Algebra Universalis, 31 (1994) 516–541 0002–5240/94/040516–26$01.50 + 0.20/0

A topological duality for some lattice ordered algebraic structures including ℓ-groups

NÉSTOR G. MARTÍNEZ*

Dedicated to my wife Eugenia

Abstract. A topological duality is developed for a wide class of lattice ordered algebraic structures by introducing in an ordered Stone space a natural binary and continuous function. In particular, duality theorems are obtained for ℓ-groups and for abelian ℓ-groups.

Introduction

Having in mind the well known topological representation of M. H. Stone for distributive lattices [9] and the duality theory of H. Priestley [7], [8], we develop a topological duality for the wide class of lattice ordered algebraic structures given by *implicative lattices* (Definition 1.1). Examples of implicative lattices are provided by structures coming from algebra, such as lattice ordered groups, and by structures coming from logic, such as Boolean algebras and Wajsberg algebras (in fact, this research can be viewed as an extension of the author's previous work [5]). Using as a basis our duality for implicative lattices we characterize ℓ-groups as implicative lattices with a distinguished element and obtain duality theorems both for ℓ-groups and for abelian ℓ-groups.

An important consequence of our results is that the algebraic structure can be restored from the lattice spectrum endowed with a natural binary and continuous function.

In §1 we introduce implicative lattices and give some examples to show the scope of our duality theory.

Presented by W. Taylor.
Received January 7, 1992; accepted in final form October 25, 1992.

* This research is a part of the Doctorial Thesis that the author presented at the Universidad de Buenos Aires, under the supervision of Prof. R. Cignoli. The author wishes to thank Prof. Cignoli for his guidance. The author also wishes to thank Keith Kearnes for his careful reading of this paper and for his deep suggestions that led to a substantial improvement of the first version.

The main result of §2 is the Representation Theorem (Theorem 2.6) which states that each implicative lattice is isomorphic to an implicative lattice of sets.

To develop our duality we need for technical reasons a slight variant of the Stone duality for distributive lattices as presented in [1]; following ideas of H. Priestley we consider in §3 *ordered Stone spaces with endpoints* (Definition 3.3) and obtain an appropriate duality theory for both distributive lattices and bounded distributive lattices based on these spaces.

In §4 (Definition 4.1) we introduce the dual spaces of implicative lattices; they are ordered Stone spaces with a binary continuous function satisfying certain algebraic and topological conditions. The main result of this section is Theorem 4.8, the duality theorem for implicative lattices.

Finally in §5 we establish the topological dualities both for ℓ-groups (Theorem 5.11) and for abelian ℓ-groups (Theorem 5.15). The dual space in the abelian case is especially nice: it happens to be a compact ordered abelian topological semi-group.

1. Implicative lattices. Definition and examples

DEFINITION 1.1. $\mathbf{A} = \langle A, \vee, \wedge, \rightarrow \rangle$ is an *implicative lattice* iff $\langle A, \vee, \wedge \rangle$ is a distributive lattice and $\rightarrow$ is a binary operation (called the *implication* of $\mathbf{A}$) satisfying the following equations:

$IL(1)$ $\quad x \rightarrow (y \wedge y') = (x \rightarrow y) \wedge (x \rightarrow y')$

$IL(2)$ $\quad (x \vee x') \rightarrow y = (x \rightarrow y) \wedge (x' \rightarrow y)$

$IL(3)$ $\quad x \rightarrow (y \vee y') = (x \rightarrow y) \vee (x \rightarrow y')$

$IL(4)$ $\quad (x \wedge x') \rightarrow y = (x \rightarrow y) \vee (x' \rightarrow y).$

EXAMPLE 1.2. Let $\mathbf{B} = \langle B, \vee, \wedge, \neg, 0, 1 \rangle$ be a Boolean algebra and consider the usual implication $x \rightarrow y = \neg x \vee y$. Then $\langle B, \vee, \wedge, \rightarrow \rangle$ is an implicative lattice.

