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Subdirectly irreducible and free Kleene–Stone algebras

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Abstract. A Kleene–Stone algebra is a bounded distributive lattice with two unary operations that make it a Kleene and a Stone algebra. In this paper, we determine all subdirectly irreducible Kleene–Stone algebras, and describe the free Kleene–Stone algebra on a finite set of generators as a product of certain free Kleene algebras endowed with a Stone negation.

There are several generalizations of Boolean algebras in the literature in which negation is replaced by several new unary operations, which satisfy some of the properties of the original operation. Some examples are: double p-algebras (Varlet [14]), Heyting algebras with pseudocomplementation, and pseudocomplemented DeMorgan algebras (Sankappanavar [10] and [11]). Kleene–Stone algebras form a subclass of pseudocomplemented DeMorgan algebras, and have received recently some attention from people working on multiple-valued logics. See Epstein and Mukaidono [4] and [5], and Takagi and Mukaidono [12] and [13]. Finite subdirectly irreducible pseudcomplemented DeMorgan algebras were characterized by Romanowska [9]. In. [11], Sankappanavar studies a variety V_0 of distributive pseudcomplemented DeMorgan algebras, lists all its subdirectly irreducibles, and describes the lattice of subvarieties of V_0. In this paper, we show that the variety of Kleene–Stone algebras is one of those subvarieties, and use this result to describe the finitely generated free Kleene–Stone algebras.

Here is a summary of our main results. There is a 5-element Kleene–Stone algebras $\mathbf{L}_5$ with the property that the subdirectly irreducible Kleene–Stone algebras are precisely the subalgebras of $\mathbf{L}_5$. See Theorem (1.3). The free Kleene–Stone algebras on a finite generating set X can be expressed as a direct product of certain free Kleene algebras endowed with a suitable Stone negation, where the product is indexed by a certain subset of the free Boolean algebra on the set X. See Theorem (2.4). This result allows us to express the free spectrum of the variety of Kleene–Stone algebras in terms of the free spectrum of the variety of Kleene algebras. See

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Theorem (2.5). We remark that no formula is known for the free spectrum of the variety of Kleene algebras. Epstein and Mukaidono [4] list the 24 elements of the free Kleene–Stone algebra in one generator.

As pointed out by one referee, the variety of Kleene–Stone algebras has the Fraser–Horn and Apple properties. In particular, the methods of Berman and Blok [2] present another context for describing the free Kleene–Stone algebras. Results related and complementary to those that we present here have been obtained, from the point of view of fuzzy functions, by Takagi and Mukaidono [12] and [13]. The results of this paper were originally announced in the "Abstracts and Summaries" section of the Bulletin of the Multiple-Valued Logic Technical Committee [7].

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1. Subdirectly irreducible algebras

In this section, we define and give properties of Kleene, Stone and Kleene–Stone algebras, and we determine the subdirectly irreducible Kleene–Stone algebras. Properties of Kleene and Stone algebras can be found in Balbes and Dwinger [1] and in Epstein and Mukaidono [5].

1.1. DEFINITION. (a) A *Kleene algebra* is a bounded distributive lattice with a unary operation $\sim$ which satisfies:

K1: $\sim\sim x = x$.

K2: $\sim(x \wedge y) = \sim x \vee \sim y$

K3: $x \wedge \sim x \leq y \vee \sim y$.

(b) A distributive lattice A with 0 is called *pseudocomplemented* provided each $x \in A$ has a pseudocomplement, i.e., a largest element disjoint from x. If A is a pseudocomplemented distributive lattice, then each $x \in A$ has a unique psuedocomplement; it will be denoted $\neg x$. We will view a pseudocomplemented distributive lattice as a distributive lattice A with 0 and a unary operation $\neg$ which satisfies:

S1: $x \wedge \neg x = 0$.

S2: $x \wedge y = 0 \Rightarrow y \leq \neg x$.

(c) A *Stone algebra* is a pseudocomplemented distributive lattice which satisfies the identity:

S3: $\neg x \vee \neg\neg x = 1$.

(d) A *Kleene–Stone algebra* is a bounded distributive lattice with two unary operations $\sim$ and $\neg$ such that it is a Kleene algebra with $\sim$ and a Stone algebra with $\neg$.

Although (S2) is only a quasi-identity, the class of pseudocomplemented distributive lattices does form a variety (Ribenboim [8]; see [1, p. 155]). It then follows that the class of Kleene–Stone algebras also forms a variety, which we denote $\mathscr{K}\mathscr{S}$.

It is easy to check that a Kleene algebra is a DeMorgan algebra and therefore a Kleene–Stone algebra is a distributive pseudcomplemented DeMorgan algebra (PCDM); see [11].

