Nova Série
BOLETIM
DA SOCIEDADE BRASILEIRA DE MATEMÁTICA

Bol. Soc. Bras. Mat., Vol. 24, N. 2, 233-259

The Dynamics of Perturbations of the Contracting Lorenz Attractor

Alvaro Rovella

Abstract. We show here that by modifying the eigenvalues $\lambda_2 < \lambda_3 < 0 < \lambda_1$ of the geometric Lorenz attractor, replacing the usual *expanding* condition $\lambda_3 + \lambda_1 > 0$ by a *contracting* condition $\lambda_3 + \lambda_1 < 0$, we can obtain vector fields exhibiting transitive non-hyperbolic attractors which are persistent in the following measure theoretical sense: They correspond to a positive Lebesgue measure set in a two-parameter space. Actually, there is a codimension-two submanifold in the space of all vector fields, whose elements are full density points for the set of vector fields that exhibit a contracting Lorenz-like attractor in generic two parameter families through them. On the other hand, for an open and dense set of perturbations, the attractor breaks into one or at most two attracting periodic orbits, the singularity, a hyperbolic set and a set of wandering orbits linking these objects.

0. Introduction

Let M be a manifold. Denote $V^r(M)$ the Banach space of C^r vector fields with uniformly bounded derivatives, endowed with the usual C^r norm. If $X \in V^r(M)$ denote $X^t: M \hookleftarrow$ the flow of diffeomorphisms generated by X. There exist various definitions of attractors. We shall use the strongest one: a set $\Lambda \subset M$ is an attractor of $X \in V^r(M)$ if it is compact, invariant under X, transitive (i.e. it contains dense orbits) and it has a compact neighborhood U such that

$$\Lambda = \bigcup_{t \geq 0} X^t(U).$$

A compact neighborhood U of Λ satisfying the above property is called a local basin of Λ.

Moreover we say that Λ is persistent (in the C^r topology) if it has a

Received 21 June 1990.

local basin U such that setting.

$$\Lambda_Y = \bigcup_{t \geq 0} Y^t(U),$$

then Λ_Y is an attractor for every Y in a C^r neighborhood of X.

Typical persistent attractors are the hyperbolic attractors. In dimension 3, a C^1-persistent attractor without singularities has to be hyperbolic [**M1**]. In every dimension > 3 examples of non hyperbolic C^1-persistent attractors without singularities are known.

Allowing singularities, there exist C^1-persistent attractors even in dimension 3. This was discovered by Guckenheimer in 1975. Motivated by an algebraically very simple differential equation on $\mathbb{R}^3$ proposed by Lorenz [**L**] as a finite dimensional approximation of the evolution equation of atmospheric dynamics, Guckenheimer produced a C^∞ vectorfield X_0 on $\mathbb{R}^3$ having a C^1 persistent attractor Λ containing a singularity with eigenvalues $\lambda_1 < \lambda_3 < 0 < \lambda_1$ and $\lambda_1 + \lambda_3 > 0$. The attractor Λ became known as the geometric Lorenz attractor, but so far it is still unknown whether the original Lorenz equations contain such an object. Richlik, [**R**], has proved its existence in a differential equation close to that of Lorenz. Beside its persistence, the geometric Lorenz attractor has other surprising properties, like having modulus of stability 2, but we shall not pursue that line of properties.

Here we shall consider a vector field almost identical to that used by Guckenheimer, but with the eigenvalues of the singularity being $\lambda_2 < \lambda_3 < 0 < \lambda_1$ and satisfying $\lambda_1 + \lambda_3 < 0$. It will be constructed so that it has an attractor Λ containing the singularity, but this attractor won't be persistent. In a neighborhood $\mathcal{U}$ there will be an open and dense set of vector fields for which the attractor breaks up into one, or at most two, attracting periodic orbits, a hyperbolic set, the singularity and wandering trajectories linking these objects. But on the other hand, Λ will have a compact neighborhood U such that

$$\Lambda = \bigcap_{t \geq 0} X_0^t(U),$$

and, for a positive measure set of vector fields $X \in \mathcal{U}$, the set

$$\Lambda_X = \bigcap_{t \geq 0} X^t(U)$$

is an attractor of X.

To give an accurate meaning to this measure theoretical property, we shall introduce a concept of full density point of a subset of a Banach space, attempting to generalize the usual concept of full density point of a subset of a finite dimensional manifold. Recall that given a subset S of a finite dimensional Riemannian manifold M, we say that x is a density point of S, if, denoting m the Lebesgue measure, and $B_r(x)$ the ball of radius r and centered at x, we have:

$$\lim_{r \to 0} \frac{m(B_r(x) \cap S)}{m(B_r(x))} = 1$$

Definition. Given a subset S of a Banach space E, we say that $x \in S$ is a point of k-dimensional full density of S if there exists a C^∞ submanifold $N \subset E$, containing x and having codimension k, such that for every k-dimensional manifold M intersecting N transversally, then every point of $N \cap M$ is a point of full density of $S \cap M$ in M.

Definition. We say that an attractor Λ of $X \in V^\ell(M)$ is k-dimensionally almost persistent, if it has a local basin U such that X is a k-dimensional full density point of the set of vector fields $Y \in V^\ell(M)$ for which $\Lambda_Y = \cap_{t \geq 0} Y^t(U)$ is an attractor.

Now we can state our result:

Theorem. *There exists a C^∞ vector field X_0 in $\mathbb{R}^3$ having an attractor Λ containing a singularity, and satisfying the following properties:*

(a) *There exist a local basin U of Λ, a neighborhood $\mathcal{U}$ of X_0, and an open and dense subset $\mathcal{U}_1$, of $\mathcal{U}$, such that for all $X \in \mathcal{U}_1$, $\Lambda_X = \cap_{t \geq 0} X^t(U)$ consists of the union of one or at most two attracting periodic orbits, a hyperbolic set of topological dimension one, a singularity, and wandering orbits linking them.*

(b) *Λ is 2-dimensionally almost persistent in the C^3 topology.*

The usual Lorenz attractor is analyzed by showing that its dynamical properties are in correspondence with those of a map of the interval

$f\colon [-1,1] \hookleftarrow$, with a graph of the form shown in figure 1, with derivative > 1. In our case, a similar reduction is possible, but it leads to a map of the form shown in figure 2, with derivative 0 at $x = 0$. This is due to having $\lambda_1 + \lambda_3 < 0$ instead of $\lambda_1 + \lambda_3 > 0$.

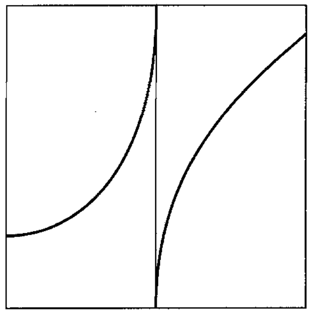

Figure 1

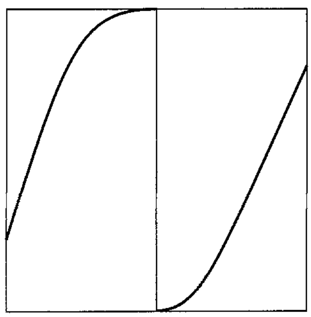

Figure 2

This kind of maps, associated to contracting Lorenz attractors was first discussed by Arneodo, Coullet and Tresser [**ACT**]. Their interest, however, was on the appearance of cascades of bifurcations as a transition to chaotic behaviour, and not on the persistence of the attractor like in the present paper.

Property (b) of the theorem follows applying to this map the methods of Benedicks and Carleson [**BC1**], [**BC2**], suitable modified.

The open and dense set in property (a), where the vector field exhibits what can be described as Axiom A dynamics, follows also from analyzing this map and exploiting its monotonicity property.

I'm grateful to Ricardo Mañé who proposed me this problem giving the right conjecture about the final result, and whose sound ideas were of invaluable help in some parts of the proof.

Maria José Pacífico and Jacob Palis gave me the necessary force to carry out this work in time. Useful conversations with them and Floris Takens clarified some obscure points.

Finally, I would like to acknowledge my gratitude to IMPA, CNPq, CAPES and FAPERJ for the financial support.