EXAMPLE 1.3. We'll say (as in [6]) that $\langle A, \vee, \wedge, \neg \rangle$ is a *De Morgan algebra* iff $\langle A, \vee, \wedge \rangle$ is a distributive lattice (not necessarily bounded) and $\neg$ is an unary operation satisfying $\neg(\neg x) = x$ and $\neg(x \vee y) = \neg x \wedge \neg y$. Let $\mathbf{A} = \langle A, \vee, \wedge, \neg \rangle$ be a De Morgan algebra and define again $x \rightarrow y = \neg x \vee y$. It follows at once that $\langle A, \vee, \wedge, \rightarrow \rangle$ is an implicative lattice.

EXAMPLE 1.4. Recall that $\mathbf{G} = \langle G, \vee, \wedge, \cdot, ^{-1}, e\rangle$ is a *lattice ordered group* (ℓ-group for short) iff $\langle G, \vee, \wedge\rangle$ is a lattice, $\langle G, \cdot, ^{-1}, e\rangle$ is a group and for each $a, b, c \in G$, $c(a \vee b) = ca \vee cb$; $(a \vee b)c = ac \vee bc$; $c(a \wedge b) = ca \wedge cb$; $(a \wedge b)c = ac \wedge bc$.

Let's define $x \to y = x^{-1}y$; it is well known ([4], p. 67) that the lattice of an ℓ-group is distributive and that for each $a, b \in G$, $(a \wedge b)^{-1} = a^{-1} \vee b^{-1}$; from this it can be readily shown that $\langle G, \vee, \wedge, \to\rangle$ is an implicative lattice.

EXAMPLE 1.5. This example follows a suggestion of the referee.

Let $\mathbf{A} = \langle A, \vee, \wedge\rangle$ be a distributive lattice and let End($\mathbf{A}$) be the set of the lattice endomorphisms of A. End($\mathbf{A}$) is a partial lattice under the operations of pointwise join and meet.

Now let's consider the lattice $\mathbf{A}^{\mathscr{V}} = \langle A, \vee', \wedge'\rangle$ where $\vee' = \wedge$ and $\wedge' = \vee$. It is not difficult to check that each lattice homomorphism ϕ from $\mathbf{A}^{\mathscr{V}}$ to a sublattice of End($\mathbf{A}$) yields an implication on $\mathbf{A}$ defined by $x \to y = \phi(x)(y)$. This happens to be the way all implications on $\mathbf{A}$ arise: if $\langle A, \vee, \wedge, \to\rangle$ is an implicative lattice, then $\to$ is a lattice endomorphism in its second variable for each fixed choice of the first variable. For each fixed second variable $\to$ is join and meet inverting in its first variable. It follows that $\{h_x : \mathbf{A} \to \mathbf{A}$ such that $x \in A$ and $h_x(y) = x \to y\}$ is a sublattice of End($\mathbf{A}$) and $\to$ determines a lattice homomorphism from $\mathbf{A}^{\mathscr{V}}$ to this sublattice given by $x \to h_x$.

2. Representation by sets

Recall that a non empty subset $\mathbf{I}$ of a lattice $\mathbf{L} = \langle L, \vee, \wedge\rangle$ is a *lattice ideal* iff $x \le y$, $y \in \mathbf{I}$ imply $x \in \mathbf{I}$ and $x, y \in \mathbf{I}$ implies $x \vee y \in \mathbf{I}$. $\mathbf{I}$ is called a *prime lattice ideal* if $\mathbf{I} \subsetneq \mathbf{L}$ and satisfies the additional condition $x \wedge y \in \mathbf{I}$ implies $x \in \mathbf{I}$ or $y \in \mathbf{I}$. A non empty subset F of L is a *lattice filter* iff $x \le y$, $x \in F$ imply $y \in F$ and $x, y \in F$ implies $x \wedge y \in F$. F is a *prime lattice filter* iff $F \subsetneq L$ and satisfies the additional condition $x \vee y \in F$ implies $x \in F$ or $y \in F$.

If A is a distributive lattice, let's denote $S(A) = \{P \subseteq A : P$ is a prime lattice filter of $A\}$ and $\overset{*}{S}(A) = S(A) \cup \{\varnothing, A\}$.

