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# RATIONAL SMOOTHNESS AND FIXED POINTS OF TORUS ACTIONS

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*Dedicated to the memory of Claude Chevalley*

**Abstract.** We obtain a criterion for rational smoothness of an algebraic variety with a torus action, with applications to orbit closures in flag varieties, and to closures of double classes in regular group completions.

## Introduction

For a complex algebraic group acting on a complex flag variety with finitely many orbits, the geometry of orbit closures is of importance in representation theory; the most interesting cases are Schubert varieties (in relation to category $\mathcal{O}$), and orbit closures of symmetric subgroups (in relation to Harish–Chandra modules), see e.g., [Ka].

In particular, it would be useful to characterize rationally smooth points of an orbit closure, i.e., those points where the local cohomology with constant coefficients is the same as for a point of a smooth variety.

Criteria for rational smoothness of Schubert varieties have been obtained by Kazhdan–Lusztig [KL1], [KL2] and then by Carrell–Peterson [C], Kumar [Ku] and Arabia [A]. The latter criteria hold, more generally, for varieties where a torus acts with isolated fixed points, such that all weights of the tangent space at such a fixed point are contained in an open half-space and have multiplicity one.

But that condition can fail for orbit closures of symmetric subgroups in flag varieties (e.g., for $\mathrm{SO}_n$ acting on the flag variety of $\mathrm{SL}_n$). In the present paper, we obtain a criterion for rational smoothness of varieties with a torus action, which applies to these orbit closures as well. Our main result can be stated as follows, in a somewhat weakened version.

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**Theorem (1.4).** *Let $X$ be a complex algebraic variety with an action of a torus $T$. Let $x \in X$ be an attractive fixed point of $T$, that is, all weights of $T$ in the tangent space $T_xX$ are contained in an open half-space. For a subtorus $T' \subset T$, let $X^{T'} \subset X$ be its fixed point set. Then we have*

$$\dim_x(X) \leq \sum_{T'} \dim_x(X^{T'})$$

*(sum over all subtori of codimension one), and this sum is finite.*

*Furthermore, $X$ is rationally smooth at $x$ if and only if the following conditions hold:*

(i) *A punctured neighborhood of $x$ in $X$ is rationally smooth.*

(ii) *For any subtorus $T' \subset T$ of codimension one, the fixed point subset $X^{T'}$ is rationally smooth at $x$.*

(iii) *We have $\dim_x(X) = \sum_{T'} \dim_x(X^{T'})$ (sum over all subtori of codimension one).*

Assume moreover that all weights in the tangent space $T_xX$ have multiplicity one. Then the subsets $X^{T'}$ identify with coordinate lines in $T_xX$, and the sum of their dimensions is the number $n(X,x)$ of closed irreducible $T$-stable curves through $x$. So we obtain $\dim_x(X) \leq n(X,x)$ with equality for rationally smooth $x$. This follows also from work of Carrell–Peterson (see [C] Theorem D), and Arabia [A], and will be generalized below (1.4 Corollary 2).

Consider now a connected semisimple group $G$, its flag variety $\mathcal{B}(G)$, and a symmetric subgroup $H \subset G$, that is, the fixed point subgroup of an involution $\theta$ of $G$. Let $T_H$ be a maximal torus of $H$, with centralizer $T$ in $G$. Then $T$ is a maximal torus of $G$, stable by $\theta$. The $T_H$-fixed points in $\mathcal{B}(G)$ are the (finitely many) $T$-fixed points, and the fixed points of subtori $T' \subset T_H$ of codimension one can be described completely in terms of the action of $\theta$ on roots of $(G,T)$ (2.5).

Then our main result leads to an inequality for the dimension of an $H$-orbit closure $X \subset \mathcal{B}(G)$, with equality if $X$ is rationally smooth at a $T_H$-fixed point (2.5); this generalizes a result of Springer [S2] concerning inner involutions. As an application, we characterize those $\mathrm{SO}_n$-orbit closures of codimension one in $\mathcal{B}(\mathrm{SL}_n)$, which are rationally smooth (2.5).

Actually, much of our analysis extends to any reductive subgroup $H \subset G$ having only finitely many orbits in $\mathcal{B}(G)$ (2.2, 2.3). However, such orbits need not admit an attractive "slice" (2.3), whereas orbits of a symmetric subgroup do admit such a slice, see [MS] 6.4.

Another application of our criterion is given in Section 3; it concerns double classes $BgB$ where $B$ is a Borel subgroup of a connected reductive group $G$, and their closures $\overline{BgB}$ in a smooth $(G\times G)$-equivariant completion of $G$ which is regular in the sense of [BDP]. We show in 3.1 that these closures admit attractive slices at all points, and that they are rationally smooth in codimension two. This generalizes classical results for Schubert varieties [KL1].

However, closures of double classes are not rationally smooth, apart from very few cases (3.3). And almost all closures of double classes are singular in codimension two (see [B1] Corollary 2.2).

Although our results are stated for complex algebraic varieties, our arguments adapt to the case of an algebraically closed field of any characteristic, with rational cohomology replaced by $l$-adic cohomology. This makes the exposition rather heavy in several places. An appendix collects results on rational smoothness and on torus actions, for which we did not find suitable references.

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## 1. A criterion for rational smoothness

### 1.1. Necessary conditions

In what follows, we consider complex algebraic varieties, that is, separated reduced schemes of finite type over $\mathbf{C}$. With this convention, varieties need not be irreducible. For such a variety $X$, we denote by $H^*(X)$ cohomology of $X$ with rational coefficients. For a point $x \in X$, we denote by

$$H_x^*(X) := H^*(X, X - \{x\})$$

cohomology with support in $\{x\}$, and by $\dim_x(X)$ the dimension of the local ring of $X$ at $x$.

**Definition.** $X$ is *rationally smooth at* $x$ if, for all $y$ in a neighborhood of $x$ in the complex topology, $H_y^m(X)$ is zero for all $m \neq 2\dim_x(X)$, and $H_y^{2\dim_x(X)}(X)$ is isomorphic to $\mathbf{Q}$.

If $X$ is rationally smooth at a point $x$, then it is irreducible at that point (see Proposition A1). The set of rationally smooth points is open for the complex topology, and contains all smooth points. More generally, quotients of smooth varieties by finite groups are rationally smooth (see e.g., Proposition A1). Other examples of rationally smooth varieties are unibranched curves.

We shall obtain necessary conditions for rational smoothness of a variety $X$ at a fixed point of an algebraic action of a torus $T$ (that is, $T$ is an algebraic group isomorphic to a product of copies of the multiplicative group $\mathbf{G}_m$). We shall always assume that $X$ is covered by open affine $T$-stable subsets. By [Su], this assumption holds for $T$-stable subvarieties of normal $T$-varieties.

**Theorem.** *Let $T$ be a torus acting on a variety $X$ with a fixed point $x$. If $X$ is rationally smooth at $x$, then, for each subtorus $T' \subset T$, the fixed point set $X^{T'}$ is rationally smooth at $x$. Furthermore, we have*

$$\dim_x(X) - \dim_x(X^T) = \sum_{T'} (\dim_x(X^{T'}) - \dim_x(X^T))$$

*(sum over all subtori $T' \subset T$ of codimension one).*

*Proof.* We use equivariant cohomology (see e.g., [H]) which we briefly review. Let $E_T \to B_T$ be a universal principal bundle for $T$. Then $T$ acts diagonally on $X \times E_T$ with a quotient denoted by $X \times_T E_T$. Let

$$H_T^*(X) := H^*(X \times_T E_T)$$

be the $T$-equivariant cohomology ring of $X$ with rational coefficients. The map $X \times_T E_T \to E_T/T = B_T$ is a fibration with fiber $X$, and $B_T$ is simply connected. Thus, there is a spectral sequence $H^p(B_T) \otimes H^q(X) \Rightarrow H_T^{p+q}(X)$ and $H_T^*(X)$ is a module over $H^*(B_T)$. The latter is the symmetric algebra of the character group of $T$, where each character has degree 2. The inclusion $i_T : X^T \to X$ induces a $H^*(B_T)$-linear map

$$i_T^* : H_T^*(X) \to H_T^*(X^T) \cong H^*(B_T) \otimes H^*(X^T).$$

By the localization theorem (see [H] or Proposition A6), $i_T^*$ becomes an isomorphism after inverting all nontrivial characters of $T$.

Let $y \in X^T$. Denote by $H_{T,y}^*(X) := H^*(X \times_T E_T, (X - \{y\}) \times_T ET)$ the equivariant cohomology of $X$ with support in $\{y\}$, and consider the map $i_{T,y}^* : H_{T,y}^*(X) \to H_{T,y}^*(X^T) = H^*(B_T) \otimes H_y^*(X^T)$. Applying the localization theorem to $X$ and $X - \{y\}$, we see that $i_{T,y}^*$ is an isomorphism after inverting all nontrivial characters. On the other hand, because $X$ is rationally smooth at $x$, the spectral sequence $H^p(B_T) \otimes H_y^q(X) \Rightarrow H_{T,y}^{p+q}(X)$ degenerates for all $y$ in a neighborhood of $x$ in $X^T$. Thus, $H_{T,y}^*(X)$ is a free $H^*(B_T)$-module generated by an element of degree $2\dim_y(X) = 2\dim_x(X)$. It follows that the space $H_y^*(X^T)$ is one dimensional, and hence that $X^T$ is rationally smooth at $x$ e.g., by Proposition A1 (this can also be deduced from Smith theory; see [Br] Chapter III, Corollary 10.11). Furthermore, identifying the $H^*(B_T)$-modules $H_{T,x}^*(X)$ and $H_{T,x}^*(X^T)$ with $H^*(B_T)$, the map $i_{T,x}^*$ becomes multiplication by a homogeneous element $f \in H^*(B_T)$ of degree $2\dim_x(X) - 2\dim_x(X^T)$. By the localization theorem, $f$ is a scalar multiple of a product of characters.

Let $\chi$ be a primitive character dividing $f$, and let $T'$ be the kernel of $\chi$, a subtorus of $T$ of codimension one. Then $i_T : X^T \to X$ factors as $i_{T,T'} : X^T \to X^{T'}$ followed by $i_{T'} : X^{T'} \to X$. By the localization theorem again, the map $i_{T',x}^* : H_{T,x}^*(X) \to H_{T,x}^*(X^{T'})$ becomes an isomorphism after inverting all characters of $T$ which restrict nontrivially to $T'$, i.e., which are not multiples of $\chi$. Furthermore, $X^{T'}$ is rationally smooth at $x$ by the first step of the proof. Thus, we can identify $H^*_{T,x}(X^{T'})$ with $H^*(B_T)$; then $i_{T',x}$ identifies with multiplication by a product of characters which are not multiples of $\chi$.

Choose a subtorus $T'' \subset T$ of dimension one such that the product map $T' \times T'' \to T$ is an isomorphism. Then the character group of $T''$ is generated by the restriction of $\chi$. Furthermore, we can take $E_T = E_{T'} \times E_{T''}$, then $X^{T'} \times_T E_T \cong B_{T'} \times (X^{T'} \times_{T''} E_{T''})$ and $X^T \times_T E_T \cong B_{T'} \times (X^T \times_{T''} E_{T''})$. Thus, we have isomorphisms

$$H^*_{T,x}(X^{T'}) \cong H^*(B_{T'}) \otimes H^*_{T'',x}(X^{T'}),\ H^*_{T,x}(X^T) \cong H^*(B_{T'}) \otimes H^*_{T'',x}(X^T)$$

compatible with $i^*_{T,T',x}$. Applying the localization theorem to the $T''$-variety $X^{T'}$, it follows that $i^*_{T,T',x} : H^*_{T,x}(X^{T'}) \to H^*_{T,x}(X^T)$ is an isomorphism after inverting $\chi$. In other words, $i^*_{T,T',x}$ identifies with multiplication by a power $\chi^{n_\chi}$, and $f$ is divisible by $\chi^{n_\chi}$ but not by $\chi^{n_\chi+1}$. Taking degrees, we obtain $2n_\chi = 2\dim_x(X^{T'}) - 2\dim_x(X^T)$. Now $f$ is a scalar multiple of $\prod_\chi \chi^{n_\chi}$ (product over all primitive characters) and our relation on dimensions follows by taking degrees. $\square$

## 1.2. An inequality for dimensions of fixed points

Let $X$ be a variety with an action of a torus $T$ and a fixed point $x$. In general, there is no inequality between $\dim_x(X) - \dim_x(X^T)$ and the sum (over all subtori of codimension one) $\sum_{T'}(\dim_x(X^{T'}) - \dim_x(X^T))$, as shown by the following

**Example.** Let $X$ be the hypersurface in $\mathbf{A}^4$ with equation $xy + zt = 0$. Let $T = \mathbf{G}_m \times \mathbf{G}_m$ act on $\mathbf{A}^4$ by $(u,v)\cdot(x,y,z,t) = (ux, u^{-1}y, vz, v^{-1}t)$. Then $X$ is $T$-stable and the origin is the unique fixed point (and the unique singular point as well). The nontrivial subsets $X^{T'}$ are: $xy = z = t = 0$ for $T' = \{1\} \times \mathbf{G}_m$, and $x = y = zt = 0$ for $T' = \mathbf{G}_m \times \{1\}$. Thus, $\sum_{T'} \dim_x(X^{T'}) = 2$, whereas $\dim_x(X) = 3$.

On the other hand, consider the action of $T = \mathbf{G}_m \times \mathbf{G}_m$ on $\mathbf{A}^4$ by $(u,v)\cdot(x,y,z,t) = (u^3x, v^3y, u^2vz, uv^2t)$. Then again $X$ is $T$-stable and the origin is the unique fixed point; but now the $X^{T'}$ are the four coordinate lines, whence $\sum_{T'} \dim_x(X^{T'}) = 4$.

