{with} \quad p_i \geqq 2 \text{ and } p_i > \tfrac{2}{3} \dim X_i$$

and for all α with $|\alpha| \leqq 2$,

$$\lim |y|^{|\alpha|} \cdot D^\alpha v_i(y) = 0 \quad \text{as} \quad y \to \infty \quad \text{in } X_i,$$

then (2.1), (2.2), (4.1), and the condition R all hold. The unique continuation result implicit here is given in [G].

VI. Concluding Remarks

We would like to mention two open problems not considered in this paper.

The first problem involves the treatment of more general perturbations of H. The perturbations treated here are not completely natural for the N-body problem (but are quite natural for the generalized N-body Schrödinger operator). A more natural class of perturbing potentials in the N-body problem would involve only a sum of two-body potentials. One would still believe that, generically, embedded eigenvalues are absent. But, in this case, the set of vectors $W\psi$, where W is the perturbation and ψ is an eigenvector of H may not be sufficiently large to achieve $(W\psi, \delta(\bar{H} - \lambda_0) W\psi) \neq 0$ with our present state of knowledge of the operator $\delta(\bar{H} - \lambda_0)$. Thus, either one needs further knowledge about the operator $\delta(\bar{H} - \lambda_0)$, or a different method is required to show that eigenvalues disappear under small perturbations.

The second problem involves the determination of the set of potentials which do produce embedded eigenvalues. There are indications that given a negative embedded eigenvalue λ_0 of $-\Delta + V$, there may be curves $W(t, \cdot)$ with $W(0, x) = 0$ such that $H_t = -\Delta + V + W(t, \cdot)$ has an eigenvalue λ_t near λ_0 for t small. It would be quite interesting to see if this were true in a general context.

Appendix A: Proof of Lemma 2.6

Suppose $|W|_1 \leqq \varepsilon$, $|\lambda - \lambda_0| \leqq \varepsilon$, and $\gamma \leqq \delta$ are so small that Theorem 3.1 applies with $\alpha = 0$. By Lemma 2.4, we can assume that the Mourre estimate holds for each λ with $|\lambda - \lambda_0| \leqq \varepsilon$. Suppose by the way of contradiction that

$$(H + W_m - \lambda_m)\psi_m = 0, \quad \|\psi_m\| = 1,$$

where $|W_m|_1 \leqq \varepsilon$, $|\lambda_m - \lambda_0| \leqq \varepsilon$ and $\|\langle x\rangle \psi_m\| \to \infty$. According to [F–H2], for each $m, e^{\beta\langle x\rangle}\psi_m \in L^2$ for some $\beta > 0$. Given this fact, it is easy to see that the estimate (3.1) applies to $\varphi = \psi_m$ so that with $\alpha = 0$, $\xi = \xi_2$, $\mu = 1$, $\gamma = \gamma_0 \leqq \delta$, we have

$$\|\xi\psi_m\| \leqq \|K\xi\psi_m\|. \tag{A.1}$$

But $\Psi_m = \xi\psi_m/\|\xi\psi_m\|$ converges weakly to zero because, for any bounded set B, $\|\chi_B \Psi_m\| \to 0$. This contradicts (A.1). ■

Appendix B: Boundedness of $R\langle x\rangle^{-s}, s>1$

Under the assumptions (2.1) and (2.2) and $\lambda_0 \notin \mathscr{T}(H)$, we will show that if $\delta(\bar{H} - \lambda_0) = 0$, then $R\langle x\rangle^{-s}$ is bounded for $s > 1$.