I. Description of the Initial Vector field

In this section we will describe the initial vector field, X_0. In the next one we will study its perturbations.

X_0 is a C^∞ vector field in $\mathbb{R}^3$ with a singularity at the origin, whose eigenvalues satisfy $-\lambda_2 > -\lambda_3 > \lambda_1 > 0$, and whose eigenvectors are supposed to have the directions of the coordinate axis. We will also assume that X_0 is linear in a neighborhood of the origin containing the cube $\{(x, y, z): |x|, |y|, |z| \leq 1\}$ Both trajectories of the unstable manifold of the singularity intersect Q, the top of the cube, as in the figure 3:

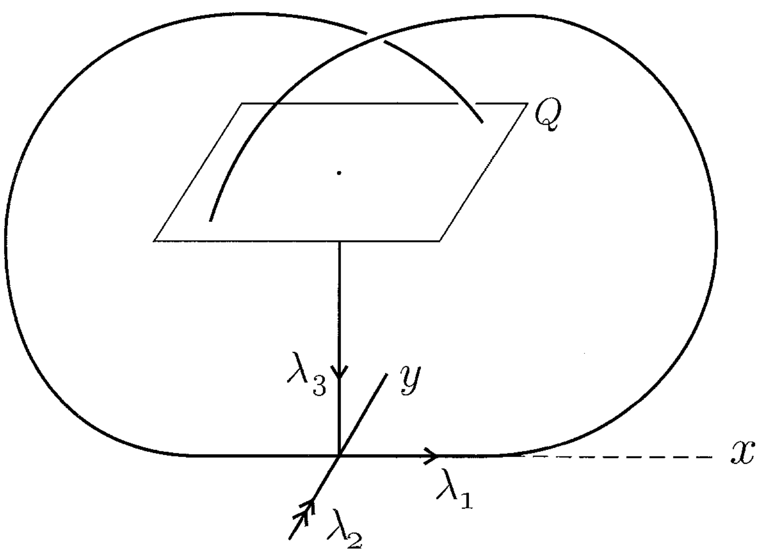

Figure 3

A local stable manifold of the singularity intersects Q at $\{x = 0\}$, so we can consider the first return map F_0 defined in $Q^* = Q \backslash \{x = 0\}$.

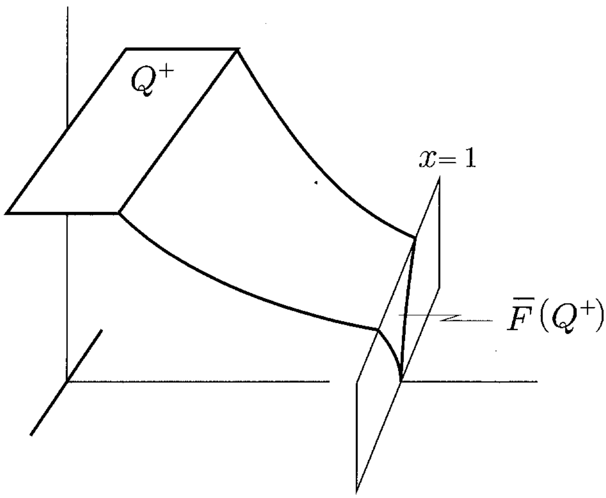

Figure 4

By a simple calculation using the form of the flow of X_0 in the linearized neighborhood, it is easy to see that the first return map $\overline{F}$, from Q^+ to $\{x = 1\}$ is:

$$\overline{F}(x, y, 1) = (1, yx^r, x^s),$$

where

$$s = -\frac{\lambda_3}{\lambda_1} \quad \text{and} \quad r = -\frac{\lambda_2}{\lambda_1}.$$

To obtain F_0, the map $\overline{F}$ must be composed with a diffeomorphism which will be supposed to carry lines $z =$ const. in $\{x = 1\}$ to lines $\{y = \text{const}\}$ in Q. Moreover, we will assume that the flow of X_0 is such that the lines with the direction of the axis OY (of the strong stable manifold of the singularity) form an invariant foliation for X_0. In particular, this implies that in Q F_0 has an invariant foliation; then F_0 has the form:

$$F_0(x, y) = (f_0(x), g_0(x, y))$$

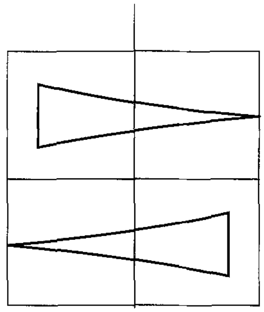

Figure 5

As the flow is smooth and has no singularities between $\{x = 1\}$ and Q, it follows from the formula for $\overline{F}$, that the order of f_0' at $x = 0$ is $s - 1$, that is:

$$\lim_{x \to 0} \frac{f_0'(x)}{|x|^{s-1}} \quad \text{is finite and} \quad \neq 0.$$

For the same reason, the orders of $\dfrac{\partial g_0}{\partial x}$ and $\dfrac{\partial g_0}{\partial y}$ at $x = 0$ are, at least, $s - 1$ and r, respectively.

Next we sill summarize the properties of X_0 just described and others that will be needed in the proofs. After this, we will briefly comment the new properties.

Properties of: X_0

1. X_0 has a singularity at the origin, whose eigenvalues satisfy:

(1.1) $-\lambda_2 > -\lambda_3 > \lambda_1 > 0$

(1.2) $r > s + 3$, where $r = -\lambda_2/\lambda_1, s = -\lambda_3/\lambda_1$.

2. There is an open set U in $\mathbb{R}^3$ containing the cube and the singularity that is positively invariant under X_0. The first return map $F_0: Q^* \to Q$ has the form

$$F_0(x, y) = (f_0(x), g_0(x, y))$$

Thus, the foliation by lines $\{y = \text{const.}\}$ of Q is invariant under F_0.

3. There is a positive number ρ that will be supposed sufficiently small such that the contraction along the invariant foliation of lines $y = \text{const.}$ in U is stronger than ρ.

4. Properties of f_0

(4.1) The order of f_0' at $x = 0$ is $s - 1 > 0$.

(4.2) f_0 has a discontinuity at $x = 0, f_0(0^+) = -1, f_0(0^-) = 1$.

(4.3) $f_0'(x) > 0 \quad x \neq 0$.

(4.4) $\max_{x>0} f_0'(x) = f_0'(1)$, $\max_{x<0} f_0'(x) = f_0'(-1)$

(4.5) The points 1 and -1 are preperiodic repelling, that is, there exist k^-, k^+, n^-, n^+ such that:

$$f_0^{k^+ + n^+}(1) = f_0^{k^+}(1), \qquad (f_0^{n^+})'(f_0^{k^+}(1)) > 1$$
$$f_0^{k^- + n^-}(-1) = f_0^{k^- +}(-1), \quad (f_0^{n^-})'(f_0^{k^-}(1)) > 1.$$

(4.6) f_0 has negative schwarzian derivative: $S(f_0) < \alpha < 0$.

Remarks.

- Properties (1.1) and (1.2) are open, so they are valid for all X near X_0.
- We will use (4.5) to prove part (b) of the theorem, and (4.6) to prove part (a).
- By (4.1), $S(f_0)(x) \to -\infty$ as $|x| \to 0$: this can be seen by direct calculation. Thus (4.6) must the verified only outside a neighborhood of $x = 0$.
- The property stated in 3 is an hypothesis on the behaviour of the vector field X_0 outside a neighbourhood of the origin. Close to the singularity, the constant of contraction of the foliation depends on the relation between the eigenvalues. Property (1.2) gives the necessary condition to obtain this contraction.

II. Existence of Foliations

In this section we will show that some of the properties of the initial vector field are still valid for C^3 perturbations. Let $\mathcal{U}$ be a small neighborhood of X_0. Then every $X \in \mathcal{U}$ has a singularity close to the origin, whose eigenvalues, $\lambda_1(X), \lambda_2(X)$ and $\lambda_3(X)$ satisfy the properties (1.1) and (1.2) of the last section. Furthermore, the trajectories $\xi_1(X)$ and $\xi_2(X)$ contained in the unstable manifold of the singularity of X, still intersect the square Q.