However, we shall obtain an upper bound for $\dim_x(X) - \dim_x(X^T)$ in terms of certain subsets of the $X^{T'}$. Observe that $T$ acts on $X^{T'}$ through its quotient $T/T'$ which we can identify with $\mathbf{G}_m$. Denote by $X^{T'}_+(x)$ (resp. $X^{T'}_-(x)$) the set of all $y \in X^{T'}$ such that $x$ is the limit of $ty$ as $t \to 0$ (resp. $t^{-1} \to 0$) where $t \in \mathbf{G}_m$. Then both $X^{T'}_+(x)$ and $X^{T'}_-(x)$ are locally closed $T$-stable subsets of $X^{T'}$, and $x$ is their unique common point.

**Theorem.** *Let $X$ be a $T$-variety with a fixed point $x$. Then, notation being as above, there are only finitely many subtori $T' \subset T$ of codimension one such that $X^{T'} \neq X^T$, and we have*

$$\dim_x(X) - \dim_x(X^T) \le \sum_{T'} (\dim_x X_+^{T'}(x) + \dim_x X_-^{T'}(x))$$

*(sum over all subtori of codimension one).*

*If moreover $X^{T'}$ is smooth at $x$, then*

$$\dim_x X_+^{T'}(x) + \dim_x X_-^{T'}(x) = \dim_x(X^{T'}) - \dim_x(X^T).$$

*In particular, if each $X^{T'}$ is smooth at $x$, then*

$$\dim_x(X) - \dim_x(X^T) \le \sum_{T'} (\dim_x(X^{T'}) - \dim_x(X^T)).$$

*Proof.* We may assume that $X$ is affine; then $X$ admits a closed equivariant embedding into a $T$-module $M$, which maps $x$ to 0. Because $M_+^{T'}(0)$ is a linear subspace of $M$, it follows that $X_+^{T'}(x) = X \cap M_+^{T'}(0)$ is closed in $X$. By definition, $X_+^{T'}(x)$ contains a unique closed orbit of $T/T'$ : the fixed point $x$. Thus, there exists a $T/T'$-module $V_+^{T'}$ and an equivariant finite surjective morphism $\pi_+^{T'} : X_+^{T'}(x) \to V_+^{T'}$ such that $\pi_+^{T'}(x) = 0$ (by a version of Noether normalization lemma, see e.g., Proposition A3). Because $X_+^{T'}(x)$ is $T$-stable and closed in $X$, we can extend $\pi_+^{T'}$ to an equivariant morphism $p_+^{T'} : X \to V_+^{T'}$. Similarly, we have $p_-^{T'} : X \to V_-^{T'}$.

Observe that there are only finitely many subtori $T' \subset T$ of codimension one, such that $V_\pm^{T'}$ is nonzero: indeed, such a subtorus is contained in the kernel of a weight of $T$ in the tangent space $T_xX$. Let $V$ denote the product of all the $V_\pm^{T'}$, and let $p : X \to V$ be the product morphism; then $p(X^T) = \{0\}$ because $p(x) = 0$ and $V^T = \{0\}$ by construction.

Shrinking $X$, we may assume that each irreducible component of $X^T$ contains $x$; in particular, $X^T$ is connected. Then we claim that $X^T$ is a connected component of the fiber $p^{-1}(0)$. Otherwise, there exists an closed irreducible $T$-stable curve $C \subset p^{-1}(0)$ such that $x$ is an isolated fixed point of $C$ (see e.g., Proposition A4). Then $T$ acts on $C$ through some nontrivial character $\chi$. Thus, $C$ is contained in $X^{T'}$ where $T' \subset T$ is the connected kernel of $\chi$. Furthermore, because $T$ acts nontrivially on $C$, this curve must be contained in $X_+^{T'}(x) \cup X_-^{T'}(x)$. But then $p(C)$ has dimension one, by construction of $p$.

From the claim, it follows that $\dim_x(X) \le \dim_x p^{-1}(0) + \dim(V) = \dim_x(X^T) + \dim(V) = \dim_x(X^T) + \sum_{T'} (\dim(V_+^{T'}) + \dim(V_-^{T'})) = \dim_x(X^T) + \sum_{T'} (\dim_x X_+^{T'}(x) + \dim_x X_-^{T'}(x))$. If moreover $X^{T'}$ is smooth at $x$, then there exists an equivariant morphism $f : X^{T'} \to T_x(X^{T'})$, $x \mapsto 0$ which is étale at $x$. It follows that $X_+^{T'}(x)$, $X_-^{T'}(x)$ and $X^T$ are smooth at $x$, and that $\dim_x(X^{T'}) = \dim T_x(X^{T'}) = \dim T_x(X^{T'})_+ + \dim T_x(X^{T'})_- + \dim T_x(X^{T'})^T = \dim_x X_+^{T'}(x) + \dim_x X_-^{T'}(x) + \dim_x(X^T)$. Here $T_x(X^{T'})_\pm$ denotes the sum of all $T/T'$-eigenspaces of $T_x(X^{T'})$ with eigenvalues of the corresponding sign. $\square$

### 1.3. Attractive fixed points

We recall the notion of an attractive fixed point, and we obtain preliminary results for rational smoothness at such a point, using ideas from the Appendix in [KL1].

**Definition.** Let $T$ be a torus acting on a variety $X$. A fixed point $x$ is called *attractive* if all weights of $T$ in the tangent space $T_xX$ are contained in an open half space.

By Proposition A2, the point $x$ is attractive if and only if there exists a one parameter subgroup $\lambda : \mathbf{G}_m \to T$ such that $\lim_{t\to 0} \lambda(t)y = x$ for all $y$ in a neighborhood of $x$. Furthermore, the set of all $y \in X$ such that $\lim_{t\to 0} \lambda(t)y = x$ is an open affine $T$-stable neighborhood of $x$. Replacing $X$ by this neighborhood, we may assume that $X$ is affine; then $X$ admits a closed $T$-equivariant embedding into $T_xX$.

Set

$$\dot{X} := X - \{x\}.$$

Choose an injective one-parameter subgroup $\lambda : \mathbf{G}_m \to T$ as above. Then all weights of the $\mathbf{G}_m$-action on $T_xX$ via $\lambda$ are positive. Thus, the quotient

$$\mathbf{P}(X) := \dot{X}/\mathbf{G}_m$$

exists and is a projective variety: indeed, it is a closed subvariety of $\mathbf{P}(T_xX)$, a weighted projective space. We can view $\mathbf{P}(X)$ as an algebraic version of the link of $X$ at $x$.

Because the set of rationally smooth points is $T$-stable and open for the complex topology, $X$ is rationally smooth at $x$ if and only if it is rationally smooth everywhere. This condition can be read on $\mathbf{P}(X)$, as follows.

**Lemma.** *Let $X$ be an affine $T$-variety with an attractive fixed point $x$ such that $\dot{X}$ is rationally smooth. Then $\mathbf{P}(X)$ is rationally smooth as well. Furthermore, $X$ is rationally smooth if and only if $\mathbf{P}(X)$ is a rational cohomology complex projective space.*

*Proof.* Observe that $\mathbf{G}_m$ acts on $\dot{X}$ with finite isotropy groups. By Proposition A5, it follows that $\dot{X}$ is covered by $\mathbf{G}_m$-stable open subsets $U$ admitting an equivariant morphism $p : U \to \mathbf{G}_m/\Gamma$ where $\Gamma \subset \mathbf{G}_m$ is a finite subgroup (depending on $U$). Let $Y$ be the fiber of $p$ at the base point of $\mathbf{G}_m/\Gamma$. Then $Y \subset X$ is a locally closed $\Gamma$-stable subvariety, and $U$ is equivariantly isomorphic to the quotient $(\mathbf{G}_m \times Y)/\Gamma$ where $\Gamma$ acts diagonally on $\mathbf{G}_m \times Y$. Thus, $\mathbf{P}(X)$ is covered by the quotients $Y/\Gamma$. Because $\dot{X}$ is rationally smooth and the map $\mathbf{G}_m \times Y \to X : (t, y) \mapsto ty$ is étale, $\mathbf{G}_m \times Y$ is rationally smooth, too (see e.g., Proposition A1). Thus, $Y$ is rationally smooth, and so is the quotient $Y/\Gamma$ by Proposition A1 again. Therefore, $\mathbf{P}(X)$ is rationally smooth.

We claim that $X$ is rationally smooth at $x$ if and only if

$$H^m(\dot{X}) = \begin{cases} \mathbf{Q} & \text{if } m = 0 \text{ or } m = 2d-1, \\ 0 & \text{otherwise,} \end{cases}$$

where $d = \dim_x(X)$. Indeed, the action of $\mathbf{G}_m$ on $X$ extends to a map $\mathbf{A}^1 \times X \to X$ sending $0 \times X$ to $x$, and restricting to the identity $1 \times X \to X$. Thus, $H^m(X) = 0$ for all $m > 0$. Now our claim follows from the long exact sequence

$$\ldots \to H^m(X) \to H^m(\dot{X}) \to H_x^{m+1}(X) \to H^{m+1}(X) \to \ldots.$$

Denote by $\pi : \dot{X} \to \mathbf{P}(X)$ the quotient map and let $\mathbf{Q}_{\dot{X}}$ be the constant sheaf on $\dot{X}$ associated with $\mathbf{Q}$. We compute the higher direct images $R^i\pi_*\mathbf{Q}_{\dot{X}}$. For this, consider the commutative square

$$\begin{array}{ccc} \mathbf{G}_m \times Y & \to & Y \\ \downarrow & & \downarrow \\ (\mathbf{G}_m \times Y)/\Gamma & \to & Y/\Gamma \end{array}$$

where $Y$ and $\Gamma$ are as above, and the downwards maps $p$, $q$ are quotients by $\Gamma$. Because $p$ and $q$ are finite, we have $R^i\pi_*(p_*^\Gamma \mathbf{Q}_{\mathbf{G}_m \times Y}) = q_*^\Gamma(R^i\mathrm{pr}_{Y*}\mathbf{Q}_{\mathbf{G}_m\times Y})$ where $p_*^\Gamma$, $q_*^\Gamma$ denote invariant direct image. But both $\mathrm{pr}_{Y*}\mathbf{Q}_{\mathbf{G}_m\times Y}$ and $R^1\mathrm{pr}_{Y*}\mathbf{Q}_{\mathbf{G}_m\times Y}$ are isomorphic to $\mathbf{Q}_Y$, and $R^i\mathrm{pr}_{Y*}\mathbf{Q}_{\mathbf{G}_m\times Y}$ vanishes for $i \geq 2$. Furthermore, $q_*^\Gamma\mathbf{Q}_Y$ is isomorphic to $\mathbf{Q}_{Y/\Gamma}$ via $q^*$, and a similar statement holds for $p_*^\Gamma$. It follows that $\pi_*\mathbf{Q}_{\dot{X}}$ and $R^1\pi_*\mathbf{Q}_{\dot{X}}$ are isomorphic to $\mathbf{Q}_{\mathbf{P}(X)}$, and that $R^i\pi_*\mathbf{Q}_{\dot{X}} = 0$ for $i \geq 2$. Thus, the Leray spectral sequence for $\pi$ reduces to a Gysin long exact sequence

$$\ldots \to H^m(\dot{X}) \to H^{m-1}(\mathbf{P}(X)) \to H^{m+1}(\mathbf{P}(X)) \to H^{m+1}(\dot{X}) \to \ldots.$$

Together with the claim, this concludes the proof. □

### 1.4. A characterization of rational smoothness at an atttractive fixed point

We obtain our main result stated in the introduction, and some useful variants as well.

**Theorem.** *Let $X$ be a $T$-variety with an attractive fixed point $x$. Then we have $\dim_x(X) \leq \sum_{T'} \dim_x(X^{T'})$ (sum over all subtori of codimension one). Furthermore, $X$ is rationally smooth at $x$ if and only if the following conditions hold:*

(i) *A punctured neighborhood of $x$ in $X$ is rationally smooth.*

(ii) *$X^{T'}$ is rationally smooth at $x$ for each subtorus $T' \subset T$ of codimension one.*

(iii) $\dim_x(X) = \sum_{T'} \dim_x(X^{T'})$.

*Proof.* The first assertion follows from Theorem 1.2: because $x$ is attractive, each $X^{T'}$ is equal to $X_+^{T'}$ or to $X_-^{T'}$ in a neighborhood of $x$. If $X$ is rationally smooth at $x$, then (i) certainly holds, and (ii), (iii) follow from Theorem 1.1. Another proof of this result, and of the converse as well, is sketched in [B2]. We reproduce this proof with some changes, so that it adapts to arbitrary characteristic.

We may assume that $X$ is affine, and we use the notation and results of 1.3. Observe that the $T$-action on $X$ induces an action on $\mathbf{P}(X)$, with fixed point set the disjoint union of the $\mathbf{P}(X^{T'})$. Indeed, $T$-fixed points in $\mathbf{P}(X)$ correspond to $T$-orbits of dimension one in $\dot{X}$.

Assume that (i), (ii) and (iii) hold. Then we claim that the rational cohomology of $\mathbf{P}(X)$ vanishes in odd degrees, and that the topological Euler characteristic $\chi(\mathbf{P}(X))$ is equal to $\dim_x(X) := d$. To check this, we use equivariant cohomology again. Notation being as in the proof of Theorem 1.1, the map $\mathbf{P}(X) \times_T E_T \to E_T/T = B_T$ is a fibration with fiber $\mathbf{P}(X)$. Because the latter is projective and rationally smooth, the associated spectral sequence degenerates (by the criterion of Deligne, see e.g., [J] Proposition 13). Thus, the $H^*(B_T)$-module $H_T^*(\mathbf{P}(X))$ is free, and $H^*(\mathbf{P}(X))$ is the quotient of $H_T^*(\mathbf{P}(X))$ by the ideal generated by all characters of $T$.