PROPOSITION 2.1. *Let* $\mathbf{A} = \langle A, \vee, \wedge, \to\rangle$ *be an implicative lattice. One can define in* $\overset{*}{S}(A)$ *a binary function* $\Phi : \overset{*}{S}(A) \times \overset{*}{S}(A) \to \overset{*}{S}(A)$ *by the stipulation*

$$\Phi(P, Q) = \bigcup_{x \in P} \{y : x \to y \in Q\} \tag{$*$}$$

Proof. If $\Phi(P, Q) = \emptyset$, $\Phi(P, Q) \in \overset{*}{S}(A)$; then we may suppose that $\Phi(P, Q) \neq \emptyset$. Note that in this case $P \neq \emptyset$ and $Q \neq \emptyset$. Let $y, y' \in A$ be such that $y \leq y'$ and $y \in \Phi(P, Q)$. Then there is $x \in P$ such that $x \to y \in Q$. From $IL(1)$ of Definition 1.1, $y \leq y'$ implies $x \to y \leq x \to y'$. Since Q is a filter, $x \to y' \in Q$ and since $x \in P$, $y' \in \Phi(P, Q)$.

If $y, y' \in \Phi(P, Q)$ let $x, x' \in P$ such that $x \to y \in Q$ and $x' \to y' \in Q$. From $IL(2)$ of Definition 1.1, $x \to y \leq (x \wedge x') \to y$ and $x' \to y' \leq (x \wedge x') \to y'$. Since Q is a filter, $((x \wedge x') \to y) \wedge ((x \wedge x') \to y') \in Q$. From $IL(1)$ of Definition 1.1, $(x \wedge x') \to (y \wedge y') \in Q$. As P is a filter, $x \wedge x' \in P$, so we have $y \wedge y' \in \Phi(P, Q)$. Then, using only $IL(1)$ and $IL(2)$ of Definition 1.1, we have proved that $\Phi(P, Q)$ is a lattice filter. Let now y, y' be such that $y \vee y' \in \Phi(P, Q)$. Then there is $x \in P$ such that $x \to (y \vee y') \in Q$. From $IL(3)$ of Definition 1.1 we have $(x \to y) \vee (x \to y') \in Q$. As $Q = A$ or Q is a prime lattice filter, it follows that $x \to y \in Q$ or $x \to y' \in Q$. Then $y \in \Phi(P, Q)$ or $y' \in \Phi(P, Q)$. □

OBSERVATION 2.2. From (∗) it follows that for all $P \in \overset{*}{S}(A)$, $\Phi(\emptyset, P) = \Phi(P, \emptyset) = \emptyset$ and, if $P \neq \emptyset$, $\Phi(P, A) = A$. Also, Φ is order preserving in each variable with respect to the set-theoretical inclusion.

OBSERVATION 2.3. Φ can not be defined in $S(A)$, as the following examples show:

Let $2 = \{0, 1\}$ with the order $0 < 1$ and define for all x, y, $x \to y = 0$. Then 2 becomes an implicative lattice. Note that $\{1\} \in S(2)$ but $\Phi(\{1\}, \{1\}) = \emptyset$.

Again in 2, now with the implication $x \to y = 1$ for all x, y, 2 is an implicative lattice and $\Phi(\{1\}, \{1\}) = 2$.

The following lemma will have a key role in the sequel.

LEMMA 2.4. *Let* $\mathbf{A} = \langle A, \vee, \wedge, \to \rangle$ *be an implicative lattice. Let* $P \in \overset{*}{S}(A)$ *and* $a \in A$; *let's define* $P_a = \{x \in A : x \to a \notin P\}$. *Then* (i) $P_a \in \overset{*}{S}(A)$; (ii) P_a *is the greatest* Q (*with respect to* $\subseteq$) *such that* $a \notin \Phi(Q, P)$; (iii) *If* $a, a' \in A$, *then either* $P_a \subseteq P_{a'}$ *or* $P_{a'} \subseteq P_a$.