In addition we can make $\mathcal{U}$ and U smaller to obtain that the open set $U \subseteq \mathbb{R}^3$ is positively invariant by the flow of each $X \in \mathcal{U}$.

Proposition. *For each $X \in \mathcal{U}$ there is a C^3 stable one dimensional foliation in U invariant under X and that varies continuously with X.*

Proof. Let $\mathcal{L} = \{(x, l): x \in U, l \text{ is a one dimensional subspace of } T_xU\}$.

Fixed a point $x \in U$ there is a diffeomorphism between the set of one dimensional subspaces of T_xU and the quotient of the unit sphere of T_xU under identification of antipodal points. This implies that $\mathcal{L}$ is locally diffeomorphic to SU, the unit tangent bundle of U. We will use this fact without specific mention.

For each $X \in \mathcal{U}$ it can be defined a vector field $\tilde{X}$ in $\mathcal{L}$ as follows: Take $x \in U$ and $v \in T_xU$ a unit vector and put

$$\tilde{X}(x, v) = (X(x), DX_x(v) - \langle DX_x(v), v\rangle v). \tag{1}$$

The first component in the definition of $\tilde{X}(x, v)$ is a vector in T_xU and the second one is a vector in T_xU orthogonal to v; so $\tilde{X}(x, v) \in T_{(x,v)}\mathcal{L}$. It is not difficult to check that the flow associated to $\tilde{X}$ is

$$\tilde{\varphi}(t, (x, v)) = \left(\varphi(t, x), \frac{(D\varphi_t)_x(v)}{\|(D\varphi_t)_x(v)\|}\right) \tag{2}$$

where φ denotes the flow of X.

Now recall from section I that the initial vector field X_0 has an invariant foliation in U defined by lines $\{y = \text{const.}\}$. The set of pairs (x, l) with $x \in U$ and l the direction of the leave passing through x define a submanifold $\mathcal{V}$ of $\mathcal{L}$. $\mathcal{V}$ is $\tilde{X}_0$-invariant because if $(x, v) \in \mathcal{V}$ than it is easy to see that $\tilde{\varphi}(t, (x, v)) \in \mathcal{V}$ for $t > 0$, by formula (2). Now we want to show that $\mathcal{V}$ is 3-normally hyperbolic.

For each $(x, v) \in \mathcal{V}$ define

$$E^u_{(x,v)} = \{0\} \times T_v(S_x)$$

where 0 is the origin in T_xU and S_x is the unit sphere in T_xU.

As $T_{(x,v)}\mathcal{V} = T_x(U) \times \{0\}$ (now 0 is the origin in $T_v(S_x)$) we have the following splitting:

$$T_{(x,v)}\mathcal{L} = T_{(x,v)}\mathcal{V} \oplus E^u_{(x,v)}$$

To show that $E^u_{(x,v)}$ is $\tilde{X}$ invariant, take a vector $w \in T_v(S_x)$ and the curve γ in $\mathcal{L}$ defined by γ in $\mathcal{L}$ defined by $\gamma(s) = (x, v + sw)$. Then we have:

$$(D\tilde{\varphi}_t)_{(x,v)}(0,w) = \frac{d}{ds}\tilde{\varphi}_t \circ \gamma\bigg|_{s=0} = \frac{d}{ds}\left(\varphi(t,x), \frac{(D\varphi_t)_x(v+sw)}{\|(D\varphi_t)_x(v+sw)\|}\right)\bigg|_{s=0}$$

$$= \left(0, \frac{(D\varphi_t)_x(w)}{\|(D\varphi_t)_x(w)\|} - \frac{\langle (D\varphi_t)_x(v), (D\varphi_t)_x(w)\rangle}{\|(D\varphi_t)_x(v)\|^3}(D\varphi_t)_x(v)\right)$$

This proves that $E^u_{(x,l)}$ is $D\tilde{\varphi}$ invariant and implies

$$\left\|(D\tilde{\varphi}_t)_{(x,v)}(0,w)\right\|^2 = \frac{\|(D\varphi_t)_x(w)\|^2}{\|(D\varphi_t)_x(v)\|^2} - \frac{\langle (D\varphi_t)_x(v), (D\varphi_t)_x(w)\rangle}{\|(D\varphi_t)_x(v)\|^4}. \tag{3}$$

On the other hand, it is easy to see that for $u \in T_xU$:

$$(D\tilde{\varphi}_t)_{(x,v)}(u,0) = ((D\varphi_t)_x(u), \gamma_{(x,v)}, (u)) \tag{4}$$

where $\gamma_{(x,v)}(u)$ is a vector tangent to S_x at the point v involving second derivatives of the map φ_t, so it is 0 in the linearized neighborhood of the origin and has norm bounded by a constant outside:

$$\|\gamma_{x,v}(u)\| \leq C\|u\|.$$

To prove that $\mathcal{V}$ is 3-normally hyperbolic we have to check that the rate of expansion in $E^u_{(x,v)}$ is three times the great expansion of vectors tangent to $\mathcal{V}$. If $(x,v) \in \mathcal{V}$, the direction of v is $(0,1,0)$, so, as we have supposed in section I that outside a neighborhood of the origin the contraction along v is given by a small number $\rho > 0$, we can diminish ρ and use formula (3) to obtain that the expansion in $E^u_{(x,v)}$ is sufficiently large if compared with that along $\mathcal{V}$. Now it remains to show this condition when x is in the linearized neighborhood of the origin. For this, it is enough to calculate the eigenvalues of $\tilde{X}$ at the point $(0,(0,1,0))$ which is the singularity of $\tilde{X}$. In fact, using (3) we obtain that the eigenvalues associated to vectors in $E_{(0,(0,1,0))}$ are $-\lambda_2 + \lambda_3$ and $-\lambda_2 + \lambda_1 (0 < -\lambda_2 + \lambda_3 < -\lambda_2 + \lambda_1)$; and using (4) it is easy to see that the eigenvalues associated to vectors in $T_{(0,(0,1,0))}\mathcal{V}$ are $\lambda_2 < \lambda_3 < 0 < \lambda_1$.

So the condition we need is

$$-\lambda_2 + \lambda_3 > 3\lambda_1$$

which is precisely hypothesis (1.2) of section I.

Once we know that $\mathcal{V}$ is 3-normally hyperbolic, we can apply well known results about such manifolds to obtain that for all $X \in \mathcal{U}$ the induced vector field $\tilde{X}$ in $\mathcal{L}$ has an invariant manifold of class C^3 and varying continuously with $\tilde{X}$ (see [**HPS**]). Now it is easy to see that this invariant manifold obtained for $\tilde{X}$ induces a C^3 invariant stable foliation for X constituted by one dimensional curves in U. This proves the proposition.

Now for each $X \in \mathcal{U}$ we construct a new square close to Q (that we will still call by Q) formed by lines of the foliation, so that the first return map F_x to Q has an invariant foliation, and we can also put coordinates (x, y) in Q such that $x = 0$ correspond to the stable manifold of the singularity and

$$F_x(x,y) = (f_x(x), g_x(x,y)).$$

The one dimensional map f_X induced by F_X through the foliation is C^3 in $x \neq 0$; 0 is the discontinuity and critical point, and we suppose that $f_X(0^+) = -1, f_X(0^-) = 1$. The order of f'_X at $x = 0$ is $s_X - 1$. Finally, the maps f_X and its three first derivatives depend continuously on X. Now we have:

Corollary. *Each f_X has negative schwarzian derivative.*

Proof. As $s_X > 1, \lim_{x\to 0} S(f_X)(x) = -\infty$ uniformly in $X \in \mathcal{U}$. Outside a neighborhood of $x = 0, Sf_X$ is close to Sf_0; as $Sf_0 < 0$, the corollary follows.

In a first version of this paper we proved that the foliations were only $C^{1+\gamma}$. It was F. Takens who suggested that C^3 foliations could be obtained. Now we can use two well known properties of maps with negative schwarzian derivative:

(i) Every attracting periodic orbit has a critical point or an extreme point of the interval in its basin. (Singer's theorem), [**S**]).