By the localization theorem in equivariant cohomology (see e.g., [H] Chapter III, or Proposition A6), the $H^*(B_T)$-module $H_T^*(\mathbf{P}(X))$ becomes isomorphic to $H_T^*(\mathbf{P}(X)^T) = H^*(B_T) \otimes H^*(\mathbf{P}(X)^T) = \bigoplus_{T'} H^*(B_T) \otimes H^*(\mathbf{P}(X^{T'}))$ after inverting all nontrivial characters of $T$. Furthermore, by the preceding discussion and rational smoothness of the $X^{T'}$, each $H^*(\mathbf{P}(X^{T'}))$ is a rational cohomology projective space; in particular, its cohomology vanishes in odd degrees. Because $H^*(B_T)$ vanishes in odd degrees too, it follows that the same holds for $H_T^*(\mathbf{P}(X))$, and for $H^*(\mathbf{P}(X))$ as well. Furthermore, we have for the Euler characteristic of $\mathbf{P}(X)$: $\chi(\mathbf{P}(X)) = \mathrm{rank}_{H^*(B_T)} H_T^*(\mathbf{P}(X)) = \mathrm{rank}_{H^*(B_T)} H_T^*(\mathbf{P}(X)^T) = \sum_{T'} \chi(\mathbf{P}(X^{T'})) = \sum_{T'} \dim(X^{T'}) = d$, which proves our claim.

Because $\mathbf{P}(X)$ is projective of dimension $d-1$, it has nontrivial rational cohomology in degrees $0, 2, \ldots, 2(d-1)$. Thus, the claim implies that $\mathbf{P}(X)$ is a rational cohomology complex projective space of dimension $d-1$, so that $X$ is rationally smooth at $x$.

Conversely, assume that $X$ is rationally smooth at $x$. Then, reversing the previous arguments, we see that rational cohomology of each $\mathbf{P}(X^{T'})$ vanishes in odd degree, and that $d = \sum_{T'} \chi(\mathbf{P}(X^{T'}))$. Because $\mathbf{P}(X^{T'})$ is a projective algebraic variety of dimension $\dim_x(X^{T'}) - 1$, it follows that $\chi(\mathbf{P}(X^{T'})) \geq \dim_x(X^{T'})$. Thus, we have $d \geq \sum_{T'} \dim_x(X^{T'})$. But the reverse inequality holds, as a consequence of Theorem 1.2, so we must have $d = \sum_{T'} \dim_x(X^{T'})$, $\chi(\mathbf{P}(X^{T'})) = \dim_x(X^{T'})$ for all $T'$. It follows that each $\mathbf{P}(X^{T'})$ is a rational cohomology projective space, and that $X^{T'}$ is rationally smooth at $x$. $\square$

The arguments above also lead to the following

**Corollary 1.** *Let $T$ be a torus acting on an irreducible variety $X$ of dimension two; let $x \in X$ be an attractive fixed point, contained in only finitely many closed irreducible $T$-stable curves. Then $X$ is rationally smooth at $x$.*

*Proof.* We may assume that $X$ is affine and that $T$ acts faithfully. Then $\dim(T) = 2$ (otherwise there are infinitely many closed irreducible $T$-stable curves through $x$, namely, the $T$-orbit closures). Thus, $X$ contains a dense $T$-orbit; in other words, the normalization of $X$ is an affine toric surface. It follows that $X$ contains exactly four $T$-orbits: the fixed point $x$, two orbits of dimension one, and the open orbit.

Thus, $\mathbf{P}(X)$ is a projective irreducible curve with a dense $T$-orbit, so that $\mathbf{P}(X)$ is homeomorphic to the projective line. Furthermore, $\dot{X}$ is covered by two affine open subsets of the form $\mathbf{G}_m \times_\Gamma C$ where $\Gamma$ is a finite group, and $C$ is an irreducible affine curve admitting a nontrivial action of $\mathbf{G}_m$ (this follows e.g., from Proposition A5). Thus, $C$ is unibranched, and $\dot{X}$ is rationally smoth. By Lemma 1.3, $X$ is rationally smooth as well. $\square$

As another consequence of (the proof of) Theorem 1.4, let us derive the following refinement of a result due to Carrell and Peterson [C] Theorem D.

**Corollary 2.** *Let $T$ be a torus acting on a variety $X$ with an isolated fixed point $x$, such that the number of closed irreducible $T$-stable curves through $x$ is finite; denote this number by $n(X,x)$. Then $\dim_x(X) \leq n(X,x)$. If moreover $X$ is rationally smooth at $x$, then $\dim_x(X) = n(X,x)$ and each closed irreducible $T$-stable curve through $x$ is exactly the fixed point set of a subtorus of codimension one in $T$.*

*Conversely, if $x$ is attractive and admits a rationally smooth punctured neighborhood, and if $\dim_x(X) = n(X,x)$, then $X$ is rationally smooth at $x$.*

*Proof.* We may assume that $X$ is affine and that $X^T = \{x\}$. Observe that each closed irreducible $T$-stable curve in $X$ is fixed pointwise by a unique subtorus $T' \subset T$ of codimension one. Furthermore, $X^{T'}$ contains only finitely closed irreducible $T$-stable curves through $x$, and all such curves must be contained in $X_+^{T'}(x) \cup X_-^{T'}(x)$. Thus, the dimension of both $X_+^{T'}(x)$ and $X_-^{T'}(x)$ is at most one, and $\dim X_+^{T'}(x) + \dim X_-^{T'}(x)$ is at most the number of closed irreducible $T$-stable curves through $x$ in $X^{T'}$. Now the inequality $\dim_x(X) \leq n(X,x)$ follows from Theorem 1.2.

If $X$ is rationally smooth at $x$, then each $X^{T'}$ is rationally smooth at $x$ as well, by Theorem 1.1. Thus, $X^{T'}$ is irreducible at $x$. It follows that the connected component of $x$ in $X^{T'}$ is either $\{x\}$ or a closed irreducible $T$-stable curve through $x$. Now the equality $\dim_x(X) = n(X,x)$ follows from Theorem 1.1.

For the converse, we argue as in the proof of Theorem 1.4: the $T$-fixed points in $\mathbf{P}(X)$ correspond to $T$-orbits of dimension one in $X$, that is, to closed irreducible $T$-stable curves through $x$. Thus, the number of $T$-fixed points in $\mathbf{P}(X)$ is $\dim_x(X) = \dim \mathbf{P}(X) + 1$. It follows that $\mathbf{P}(X)$ is a rational cohomology complex projective space. $\square$

*Remark.* The assumption that $x$ admits a rationally smooth punctured neighborhood cannot be omitted, as shown by the following example.

Let $X$ be the hypersurface in $\mathbf{A}^5$ with equation $x^2 + yz + xtw = 0$. Then $X$ is irreducible, with singular locus $x = y = z = tw = 0$, a union of two lines meeting at the origin. Let $T = \mathbf{G}_m \times \mathbf{G}_m$ act on $\mathbf{A}^5$ by $(u, v) \cdot (x, y, z, t, w) = (u^2v^2x, u^3vy, uv^3z, u^2t, v^2w)$. Then the origin of $\mathbf{A}^5$ is an attractive fixed point, $X$ is $T$-stable of dimension four, and $X$ contains four closed irreducible $T$-stable curves: the coordinate lines, except for the $x$-axis. But $X$ is not rationally smooth at the origin. To see this, consider the action of $\mathbf{G}_m$ on $\mathbf{A}^5$ by $u \cdot (x, y, z, t, w) = (x, uy, u^{-1}z, t, w)$. Then $X$ is $\mathbf{G}_m$-stable and $X^{\mathbf{G}_m}$ is defined by $y = z = x^2 + xtw = 0$. Thus, $X^{\mathbf{G}_m}$ is reducible at the origin. By Theorem 1.1, $X$ is not rationally smooth.

## 2. Rational smoothness of orbit closures in flag varieties

### 2.1. Attractive slices

We shall apply our criterion of rational smoothness to certain orbit closures. For this, we need the following notion, a variant of [MS] 2.3.2.

**Definition.** Let $X$ be a variety with an action of a linear algebraic group $H$ and let $x \in X$. A *slice* to the orbit $Hx$ at $x$ is a locally closed affine subvariety $S \subset X$ which satisfies the following conditions:

(a) $x$ is an isolated point of $S \cap Hx$.

(b) $S$ is stable under a maximal torus $T$ of the isotropy group $H_x$.

(c) The morphism

$$\begin{array}{rccc} \alpha: & H \times S & \to & X \\ & (h, s) & \mapsto & hs \end{array}$$

is smooth at $(1, x)$.

The slice $S$ is *attractive* if

(d) $x$ is an attractive fixed point for the $T$-action on $S$.

It is easy to see that there always exists a slice $S$. If moreover $S$ is attractive, then $S \cap Hx = \{x\}$ and the morphism $\alpha$ is smooth everywhere.

**Proposition.** *Let $X$ be a variety with an action of a linear algebraic group $H$, let $x \in X$ and let $S$ be a slice to $Hx$ at $x$. If $X$ is rationally smooth at $x$ (and hence at all points of $Hx$) and if $x$ is an isolated $T$-fixed point of $S$, then the $T$-variety $S$ satisfies conditions* (i), (ii) *and* (iii) *of Theorem* 1.4 *at $x$. The converse holds if the $T$-variety $S$ is attractive.*

*Proof.* The map $\alpha$ is $H$-equivariant; thus, it is smooth at all points $(h, x)$ where $h \in H$, and the image of $\alpha$ is a neighborhood of $Hx$ in $X$. Using Proposition A1, we see that $X$ is rationally smooth along $Hx$ if and only if

$S$ is rationally smooth at $x$. Now the first assertion follows from Theorem 1.1, and the second one from Theorem 1.4. $\square$

As a first application, we give a direct proof of a criterion for rational smoothness of Schubert varieties, obtained by Carrell and Peterson using Kazhdan–Lusztig theory (see [C] Theorem E).

Let $G$ be a connected semisimple group, $B \subset G$ a Borel subgroup, and $T \subset B$ a maximal torus with Weyl group $W$. The $T$-fixed points in the flag variety $G/B$ are indexed by $W$. For $w \in W$ we still denote by $w$ the corresponding fixed point, and by $X(w) = \overline{BwB}/B$ the corresponding Schubert variety; then the dimension of $X(w)$ is the length of $w$, denoted by $\ell(w)$. Let $x \in W$. Then $x \in X(w)$ if and only if $x \leq w$ for the Bruhat ordering on $W$.

We now recall the construction of slices to Schubert varieties, and the description of their $T$-stable curves. By the Bruhat decomposition, the map

$$\begin{array}{ccc} (U \cap xU^-x^{-1}) \times (U^- \cap xU^-x^{-1}) & \to & \mathcal{B}(G) \\ (g,h) & \mapsto & ghx \end{array}$$

is an open immersion, and its restriction

$$\begin{array}{ccc} U \cap xU^-x^{-1} & \to & Bx \\ g & \mapsto & gx \end{array}$$

is an isomorphism. Set

$$S := X(w) \cap (U^- \cap xU^-x^{-1})x.$$

Then $S$ is a $T$-stable attractive slice to $Bx$ at $x$ in $X(w)$.

Let $R \subset W$ be the set of reflections. For $r \in R$, let $T^r$ be its fixed point set in $T$, and let $G_r$ be the derived subgroup of the centralizer $G^{T^r}$; then $G_r$ is a connected semisimple group of rank one. Set

$$C(x,r) := G_r x.$$

Then the $C(x,r)$ $(r \in R)$ are the closed irreducible $T$-stable curves through $x$ in $G/B$. Furthermore, $rx \leq w$ if and only if $C(x,r)$ is contained in $X(w)$. More precisely, we have $x < rx \leq w$ (resp. $rx < x$) if and only if $C(x,r) \subset S$ (resp. $C(x,r) \subset Bx$); see [C] Theorem F.

Now, combining the proposition above with Corollary 1.4.2, we obtain the following

**Corollary.** *Let $x$, $w$ in $W$ such that $x \leq w$, and let $n(x,w)$ be the number of $r \in R$ such that $rx \leq w$. Then $l(w) \leq n(x,w)$. Furthermore, $X(w)$ is rationally smooth at $x$ if and only if $l(w) = n(y,w)$ for all $y \in W$ such that $x \leq y < w$.*

The first part of this result was conjectured by Deodhar and proved by Carrell–Peterson (see [C] Theorem A), Dyer [D] and Polo [Po]; the second part is due to Carrell–Peterson (see [C] Theorem E).

### 2.2. Orbits of spherical subgroups in flag varieties

We still consider a connected semisimple group $G$ and we denote by $\mathcal{B}(G)$ its flag variety. Let $H \subset G$ be a spherical subgroup, that is, $\mathcal{B}(G)$ contains only finitely many $H$-orbits. Let $H^0$ be the connected component of 1 in $H$; then $H^0$ is spherical in $G$, too.

Easy but useful properties of $H$-orbits in $\mathcal{B}(G)$ are given by the following

**Proposition.** (i) *Each closed orbit is isomorphic to a finite union of copies of the flag variety $\mathcal{B}(H^0)$.*

(ii) *Let $X \subset \mathcal{B}(G)$ be an orbit closure and $X_0 \subset X_1 \subset \ldots \subset X_\ell = X$ a maximal chain of orbit closures. Then $\ell = \dim(X) - \dim \mathcal{B}(H^0)$.*

(iii) *Let $\tilde{H} \supset H$ be a subgroup of $G$ which normalizes $H$ and such that $\tilde{H}/H$ is connected. Then $H$ and $\tilde{H}$ have the same orbits in $\mathcal{B}(G)$.*

*Proof.* (i) Let $x \in \mathcal{B}(G)$ be such that $Hx$ is closed. Then the variety $Hx$ is complete; thus, the same holds for its component $H^0x$. Moroever, the isotropy group $H_x = H \cap G_x$ is solvable. Thus $H_x^0$ is a Borel subgroup of $H$.