According to [M2] and [P–S–S], the norm limits

$$\lim_{\varepsilon\downarrow 0} \langle A\rangle^{-s}(\bar{H} - \lambda_0 \pm i\varepsilon)^{-1}\langle A\rangle^{-s}, \tag{B.1}$$

and

$$\lim_{\varepsilon\downarrow 0} P_{\mp}(\bar{H} - \lambda_0 \mp i\varepsilon)^{-1}\langle A\rangle^{-2s} \tag{B.2}$$

exist in $L^2(\mathbb{R}^n)$ for any $s > 1/2$. Here $\langle A\rangle = (1 + |A|^2)^{1/2}$. It easily follows from $\delta(\bar{H} - \lambda_0) = 0$ that the limits in (B.1) are equal. (Note that $\delta(\bar{H} - \lambda_0)$ is an operator from L_s^2 to L_{-s}^2 so that this is not immediate from the definition.)

We will show that for $s > 1/2$,

$$|(\varphi, R\langle x\rangle^{-2s}\psi)| \leqq c\|\varphi\|\cdot\|\psi\| \tag{B.3}$$

for all $\varphi \in \mathscr{S}(\mathbb{R}^n)$, the Schwartz space of rapidly decreasing functions. This will prove the result.

Using the resolvent equation we find

$$\begin{aligned}
&|(\varphi, R\langle x\rangle^{-2s}\psi)| \\
&\quad = \lim_{\varepsilon\downarrow 0} |(\varphi, \{(\bar{H} - i)^{-1} + (\lambda_0 + i\varepsilon - i)(\bar{H} - \lambda_0 - i\varepsilon)^{-1}(\bar{H} - i)^{-1}\}\langle x\rangle^{-2s}\psi)| \\
&\quad \leqq c\|\varphi\|\cdot\|\psi\| + c\lim_{\varepsilon\downarrow 0} |(\varphi, (\bar{H} - \lambda_0 - i\varepsilon)^{-1}\langle A\rangle^{-2s}\psi')|,
\end{aligned}$$

where

$$\psi' = \langle A\rangle^{2s}(\bar{H} - i)^{-1}\langle x\rangle^{-2s}\psi.$$

By the proof of Lemma 4.1 and interpolation ($0 \leqq s \leqq 1$), one easily proves that $\|\psi'\| \leqq c\|\psi\|$. Thus, to prove (B.3), we need only show that for all $\varphi \in \mathscr{S}(\mathbb{R}^n)$ and all ψ,

$$\lim_{\varepsilon\downarrow 0} |(\varphi, (\bar{H} - \lambda_0 - i\varepsilon)^{-1}\langle A\rangle^{-2s}\psi)| \leqq c\|\varphi\|\cdot\|\psi\|.$$

We have

$$\begin{aligned}
&\lim_{\varepsilon\downarrow 0} |(\varphi, (\bar{H} - \lambda_0 - i\varepsilon)^{-1}\langle A\rangle^{-2s}\psi)| \\
&\quad = \left|\left(\langle A\rangle^{s}\varphi, \lim_{\varepsilon\downarrow 0} \langle A\rangle^{-s}(\bar{H} - \lambda_0 - i\varepsilon)^{-1}\langle A\rangle^{-2s}\psi\right)\right|
\end{aligned}$$

$$\leq \left| \left(P_- \langle A \rangle^s \varphi, \lim_{\varepsilon \downarrow 0} \langle A \rangle^{-s} (\bar{H} - \lambda_0 - i\varepsilon)^{-1} \langle A \rangle^{-2s} \psi \right) \right|$$

$$+ \left| \left(P_+ \langle A \rangle^s \varphi, \lim_{\varepsilon \downarrow 0} \langle A \rangle^{-s} (\bar{H} - \lambda_0 + i\varepsilon)^{-1} \langle A \rangle^{-2s} \psi \right) \right|$$

$$= \left| \left(\varphi, \lim_{\varepsilon \downarrow 0} P_- (\bar{H} - \lambda_0 - i\varepsilon)^{-1} \langle A \rangle^{-2s} \psi \right) \right|$$

$$+ \left| \left(\varphi, \lim_{\varepsilon \downarrow 0} P_+ (\bar{H} - \lambda_0 + i\varepsilon)^{-1} \langle A \rangle^{-2s} \psi \right) \right| \leq c \| \varphi \| \cdot \| \psi \|. \quad \blacksquare$$

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