Proof. If $x \in P_a$, $x \to a \notin P$. Let $x' \geq x$; from $IL(2)$ of Definition 1.1, $x' \to a \leq x \to a$, then $x' \to a \notin P$. If $x, x' \in P_a$, $x \to a \notin P$ and $x' \to a \notin P$; as $P \in \overset{*}{S}(A)$, $(x \to a) \vee (x' \to a) \notin P$. From $IL(4)$ of Definition 1.1, $(x \wedge x') \to a \notin P$. If $x \vee x' \in P_a$, again from $IL(2)$, $(x \to a) \wedge (x' \to a) \notin P$. As P is a filter, $x \to a \notin P$ or $x' \to a \notin P$. Then $x \in P_a$ or $x' \in P_a$. This proves (i).

If $a \in \Phi(P_a, P)$, there is $x \in P_a$ such that $x \to a \in P$, which is a contradiction. Then $a \notin \Phi(P_a, P)$. Now let $Q \in \overset{*}{S}(A)$ be such that $a \notin \Phi(Q, P)$. For all $x \in Q$, $x \to a \notin P$. Then $Q \subseteq P_a$.

To prove (iii) let $a, a' \in A$ and consider $P_{(a \vee a')} = \{x : x \to (a \vee a') \notin P\}$. From $IL(3)$ of Definition 1.1, $P_{(a \vee a')} \subseteq P_a$ and $P_{(a \vee a')} \subseteq P_{a'}$. Now suppose there exist $b \in P_a \backslash P_{a'}$ and $b' \in P_{a'} \backslash P_a$. Then $b \to a \notin P$ and $b' \to a' \notin P$. From the properties of $\to$, $(b \vee b') \to a \notin P$ and $(b \vee b') \to a' \notin P$, and we obtain $(b \vee b') \notin P$. Then $(b \vee b') \in P_{(a \vee a')}$.

Since $P_{(a \vee a')} \in \overset{*}{S}(A)$, then either $b \in P_{(a \vee a')}$ or $b' \in P_{(a \vee a')}$. Both cases lead to contradiction, because $b \notin P_a$, and $b' \notin P_a$. □

THEOREM 2.5. *Let* $\mathbf{A} = \langle A, \vee, \wedge, \to \rangle$ *be an implicative lattice and* Φ *the binary function* (*) *of Proposition* 2.1. *For each* $P \in \overset{*}{S}(A)$ *let the function* $\Phi_P : \overset{*}{S}(A) \to \overset{*}{S}(A)$ *be defined by* $\Phi_P(Q) = \Phi(P, Q)$. *For each* $a \in A$ *let's denote* $\sigma(a) = \{P \in \overset{*}{S}(A) : a \in P\}$. *Then*

(1) $\sigma(a \to b)$ *can be obtained from* $\sigma(a)$, $\sigma(b)$ *and* Φ *by the formula*

$$\sigma(a \to b) = \bigcap_{P \in \sigma(a)} \Phi_P^{-1}[\sigma(b)] \qquad (**)$$

(2) $\sigma(a \to b)$ *is the greatest subset* $W \subseteq \overset{*}{S}(A)$ *such that* $\sigma(a) \times W \subseteq \Phi^{-1}[\sigma(b)]$.

Proof. Let $Q \in \sigma(a \to b)$ and $P \in \sigma(a)$. Then $a \to b \in Q$ and $a \in P$; from this we get $b \in \Phi(P, Q)$; therefore, $b \in \Phi_P(Q)$ and $Q \in \Phi_P^{-1}[\sigma(b)]$. For the opposite inclusion choose $Q \in \Phi_P^{-1}[\sigma(b)]$ and assume for the moment that $a \to b \notin Q$. Then $a \in Q_b = \{x : x \to b \in Q\}$. From Lemma 2.4 (i) above we have $Q_b \in \sigma(a)$. Now, by the choice of Q we have $Q \in \Phi_{Q_b}^{-1}[\sigma(b)]$, which means that $\Phi(Q_b, Q) \in \sigma(b)$. But this contradicts Lemma 2.4 (ii).