(ii) Every compact invariant set with all its periodic points hyperbolic repelling and without critical points, is hyperbolic. (Guckenheimer's theorem, [**G**]).

III. Proof of Part (a) of the Theorem

We want to prove that if $\mathcal{U}$ is a small neighborhood of X_0, then there exists $\mathcal{U}_1$, open and dense in $\mathcal{U}$, such that for all $X \in \mathcal{U}_1$ then non-wandering set of Λ_X is hyperbolic.

Lemma 1. *All $X \in \mathcal{U}$ can be perturbed so that the two trajectories of the unstable manifold of the singularity have attracting periodic orbits as w-limit.*

Suppose this lemma proved and let's see how part (a) of the theorem follows from it. Consider the one dimensional maps f_X induced by the vector fields $X \in \mathcal{U}$. The points 1 and -1 (the critical values) correspond to the separatrices of the unstable manifold. From lemma 1 it follows that there exists $\mathcal{U}_1$ residual in $\mathcal{U}$ such that for each $X \in \mathcal{U}_1$:

- f_X has one or at most two attracting periodic orbits whose basins contain the critical points of f_X.
- Every periodic orbit of f_X is hyperbolic.

As each f_X has negative schwarzian derivative, Singer's theorem implies that each f_X has at most two attracting periodic orbits. In addition, if $X \in \mathcal{U}_1$, the complementary set of the basins of the attracting periodic orbits is hyperbolic. Hence $\mathcal{U}_1$ is actually open and all $X \in \mathcal{U}_1$ satisfies part (a) of the theorem.

Proof of Lemma 1. We will consider the one-dimensional maps induced by each $X \in \mathcal{U}$. What we must prove is that every X can be approximated by $Y \in \mathcal{U}$ such that the points 1 and -1 are both attracted by attracting orbits of f_Y.

The transformations:

$$Y \in \mathcal{U} \to \omega_{f_Y}(0^+) \subset [-1,1]$$
$$Y \in \mathcal{U} \to \omega_{f_Y}(0^-) \subset [-1,1]$$

where $\omega_{f_Y}(x)$ denotes the ω-limit set of x under f_Y, are lower semicontinuous, as it is easy to verify, if considered with topology C^3 in the domain and the Hausdorff topology for closed sets. Therefore, they are continuous in a residual of $\mathcal{U}$, and we can assume that X pertains to this residual and has all its periodic orbits hyperbolic.

Now we will suppose that $\omega_{f_X}(1)$ is not an attracting periodic orbit; to prove the lemma we must find Y close to X such that $\omega_{f_Y}(1)$ is an attracting periodic orbit.

Claim. If $\omega_{f_X}(1)$ is not an attracting periodic orbit, then $0 \in \omega_{f_X}(1)$

If $0 \notin \omega_{f_X}(1)$ then $\omega_{f_X}(1)$ is hyperbolic. As hyperbolic sets have empty interior, there exists a neighborhood V of $\omega_{f_X}(1)$ such that for a residual set of $x \in V$, $f_X^j(x) \notin V$ for infinitely many $j \geq 0$.

Note that 0^- is the unique preimage of 1, so, from $0 \notin \omega_{f_X}(1)$ it follows that $1 \notin \omega_{f_X}(1)$. Thus we can perturb f_X in a neighborhood of 1, disjoint of V, such that for the new map, f_Y, we have $f_Y^j(1) \notin V$ infinitely many times. So $\omega_{f_Y}(1)$ is not contained in V and this gives a contradiction because we supposed that X was a point of continuity of the map $X \to \omega_{f_X}(1)$. This proves the claim.

So, $0 \in \omega_{f_X}(1)$ if $\omega_{f_X}(1)$ is not an attracting periodic orbit. Suppose first that 0 can be accumulated by $\omega_{f_X}(1)$ from the left, that is: there exists a sequence $k_n \to \infty$ such that $f_X^{k_n}(1) < 0 \quad \forall n$, and $f_X^{k_n}(1) \to 0$.

It is not difficult to see that given $\delta > 0$ there exists $Y \in \mathcal{U}$ at a distance less than δ from such X such that:

$$f_Y(x) \geq f_X(x) \quad \forall x; f_Y(x) > f_X(x) + \delta/2 \quad \forall |x| > x_0$$

where x_0 is chosen so that $|f_Y(z)| < \varepsilon$ implies $|z| > x_0$, for all $Y \in \mathcal{U}$ and ε small enough.

Claim. There exists $j > 0$ such that $f_X^j(1) < 0 < f_Y^j(1)$.

Assuming the contrary we prove by induction that $f_Y^j(1) > f_X^j(1)$ for all $j > 0$: this is very simple because $f_Y > f_X$ and both maps are increasing in $[-1, 0]$ and in $[0, 1]$. Furthermore, it follows that $f_Y^{j+1}(1) - f_X^{j+1}(1) > \delta/2$ if $f_X^{j+1}(1) \in (-\varepsilon, 0), \varepsilon$ a small constant. Now take n_k such that $f_X^{n_k}(1) \in (-\delta, 0)$, and note that:

$$-f_X^{n_k}(1) \geq f_Y^{n_k}(1) - f_X^{n_k}(1) > \delta/2.$$

This contradicts the fact that $f_X^{n_k}(1) \to 0$ and proves the claim.

This implies that there exists Y_0 at a distance of X less than δ such that $f^j_{Y_0}(1) = 0$; so f_{Y_0} has a super attractive periodic orbit. Now it is easy to see that we can find Y close to Y_0 and such that f_Y has an hyperbolic periodic attractor whose basin contains 1. This proves the lemma under the assumption that $\omega_{f_X}(1)$ accumulates on 0 from the left. Suppose now that this doesn't occurs; so, as $0 \in \omega_{f_X}(1), \omega_{f_X}(1)$ must accumulate on 0 from the right.

This implies that $-1 \in \omega_{f_X}(1)$, thus, as we are supposing that $\omega_{f_X}(1)$ is not an attracting periodic orbit, then $\omega_{f_X}(-1)$ is not an attracting periodic orbit. So we can use the first claim to obtain that $0 \in \omega_{f_X}(-1)$.

Now $\omega_{f_X}(1) \subset \omega_{f_X}(-1)$ and so 0 is accumulated by $\omega_{f_X}(-1)$ from the right. Next, as in the second claim, we perturb to obtain that -1 is being attracted by a periodic attractor. This is an open condition, so we repeat the argument, but now beginning with a vector field X such that $\omega_{f_X}(-1)$ is an attracting periodic orbit, and so the proof finishes, because we have two cases: $w_{f_X}(1)$ *is an attracting periodic orbit*, or 0 *is accumulated from the left by* $\omega_{f_X}(1)$, and in both cases we showed how to obtain the lemma.

IV. Proof of Part (b) of the Theorem

Each X in a small neighborhood of X_0, induces a map of the interval, f_X. For X_0 this map was denoted by f_0: it is defined in $[-1, 1]$, being that $f_0(0^-) = 1$ and $f_0(0^+) = -1$; the points 1 and -1 are preperiodic repelling. Let k^- be such that $f_0^{k^-}(-1)$ and $f_0^{k^+}(1)$ are periodic of periods n^- and n^+. Let's define

$$N = \left\{ \begin{array}{l} X \in \mathcal{U}: f_X^{k^+}(1) \text{ and } f_X^{k^-}(-1) \text{ are periodic} \\ \text{with periods } n^+ \text{ and } n^- \text{ respectively} \end{array} \right\}.$$

If $\mathcal{U}$ is small enough, we have that N is a submanifold of codimension 2 containing X_0, and that $f_X^{k^+}(1)$ and $f_X^{k^-}(-1)$ are preperiodic repelling.

Let M be a C^3 bidimensional submanifold of $\mathcal{U}$ intersecting N transversally. We must prove that all vector field in $N \cap M$ is a full density point of the set of $Y \in M$ such that Λ_Y is an attractor.