(ii) Choose a Borel subgroup $B$ of $G$, then the partially ordered sets of $H$-orbit closures in $\mathcal{B}(G)$ and of $B$-orbit closures in $G/H$ are isomorphic. Let $Y_0 \subset Y_1 \subset \ldots \subset Y_\ell = Y$ be a maximal chain of $B$-orbit closures in $G/H$. Then $Y_{\ell-1}$ is an irreducible component of the complement of the open $B$-orbit in $Y_\ell$. Because that orbit is affine, we have $\dim(Y_{\ell-1}) = \dim(Y_\ell) - 1$. It follows that $\dim(Y_0) = \dim(Y) - \ell$. Back to $H$-orbits in $\mathcal{B}(G)$, we thus have $\dim(X_0) = \dim(X) - \ell$. Furthermore, $X_0$ is a closed orbit, whence $\dim(X_0) = \dim \mathcal{B}(H^0)$.

(iii) Let $\mathcal{O} \subset \mathcal{B}(G)$ be an $H$-orbit and let $c$ be its codimension in $\mathcal{B}(G)$. We show that $\mathcal{O}$ is $\tilde{H}$-stable, by induction on $c$.

If $c = 0$, then $\mathcal{O}$ is open in $\mathcal{B}(G)$. Choose $x \in \mathcal{O}$; then $Hx$ is an open subset of $\tilde{H}x$, whence the product $H\tilde{H}_x$ is open in $\tilde{H}$. But $H\tilde{H}_x$ is a closed subgroup of $\tilde{H}$ containing $H$, because $\tilde{H}$ normalizes $H$. Thus, $H\tilde{H}_x$ is a union of components of $\tilde{H}$, and $Hx$ is a union of components of $\tilde{H}x$. But $\tilde{H}/H$ is connected, whence $Hx = \tilde{H}x$.

For arbitrary $c$, observe that the closure $\overline{\mathcal{O}}$ is a union of components of the set of $x \in \mathcal{B}(G)$ such that the codimension of $Hx$ in $\mathcal{B}(G)$ is at least $c$. The latter set is closed and $\tilde{H}$-stable because $\tilde{H}$ normalizes $H$. As $\overline{\mathcal{O}}$ is $H$-stable and $\tilde{H}/H$ is connected, it follows that $\overline{\mathcal{O}}$ is $\tilde{H}$-stable. Now the argument above shows that $\mathcal{O}$ is $\tilde{H}$-stable. □

**Definition.** The *rank* $\ell(X)$ of an $H$-orbit closure $X \subset \mathcal{B}(G)$ is the codimension in $X$ of any closed orbit, or equivalently, the common length of all maximal chains $X_0 \subset X_1 \subset \ldots \subset X_\ell = X$ of orbit closures.

In the case where $H = B$ as in 2.1, the closed orbits are fixed points, and the rank of $X = X(w)$ is the length of $w$.

For a *reductive* spherical subgroup $H \subset G$ and an $H$-orbit closure $X$ in $\mathcal{B}(G)$, we shall show that $\ell(X)$ satisfies an inequality similar to Corollary 2.1, with equality if $X$ is rationally smooth. For this, we shall analyze the fixed points in $X$ of a maximal torus of $H$, and of its codimension one subtori.

### 2.3. Fixed points in flag varieties

Let $H \subset G$ be a reductive spherical subgroup, and let $T_H \subset H$ be a maximal torus. For a subtorus $T' \subset T_H$, we denote by $G^{T'}$ (resp. $H^{T'}$) its centralizer in $G$ (resp. $H$) and by $\mathcal{B}(G)^{T'}$ its fixed point in $\mathcal{B}(G)$. It is well known that $G^{T'}$ is connected and reductive and that $\mathcal{B}(G)^{T'}$ contains only finitely many orbits of $G^{T'}$, each of them being isomorphic to the flag variety $\mathcal{B}(G^{T'})$. The torus $T'$ is *regular in* $G$ if $\mathcal{B}(G)^{T'}$ is finite, or equivalently, $G^{T'}$ is a maximal torus of $G$.

**Lemma.** *Notation and assumptions being as above, $T_H$ is regular in $G$. Furthermore, each $H^{T'}$ is a reductive spherical subgroup of $G^{T'}$.*

*Proof.* Because $H^0$ acts on $\mathcal{B}(G)$ with only finitely many orbits, $(H^0)^{T'}$ acts on $\mathcal{B}(G)^{T'}$ with only finitely many orbits as well; see [R]. It follows that $(H^0)^{T'}$ is spherical in $G^{T'}$. In particular, $(H^0)^{T_H} = T_H$ is spherical and central in $G^{T_H}$. Thus, $G^{T_H}$ is a torus, and $T_H$ is regular in $G$. □

Now assume that the codimension of $T'$ in $T_H$ is one, and that $T'$ is singular in $G$. Then $T' \subset H^{T'} \subset G^{T'}$ and the quotient $H^{T'}/T'$ has rank at most one. Let $G'$ be the quotient of $G^{T'}$ by its center, and let $H'$ be the image of $H^{T'}$ in $G'$. Then $G^{T'}$ and $G'$ have the same flag variety which we denote by $\mathcal{B}'$. Furthermore, $H'$ is a reductive spherical subgroup, of rank at most one, of the nontrivial connected adjoint semisimple group $G'$. Thus, $H'^0$ is either the multiplicative group or (P)$\mathrm{SL}_2$. Because $H'$ has finitely many orbits in $\mathcal{B}'$, we have $\dim(\mathcal{B}') \leq 1$ in the former case, and $\dim(\mathcal{B}') \leq 3$ in the latter case. Thus, $G'$ is isomorphic to $(\mathrm{PSL}_2)^n$ with $n \leq 3$, or to $\mathrm{PSL}_3$. A closer look leads to the following classification.

(1) $H' = G' = \mathrm{PSL}_2$. Then $\mathcal{B}'$ is projective line $\mathbf{P}^1$ with transitive action of $H'$.

(2) $H'^0$ is a one dimensional torus of $G' = \mathrm{PSL}_2$. Then $\mathcal{B}' = \mathbf{P}^1$, and the $H'^0$-orbits in $\mathcal{B}'$ are two fixed points and their complement. If $H'$ is not connected, then it is the normalizer of $H'^0$, and it exchanges both $H'^0$-fixed points in $\mathcal{B}'$.

(3) $H' = \mathrm{PSL}_2$, the diagonal in $G' = \mathrm{PSL}_2 \times \mathrm{PSL}_2$. Then $\mathcal{B}' = \mathbf{P}^1 \times \mathbf{P}^1$ with diagonal action of $H'$. The $H'$-orbits in $\mathcal{B}'$ are the diagonal and its complement.

(4) $H' = \mathrm{PSL}_2 = \mathrm{SO}_3$ embedded into $\mathrm{PSL}_3 = G'$. We can consider $\mathcal{B}'$ as the variety of flags in the projective plane $\mathbf{P}^2$, and $H'$ as the stabilizer in $\mathrm{PSL}_3$ of a smooth conic $C_0$. Then the $H'$-orbits in $\mathcal{B}'$ are given by the position of a flag $(p, d)$ (where $p$ is a point of $\mathbf{P}^2$ and $d$ a line containing

$p$) with respect to $C_0$. So there is a unique closed orbit: the set of flags $(p,d)$ such that $d$ is tangent to $C_0$ at $p$. This orbit is isomorphic to $\mathbf{P}^1$. And there are two orbit closures of dimension two, defined by $p$ is in $C_0$, resp. $d$ is tangent to $C_0$. It is easy to see that the maps $(p,d) \mapsto p$, resp. $(p,d) \mapsto d$ identify these orbit closures to the rational ruled surface of index two, denoted by $\mathbf{F}_2$.

(5) $H'^0 = \mathrm{SL}_2$ and $G' = \mathrm{PSL}_3$ where $H'^0$ is embedded as the image of matrices of the form $\begin{bmatrix} 1 & 0 & 0 \\ 0 & a & c \\ 0 & b & d \end{bmatrix}$ with $ad-bc=1$. Denote by $\tilde{H}'$ the normalizer of $H'^0$ in $G'$. Then $\tilde{H}'$ is the image of matrices of the form $\begin{bmatrix} t & 0 & 0 \\ 0 & a & c \\ 0 & b & d \end{bmatrix}$ with $t(ad-bc)=1$. Observe that $\tilde{H}'$ normalizes $H'$, and that the quotient $\tilde{H}'/H'$ is the multiplicative group. Thus, $H'$ and $\tilde{H}'$ have the same orbits in $\mathcal{B}'$, by Proposition 2.2. Furthermore, $\tilde{H}'$ is the stabilizer in $G'$ of a point $p_0$ in $\mathbf{P}^2$, represented by the first basis vector of $\mathbf{C}^3$, and of a line $l_0$ in $\mathbf{P}^2$, represented by the first dual basis vector. Thus, $\tilde{H}'$ has three closed orbits in $\mathcal{B}'$: the set of flags $(p,d)$ such that $p = p_0$ (resp. $d = d_0$; $p \in d_0$ and $d \in p_0$). These orbits are isomorphic to $\mathbf{P}^1$. Furthermore, there are two $\tilde{H}'$-orbit closures of dimension two, consisting of flags $(p,d)$ such that $p_0 \in d$ (resp. $p \in d_0$). The maps $(p,d) \mapsto p$ (resp. $(p,d) \mapsto d$) identify theses orbit closures to the blow-up of $\mathbf{P}^2$ at the point $p_0$ (resp. the blow-up of the dual projective plane at the point $d_0$). Thus, both orbit closures are isomorphic to the rational ruled surface $\mathbf{F}_1$ of index 1.

(6) $H' = \mathrm{PSL}_2$, the small diagonal in $G' = \mathrm{PSL}_2 \times \mathrm{PSL}_2 \times \mathrm{PSL}_2$. Then $\mathcal{B}' = \mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ with diagonal action of $H'$. The $H'$-orbit closures in $\mathcal{B}'$ are the small diagonal $\mathbf{P}^1$, three partial diagonals isomorphic to $\mathbf{P}^1 \times \mathbf{P}^1$, and $\mathcal{B}'$.

*Remarks.* (i) For a symmetric subgroup $H$ of $G$, we shall see in 2.5 that only types (1) to (4) can occur. It can be checked that the same holds if $G$ is simple and $H^0 \subset G$ is a maximal connected reductive spherical subgroup; for this, one uses Krämer's classification of reductive spherical subgroups of simple groups [Kr]. But types (5) and (6) do occur in general, e.g., type (5) for $H = \mathrm{Sp}_{2n} \subset \mathrm{SL}_{2n+1} = G$, and type (6) for $H = \mathrm{SO}_{2n+1} \subset \mathrm{SO}_{2n+1} \times \mathrm{SO}_{2n+2} = G$ where $H$ is embedded in $G$ by $h \mapsto (h,(h,1))$, or for $H = \mathrm{G}_2 \subset \mathrm{SO}_8 = G$ embedded by its defining representation.

(ii) By [MS] 6.4, all orbits of symmetric subgroups in flag varieties admit attractive slices. But this fails for arbitrary reductive subgroups: consider for example, $G = \mathrm{PSL}_3$ and $H = \mathrm{SL}_2$ as in type (5). Then we can take for $T_H$ the image of diagonal matrices with eigenvalues $(1,t,t^{-1})$ where $t \in \mathbf{G}_m$. Let $x \in \mathcal{B}(G)$ be the standard flag in $\mathbf{C}^3$. Then the weights of the $T_H$-action on the normal space $T_x\mathcal{B}(G)/T_xHx$ are 1 and $-1$. Thus, $Hx$

admits no attractive slice at $x$. Furthermore, both $H$-orbits of dimension 2 have unipotent isotropy groups, so that they admit no attractive slice either.

## 2.4. A criterion for rational smoothness

Notation and assumptions being as in 2.3, we shall describe fixed point subsets in an $H$-orbit closure $X \subset \mathcal{B}(G)$, and deduce a necessary condition for rational smoothness of $X$ at a $T_H$-fixed point. We begin with the following result, which is easily checked by inspection using the discussion in 2.3.

**Lemma.** *For any subtorus $T' \subset T_H$ of codimension one, each irreducible component of $X^{T'}$ is smooth, and is either a point (this may occur in type* (1)*), or $\mathbf{P}^1$ (this may occur in all types), or $\mathbf{P}^1 \times \mathbf{P}^1$ (in types* (3) *and* (6)*), or $\mathbf{F}_1$ (in case* (5)*), or $\mathbf{F}_2$ (in type* (4)*), or $\mathcal{B}(\mathrm{PSL}_3)$ (in types* (4) *and* (5)*), or $\mathbf{P}^1 \times \mathbf{P}^1 \times \mathbf{P}^1$ (in type* (6)*).*

For a subtorus $T' \subset T_H$ of codimension one, let $\ell_{T'}(X, x)$ be the sum of the ranks of the irreducible components of the $H^{T'}$-varieties $X^{T'}$ which contain $x$. Observe that $\ell_{T'}(X, x)$ is 0 in type (1), at most 1 in types (2) and (3), at most 2 in types (4) and (5), and at most 3 in type (6).