To prove (2) note that from the above discussion $\sigma(a) \times \sigma(a \to b) \subseteq \Phi^{-1}[\sigma(b)]$ and suppose $W \nsubseteq \sigma(a \to b)$. Then, there is $Q \in W$ such that $a \to b \notin Q$. As $(Q_b, Q) \in \sigma(a) \times W$, $b \in \Phi(Q_b, Q)$, a contradiction. □

THEOREM 2.6: *Representation Theorem*

Let $\mathbf{A} = \langle A, \vee, \wedge, \to \rangle$ *be an implicative lattice and consider the family of sets* $\sum(\overset{*}{S}(A)) = \{\sigma(a) : a \in A\}$ *equipped with set-theoretical union and intersection and with the implication* $\sigma(a) \Rightarrow \sigma(b) = \bigcap_{P \in \sigma(a)} \Phi_P^{-1}[\sigma(b)]$.

Then $\langle \sum(\overset{*}{S}(A)), \cup, \cap, \Rightarrow \rangle$ *is an implicative lattice and the map* $a \mapsto \sigma(a)$ *is an isomorphism of implicative lattices.*

Proof. As $\sigma(a) \cup \sigma(b) = \sigma(a \vee b)$ and $\sigma(a) \cap \sigma(b) = \sigma(a \wedge b)$, $\langle \sum(\overset{*}{S}(A)), \cup, \cap \rangle$ is a distributive lattice.

If $a \neq a'$ we may suppose, for example, that $a \nleq a'$. From the Prime Filter Theorem, there is a prime lattice filter P such that $a \in P$ and $a' \notin P$. Then $\sigma(a) \neq \sigma(a')$.

We have that $\langle A, \vee, \wedge \rangle$ and $\langle \sum (\overset{*}{S}(A)), \cup, \cap \rangle$ are isomorphic as distributive lattices. But from Theorem 2.5 it also holds that $\sigma(a \to b) = \sigma(a) \Rightarrow \sigma(b)$; then $\langle \sum (\overset{*}{S}(A)), \cup, \cap, \to \rangle$ is an implicative lattice isomorphic to $\mathbf{A}$. □

3. A topological duality for distributive lattices

Recall that a *duality* (or a *coequivalence*) between two categories $\mathscr{A}$ and $\mathscr{B}$ is a contravariant function $\mathscr{F} : \mathscr{A} \to \mathscr{B}$ satisfying:

(i) For each object B of $\mathscr{B}$ there is an object A of $\mathscr{A}$ such that $F(A)$ and B are isomorphic.

(ii) For each pair of objects A, B of $\mathscr{A}$, the function from $[A, B]_{\mathscr{A}}$ to $[F(B), F(A)]_{\mathscr{B}}$ induced by F is one-one and onto.

In order to obtain a topological duality for implicative lattices we will first develop an appropriated version of the well known Stone duality for distributive lattices as presented in [1].

DEFINITION 3.1. Recall from [1] that a *Stone space* is a topological space X satisfying:

(a) X is T_0 *space* (i.e., for any two distinct points of X there is an open set containing one and not the other).

(b) The family of compact and open subsets of X is a basis for X and a distributive lattice under set-theoretical union and intersection.

(c) If $(U_s)_{s \in S}$ and $(V_t)_{t \in T}$ are non empty families of non empty compact open sets and $\bigcap_{s \in S} U_s \subseteq \bigcup_{t \in T} V_t$, then there exist finite subsets $S' \subseteq S$, $T' \subseteq T$ such that $\bigcap_{s \in S'} U_s \subseteq \bigcup_{t \in T'} V_t$.

With each distributive lattice L one can associate a *Stone space* $S(L)$ whose points are the prime filters of L (in [1], the prime ideals of L) with the topology determinated by the basis $\{\varnothing\} \cup \{\hat{a}: a \in L\}$, where $\hat{a}$ is the set of prime lattice filters containing a (in [1], the set of prime lattice ideals not containing a).

If X is a Stone space, X can be endowed with the following partial order: $x \le y$ iff $x \in \mathrm{Cl}(\{y\})$. We shall say that $\langle X, \tau, \le \rangle$ is an *ordered Stone space* if $\langle X, \tau \rangle$ is a Stone space and $\le$ is defined in this way. For $S(L)$ this order coincides with the natural one: $P \le Q$ iff $P \subseteq Q$.