Let's take $Y_0 \in M \cap N$, and $\{Y_a\}_{a \geq 0}$, a one parameter family contained in M such that the functions $a \to f_{Y_a}(\mp 1)$ have derivative 1 at $a = 0$. We will prove that $a = 0$ is a full density point of the set of parameters for which Λ_{Y_a} is an attractor. This, as the next lemma shows, implies that Y_0 is a 2-dimensional full density point of the vector fields Y in M such that Λ_Y is an attractor.

Lemma. *Let $A \subseteq \mathbb{R}^2$, and for each $\theta \in [0, 2\pi)$ define $A_\theta = \{re^{i\theta}: r \geq 0\} \cap A$. Suppose that for all $\theta \in [0, 2\pi)$, $\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} m_1(A_\theta \cap B_\varepsilon) = 1$, where m_1 denotes Lebesgue measure in $\mathbb{R}$ and B_ε is the ball of center $(0,0)$ and radius ε in $\mathbb{R}^2$. Then:*

$$\lim_{\varepsilon \to 0} \frac{m_2(A \cap B_\varepsilon)}{m_2(B_\varepsilon)} = 1,$$

where m_2 is the two-dimensional Lebesgue measure.

Proof. Fix any $\delta > 0$ and define

$$C_{n_0}^{\delta} = \{\theta \in [0, 2\pi): m_1(A \cap B_{1/n}) \geq \frac{1 - \delta}{n} \forall n \geq n_0\}.$$

Observe that $\{C_{n_0}^{\delta}\}_{n_0 \geq 1}$ is an increasing sequence of sets which union is $[0, 2\pi)$, so $m_1(C_{n_0}^{\delta}) \to 2\pi$ for all δ. Then, denoting by $\chi_A(r, \theta)$ the characteristic function of A:

$$\begin{aligned} m_2(A \cap B_{1/n}) &= \int_0^{2\pi} d\theta \int_0^{1/n} r\chi_A(r,\theta) dr \\ &\geq \int_{C_n^{\delta}} d\theta \int_0^{1/n} r\chi_A(r,\theta) dr \\ &\geq \int_{C_n^{\delta}} d\theta \int_0^{\frac{1-\delta}{n}} r dr = m_1(C_n^{\delta}) \frac{1}{2} \frac{(1-\delta)^2}{n^2}. \end{aligned}$$

The result follows easily.

Thus, as was pointed above, we can consider one parameter families.

Theorem 2. *There exists a set E of parameters such that:*

- *For all $a \in E$ the points 1 and -1 have positive Lyapunov exponents under f_{Y_a}, that is: there exists $\lambda_1 > 1$ such that $(f_{Y_a}^n)'(\pm 1) > \lambda_1^n \quad \forall n \geq 0$.*
- *For all $a \in E$ the positive orbits of the points 1 and -1 under f_{Y_a} are dense in $[-1, 1]$.*

- 0 *is a full density point of* E:

$$\lim_{a\to 0}\frac{1}{a}m_1(E\cap[0,a))=1$$

Now observe that this theorem implies part (b) of our theorem. In fact, for each $a\in E$, Λ_{Y_a} is an attractor: its transitiveness follows from that of f_{Y_a} because the foliation that gives rise to f_{Y_a} is stable.

Benedicks and Carleson proved a version of theorem 2 for the quadratic family. Here we will follow their arguments, only proving those facts that have essential differences. For the rest we refer to [**BC1**], [**BC2**] and [**MV**].

Proof of Theorem 2. To clarify the notation, denote by φ_a the function f_{Y_a}. Before beginning with the proof, we recall some properties of the maps φ_a that will be needed in the sequel.

V.1 There exist positive constants K_1, K_2, independent of a (and of θ), such that:

$$K_2|x|^{s-1}\le\varphi_a'(x)\le K_1|x|^{s-1}$$

for all a, x, where $s=s(a)>1$. To simplify the notation, we will take s independent of a.

V.2 $\varphi_a\in C^3$. Their derivatives depend continuously on a. So φ_a has negative schwarzian derivative, for sufficiently small a.

V.3 The functions $a\to\varphi_a(1)$ and $a\to\varphi_a(-1)$ have derivative 1 at $a=0$. This is the condition of transversality.

V.4 With the purpose of simplify the notation, we will suppose that the points -1 and 1 are fixed by φ_0, so $\varphi_0'(1)>1$ and $\varphi_0'(-1)>1$.

We will begin proving that maximal orbits outside a neighborhood of the critical points have exponential growth of the derivative.

Lemma 1. *There exists* $\lambda_0>1$ *with the following property: given* $\delta>0$ *there exists* $a_0(\delta)>0$ *such that, if*

$$0\le a\le a_0(\delta);\quad \left|\varphi_a^j(x)\right|>\delta\ \ \forall 0\le j\le k-1\quad \textit{and}\quad \left|\varphi_a^k(x)\right|\le\delta,$$

then:

$$(\varphi_a^k)'(x)>\lambda_0^k.$$

This lemma is one of the basic facts that support the proof of Benedicks and Carleson. Their initial map, $1-2x^2$, is C^1 conjugated to an expansive map, $1-2|x|$: they used this in the proof of the lemma. We don't have this fact in our case, hence the proof won't be so simple. It will require two new lemmas.

Lemma 1.1. *There exist $\delta_0 > 0$ and $\lambda' > 1$ only depending on the initial vector field X_0, and satisfying the following property: Given $\delta > 0$, there exists $a_2(\delta)$ such that for $|y| \in (\delta, \delta_0)$ and $0 \leq a \leq a_2(\delta)$, there exists a time $\ell > 0$ such that $|\varphi_a^j(y)| > \delta_0$ for $1 \leq j \leq \ell$ and $(\varphi_a^\ell)'(y) \geq \lambda'^\ell$.*

The orbit of the point x of lemma 1, cannot enter in $(-\delta, \delta)$ until k, but it could intersect $(-\delta_0, \delta_0)$ before this time; lemma 1.1 controls the derivative in a piece of orbit beginning at this return to $(-\delta_0, \delta_0)$.

Lemma 1.2. *There exist $a_1(\delta_0) > 0$ and $\lambda_0(\delta_0) > 1$ such that, if $a \leq a_1(\delta_0), |\varphi_a^j(y)| > \delta_0 \quad \forall 0 \leq j < k_0$ and $\left|\varphi_a^{k_0}(y)\right| \leq \delta_0$, then:*

$$(\varphi_a^{k_0})'(y) > (\lambda_0(\delta_0))^{k_0}.$$

This seems lemma 1, but permitting a_1 and λ_0 depend on δ_0. However, δ_0 was fixed in lemma 1.1, so it is easy to see that lemma 1 follows.

Proof of Lemma 1.1. Let

$$M_\varepsilon^\pm(a) = \max_{|z\mp 1|<\varepsilon} \varphi_a'(z), \quad \text{and} \quad m_\varepsilon^\pm(a) = \min_{|z\mp 1|<\varepsilon} \varphi_a'(z).$$

As $M_0^+(a) = m_0^+(a)$ and $M_0^-(a) = m_0^-(a)$, there exist $a' > 0$ and $\varepsilon > 0$ independent of a, such that

$$(M_\varepsilon^+(a))^{\frac{s-1}{s}} < m_\varepsilon^+(a), \quad \forall a < a', \tag{1}$$

and a similar formula for M_ε^- and m_ε^-.

Let $\delta_0 > 0$ be such that $|\varphi_a(y) \pm 1| < \varepsilon$ if $|y| < \delta_0$. Let $|y| \in (\delta, \delta_0)$, for example, $\delta < y < \delta_0$. We define:

$$\ell(y,a) = \min\{j \geq 1 \colon \varphi_a^j(y) \geq -1+\varepsilon\}.$$

It follows that:

$$(\varphi_a^\ell)'(y) = \varphi_a'(y) \cdot (\varphi_a^{\ell-1})'(z) \geq K_1 y^{s-1} m_\varepsilon^{\ell-1} \tag{2}$$

where $z = \varphi_a(y), m_\varepsilon = m_\varepsilon^-(a), \ell = \ell(y,a)$ and K_1 comes from V.1.