**Proposition.** (i) *For any $x \in X^{T_H}$, we have $\ell(X) \leq \sum_{T'} \ell_{T'}(X, x)$ with equality if $X$ is rationally smooth at $x$.*

(ii) *If moreover $X$ is irreducible and $\ell(X) \leq \ell(Hx) + 2$, then $X$ is rationally smooth at $x$.*

*Proof.* (i) By Theorems 1.1 and 1.2, we have $\dim \mathcal{B}(H^0) = \sum_{T'} \dim \mathcal{B}(H^{T',0})$ and $\dim_x(X) \leq \sum_{T'} (\dim_x X_+^{T'}(x) + \dim_x X_-^{T'}(x))$. Furthermore, we claim that $\dim_x X_+^{T'}(x) + \dim_x X_-^{T'}(x) \leq \dim \mathcal{B}(H^{T',0}) + \ell_{T'}(X, x)$. Indeed, if $X^{T'}$ is irreducible at $x$, then it is smooth at $x$ by (i). Thus, we have $\dim_x X_+^{T'}(x) + \dim_x X_-^{T'}(x) = \dim_x(X^{T'}) = \dim \mathcal{B}(H^{T',0}) + \ell(X^{T'})$ where the first equality follows from Theorem 1.2, and the second one is the definition of the rank. If $X^{T'}$ is reducible at $x$, then we are in case (4), (5) or (6), and moreover $H'x$ is closed in $\mathcal{B}'$. In cases (4) and (6), $x$ is attractive in $\mathcal{B}'$ and the claimed inequality is obvious; in case (5), it is checked by inspection. It follows that $\ell(X) \leq \sum_{T'} \ell_{T'}(X, x)$.

If moreover $X$ is rationally smooth at $x$, then each $X^{T'}$ is irreducible at $x$, and hence smooth at $x$. We conclude by Theorem 1.1.

(ii) Let $\Sigma$ be a $T_H$-stable slice to $Hx$ at $x$ in $\mathcal{B}(G)$; then $S := \Sigma \cap X$ is a slice to $Hx$ in $X$. If $\ell(X) = \ell(Hx) + 1$, then $S$ is an irreducible curve with nontrivial action of $T_H$ (because $T_H$ is regular in $G$). Thus, $S$ is unibranched at $x$, and hence rationally smooth. If $\ell(X) = \ell(Hx) + 2$ and $T_H$ acts on $S$ with a dense orbit, then $S$ is rationally smooth by Corollary 1.4.1. Finally, if $\ell(X) = \ell(Hx) + 2$ but $T_H$ has no dense orbit in $S$, then $S$ is fixed pointwise by a subtorus $T' \subset T_H$ of codimension one. Thus, $S \subset \Sigma^{T'}$ and the latter is a slice to $H^{T'}x$ in $\mathcal{B}^{T'}$. Because $\dim(S) = 2$, it follows from the classification in 2.3 that $S = \Sigma^{T'}$, whence $S$ is smooth. $\square$

### 2.5. The symmetric case

Consider now a connected semisimple group $G$ with an involutive automorphism $\theta$. Then the fixed point set $H = G^\theta$ is called a symmetric subgroup; it is a reductive spherical subgroup of $G$. We refer to [S1] for this and for other results on symmetric spaces, to be used below.

We shall obtain a precise version of Proposition 2.4 (i), in terms of the combinatorics of $H$-orbits in $\mathcal{B}(G)$. We begin by relating the approach of 2.3 to the structure of symmetric spaces.

Let $T_H \subset H$ be a maximal torus. Then its centralizer $T$ is a $\theta$-stable maximal torus of $G$. Thus, $\theta$ acts on the Weyl group $W$, on the subset $R$ of reflections, and on the set $\Phi$ of roots of $(G, T)$ as well. For $\alpha \in \Phi$, let $r_\alpha \in R$ be the corresponding reflection, and $G_\alpha \subset G$ the corresponding semisimple group of rank one. Then $G_\alpha$ contains a representative of $r_\alpha$. Finally, set $T_H^\alpha = (T_H \cap \ker(\alpha))^0$. Then the $T_H^\alpha$ are exactly the codimension one subtori of $T_H$ which are singular in $G$. Define the type of $\alpha$ (or of the corresponding reflection $r_\alpha$) as the type of $T_H^\alpha$ in the classification of 2.3.

**Lemma.** (i) *There exists a $\theta$-stable Borel subgroup $B$ of $G$ containing $T$; then $B^{\theta,0}$ is a Borel subgoup of $H$. Any two such Borel subgroups of $G$ are conjugated by $W^\theta$.*

(ii) *Let $\alpha \in \Phi$; then*

*$\alpha$ has type* (1) *if and only if $G_\alpha$ is contained in $H$ (in particular, $\theta(\alpha) = \alpha$).*

*$\alpha$ has type* (2) *if and only if $\theta(\alpha) = \alpha$, $G_\alpha$ is not contained in $H$, and $\alpha \neq \beta + \theta(\beta)$ for all $\beta \in \Phi$.*

*$\alpha$ has type* (3) *if and only if $\theta(\alpha) \neq \alpha$, and $\alpha + \theta(\alpha) \notin \Phi$.*

*$\alpha$ has type* (4) *if and only if: $\alpha + \theta(\alpha) \in \Phi$, or $\alpha = \beta + \theta(\beta)$ for some $\beta \in \Phi$. And there are no roots of type* (5) *or* (6).

*Proof.* (i) There exists a pair $(B_0, T_0)$ where $B_0$ is a $\theta$-stable Borel subgroup of $G$, and $T_0$ is a $\theta$-stable maximal torus of $B_0$. Let $U_0$ be the unipotent radical of $B_0$, and let $B_0^-$ be the opposite Borel subgroup, with unipotent radical $B_0^-$. Then the product map $U_0^- \times T_0 \times U_0 \to G$ is an open immersion. Thus, the same holds for the product map $(U_0^-)^{\theta,0} \times T_0^{\theta,0} \times U_0^{\theta,0} \to H$. It follows that $B_0^{\theta,0}$ and $(B_0^-)^{\theta,0}$ are opposite Borel subgroups of $H$. In particular, $T_0^{\theta,0}$ is a maximal torus of $H$. Thus, we can write $T_H = hT_0^{\theta,0}h^{-1}$ for some $h \in H$. Taking centralizers in $G$, we obtain $T = hT_0h^{-1}$; then we can take $B = hB_0h^{-1}$. If $B'$ is another Borel subgroup containing $T$, there exists a unique $w \in W$ such that $B' = wBw^{-1}$; now $B'$ is $\theta$-stable if and only if $\theta(w) = w$.

(ii) Let $T' = T_H^\alpha$. Then $\theta$ acts on the group $G^{T'}$ and on its quotient $G'$ by its center. Let $H'$ be the image of $H$ in $G'$; then $H'$ is a subgroup of finite index in $G'$. It follows that $(G', H')$ is not of type (5) or (6), because $SL_2$ is not a subgroup of finite index of a symmetric subgroup of $PSL_3$ or of $SL_2 \times SL_2 \times SL_2$. The description of types (1) to (4) follows from the discussion in [S1] §2. $\square$

For $B$ as in the lemma above, the pair $(T, B)$ is called *standard.* We then identify $\mathcal{B}(G)$ with $G/B$; the point $x \in (G/B)^T$ is identified with an element of $W$, still denoted by $x$.

Recall that $\alpha \in \Phi$ is called *compact imaginary* (resp. *noncompact imaginary*; *real*; *complex*) if $G_\alpha$ is contained in $H$ (resp. $\theta(\alpha) = \alpha$ but $G_\alpha$ is not contained in $H$; $\theta(\alpha) = -\alpha$; $\theta(\alpha) \neq \pm\alpha$). In our case, there are no real roots, because the set of roots of $(B, T)$ is $\theta$-stable. Furthermore, reflections of type (1) (resp. (2); (3) and (4)) correspond to compact imaginary roots (resp. certain noncompact imaginary roots; complex roots).

We now recall the parametrization of $H$-orbits in $G/B$; our notation differs from that in [S1] by an inverse, because $B$-orbits in $G/H$ are considered there. Let $N$ be the normalizer of $T$ in $G$; then $N$ is $\theta$-stable. Set

$$\mathcal{V} := \{g \in G \mid g^{-1}\theta(g) \in N\}.$$

Then $\mathcal{V}$ is stable by the $(H \times T)$-action: $(h, t)g = hgt^{-1}$, and each $(H \times B)$-orbit in $G$ meets $\mathcal{V}$ in a unique $(H \times T)$-orbit. As a consequence, $H$-orbits in $G/B$ are parametrized by the set of double classes $V := H\backslash\mathcal{V}/T$.

There is a base point $v_0 \in V$, the image of $1 \in N$; the corresponding $H$-orbit is closed, e.g., by the lemma above. Observe that $\mathcal{V}$ is stable under right multiplication by $N$; this defines an action of $W$ on $V$, denoted by $(w, v) \mapsto w \cdot v$.

For $v \in V$, we denote by $X(v) \subset G/B$ the corresponding $H$-orbit closure, and by $\ell(v)$ its rank. We write $v' \leq v$ if $X(v') \subset X(v)$. This defines a partial order on $V$, which is studied in [RS].

Finally, we shall need the following result, see [S2] 2.5: For any $r \in R$ of type (2), there exists $g(r) \in G_r$ such that $g(r)^{-1}\theta(g(r))$ is a representative of $r$ in $N$. In particular, $g(r) \in \mathcal{V}$. Let $v(r)$ be the image of $g(r)$ in $V$.

**Theorem.** *Let $v \in V$ and let $x \in W$ such that $x \cdot v_0 \leq v$. Let $n_2(v, x)$ be the number of reflections $r$ of type* (2) *such that $x \cdot v(r) \leq v$. For $t = 3$, 4, let $n_t(v, x)$ be the number of reflections $r$ of type $(t)$ such that $rx \cdot v_0 \leq v$. Then we have*

$$\ell(v) \leq n_2(v, x) + \frac{1}{2}n_3(v, x) + n_4(v, x)$$

*with equality if $X(v)$ is rationally smooth at $x$.*

*Proof.* We wish to apply Proposition 2.4 (i) combined with the lemma above. For this, given a subtorus $T' \subset T_H$ of codimension one, we analyze the contribution of $T'$ to the formula in that proposition. We denote by $\ell_{T'}(v, x)$ the sum of the ranks of the irreducible components of $X(v)^{T'}$ which contain $x$, and by $X(v)^{T'}_x$, the union of these components, i.e., the connected component of $x$ in $X(v)^{T'}$.

If $T' = T_H^r$ for $r$ of type (2), the component of $x$ in $(G/B)^{T'}$ is the curve $C(x, r)$ considered in 2.1. By [S2] 3.1, this curve is contained in $X(v)$ if and only if $x \cdot v(r) \leq v$. In other words, we have $\ell_{T'}(v,x) = 1$ if $x \cdot v(r) \leq v$, and $\ell_{T'}(v,x) = 0$ otherwise.

If $T' = T_H^r$ for $r$ of type (3) or (4), observe that $X(v) \cap G^{T'}x$ is connected, by the explicit description in 2.3. Thus, we have $X(v) \cap G^{T'}x = X(v)_x^{T'}$. Now $rx \cdot v_0 \leq v$ iff $rx \in X(v)$ iff $rx \in X(v)_x^{T'}$ (because $rx \in G^{T'}x$ anyway). For $T'$ of type (3), one checks that $\ell_{T'}(v,x)$ is the half of the number of $r \in R$ such that $T_H^r = T'$ and that $rx \in X(v)_x^{T'}$.

If $r$ has type (4), then one checks that $\ell_{T'}(v,x)$ is at most the number of $r$ as above, with equality if $X(v)_x^{T'}$ is irreducible.

**Examples.** 1) In the case where $T_H$ is a maximal torus of $G$ (that is, $\theta$ is inner), only types (1) and (2) occur, and we recover the following result of Springer [S2]: the rank of $X(v)$ is at most the number of noncompact imaginary reflections $r$ such that $x \cdot v(r) \leq v$, with equality if $X(v)$ is rationally smooth at $x$.

2) Consider $G = \mathrm{SL}_n$ with the involution $\theta$ such that $\theta(g) = {}^t g^{-1}$. Then $H = \mathrm{SO}_n$. The flag variety $\mathcal{B}(\mathrm{SL}_n)$ contains $n-1$ irreducible $H$-stable divisors $D_1, \ldots, D_{n-1}$, where each $D_m$ consists of those complete flags $(V_1 \subset \ldots \subset V_{n-1})$ in $\mathbf{C}^n$ such that the restriction of the standard quadratic form to $V_m$ is degenerate.

For $n \geq 4$, we claim that $D_1$ and $D_{n-1}$ are smooth; $D_2$ and $D_{n-2}$ are rationally smooth, but singular; and no other $D_m$ is rationally smooth (see [Ku] for a similar result concerning Schubert divisors in arbitrary flag varieties).

To check this, consider first the case where $n = 2n'$ is even. Choose a basis of $\mathbf{C}^n$ with coordinates $x_1, \ldots, x_n$ such that the quadratic form is $x_1 x_{2n'} + x_2 x_{2n'-1} + \ldots + x_{n'} x_{n'+1}$. Let $T$ (resp. $B$) be the group of diagonal (resp. upper triangular) matrices in this basis. Then $(T,B)$ is a standard pair, and $\theta$ acts on $T$ by $\theta(t_1, \ldots, t_n) = (t_n^{-1}, \ldots, t_1^{-1})$. The roots of $(B,T)$ are the $\alpha_{i,j}$ $(1 \leq i < j \leq n)$ where $\alpha_{i,j}(t_1, \ldots, t_n) = t_i t_j^{-1}$. The roots of type (3) are the $\alpha_{i,j}$ where $i + j \neq 2n' + 1$, and all other roots have type (2). Let $x$ be the standard flag in our basis. Using either [RS] 10.3 or geometric arguments, one checks that

$$n_2(D_m, x) = \begin{cases} n' - 1 & \text{if } m = 1 \text{ or } m = 2n' - 1, \\ n' & \text{otherwise,} \end{cases}$$

$$n_3(D_m, x) = \begin{cases} 2n'(n'-1) - 2 & \text{if } m = 2 \text{ or } m = 2n' - 2, \\ 2n'(n'-1) & \text{otherwise.} \end{cases}$$

On the other hand, $\ell(D_m) = \ell(\mathcal{B}(\mathrm{SL}_n)) - 1 = n'^2 - 1$. By the Theorem above, it follows that $D_3, \ldots, D_{2n'-3}$ are not rationally smooth.