Note that Theorem 2.4 enables one to express $\sigma(a \to b)$ in terms of $\sigma(a)$ and $\sigma(b)$ together with the binary function Φ. Thus, we want to define our representation space as a space endowed with a binary function. From Observation 2.3 we have that the function Φ of Proposition 2.1 need not be defined at every point of the Stone space $S(L)$ of a lattice L. Thus, we will topologize $\overset{*}{S}(L)$.

PROPOSITION 3.2. *Let L be a distributive lattice. Then*

(i) *The family $\{\sigma(a): a \in L\} \cup \{\varnothing, \overset{*}{S}(L)\}$ is a basis of a topology $\overset{*}{\tau}$ for $\overset{*}{S}(L)$.*

(ii) *The compact and open subsets of $\langle \overset{*}{S}(L), \overset{*}{\tau} \rangle$ are exactly the members of $\{\varnothing, \overset{*}{S}(L)\} \cup \{\sigma(a): a \in L\}$.*

(iii) *$\overset{*}{S}(L) = \langle \overset{*}{S}(L), \overset{*}{\tau}, \subseteq, \varnothing, L \rangle$ is an ordered Stone space such that $\varnothing \subseteq P \subseteq L$ for all $P \in \overset{*}{S}(L)$.*

Proof. (i) follows from the fact that $\{\varnothing, \overset{*}{S}(L)\} \cup \{\sigma(a): a \in L\}$ is a lattice with respect to union and intersection.

For (ii), let U be a compact and open subset such that $U \neq \varnothing$ and $U \neq \overset{*}{S}(L)$. As U is open, U is obtained as a union of members of $\{\varnothing, \overset{*}{S}(L)\} \cup \{\sigma(a): a \in L\}$; since $U \neq \varnothing$ and $U \neq X$, U can be written as $U = \bigcup_{s \in S} \sigma(a_s)$ with $S \neq \varnothing$.

From the compactness of U, it follows that $U = \bigcup_{s \in S'} \sigma(a_s)$ for a suitable finite subset S' of S; then $U = \sigma(V_{s \in S'} a_s)$.

In order to prove that the members of $\{\varnothing, \overset{*}{S}(L)\} \cup \{\sigma(a): a \in L\}$ are compact and open subsets we need to prove the following statement:

(x): If L is a distributive lattice and S, T are non empty subsets of L such that $\bigcap_{a \in S} \sigma(a) \subseteq \bigcap_{b \in T} \sigma(b)$, there exist finite subsets $S' \subseteq S$, $T' \subseteq T$ such that

$$\bigcap_{a \in S'} \sigma(a) \subseteq \bigcup_{b \in T'} \sigma(b).$$

The proof parallels [1, p. 72] but we include it for the sake of completeness.

Let $[S)$ and $(T]$ respectively be the lattice filter generated by S and the lattice ideal generated by T. If $[S) \cap (T] = \varnothing$, by the Prime Filter Theorem there exists a prime lattice filter P such that $[S) \subseteq P$ and $P \cap T = \varnothing$. Since $S \subseteq P$ and $P \cap T = \varnothing$, $P \in \bigcap_{a \in S} \sigma(a) \setminus \bigcup_{b \in T} \sigma(b)$, which leads to a contradiction. Hence $[S) \cap (T] \neq \varnothing$.

Let $c \in [S) \cap (T]$ and let $S' \subseteq S$, $T' \subseteq T$ be finite subsets of S and T such that $\bigwedge S' \leq c \leq \bigvee T'$. By Theorem 2.5, $a \to \sigma(a)$ is a lattice isomorphism; then $\bigcap_{a \in S'} \sigma(a) \subseteq \bigcup_{b \in T'} \sigma(b)$.

It follows that the sets $\sigma(a)$ such that $a \in L$ are compact. Trivially, the subset $\varnothing$ is compact and $\overset{*}{S}(L)$ is also compact because it is the only open subset that contains the point $\varnothing$ of $\overset{*}{S}(L)$.