1.2. LEMMA. *Let A be a Kleene–Stone algebra, and $a \in A$.*
(a) *If a is complemented, then its complement is $\sim a = \neg a$.*
(b) *$\neg a$ is complemented.*
(c) $\neg a \leq \sim a$
(d) $\neg \sim a \leq a$
(e) $\neg a = \neg \sim \neg a$
(f) $a \wedge \neg \sim a = \neg \sim (a \wedge \neg \sim a)$

Proof. (a), (b), and (c) are well-known facts. See for example [5].
(d) From (c) and K1, $\neg \sim a \leq \sim \sim a = a$.
(e) Follows from (b) and (a).
(f) From (d) we get a $\wedge \neg \sim a = \neg \sim a$. From (e), $\neg \sim a = \neg \sim \neg \sim a$. □

Clearly, a bounded distributive lattice in which 0 is meet irreducible, is a Stone algebra with Stone negation given by:

$$\neg x = \begin{cases} 0 & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases} \tag{1}$$

This can be checked directly, or obtained from the structure theorem for Stone algebras of Grätzer and Schmidt [6, Theorem 2]. A finitely generated free Kleene algebra has a unique atom $\bigwedge_{x \in X} (x \wedge \sim x)$, where X is the generating set. It follows that in a free Kleene algebra A, 0 is meet irreducible since each element of A belongs to a finitely generated free subalgebra of A. Therefore A is a Kleene–Stone algebra with $\neg$ as given in (1).

We now characterize the subdirectly irreducible Kleene–Stone algebras, using two results about pseudocomplemented DeMorgan algebras from [11].

1.3. THEOREM. *The subdirectly irreducible Kleene–Stone algebras are the subalgebras of the five element chain $\mathbf{L}_5 = \{0, b, h, t, 1\}$, with $0 < b < h < t < 1$ and unary operations given by:*

	0	b	h	t	1
$\sim$	1	t	h	b	0
$\neg$	1	0	0	0	0

Therefore $\mathscr{K}\mathscr{S} = V(\mathbf{L}_5)$.

Proof. Clearly $\mathbf{L}_5$ is a bounded distributive lattice and $\sim$ satisfies K1–K3. Since 0 is meet irreducible, $\mathbf{L}_5$ is a Kleene–Stone algebra, and $V(\mathbf{L}_5)$ is a subvariety of $\mathcal{KS}$. In [11, Theorem 7.3] $V(\mathbf{L}_5)$ is characterized as the variety of distributive pseudocomplemented DeMorgan algebras that satisfy (1.2.f), (1.2.d) and K3. Therefore $\mathcal{KS}$ is a subvariety of $V(\mathbf{L}_5)$. The result now follows from [11, Theorem 6.5]. □

2. Free algebras

In this section, we describe the free Kleene–Stone algebra on a finite set of generators in terms of free Kleene algebras. The description can be easily extended to an infinite free generating set, but we only present the simpler case, since our main concern is the free spectrum of $\mathcal{KS}$.

Throughout this section, X denotes a finite set, and $F_{\mathcal{B}}(X)$ denotes the free Boolean algebra on the set X. In $F_{\mathcal{B}}(X)$, a *literal* is a term of the form x or x' for $x \in X$, where $'$ denotes complementation in $F_{\mathcal{B}}(X)$; a *monomial* is a conjunction of literals (including the empty conjunction, which, by convention, is the constant 1). Let M denote the set of all non-zero monomials in $F_{\mathcal{B}}(X)$. Given $m \in M$, let $\operatorname{supp}(m) = \{x \in X \mid m \leq x \text{ or } m \leq x'\}$, let $X_m = X \backslash \operatorname{supp}(m)$, and let A_m be the free Kleene algebra on the set X_m. As noted in Section 1 each A_m is a Kleene–Stone algebra, and hence so is the product $A = \prod_{m \in M} A_m$. We will show that A is the free Kleene–Stone algebra on the set X.

First, we embed X in A. For $a \in A$, let $\pi_m(a) \in A_m$ denote the image of a under the projection map $\pi_m : A \to A_m$. Given $x \in X$, define $\hat{x} \in A$ as follows:

$$\pi_m(\hat{x}) = \begin{cases} 1, & \text{if } m \leq x \\ 0, & \text{if } m \leq x' \\ x, & \text{if } m \in X_m, \end{cases}$$

and let $\hat{X} = \{\hat{x} \mid x \in X\}$.