In the case where $n = 2n' + 1$ is odd, we replace the quadratic form by $x_1 x_{2n'+1} + \ldots + x_{n'} x_{n'+2} + x_{n'+1}^2$. Then the discussion is similar, but now the roots of type (3) are the $\alpha_{i,j}$ with $i \neq n'+1$, $j \neq n'+1$ and $i+j \neq 2n'+2$, whereas all other roots have type (4). We have $\ell(D_m) = n'^2 + n' - 1$ for all $m$, and one checks that

$$n_3(D_m, x) = \begin{cases} 2n'(n'-1) - 2 & \text{if } m = 2 \text{ or } m = 2n' - 1, \\ 2n'(n'-1) & \text{otherwise,} \end{cases}$$

$$n_4(D_m, x) = \begin{cases} 2n' - 1 & \text{if } m = 1 \text{ or } m = 2n', \\ 2n' & \text{otherwise.} \end{cases}$$

Again, it follows that $D_3, \ldots, D_{2n'-2}$ are not rationally smooth.

It remains to check our assertion for $D_1$ and $D_2$ (because $\theta$ acts on $\mathcal{B}(G)$ and exchanges $D_m$ and $D_{n-m}$). For this, let $\pi_m : \mathcal{B}(\mathrm{SL}_n) \to \mathrm{Gr}_{n,m}$ be the canonical map to the Grassmanian of $m$-dimensional subspaces. Then $D_m$ is the preimage of the divisor $E_m$ of degenerate subspaces, under the fibration $\pi_m$. For $m = 1$, because $\mathrm{Gr}_{n,1} = \mathbf{P}^{n-1}$ and $E_1$ is a smooth quadric, $D_1$ is smooth. For $m = 2$, let $\mathrm{Gr}_{n,2}^{is} \subset \mathrm{Gr}_{n,2}$ be the subvariety of totally isotropic planes. Then $E_2$ contains $\mathrm{Gr}_{n,2}^{is}$ as its closed $\mathrm{SO}_n$-orbit. Furthermore, one checks that a slice to $\mathrm{Gr}_{n,2}^{is}$ in $\mathrm{Gr}_{n,2}$ at the point $\mathrm{span}(e_1, e_2)$ is

$$S := \{\mathrm{span}(e_1 + ae_{n-1} + be_n, e_2 + ce_{n-1} + ae_n) \mid a, b, c \in \mathbf{C}\}$$

where $(e_1, \ldots, e_n)$ is the basis introduced above. Thus, a slice to $\mathrm{Gr}_{n,2}^{is}$ in $E_2$ is $S \cap E_2$, isomorphic to the quadratic cone $(a^2 - bc = 0)$. We conclude that $E_2$ is rationally smooth but singular along $\mathrm{Gr}_{n,2}^{is}$. Thus, $D_2$ is rationally smooth but singular as well.

This result, combined with Proposition 2.4 (ii), implies e.g., that all $\mathrm{SO}_n$-orbit closures in $\mathcal{B}(\mathrm{SL}_n)$ are rationally smooth for $n = 4$. This is no longer true for $n = 5$, an example being $D_2 \cap D_3$.

*Remark.* Back to the case of an arbitrary symmetric subgroup, consider a point $x \in X$ not necessarily fixed by a maximal torus of $H$. Then the orbit $Hx$ admits an attractive slice at $x$, by [MS] 6.4. Thus, a criterion for rational smoothness of $X$ along $Hx$ can be derived from Proposition 2.1. This leads to the following question: For a subtorus $T'$ of codimension one in a maximal torus of $H_x$, when is $X^{T'}$ rationally smooth at $x$?

## 3. Closures of double classes in regular group completions

### 3.1. Construction of slices

Let $G$ be a connected reductive group. Then $G \times G$ acts on $G$ by $(g_1, g_2)\gamma = g_1 \gamma g_2^{-1}$. This identifies $G$ with the homogeneous space $(G \times G)/\mathrm{diag}\, G$ where $\mathrm{diag}\, G$ denotes the diagonal in $G \times G$. Let $T \subset G$ be a maximal torus, $W$ its Weyl group, and $B$, $B^-$ two opposite Borel subgroups containing $T$. Then $B \times B^-$ acts on $G$ as above, the orbits being the double classes $BwB^-$ where $w \in W$. In particular, the open orbit is $BB^-$.

Let $X$ be a $(G \times G)$-equivariant completion of $G$ which is regular in the sense of [BDP]. Then $B \times B^-$ acts on $X$ with finitely many orbits, whose study was initiated in [B1]. We shall construct attractive slices to these orbits. For this, we need more notation and results, adapted from [B1] 2.1.

Each $(G \times G)$-orbit $\mathcal{O} \subset X$ contains a unique point $y$ such that $(B \times B^-)y$ is open in $\mathcal{O}$, and $y$ is the limit of a one parameter subgroup of $T$. We refer to $y$ as the base point of $\mathcal{O}$.

Furthermore, $\mathcal{O}$ determines two opposite parabolic subgroups $P \supset B$ and $Q \supset B^-$, with unipotent radicals $R_u(P)$, $R_u(Q)$ and common Levi subgroup $L = P \cap Q$, by requiring that the stabilizer $(G \times G)_y$ is the semidirect product of $R_u(Q) \times R_u(P)$ and $(\operatorname{diag} L)(T \times 1)_y$. In particular, $(T \times T)_y = (\operatorname{diag} T)(T \times 1)_y$ is a maximal torus in $(G \times G)_y$. In fact, $(T \times 1)_y = (Z \times 1)_y$, where $Z$ denotes the connected center of $L$.

Let $\Phi$ be the root system of $(G, T)$; then we have the subsets $\Phi^+$ (resp. $\Phi_L$) of roots of $(B, T)$ (resp. $(L, T)$). Let $W^L$ be the set of all $w \in W$ such that $w(\Phi_L^+)$ is contained in $\Phi^+$. Then each $(B \times B^-)$-orbit in $\mathcal{O} = (G \times G)y$ can be written uniquely as $(B \times B^-)(w, \tau)y$ for $w \in W$ and $\tau \in W^L$.

Choose representatives $\tilde{w}$, $\tilde{\tau}$ in the normalizer of $T$, and set $x := (\tilde{w}, \tilde{\tau})y$. Then $(T \times T)_x = (w, \tau)(T \times T)_y(w^{-1}, \tau^{-1})$ is a maximal torus in $(G \times G)_x$ and thus in $(B \times B^-)_x$. The codimension of $(B \times B^-)x$ in $(G \times G)x$ is $\ell(w) + \ell(\tau)$.

For simplicity, set

$$Z_y := (Z \times 1)_y;$$

then $Z_y$ is the isotropy group of $y$ for the left action of $T$ on $\overline{T}$. Set

$$\Sigma(y) := \{z \in \overline{T} \mid y \in \overline{Z_y z}\}.$$

Since $\overline{T}$ is a smooth toric variety, $\Sigma(y)$ is a $Z_y$-stable slice to $Ty$ at $y$ in $\overline{T}$. Since $X$ is regular, $\Sigma(y)$ is a slice to $\mathcal{O}$ in $X$ as well. Furthermore, $\Sigma(y)$ is isomorphic to the affine space $\mathbf{A}^d$ where $d = \operatorname{codim}_{\overline{T}}(Ty) = \operatorname{codim}_X(\mathcal{O})$, and $Z_y$ acts linearly on $\mathbf{A}^d$ by $d$ independent characters. Thus, $\Sigma(y)$ contains exactly $d$ closed irreducible $Z_y$-stable curves through $y$, the coordinate lines $C_1(y), \ldots, C_d(y)$.

For any $\alpha \in \Phi$, let $U_\alpha \subset G$ be the corresponding unipotent subgroup. If $w^{-1}(\alpha) \in \Phi^+ \cup \Phi_L$, then $U_{w^{-1}(\alpha)}$ does not fix $y$, whence $U_\alpha \times 1$ does not fix $x$. Thus,

$$C(x, \alpha) := (U_\alpha \times 1)x$$

is an irreducible locally closed curve through $x$, stable by $(T \times T)_x$. We define similarly

$$C(x, \alpha)^- := (1 \times U_\alpha)x$$

for $\alpha \in \Phi$ such that $\tau^{-1}(\alpha) \in \Phi^- \cup \Phi_L$. Finally, we set

$$C_i(x) := (\tilde{w}, \tilde{\tau})C_i(y)$$

for $1 \le i \le d$. Now we can state the following

**Theorem.** *Notation being as above, the map*

$$\begin{array}{ccc} (U^- \cap wUw^{-1}) \times (U \cap \tau U^- \tau^{-1}) \times \Sigma(y) & \to & X \\ (g, h, z) & \mapsto & (g\tilde{w}, h\tilde{\tau})z \end{array}$$

*is an embedding, and its image $S$ is an attractive $(T \times T)_x$-stable slice to $(B \times B^-)x$ at $x$ in $X$. Furthermore, the closed irreducible $(T \times T)_x$-stable curves through $x$ in $S$ are the $C(x,\alpha)$ $(\alpha \in \Phi^- \cap w(\Phi^+))$, the $C(x,\alpha)^-$ $(\alpha \in \Phi^+ \cap \tau(\Phi^-))$, and the $C_i(x)$ $(1 \leq i \leq d)$.*

*Proof.* After multiplication by $(\tilde{w}, \tilde{\tau})^{-1}$, we reduce to the somewhat simpler study of $X$ along the orbit $(w^{-1}Bw, \tau^{-1}B^-\tau)y$. For this, set

$$\tilde{S} := (U \cap w^{-1}U^-w) \times (U^- \cap \tau^{-1}U\tau) \times \Sigma(y), \ \tilde{y} := (1, 1, y).$$

Consider the map

$$\begin{array}{cccc} \pi : & \tilde{S} & \to & X \\ & (g, h, z) & \mapsto & (g, h)z. \end{array}$$

The group $(T \times T)_y$ acts on $\tilde{S}$ by $(u, v) \cdot (g, h, z) = (ugu^{-1}, vhv^{-1}, uv^{-1}z)$ with fixed point $\tilde{y}$, and $\pi$ is equivariant. Identifying $\tilde{S}$ with the affine space of dimension $\ell(w) + \ell(\tau) + d$, the action of $(T \times T)_y$ is linear, with weights: $(\alpha, 0)$ $(\alpha \in \Phi^+ \cap w^{-1}(\Phi^-))$, $(0, -\alpha)$ $(\alpha \in \Phi^- \cap \tau^{-1}(\Phi^+))$, and the weights of $C_1(y), \ldots, C_d(y)$. Furthermore, the multiplicity of each weight is one, and $(T \times T)_y = (\operatorname{diag} T)Z_y$ where $Z_y$ acts on $C_1(y), \ldots, C_d(y)$ through $d$ linearly independent weights. It follows that $\tilde{y}$ is attractive, and that the $(T \times T)_y$-stable curves in $\tilde{S}$ are the $(U_\alpha \times 1)y$ $(\alpha \in \Phi^+ \cap w^{-1}(\Phi^-))$, the $(1 \times U_\alpha)y$ $(\alpha \in \Phi^- \cap \tau^{-1}(\Phi^+))$, and $C_1(y), \ldots, C_d(y)$.

Furthermore, from the description of $(G \times G)_y$ and the fact that $\Sigma(y)$ is transversal to $(G \times G)y$ at $y$, it follows that $\pi$ is étale at $\tilde{y}$, and that $\pi^{-1}(\pi(\tilde{y})) = \{\tilde{y}\}$. Because $\tilde{y}$ is attractive, $\pi$ is an isomorphism onto its image, a locally closed subvariety of $X$.

Finally, we check that the action map $w^{-1}Bw \times \tau^{-1}B^-\tau \times \pi(\tilde{S}) \to X$ is smooth at $(1, 1, y)$; this follows from the decompositions of tangent spaces

$$T_yX = T_y(G \times G)y \oplus T_y\Sigma(y) = T_y(B \times B^-)y \oplus T_y\Sigma(y)$$

$$= T_y(w^{-1}Bw \times \tau^{-1}B^-\tau)y \oplus T_y((U \cap w^{-1}U^-w) \times (U^- \cap \tau^{-1}U\tau)y) \oplus T_y\Sigma(y)$$

$$= T_y(w^{-1}Bw \times \tau^{-1}B^-\tau)y \oplus T_yS(y),$$

which follow in turn from the structure of $(G \times G)_y$ described above. □

Applying Corollary 1.4.1, we obtain immediately the following

**Corollary.** *Any $(B \times B^-)$-orbit closure in a regular completion of $G$ is rationally smooth in codimension two.*

In contrast, $(B \times B^-)$-orbit closures in regular completions are singular in codimension two, apart from very few exceptions (see [B1] Corollary 2.2).

### 3.2. More on slices and closures of double classes

We just saw that closures of double classes in regular group completions admit attractive slices at all points; furthermore, these slices contain only finitely many invariant curves. Therefore, we can obtain a criterion for rational smoothness of these closures, similar to that for Schubert varieties (Corollary 2.1). To make this explicit, we need to know more about invariant curves, and to describe the inclusion relations between closures of double classes as well.