On one hand, if $\nu_a = \varphi_a(-1)+1, M_\varepsilon = M_\varepsilon^-(a)$ and $|z+1| < \varepsilon$ we get:

$$\varphi_a(z)+1 \le \nu_a + M_\varepsilon(z+1),$$

because $\varphi_a(-1)+1 = \nu_a$ and $\varphi_a'(z) \le M_\varepsilon$ for $|z+1| < \varepsilon$. If we put $z = \varphi_a(y)$, it follows, by definition of $\ell = \ell(a,y)$:

$$\varphi_a^\ell(z) \le \nu_a \sum_{i=0}^{\ell-1} M_\varepsilon^i + M_\varepsilon^\ell(z+1) - 1.$$

Then, as $\varphi_a^\ell(z) \ge -1+\varepsilon$, it follows that:

$$z+1 \ge \left(\varepsilon - \nu_a \sum_{i=0}^{\ell-1} M_\varepsilon^i\right) M_\varepsilon^{-\ell} \tag{3}$$

But on the other hand, property V.1 implies that $z+1 \le K_2 y^s/s$. Putting this in (2) and using (3) we obtain:

$$\begin{aligned}(\varphi_a^\ell)'(y) &\ge K_1 \left(\frac{s}{K_2}(z+1)\right)^{\frac{s-1}{s}} m_\varepsilon^{\ell-1} \\ &\ge \frac{K_1}{m_\varepsilon}\left(\frac{s}{K_2}\right)^{\frac{s-1}{s}} \left[\varepsilon - \nu_a \sum_{i=0}^{\ell-1} M_\varepsilon^i\right] \left(\frac{m_\varepsilon}{M^{s-1/s}\varepsilon}\right)^\ell .\end{aligned} \tag{4}$$

Now, as $\ell(a,y) < \ell(a,\delta)$, the sum $\sum_{i=0}^{\ell-1} M_\varepsilon^i$ is bounded independently of y. As $\nu_a \to 0$ when $a \to 0$ we can choose $a_0(\delta)$ such that

$$\nu_a \sum_0^{\ell-1} M_\varepsilon^i < \frac{\varepsilon}{2}.$$

Then it follows from (4), that:

$$(\varphi_a^\ell)'(y) \ge \frac{\varepsilon}{2}\frac{K_1}{m_\varepsilon}\left(\frac{s}{K_2}\right)^{\frac{s-1}{s}} \left(\frac{m_\varepsilon}{M_\varepsilon^{s-1/s}}\right)^\ell . \tag{5}$$

By (1), there exists $\lambda_1 > 1$ such that $\lambda_1 < \dfrac{m_\varepsilon}{M_\varepsilon^{s-1/s}}$.

Finally, $\ell(a,\delta_0)$ can be made large, by choosing a and δ_0 small. Then, as $\ell = \ell(a,y) > \ell(a,\delta_0)$ we can obtain, from (5), that

$$(\varphi_a^\ell)'(y) \ge \lambda'^\ell$$

Observe that the proof of lemma 1.1 implies that δ_0 and λ' can be chosen independent of the family contained in $\mathcal{U}$.

Proof of Lemma 1.2.

Claim. There exists $\mathcal{U}_{\delta_0}$ such that if $X \in \mathcal{U}_{\delta_0}$ and f_X has a non-repelling periodic orbit Γ, then $\Gamma \cap (-\delta_0, \delta_0) \neq \phi$. In other words, all attracting or non-hyperbolic periodic orbit for f_X with $X \in \mathcal{U}_{\delta_0}$, must intersect $(-\delta_0, \delta_0)$.

Suppose that this is not true, then there exists a sequence $X_n \to X_0$, periodic points p_n of period k_n for f_{X_n}, with $(f_{X_n}^{k_n})'(p_n) \leq 1$, and such that $\{f_{X_n}^j(p_n): 0 \leq j \leq k_n - 1\}$ does not intersect $(-\delta_0, \delta_0)$.

Let $P_n = \{f_{X_n}^j(p_n): 0 \leq k_n - 1\}$, and let Λ be the set of limit points of $\cup_{n \geq 1} P_n$. It is easy to see that Λ is invariant under f_0 and that $\Lambda \cap (-\delta, \delta) = \phi$. Then, as f_0 has negative schwarzian derivative, Λ must be hyperbolic. Then there exist $m > 0$, $\lambda > 1$ and a neighborhood V of Λ, such that $(f_Y^m)'(x) > \lambda$ for all Y near X_0 and $x \in V$. But this is a contradiction because $P_n \subset V$ for all n large.

Claim. There exists $m > 0$, $\lambda > 1$ such that, if a is sufficiently small and $\varphi_a^j(x) \notin (-\delta_0, \delta_0)$ for all $0 \leq j \leq m - 1$, then $(\varphi_a^m)'(x) \geq \lambda^m$.

Reasoning as in the previous claim, we can find a set Λ, disjoint form $(-\delta_0, \delta_0)$, φ_0-invariant and closed. Its periodic orbits are repelling, by the first claim, and so the conclusion follows as before.

This claim implies that $(\varphi_a^k)'(x) > \lambda^k$ for all $k \geq m$ such that $|\varphi_a^j(x)| > \delta_0$ for all $j < k$. It only remains to prove that this is also true for $k < m$.

As f_0 has negative schwarzian derivative, and the images of the critical points are fixed by f_0, there exists $\mu > 1$, independent of k, such that:

$$f_0^k(x) \in (-\delta_0, \delta_0) \text{ implies} (f_0^k)'(x) \geq \mu \tag{$*$}$$

This is easy to prove looking at the picture of the graph of f_0^k restricted to the maximal interval of continuity of f_0^k that contains x, as in figure 6 below. If we denote by $[a, b]$ this interval, then:

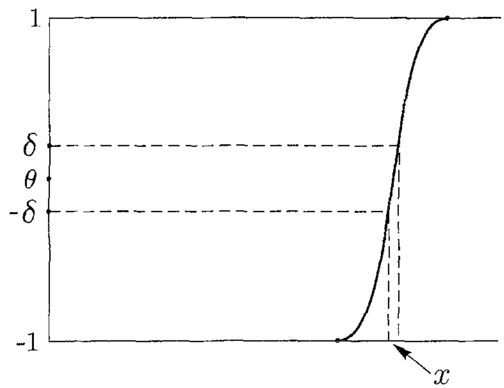

Figure 6

$$f_0^k(b) - f_0^k(x) \geq 1 - \delta_0, \quad \text{and}$$
$$f_0^k(x) - f_0^k(a) \geq 1 - \delta_0,$$

so there exists $\mu > 1$ such that:

$$\frac{f_0^k(b) - f_0^k(x)}{b - x} > \mu, \frac{f_0^k(x) - f_0^k(a)}{x - a} > \mu.$$

This implies that $(f_0^k)'(x) > \mu$, because the contrary assumption violates the minimum principle (if g has negative schwarzian, then g' cannot have a positive minimum). Then property $(*)$ of f_0 is proved, and, once a value m is fixed, it extends to a neighborhood of X_0, for all $k < m$, that is:

$$(f_X^k)'(x) > \mu \quad \text{if} \quad f_X^k(x) \in (-\delta_0, \delta_0)$$

for all $k < m$ and X in a neighborhood of X_0 that doesn't depends on δ (only on m and δ_0). Taking $\lambda > 1$ such that $\lambda^m \leq \mu$, the proof of lemma 1.2 is complete.

Now, as in [**BC1**] or [**BC2**], we will exclude the parameters that don't verify the following *basic assumption*:

$$\left|\varphi_a^j(1)\right| \geq e^{-\alpha j}, \left|\varphi_a^j(-1)\right| \geq e^{-\alpha j} \tag{BA}$$

(α is a positive small constant).