Let $m \in M$. Since A_m is the free Kleene algebra on X_m, the restriction to X_m of the function $x \mapsto \hat{x} : X \to A$ extends to a morphism of Kleene algebras from A_m to A. This morphism will be denoted $\iota_m : A_m \to A$. Note that the composition $\pi_m \circ \iota_m$ is the identity morphism on A_m. We remark that ι_m is usually not a morphism of Stone algebras. Also, define $p_m, q_m \in A$ as follows:

$$p_m = \Bigl(\bigwedge_{m \leq x} (\neg \sim \hat{x})\Bigr) \wedge \Bigl(\bigwedge_{m \leq x'} (\neg \hat{x})\Bigr) \quad \text{and} \quad q_m = p_m \wedge \Bigl(\bigwedge_{l < m} (\neg p_l)\Bigr).$$

Note that p_m, q_m and each $\iota_m(t)$ for $t \in A_m$ are in the subalgebra of A generated by $\hat{X}$.

2.1. LEMMA. *For all m, $l \in M$,*

(a) $\pi_l(p_m) = \begin{cases} 1, & \text{if } l \leq m \\ 0, & \text{otherwise.} \end{cases}$

(b) $\pi_l(q_m) = \begin{cases} 1, & \text{if } l = m \\ 0, & \text{otherwise.} \end{cases}$

Proof. (a) Since the projection map $\pi_m : A \to A_m$ is a morphism of Kleene–Stone algebras, we have

$$\pi_l(p_m) = \left(\bigwedge_{m \leq x} (\neg \sim \pi_l(\hat{x})) \right) \wedge \left(\bigwedge_{m \leq x'} (\neg \pi_l(\hat{x})) \right).$$

First assume $l \leq m$. If $m \leq x$, then $l \leq x$, so that $\pi_l(\hat{x}) = 1$, which gives $\neg \sim \pi_l(\hat{x}) = 1$. If $m \leq x'$, then $l \leq x'$, so that $\pi_l(\hat{x}) = 0$, which gives $\neg \pi_l(\hat{x}) = 1$. Thus, $\pi_l(p_m) = 1$.

Next assume $l \nleq m$. Then there exists $x \in X$ such that either $m \leq x$ and $l \nleq x$ or $m \leq x'$ and $l \nleq x'$. In the first case, $\pi_l(\hat{x}) < 1$ so $\neg \sim \pi_l(\hat{x}) = 0$. In the second case, $\pi_l(\hat{x}) > 0$ so $\neg \pi_l(\hat{x}) = 0$. Thus, $\pi_l(p_m) = 0$.

Part (b) follows from part (a). □

2.2. PROPOSITION. *A is generated by $\hat{X}$ as a Kleene–Stone algebra.*

Proof. In fact, we will prove that for each $a \in A$, $a = \bigvee_{m \in M} (\iota_m(\pi_m(a)) \wedge q_m)$. The proposition follows, since for each $m \in M$, q_m and the image of ι_m are in the subalgebra of A generated by $\hat{X}$. The equality above can be checked componentwise: $a = b$ in A iff for all $l \in M$, $\pi_l(a) = \pi_l(b)$. Let $l \in M$. Then

$$\begin{aligned} \pi_l\left(\bigvee_{m \in M} (\iota_m(\pi_m(a)) \wedge q_m) \right) &= \bigvee_{m \in M} (\pi_l(\iota_m(\pi_m(a))) \wedge \pi_l(q_m)) \\ &= (\pi_l(\iota_l(\pi_l(a))) \wedge \pi_l(q_l)) \\ &= \pi_l(a). \end{aligned}$$

(The first equality uses the fact that π_l is a morphism of distributive lattices. The second and third equality use part (b) of Lemma 2.1. The third equality also uses the fact that $\pi_l \circ \iota_l$ is the identity on A_l.) This completes the proof of Proposition 2.2. □

Next we show that A is freely generated by $\hat{X}$. We start with the algebra $\mathbf{L}_5$ defined in Theorem 1.3.

2.3. LEMMA. *If* $\phi : X \to \mathbf{L}_5$ *is a function, then there is a unique morphism* $\Phi : A \to \mathbf{L}_5$ *of Kleene–Stone algebras such that* $\Phi(\hat{x}) = \phi(x)$, *for each* $x \in X$.

Proof. We associate to ϕ the following monomial $m \in M$:

$$m = \left(\bigwedge_{\phi(x)=1} x\right) \wedge \left(\bigwedge_{\phi(x)=0} x'\right).$$

Let $\Phi_m : A_m \to \mathbf{L}_5$ denote the morphism of Kleene algebras induced by the restriction of ϕ to X_m, and let $\Phi = \Phi_m \circ \pi_m$. Clearly, Φ is a morphism of Kleene algebras.

To show that Φ is a morphism of Stone algebras, it suffices to show that Φ_m is a morphism of Stone algebras, so let $t \in A_m$. If $t = 0$, then

$$\Phi_m(\neg 0) = \Phi_m(1) = 1 = \neg 0 = \neg \Phi_m(0).$$

If $t \neq 0$, then, since $\Phi_m(X_m) \subseteq \mathbf{L}_5 \backslash \{0, 1\}$ and in $\mathbf{L}_5$, 0 is meet irreducible, we must have $\Phi_m(t) \neq 0$. Thus

$$\Phi_m(\neg t) = \Phi_m(0) = 0 = \neg \Phi_m(t),$$

as required.