Notation being as in 3.1, we begin by analyzing the closed irreducible $(T \times T)_x$-stable curves through $x$ in the slice $S$. Because $X$ is regular, the $(G \times G)$-orbit $\mathcal{O}$ of codimension $d$ is contained in the closure of $d$ orbits $\mathcal{O}_1, \dots, \mathcal{O}_d$ of codimension $d-1$. Furthermore, we can index these orbits so that the base point $y_i$ of each $\mathcal{O}_i$ belongs to the curve $C_i(y)$. Thus, we have $C_i(y) = \overline{Z_y y_i} = Z_y y_i \cup \{y\}$, and $C_i(x) - \{x\}$ is contained in $(B \times B^-)(w, \tau) y_i$.

The behaviour of the other curves is given by the following

**Proposition.** *Notation being as above, the curve $C(x,\alpha) - \{x\}$ is contained in $(B \times B^-)(r_\alpha w, \tau) y$ for any $\alpha \in \Phi^- \cap w(\Phi^+)$. Similarly, $C(x,\alpha)^- - \{x\}$ is contained in $(B \times B^-)(w, r_\beta \tau) y$ for any $\alpha \in \Phi^+ \cap \tau(\Phi^-)$.*

*Proof.* Set $\dot{U}_\alpha := U_\alpha - \{1\}$; then $C(x,\alpha) - \{x\} = \dot{U}_\alpha x$ and $\dot{U}_\alpha \subset U_{-\alpha} r_\alpha T U_{-\alpha} = U_{-\alpha} T r_\alpha w U_{-w^{-1}(\alpha)} w^{-1} \subset B r_\alpha w U_{-w^{-1}(\alpha)} w^{-1}$. Set $\beta := w^{-1}(\alpha)$; then $\beta \in \Phi^+$. If $\beta \notin \Phi_L^+$, then $U_{-\beta} \times 1$ fixes $y$ and the assertion follows. Otherwise, $(U_{-\beta} \times 1) y = (1 \times U_{-\beta}) y$ because $\beta \in \Phi_L^+$. Thus, we have $(\dot{U}_\alpha \times 1) x \subset (B r_\alpha w U_{-\beta}, \tau) y = (B r_\alpha w, \tau U_{-\beta}) y \subset (B \times B^-)(r_\alpha w, \tau) y$ because $\tau U_{-\beta} = U_{-\tau(\beta)} \tau$ is contained in $B^- \tau$. The proof of the second assertion is similar. □

We now describe the inclusion order between closures of $(B \times B^-)$-orbits in $X$. This is given by the lemma below, where $w_{0,L}$ denotes the longest element in $W_L$. A closely related statement is obtained in [PPR] for reductive algebraic monoids; the latter can be considered as affine embeddings of connected reductive groups.

**Lemma.** *Notation being as above, the closure of $(B \times B^-)(w, \tau) y$ in $\mathcal{O} = (G \times G) y$ is the union of the $(B \times B^-)(w', \tau') y$, where $w', \tau' \in W$ satisfy $w' \geq w$ and $\tau' w_{0,L} \geq \tau w_{0,L}$.*

*If moreover $\mathcal{O}' \subset \overline{\mathcal{O}}$ is a $(G \times G)$-orbit with base point $y'$ and associated Levi subgroup $L'$, then*

$$\overline{(B \times B^-)(w, \tau) y} \cap \mathcal{O}' = \bigcup \overline{(B \times B^-)(wv, \tau v) y'}$$

*(decomposition into irreducible components), where the union is over all $v \in W_L$ such that $\tau v \in W^{L'}$ and $\ell(w) = \ell(wv) + \ell(v)$.*

*Proof.* Consider the $(B^- \times B)$-orbits in $\mathcal{O}$. We claim that the orbit $(B^- \times B)(1, w_{0,L}) y$ is closed. Indeed, setting $B_L := B \cap L$ and $B_L^- := B^- \cap L$, we have $B^- = B_L^- R_u(Q)$ and $B = B_L R_u(P)$, whence $(B^- \times B)(1, w_{0,L})y = (B_L^- \times B_L)(1, w_{0,L})y = (1, w_{0,L})(B_L^- \times B_L^-)y = (1, w_{0,L})(1 \times B_L^-)y$ and $(1 \times B_L^-)y$ identifies with the image of $B_L^-$ in $L/Z_y$, which is closed in there.

Now we have $B^-\tau = B^-\tau B_L^-$ (because $\tau B_L^- \tau^{-1} \subset B^-$), whence

$$(B \times B^-)(w, \tau)y = (B \times B^-)(w, \tau w_{0,L})(1, w_{0,L})B_L^- y.$$

Equivalently, $(B \times B^-)(w, \tau)y = (B \times B^-)(w, \tau w_{0,L})(B^- \times B)y$. So the canonical map from

$$\overline{(B \times B^-)(w, \tau w_{0,L})(B^- \times B)} \times_{B^- \times B} (B^- \times B)(1, w_{0,L})y$$

to $\overline{(B \times B^-)(w, \tau)y}$ is dominant and proper, hence surjective. By the Bruhat decomposition, the closure in $G$ of $(B \times B^-)(w, \tau w_{0,L})(B^- \times B)$ is the union of the double classes $(B \times B^-)(w', \tau' w_{0,L})(B^- \times B)$ with $w' \geq w$ and $\tau' w_{0,L} \geq \tau w_{0,L}$. This implies the first assertion, whereas the second assertion follows from [B1] Theorem 2.1. □

### 3.3. Singularities of closures of double classes

Using the combinatorics of 3.2, we show that the closure of a double class $BwB^-$ at a fixed point of $B \times B^-$ contains in general all closed irreducible $(T \times T)$-stable curves through that point (this improves on [B1] Theorem 2.2, with a more natural proof). Thus, this closure is not rationally smooth, as a rule.

An exception to that rule is the case where $G = \mathrm{PGL}(2)$. Indeed, that group has a unique regular completion $X$, the projectivization of the space of $2 \times 2$ matrices. Furthermore, the closure in $X$ of the standard Borel subgroup $B$ is isomorphic to $\mathbf{P}^2$ and hence smooth; it contains only two closed irreducible $(T \times T)$-stable curves through the $(B \times B)$-fixed point.

Similarly, the group $\mathrm{SL}_2$ has a unique regular completion $X$, a quadric in the projective completion of the space of $2 \times 2$ matrices. Furthermore, the closure in $X$ of the standard Borel subgroup $B$ is a nondegenerate quadratic cone of dimension two. Thus, $\overline{B}$ is singular, but rationally smooth; again, it contains only two closed irreducible $(T \times T)$-stable curves through the $(B \times B)$-fixed point.

We shall see that all exceptions arise from both examples above. To state our result in a precise way, we need the following

**Definition.** A simple root $\alpha$ is called *isolated* if $\alpha$ is not connected to any simple root in the Dynkin diagram of $G$. In particular, $G$ has no isolated simple root if and only if the quotient of $G$ by its center contains no direct factor isomorphic to PGL(2).

**Theorem.** *Let $X$ be a regular completion of $G$, let $w \in W$ and let $x \in X$ be a fixed point of $B \times B^-$. If $G$ has no isolated simple root, then $\overline{BwB^-}$ contains all closed irreducible $(T \times T)$-stable curves through $x$. In particular,* *the tangent space to $\overline{BwB^-}$ at $x$ is the whole tangent space to $X$ at $x$, and $\overline{BwB^-}$ is not rationally smooth there unless $w = 1$, that is, $\overline{BwB^-} = X$.*

*Proof.* Since $\overline{BwB^-}$ contains $Bw_0B^-$, we may assume that $w = w_0$. Then the slice $S$ at $x$ is a $(T \times T)$-stable open neighborhood of $x$. Furthermore, the closed irreducible $(T \times T)$-stable curves through $x$ in $S$ are: the $C(x, \alpha) = (U_\alpha \times 1)x$ ($\alpha \in \Phi^-$), the $C(x, \alpha)^- = (1 \times U_\alpha)x$ ($\alpha \in \Phi^+$), and $C_1(x), \ldots, C_l(x)$ where $l$ is the rank of $G$. Furthermore, $C(x, \alpha) - \{x\}$ is contained in $(B \times B^-)(r_\alpha, 1)x$ by Proposition 3.2, and similarly for $C(x, \alpha)^-$.

Let $z$ be the base point of the closed orbit $Z := (G \times G)x$. Then $x = (w_0, w_0)z$ where $w_0 \in W$ is the longest element. We have $(B \times B^-)(r_\alpha, 1)x = (B \times B^-)(r_\alpha w_0, w_0)z \subset \overline{Bw_0B^-}$ where the inclusion follows from Lemma 3.2. Thus, $C(x, \alpha)$ is contained in $\overline{Bw_0B^-}$. The argument for $C(x, \alpha)^-$ is similar.

Consider now a curve $C_i(x)$ where $1 \le i \le l$. By Proposition 3.2, there exists a $(G \times G)$-orbit $\mathcal{O}_i$ with base point $z_i$ such that $\dim(\mathcal{O}_i) = \dim(Z) + 1$ and that $C_i(x) - \{x\}$ is contained in $(T \times T)(w_0, w_0)z_i$. Let $P$, $Q$, $L$, $Z$ be associated to $\mathcal{O}_i$ as in 3.1. Then $\dim(Z) \ge \dim(Z_{y_i}) = \dim(T) - 1$. Thus, either $P = B$, or $P$ is a minimal parabolic subgroup containing $B$.

In the former case, $(G \times G)_{y_i}$ is the kernel of a character of $B^- \times B$. Arguing as above, we obtain that $C_i(x)$ is contained in $\overline{BwB^-}$.

In the latter case, let $\alpha$ be the simple root corresponding to $P$, and set $W^\alpha := \{w \in W \mid w(\alpha) \in R^+\}$. Then we have by Lemma 3.2, $\overline{Bw_0B^-} \cap \mathcal{O}_i = \bigcup_{v \in W^\alpha} \overline{(B \times B^-)(w_0 v, v)z_i}$. Choose a simple root $\beta$ which is connected to $\alpha$ in the Dynkin diagram. Then $r_\alpha r_\beta$ and $w_0 r_\alpha r_\beta r_\alpha$ are in $W^\alpha$. Thus,

$$\overline{Bw_0B^-} \supset \overline{(B \times B^-)(r_\alpha r_\beta r_\alpha, w_0 r_\alpha r_\beta r_\alpha)z_i} \supset (B \times B^-)(w_0, w_0)z_i,$$

where the first inclusion follows from Lemma 3.2, and the second one from that lemma applied to $w = r_\alpha r_\beta r_\alpha$, $\tau = w_0 r_\alpha r_\beta r_\alpha$, $w' = \tau' = w_0$. Indeed, $w' \ge w$ is clear, and $\tau' r_\alpha = w_0 r_\alpha \ge w_0 r_\alpha r_\beta = \tau r_\alpha$ because $r_\alpha \le r_\alpha r_\beta$.

So we conclude that $C_i(x)$ is contained in $\overline{Bw_0B^-}$. The remaining assertions follow now from Corollary 1.4.2. □

## Appendix

**Proposition A1.** *Let $X$ be an algebraic variety of dimension $d$ and let $x \in X$.*

(i) *The dimension of the space $H_x^{2d}(X)$ is the number of $d$-dimensional irreducible components of $X$ through $x$.*

(ii) *If $X$ is rationally smooth at $x$, then it is irreducible at $x$.*

(iii) *Let $\pi : X \to Y$ be the quotient by the action of a finite group $G$. If $X$ is rationally smooth at $x$, then $Y$ is rationally smooth at $\pi(x)$.*

(iv) *Let $\pi : X \to Y$ be a smooth morphism. Then $X$ is rationally smooth at $x$ if and only if $Y$ is rationally smooth at $\pi(x)$.*

*Proof.* (i) Let $\mathcal{T}_{X,\mathbf{Q}}$ be the dualizing complex of $X$ for sheaves of $\mathbf{Q}$-vector spaces [V]. For each integer $m$, the homology sheaf $\mathcal{H}_m(\mathcal{T}_{X,\mathbf{Q}})$ is associated with the presheaf $U \mapsto H_c^m(U)^*$ (the dual of cohomology with compact supports). This presheaf vanishes for $m > 2d$, and is a sheaf for $m = 2d$. Furthermore, by [V] Corollaire 2.6.5, the stalk of $\mathcal{T}_{X,\mathbf{Q}}$ at $x$ is the dual of $R\Gamma_x(\mathbf{Q}_X)$ where $\mathbf{Q}_X$ denotes the constant sheaf on $X$ associated with $\mathbf{Q}$. It follows that $U \mapsto H_c^{2d}(U)$ is a sheaf, and that its stalk at $x$ is $H_x^{2d}(X)$. This implies our assertion.

(ii) It follows from (i) that $X$ has a unique irreducible component $Y$ of dimension $d$ which contains $x$. If $X$ has another irreducible component $Z$ of dimension $e < d$ which contains $x$, then we can choose a smooth point $z \in Z - Y$ arbitrarily close to $x$. Now $H_z^{2e}(X) = H_z^{2e}(Z)$ is nonzero, a contradiction.