Consider a time k for which $\left|\varphi_a^k(1)\right| < \delta$, and suppose that the parameter a is not excluded by application of the (BA). Then we define the *binding period* associated to a and k as the maximal period $1 \leq j \leq p$ such that, for some small $\beta > 0$:

$$\left|\varphi_a^{k+j}(1) - \varphi_a^{j-1}(-1)\right| < e^{-\beta j}, \quad \text{if} \varphi_a^k(1) > 0, \quad \text{and}$$
$$\left|\varphi_a^{k+j}(1) - \varphi_a^{j-1}(1)\right| < e^{-\beta j}, \quad \text{if} \varphi_a^k(1) < 0.$$

Then, during the binding period, the orbit of $\varphi_a^{k+1}(1)$ is close to that of 1 (or -1). Thus, the arguments will contain an *induction hypothesis.*

Let λ_1 be such that $1 < \lambda_1 < \lambda_0$. We assume that

$$(\varphi_a^j)'(1) > \lambda_1^j, (\varphi_a^j)'(-1) > \lambda_1^j, \quad \forall 1 \leq j < k$$

Thus, the first item of theorem 2, is proved for those parameters for which the induction can be completed until k.

Lemma 2. *Let k be such that $\varphi_a^k(1) \in (e^{-\alpha k}, \delta)$, and assume the induction hypothesis valid until $k-1$.*

Then there exist positive constants ρ and τ depending only on α and β, such that:

(a) $\dfrac{(\varphi_a^j)'(\xi)}{(\varphi_a^j)'(\eta)} \in (\rho^{-1}, \rho) \quad \forall \xi, \eta \in [-1, \varphi_a^{k+1}(1)]$, *for all $j \leq p$, where p is the binding period associated to k and a.*

(b) $p \in \left[\dfrac{r_0}{\rho + \log 3} - 1, \dfrac{r_0 s - \log \dfrac{\rho K_2}{s}}{\beta + \log \lambda_1}\right]$, *where* $e^{-r_0} = \left|\varphi_a^k(1)\right|$.

(c) $(\varphi_a^{p+1})'(\varphi_a^k(1)) \geq \tau \exp\left[\left(\dfrac{\log \lambda_1}{s} - \dfrac{s-1}{s}\beta\right)(p+1)\right] > 1.$

Similar results can be obtained when $\varphi_a^k(1) \in (-\delta, -e^{\alpha k})$ or for the orbit of the point -1.

Proof.

$$\frac{(\varphi_a^j)'(\xi)}{(\varphi_a^j)'(1)} = \prod_{i=0}^{j-1} 1 + \frac{\varphi_a'(\varphi_a^i(\xi)) - \varphi_a'(\varphi_a^i(1))}{\varphi_a'(\varphi_a^i(1))}.$$

To give a proof of part (a) we need the next sum bounded

$$\sum_{i=0}^{j-1} \frac{\varphi_a'(\varphi_a^i(\xi)) - \varphi_a'(\varphi_a^i(1))}{\varphi_a'(\varphi_a^i(1))}. \tag{1}$$

As $\varphi_a^i(1) \geq e^{-\alpha i}$, $\varphi_a'(\varphi_a^i(1)) \geq K_2 e^{-\alpha(s-1)}$. On the other hand,

$$\left|\varphi_a^i(\xi) - \varphi_a^i(1)\right| \leq \left|\varphi_a^i(\varphi_a^{k+1}(1)) - \varphi_a^i(1)\right| < e^{-\beta(i+1)}$$

and as φ_a is C^2:

$$\left|\varphi_a'(\varphi_a^i(\xi)) - \varphi_a'(\varphi_a^i(1))\right| \leq A\left|\varphi_a^i(\xi) - \varphi_a^i(1)\right| \leq Ae^{-\beta(i+1)}$$

Then, the sum (1) can be bounded above by:

$$\sum_{i=0}^{j-1} \frac{Ae^{-\beta(i+1)}}{K_2 e^{-\alpha i(s-1)}} = \frac{Ae^{-\beta}}{K_2} \sum_{i=0}^{j-1} \exp[-\beta + \alpha(s-1)]i \leq \rho$$

where $\beta > \alpha(s-1)$ is the first condition we impose to α and β.

Let's prove (b).

Fixed $j \leq k-1$, there exists $\eta \in (-1, \varphi_a^{k+1}(1))$, such that:

$$\begin{aligned}\left|\varphi_a^j(\varphi_a^{k+1}(1)) - \varphi_a^j(-1)\right| &= (\varphi_a^j)'(\eta)\left|\varphi_a^{k+1}(1) + 1\right| \\ &\leq \rho(\varphi_a^j)'(1)\frac{K_2}{s}(\varphi_a^k(1))^s \geq \rho\lambda_1^j e^{-r_0 s}\frac{K_2}{s}\end{aligned}$$

This, together with the definition of binding period, imply the following assertion:

$$\text{if } j \leq p \text{ and } j < k, \text{ then } e^{-\beta j} \geq \rho\lambda_1^j e^{-r_0 s}\frac{K_2}{s}, \tag{2}$$

that is:

$$j \leq \frac{r_0 s - \log \rho K_2/s}{\beta + \log \lambda_1} < k/2$$

(the last inequality because $r_0 < \alpha k$, and if α is small)

If $p \geq k$, then j can be substituted in the assertion (2) by $k-1$, obtaining $k - 1 < k/2$, that is not possible. Thus, $p < k$, and so j can be substituted by p in (2), and we obtain:

$$p \leq \frac{r_0 s - \log \rho K_2/s}{\beta + \log \lambda_1}.$$

This gives one of the estimates of p. To get the other suppose that $\varphi_a'(x) \leq 3$ for all x. Then:

$$\begin{aligned}3^{p+1}\left|\varphi_a^k(1)\right| &\geq (\varphi_a^{p+1})'(\eta)\left|\varphi_a^k(1)\right| \\ &= \left|\varphi_a^{p+1}(\varphi_a^k(1)) - \varphi_a^{p+1}(0^+)\right| > e^{-\beta(p+1)},\end{aligned}$$

for some $\eta \in (0, \varphi_a^k(1))$.

From this it follows that

$$-\beta(p+1) < (p+1)\log 3 + \log\left|\varphi_a^k(1)\right| = (p+1)\log 3 - r_0.$$

This implies (b).

Finally, let $t = \dfrac{s}{s-1}$.

$$\begin{aligned}[(\varphi_a^{p+1})'(\varphi_a^k(1))]^t &= \varphi_a'(\varphi_a^k(1))^t[(\varphi_a^p)'(\varphi_a^{k+1}(1))]^t \\ &\geq K_2\varphi_a^k(1)^s[(\varphi_a^p)'(\varphi_a^{k+1}(1))]^t \\ &\geq \frac{sK_2}{K_1}(1+\varphi_a^{k+1}(1))[(\varphi_a^p)'(\varphi_a^{k+1}(1))]^{\frac{1}{s-1}}\rho^{-1}(\varphi_a^p)'(\eta)\end{aligned}$$

where η is such that

$$\left|\varphi_a^p(-1) - \varphi_a^{p+k+1}(1)\right| = (\varphi_a^p)'(\eta)(\varphi_a^{k+1}(1)+1).$$

Following:

$$\begin{aligned}[(\varphi_a^{p+1})'(\varphi_a^k(1))]^t &\geq \frac{sK_2}{K_1}\left|\varphi_a^{p+k+1}(1) - \varphi_a^p(-1)\right|\rho^{-t}\lambda_1^{p/s-1} \\ &\geq \frac{sK_2}{K_1\rho^t}\exp(-\beta(p+1) + \frac{p}{s-1}\log\lambda_1) \\ &\geq \tau\exp\left(\frac{\log\lambda_1}{s-1} - \beta\right)(p+1).\end{aligned}$$

Finally, if β is small, the coefficient of $(p+1)$ in the exponential is positive. Then the last inequality in (c) follows by making δ small (this implies that r_0 is large, and so p is large).

The proof of lemma 2 is complete.

We say that a time j is *free* for the point 1 and the parameter a if $|\varphi_a^j(1)| > \delta$ and j does not belong to any binding period; j is a *return* for the point 1 and the parameter a if $|\varphi_a^j(1)| \leq \delta$ and j does not belong to any binding period.

Let $H_j^+(a) = \#\{i \leq j : i \text{ is free for 1 and } a\}$.

We will exclude the parameters a that don't satisfy the *free period assumption*:

$$H_j^+(a) \geq (1-\varepsilon_0)j, \quad H_j^-(a) \geq (1-\varepsilon_0)j \qquad \text{(FA)}$$

$H_j^-(a)$ is defined in the obvious way; ε_0 is a small positive constant.