To see that Φ behaves correctly on $\hat{X}$, note that if $x \in X_m$, then

$$\Phi(\hat{x}) = \Phi_m(\pi_m(\hat{x})) = \Phi_m(x) = \phi(x).$$

If $x \notin X_m$, then either $m \leq x$ or $m \leq x'$. In the first case, $\phi(x) = 1$; in the second case, $\phi(x) = 0$. If $m \leq x$, then $\pi_m(\hat{x}) = 1$, and if $m \leq x'$, then $\pi_m(\hat{x}) = 0$. In both cases, it follows that $\Phi(\hat{x}) = \phi(x)$.

Finally, Φ is uniquely determined, since $\hat{X}$ generates A as a Kleene–Stone algebra. □

2.4. THEOREM. *A is the free Kleene–Stone algebra on* $\hat{X}$.

Proof. Let B be a Kleene–Stone algebra, and let $\phi : X \to B$ be a function. By Theorem (1.3), B is a subalgebra of a direct product of copies of $\mathbf{L}_5$. By Lemma (2.3), the composition of ϕ with each projection map in the direct product extends uniquely to a morphism from A to the corresponding factor. The universal property of the direct product gives a unique morphism $\Phi : A \to B$ such that $\Phi(\hat{x}) = \phi(x)$ for each $x \in X$, as required.

2.5. THEOREM. *The free spectrum of the variety* $\mathcal{KS}$ *of Kleene–Stone algebras, is given by:*

$$|F_{\mathcal{KS}}(n)| = \prod_{i=0}^{n} |F_{\mathcal{K}}(i)|^{\binom{n}{i}2^{n-i}}$$

where $F_{\mathcal{K}}(i)$ *is the cardinality of the free Kleene algebra on i generators.*

Proof. This follows from the fact that X_m (the set of free generators of A_m as a free Kleene algebra) has i elements if m is the conjunction of $n - i$ literals, and the fact that there are exactly $\binom{n}{i}2^{n-i}$ such monomials in M.

From Theorem 2.5 it follows that

$$\lg |F_{\mathcal{KS}}(n)| = \sum_{i=0}^{n} \binom{n}{i} 2^{n-i} \lg |F_{\mathcal{K}}(i)|$$

and therefore, any bound of the form $\alpha a^n \leq \lg |F_{\mathcal{K}}(n)| \leq \beta b^n$ for $\lg |F_{\mathcal{K}}(n)|$ yields a bound of the form $\alpha(a+2)^n \leq \lg |F_{\mathcal{KS}}(n)| \leq \beta(b+2)^n$ for $\lg |F_{\mathcal{KS}}(n)|$.

For example, it is easy to show that $2^n \leq \lg |F_{\mathcal{K}}(n)| \leq 3^n$ (cf. [3]) and therefore we immediately get $4^n \leq \lg |F_{\mathcal{KS}}(n)| \leq 5^n$. In fact, we can do better.

2.6. THEOREM. *There is a constant* $C > 0$ *such that if* $n \geq 1$, *then* $Cn^{-1/2}5^n \leq \lg |F_{\mathcal{KS}}(n)| \leq 5^n$.

Proof. It is not difficult to show that there exists a constant $C'' > 0$ such that if $n \geq 1$, then

$$C''\left(\frac{27}{4}\right)^{n/3}\left(\frac{9}{4\pi n}\right)^{1/2} \leq \binom{n}{\lceil \frac{2n}{3} \rceil}.$$

(This can be done using induction in three separate cases according to n's congruence class modulo 3, or using Stirling's approximation formula.) Combining this with the following bound from [3]

$$\lg |F_{\mathcal{K}}(n)| \geq 2^{\lceil 2n/3 \rceil}\binom{n}{\lceil \frac{2n}{3} \rceil}.$$

we obtain

$$\begin{aligned}\lg |F_{\mathscr{K}\mathscr{S}}(n)| &\geq \sum_{i=1}^{n} \binom{n}{i} 2^{n-i} 2^{\lceil 2i/3 \rceil} C'' \left(\frac{27}{4}\right)^{i/3} \left(\frac{9}{4\pi i}\right)^{1/2} \\ &\geq C' n^{-1/2} 2^{n} \sum_{i=1}^{n} \binom{n}{i} 2^{-i/3} \left(\frac{27}{4}\right)^{i/3} \\ &= C' n^{-1/2} 2^{n} \left(\left(\frac{5}{2}\right)^{n} - 1 \right) \\ &\geq C n^{-1/2} 5^{n},\end{aligned}$$

with $C = 3C'/5$. □

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