(iii) Denote by $\mathbf{Q}_X$ the constant sheaf on $X$ associated with $\mathbf{Q}$. Then $G$ acts on the direct image $\pi_*\mathbf{Q}_X$ and the subsheaf of invariants $\pi_*^G\mathbf{Q}_X$ is isomorphic to $\mathbf{Q}_Y$ via the map $\mathbf{Q}_Y \to \pi_*\mathbf{Q}_X$ (indeed, this map induces an isomorphism on stalks). Furthermore, $R^i\pi_*\mathbf{Q}_X = 0$ for $i \geq 1$. It follows that $\pi_* : H^*(X) \to H^*(Y)$ restricts to an isomorphism $H^*(X)^G \cong H^*(Y)$. Considering the isomorphisms above for $X$ and $X - \pi^{-1}\pi(x) = X - Gx$, we obtain an isomorphism $H^*_{Gx}(X)^G \cong H^*_{\pi(x)}(Y)$. Furthermore, the left hand side is isomorphic to $(\bigoplus_{g\in G/G_x} H^*_{gx}(X))^G \cong H_x^*(X)^{G_x}$. Since $X$ is rationally smooth at $x$, the vector space $H_x^*(X)$ is one dimensional, concentrated in degree $2\dim_x(X)$, and $G_x$ acts trivially there. Thus $Y$ is rationally smooth at $\pi(x)$.

(iv) Shrinking $X$ and $Y$ if necessary, we can factor $\pi$ as an étale morphism $f : X \to Y \times \mathbf{A}^n$ followed by projection $g : Y \times \mathbf{A}^n \to Y$. By excision, we have $H_x^m(X) \cong H^m_{f(x)}(Y \times \mathbf{A}^n)$. Furthermore, by the Künneth isomorphism, we have $H^m_{(y,z)}(Y \times \mathbf{A}^n) \cong H_y^{m-2n}(Y)$. It follows that $H_x^m(X)$ is isomorphic to $H^{m-2n}_{\pi(x)}(Y)$. □

**Proposition A2.** *For a torus $T$ acting on a variety $X$ with a fixed point $x$, the following conditions are equivalent:*

(i) *The weights of $T$ in the tangent space $T_xX$ are contained in an open half space.*

(ii) *There exists a one-parameter subgroup $\lambda : \mathbf{G}_m \to T$ such that, for all $y$ in a neighborhood of $x$, we have $\lim_{t\to 0}\lambda(t)y = x$.*

*If* (ii) *holds, then the set*

$$X_x := \{y \in X \mid \lim_{t\to 0}\lambda(t)y = x\}$$

*is an open affine $T$-stable neighborhood of $x$, which admits a closed $T$-equivariant embedding into $T_xX$.*

*Proof.* For equivalence of (i) and (ii), we can replace $X$ by any open affine $T$-stable neighborhood of $x$, and thus suppose that $X$ is affine. Let $A$ be the algebra of regular functions on $X$, and let $m_x$ be the maximal ideal of

$A$ corresponding to $x$. Then $T$ acts on $A$ so that $m_x$ is $T$-stable, and $T_xX$ is the dual space of $m_x/m_x^2$.

If (i) holds, then we can find a one parameter subgroup $\lambda$ which is positive on all weights of $T_xX$. Then $\lambda$ is negative on all weights of $m/m^2$ and thus, of $m^n/m^{n+1}$ for all positive integers $n$. Because $A \cong \oplus_{n\geq 0} m^n/m^{n+1}$ as a $T$-module, the action of $\lambda$ on $A$ has negative weights, and $A^\lambda = \mathbf{C}$. It follows that $\lim_{t\to 0}\lambda(t)y = x$ for all $y \in X$.

Conversely, if (ii) holds, then the algebra $A$ is negatively graded via $\lambda$. Thus $T_xX$ is positively graded via $\lambda$.

For arbitrary $x$, observe that $X_x$ is contained in any open $T$-stable neighborhood of $x$ in $X$. Thus, to check that $X_x$ is open and affine, we may assume that $X$ is affine; now $X_x = X$ by the argument above. Let $V$ be a $T$-stable complement to $m_x^2$ in $m_x$. Then $V$ generates the algebra of regular functions on $X_x$ (this follows from the graded version of Nakayama's lemma; see e.g., [E] p. 135). Thus the corresponding map $X_x \to V^*$ is a closed equivariant embedding. Furthermore, $V$ is isomorphic to $(T_xX)^*$. □

**Proposition A3.** *Let $X$ be an affine variety with a $\mathbf{G}_m$-action and an attractive fixed point $x$. Then there exists a $\mathbf{G}_m$-module $V$ and a finite equivariant surjective morphism $\pi : X \to V$ such that $\pi^{-1}(0) = \{x\}$ (as a set).*

*Proof.* Let $A$ be the algebra of regular functions over $X$. Then $A = \bigoplus_{n=0}^{\infty} A_n$ is positively graded by the $\mathbf{G}_m$-action. For any positive integer $r$, set $A^{(r)} := \bigoplus_{n=0}^{\infty} A_{nr}$. Then $A$ is a finite module over $A^{(r)}$, and there exists $r$ such that $A^{(r)}$ is generated by its elements of minimal degree. So we can assume that $A$ is generated by its elements of degree 1.

For any irreducible component $Y$ of $X$, the set of $f \in A_1$ such that $f(Y) = 0$ is a proper linear subspace of $A_1$. So there exists $f \in A_1$ such that $f(Y) \neq 0$ for all such $Y$. Let $X' \subset X$ be the zero set of $f$; then $x \in X'$ and $\dim(X') = d - 1$ where $d = \dim(X)$. So we construct inductively $f = f_1, f_2, \ldots, f_d \in A_1$ such that $x$ is their unique common zero. Consider the morphism $\pi = (f_1, f_2, \ldots, f_d) : X \to \mathbf{A}^d$. Then $\pi$ is equivariant for the $\mathbf{G}_m$-action on $\mathbf{A}^d$ by multiplication, and $\pi^{-1}(0) = \{x\}$: the quotient of $A$ by its ideal generated by $f_1, \ldots, f_d$ is finite dimensional. By the graded version of Nakayama's lemma, it follows that $\pi$ is finite. Because $\dim(X) = d$, the map $\pi$ is dominant, and hence surjective. □

**Proposition A4.** *Let $X$ be a connnected variety with a nontrivial action of a torus $T$ and a fixed point $x$. Then there exists a closed irreducible $T$-stable curve $C \subset X$ which contains $x$ as an isolated fixed point.*

*Proof.* By induction on the dimension of $X$ at $x$, the case of dimension one being trivial. We may assume that $X$ is affine and irreducible. Let $\pi : X \to X/\!\!/T$ be the quotient in the sense of geometric invariant theory. Then $\pi$ is surjective, and its fibers are connected; because $T$ acts nontrivially on $X$, these fibers are infinite. In particular, $\pi^{-1}\pi(x) = \{y \in X \mid x \in \overline{Ty}\}$

is infinite. Let $y \in \pi^{-1}\pi(x)$, $y \neq x$. If $\dim(Ty) = 1$, we can take $C = \overline{Ty}$; otherwise, we can choose $z \in \overline{Ty} - Ty$, $z \neq x$. Then $x \in \overline{Tz}$ with $\dim(Tz) < \dim(Ty)$, and we conclude by induction. □

**Proposition A5.** *Let $T$ be a torus acting on a variety $X$ and let $\mathcal{O} \subset X$ be an orbit. Then $\mathcal{O}$ admits an open affine $T$-stable neighborhood $U$ in $X$, with an equivariant retraction $\pi : U \to \mathcal{O}$.*

*Proof.* We may assume that $X$ is affine. Let $f$ be a regular function on $X$ which vanishes identically on $\overline{\mathcal{O}} - \mathcal{O}$ but not on $\mathcal{O}$, and which is an eigenvector of $T$. Then $f$ has no zero in the orbit $\mathcal{O}$, and therefore $\mathcal{O}$ is closed in the open affine $T$-stable subset $X \cap (f \neq 0)$. Thus, we may assume that $\mathcal{O}$ is closed in $X$.

The orbit $\mathcal{O}$ is isomorphic to a torus. Choose such an isomorphism $f : \mathcal{O} \to \mathbf{G}_m^n$. Then the coordinate functions $f_1, \ldots, f_n$ are eigenvectors of $T$. Since $\mathcal{O}$ is closed in $X$, we can extend $f_1, \ldots, f_n$ to regular functions on $X$, eigenvectors of $T$. They define an equivariant morphism $F : X \to \mathbf{A}^n$ which maps $\mathcal{O}$ isomorphically to $\mathbf{G}_m^n$. Then we can take $U = F^{-1}(\mathbf{G}_m^n)$. □

**Proposition A6.** *Let $T$ be a torus acting on a variety $X$. Let $T' \subset T$ be a subtorus, and $i_{T'} : X^{T'} \to X$ the inclusion of the fixed point set. Then the map*

$$i_{T'}^* : H_T^*(X) \to H_T^*(X^{T'})$$

*becomes an isomorphism after inverting finitely many characters of $T$ which restrict nontrivially to $T'$.*

*Proof.* Observe that the kernel and cokernel of $i_{T'}^*$ are both modules over $H_T^*(X - X^{T'})$. Thus, it is enough to prove that $H_T^*(X - X^{T'})$ is killed by a product of characters which restrict nontrivially to $T'$. In other words, we may assume that $T'$ fixes no point of $X$.

Let $U \subset X$ and $\mathcal{O}$ be as in Proposition 4 above. Then $H_T^*(U)$ is a module over $H_T^*(\mathcal{O})$ and the latter is killed by all characters which restrict trivially to the isotropy group $\Gamma$ of $\mathcal{O}$. Since $T'$ fixes no point of $\mathcal{O}$, we can find a character $\chi$ which restricts trivially to $\Gamma$ but not to $T'$. Now the kernel and cokernel of the map $H_T^*(X) \to H_T^*(U)$ are modules over $H_T^*(X - U)$, and we conclude by Noetherian induction. □

## References


[A] A. Arabia, *Classes d'Euler équivariantes et points rationnellement lisses*, Annales de l'Institut Fourier **48** (1998), 861–912.

[BDP] E. Bifet, C. De Concini, and C. Procesi, *Cohomology of regular embeddings*, Adv. Math. **82** (1990), 1–34.

[Br] G. E. Bredon, *Introduction to Compact Transformation Groups*, Academic Press, New York, 1972. Russian translation: Г. Бредон, *Введение в теорию компактных групп преобразований*, М., Наука, Гл. ред. физ.-мат. лит., 1980.

[B1] M. Brion, *The behaviour at infinity of the Bruhat decomposition*, Comment. Math. Helv. **73** (1998), 137–174.

[B2] M. Brion, *Equivariant cohomology and equivariant intersection theory*, In: Representation Theories and Algebraic Geometry (A.Broer and A. Daigneault, eds.); Kluwer, 1998, pp. 1–37.

[C] J. B. Carrell, *The Bruhat graph of a Coxeter group, a conjecture of Deodhar, and rational smoothness of Schubert varieties*, In: Algebraic Groups and their Generalizations (W. Haboush and B. Parshall, eds.); American Mathematical Society, 1994, pp. 53–62.

[D] M. Dyer, *The nil Hecke ring and Deodhar's conjecture on Bruhat intervals*, Invent. Math. **111** (1993), 571–574.

[E] D. Eisenbud, *Commutative Algebra with a View Towards Algebraic Geometry*, Springer, New York, 1994.

[H] W. Y. Hsiang, *Cohomology Theory of Topological Transformation Groups*, Springer, New York, 1975. Russian translation: У. И. Сян, *Когомологическая теория топологических групп преобразоеваний*, М., Мир, 1979.

[J] R. Joshua, *Vanishing of odd-dimensional intersection cohomology*, Math. Z. **195** (1987), 239–253.

[Ka] M. Kashiwara, *Representation theory and D-modules on flag manifolds*, in: Orbites et Faisceaux Pervers; vol. 173–174, Astérisque, 1989, pp. 55–109.

[KL1] D. Kazhdan and G. Lusztig, *Representations of Coxeter groups and Hecke algebras*, Invent. Math. **53** (1979), 165–184.

[KL2] D. Kazhdan and G. Lusztig, *Schubert varieties and Poincaré duality*, In: Proc. Symp. Pure Math.; vol. 36, American Mathematical Society, 1980, pp. 185–203.

[Kr] M. Krämer, *Sphärische Untergruppen in kompakten zusammenhängenden Liegruppen*, Compositio Math. **38** (1979), 129–153.

[Ku] S. Kumar, *The nil Hecke ring and singularity of Schubert varieties*, Invent. Math. **123** (1996), 476–506.

[MS] J. G. M. Mars and T. A. Springer, *Hecke algebra representations related to spherical varieties*, Represent. Theory **2** (1998), 33–69.

[PPR] E. A. Pennel, M. S. Putcha and L. E. Renner, *Analogues of the Bruhat–Chevalley order for reductive monoids*, Journal of Algebra **196** (1997), 339–368.

[Po] P. Polo, *On Zariski tangent spaces of Schubert varieties, and a proof of a conjecture of Deodhar*, Indag. Math. (N. S.) **4** (1994), 483–493.

[R] R. W. Richardson, *On orbits of algebraic groups and Lie groups*, Bull. Austral. Math. Soc. **25** (1982), 1–28.

[RS] R. W. Richardson and T. A. Springer, *The Bruhat order on symmetric varieties*, Geometriae Dedicata **35** (1990), 389–436.

[S1] T. A. Springer, *Some results on algebraic groups with involutions*, In: Algebraic groups and related topics; Kinokuniya–North Holland, 1985, pp. 525–543.

[S2] T. A. Springer, *A combinatorial result on K-orbits on a flag manifold*, In: Proceedings of the Sophus Lie Memorial Conference; Scandinavian University Press, 1994, pp. 363–372.

[Su] H. Sumihiro, *Equivariant completion*, J. Math. Kyoto University **14** (1974), 1–28.

[V] J–L. Verdier, *Classe d'homologie associée à un cycle*, in: Séminaire de Géométrie Analytique; vol. 36–37, Astérisque, 1976, pp. 101–151.