Now we prove that for a parameter a not excluded by application of (BA) or (FA) until time k, the induction can be completed.

In fact, during the free periods, lemma 1 implies that the derivative has exponential growth at a rate of λ_0, while for binding periods, lemma 2 says that we don't have loose of derivative. So we obtain that $(\varphi_a^k)'(1) \geq \lambda_0^{H_k^+(a)} e^{-\alpha k} \geq e^{[(1-\varepsilon_0)\log\lambda_0 - \alpha]k} \geq \lambda_1^k$ where λ_1 can be taken as close to λ_0 as we wish by making α and ε_0 small.

The same estimates hold for the point -1.

Let E be the set of parameters never excluded. Then as we have just shown, E satisfies the first item of Theorem 2. Now it must be proved that 0 is a full density point of E.

Let $E_n^+(E_n^-)$ be the set of parameter values satisfying (BA) until n for the point 1 (resp. -1). Suppose that n is a return for some $a \in E_{n-1}^+$; then, according to the location of $\varphi_a^n(1)$, this parameter should be excluded or not. Those parameters in E_{n-1}^+ that are not excluded at time n will be divided into small intervals, so forming a partition of E_n^+ (for the detailed definition see [**BC1**], [**BC2**] or [**MV**]).

It can be proved, as in the mentioned papers, that:

$$\frac{(\varphi_a^k)'(1)}{(\varphi_b^k)'(1)} \leq B \tag{1}$$

for all $1 \leq k \leq n$, where a, b are in the same interval of the partition of E_{n-1}^+; the constant B depends only on α, β but not on n, a or b.

This fact is the principal reason for introducing the partitions. Now the measure of the set of parameters excluded by application of (BA) at step n, can be estimated:

Lemma 3. *$m(E_{n-1}^+ \backslash E_n^+) \leq Ce^{-\psi\alpha n} m(E_{n-1}^+)$, where the constant C depends only on α and β, and $\psi > 0$ is any number less than*

$$\frac{\log\lambda_1}{s} - \frac{s-1}{s}\beta > 0$$

(remember that this was the coefficient of p in the estimates obtained in part (c) of Lemma 2).

To prove this lemma a distortion property like (1) is needed for derivatives with respect to the parameter. This follows by putting (1) together with the general fact that under expansiveness, derivatives with respect to the parameter and with respect to the variable are similar. For this it is used (V.3).

To estimate the measure of the parameter values excluded by the application of (FA), Benedicks and Carleson introduce a large deviations argument and prove the following lemma, that can be translated to our family of maps without essential modifications.

Lemma 4. *There exists $\gamma_0 > 0$, an absolute constant, such that*

$$m\{a \in E_n^+ : H_n^+(a) < (1-\varepsilon_0)n\} \leq a_0 e^{-\gamma_0 \varepsilon_0 n}$$

where $a_0 = a_0(\delta)$ is given by Lemma 1.

For the proof of Lemma 4 we refer to [**BC2**].

Now we will conclude the proof that $m(E)$ is positive.

Let

$$\begin{aligned} F_n^+ &= E_n^+ \setminus \{a \in E_n^+ : H_n^+(a) < (1-\varepsilon_0)n\} \\ F_n^- &= E_n^- \setminus \{a \in E_n^- : H_n^-(a) < (1-\varepsilon_0)n\} \\ F_n &= F_n^+ \cap F_n^- \\ F^+ &= \cap_{n\geq 0} F_n^+ \\ F^- &= \cap_{n\geq 0} F_n^- \end{aligned}$$

Then the intersection of the F_n gives the set of parameters never excluded, that is, $E = \cap_{n\geq 0} F_n = F^+ \cap F^-$.

By Lemma 4, $m(E_n^+ \setminus F_n^+) \leq a_0 e^{-\gamma_0 \varepsilon_0 n}$.

By Lemma 3, $m(E_{n-1}^+ \setminus E_n^+) \leq C e^{-\psi \alpha n} \cdot a_0$.

Thus we obtain

$$\begin{aligned} m(F_{n-1}^+ \setminus F_n^+) &\leq m(E_{n-1}^+ \setminus F_n^+) \\ &\leq m(E_{n-1}^+ \setminus E_n^+) + m(E_n^+ \setminus F_n^+) \leq C_0 e^{-\gamma_1 n} a_0. \end{aligned} \tag{$*$}$$

where $C_0 > 0$ and $\gamma_1 > 0$ are independent of a_0 and n.

As $\varphi_0(1) = 1$ and $\varphi_0(-1) = -1$, it is easy to see that exists a natural-valued function N such that

- $N(a_0) \to \infty$ as $a_0 \to 0$
- $|\varphi_a^j(1)| > \delta$ and $|\varphi_a^j(-1)| > \delta$ for every $a \leq a_0$ and $j \leq N(a_0)$.

Thus $F_j^{\pm} = [0, a_0]$ for all $j \leq N(a_0)$.

Therefore, using $(*)$ it follows that:

$$\begin{aligned} m(F^+) &\geq m([0, a_0]) - \sum_{n \geq 1} m(F_{n-1}^+ \backslash F_n^+) \\ &= a_0 - \sum_{n \geq N+1} m(F_{n-1}^+ \backslash F_n^+) \\ &\geq a_0 \left(1 - \sum_{n \geq N+1} C_0 \varepsilon^{-\gamma_1 n}\right), \quad \text{where} \quad N = N(a_0) \end{aligned}$$

Now, as C_0 and γ_1 don't depend on a_0, we obtain that:

$$\frac{m(F^+)}{a_0} \longrightarrow 1 \quad \text{as} \quad a_0 \to 0.$$

The same can be said about F^-, then it follows that

$$\frac{m(E)}{a_0} \longrightarrow 1 \text{ as } \quad a_0 \to 0.$$

Finally, the constants C_0 and γ_0 depend only on the number α, β and δ.

Thus, a set E verifying the first and the last items of Theorem 2 has been founded.

It remains to prove the transitiveness of the maps φ_a for almost every $a \in E$. This was done in the last chapter of [**BC2**], where the density of the unstable manifold of a fixed point was used. Our transformations haven't fixed points for $a \neq 0$, but have two-periodic points with dense unstable manifold in $[-1, 1]$, and so the argument of [**BC2**] can be adapted.

References

[**ACT**] Arneodo, A., Coullet, P., Tresser, C. (1981), *A possible new mechanism for the onset of turbulence.* Physics Letter, 81A/4.

[**BC1**] Benedicks, M., Carleson, L. (1985), *On iterations of* $1 - ax^2$ on $(-1, 1)$. Ann. Math., 122.

[**BC2**] Benedicks, M., Carleson, L. (1991), *The dynamics of the Hénon map.* Ann. Math., 133.

[**G**] Guckenheimer (1979), *On sensitive dependence on initial conditions for one dimensional maps.* Comm. in Math. Phys., 70.

[**GW**] Guckenheimer, Williams (1979), *Structural stability of Lorenz attractors.* Publ. Math. IHES, 50.

[**HPS**] Hirsch, Pugh, Schub (1977), *Invariant manifold.* Springer Lecture notes, 583.

[**L**] Lorenz (1963), *Deterministic non-periodic flow.* Journal of Atmosphere Sciences, 20.

[**MV**] Mora, L., Viana, M., *Abundance of strange attractors.* To appear in Acta Mathematica.

[**M1**] Mañé, R. (1978), *Persistent manifolds are normally hyperbolic.* Trans. A.M.S., 246.

[**M2**] Mañé, R. (1985), (1987), *Hyperbolicity, sinks and measure in one dimensional dynamics.* Comm. in Math. Phys., 100, 495-524 and Erratum-Comm. in Math. Phys., 112.

[**R**] Richlik, M. R. (1990), *Lorenz attractor through Sil'nikov-type bifurcation.* Part I. Erg. th. and Dyn. Syst., 10.

[**S**] Singer (1978), *Stable orbits and bifurcations of maps of the internal.* SIAM. J. Appl. Math., 35.

Alvaro Rovella
Fac. Ingeneria - IMERL
Montevideo, Uruguay