Let's now prove that $\langle \overset{*}{S}(L), \overset{*}{\tau} \rangle$ is a Stone space. In the light of condition (a) of Definition 3.1, let $P, Q \in \overset{*}{S}(L)$ be such that $P \neq Q$. Suppose, for example, that $a \in P \setminus Q$. Then $P \in \sigma(a)$ and $Q \notin \sigma(a)$. Condition (b) is verified from (ii) above and the fact that $\{\emptyset, \overset{*}{S}(L)\} \cup \{\sigma(a): a \in A\}$ is a lattice with respect to union and intersection. Finally, condition (c) is also verified as a consequence of the statement (x) above and the fact that if U is a non empty compact and open subset, then either $U = \overset{*}{S}(L)$ or $U = \sigma(a)$ for some $a \in A$. □

DEFINITION 3.3. We say that $\langle X, \tau, \leq, p_m, p_M \rangle$ is an *ordered Stone space with endpoints* if $\langle X, \tau, \leq \rangle$ is an ordered Stone space and $p_m \leq p \leq p_M$ for all $p \in X$.

From Proposition 3.2, we have that if L is a distributive lattice, then $\overset{*}{S}(L) = \langle \overset{*}{S}(L), \overset{*}{\tau}, \subseteq, \emptyset, L \rangle$ is an ordered Stone space with endpoints.

OBSERVATION 3.4. As $\langle \overset{*}{S}(L), \tau \rangle$ is a Stone space, from the Stone Representation Theorem [1, p. 77], $\overset{*}{S}(L)$ is homeomorphic to the Stone space $S(L')$ of some distributive lattice L'. It can be shown that L' is the lattice obtained by adding to L an upper bound (if L hasn't one), or a new one if L has a maximum, and a lower bound (if L hasn't one), or a new one if L has a minimum.

OBSERVATION 3.5. If $\langle X, \tau, \leq, p_m, p_M \rangle$ is an ordered Stone space with endpoints, the compact and open subsets U of X are *increasing sets* (i.e., $x \in U$ and $x \leq y$ imply $y \in U$). In fact, $x \leq y$ iff $x \in \mathrm{Cl}(\{y\})$; since U is open, if $x \in U$, $y \in U$.

It follows that X is the only member of the basis that contains the point p_m. Further, all ordered Stone spaces with endpoints are compact spaces.

DEFINITION 3.6. We say that a compact and open subset U of an ordered Stone space with endpoints X is *proper* if $U \neq \emptyset$ and $U \neq X$. We denote by $\sum(X)$ the family of proper compact and open subsets of X.

PROPOSITION 3.7. *If* $\langle X, \tau, \leq, p_m, p_M \rangle$ *is an ordered Stone space with endpoints,* $\langle \sum(X), \cup, \cap \rangle$ *is a distributive lattice.*

Proof. Let $U, V \in \sum(X)$ and suppose $U \cup V = X$; then $p_m \in U$ or $p_m \in V$; suppose, for example, $p_m \in U$. From Observation 3.5, since U is increasing, it follows that $U = X$, a contradiction. Also, if $U, V \in \sum(X)$, then $U \cap V \neq \emptyset$, because $p_M \in U \cap V$. □

We are now in a position to prove the following:

THEOREM 3.8. *For each distributive lattice* L, $\sum(\overset{*}{S}(L))$ *is isomorphic to* L, *and for each ordered Stone space with endpoints* X, *there is an order preserving homeomorphism from* X *onto* $\overset{*}{S}(\sum(X))$ *that also preserves endpoints.*

Proof. From Proposition 3.2 (ii), $U \subseteq \overset{*}{S}(L)$ is compact and open iff $U = \varnothing$ or $U = \overset{*}{S}(L)$ or $U = \sigma(a)$ for some $a \in L$. For all $a \in A$, $\sigma(a) \neq \varnothing$ and $\sigma(a) \neq \overset{*}{S}(L)$ ($L \in \sigma(a)$ and $\varnothing \notin \sigma(a)$). Then $\sum(\overset{*}{S}(L))$ coincides with the family $\{\sigma(a) : a \in L\}$. From Theorem 2.5, $\sum(\overset{*}{S}(L)) \cong L$.

Let's now prove that there is an order preserving homeomorphism between X and $\overset{*}{S}(\sum(X))$ that also preserves endpoints.

For each $x \in X$, let $\delta(x) = \{U \in \sum(X) : x \in U\}$. It is easy to prove that $\delta(x) \in \overset{*}{S}(\sum(X))$. Then, a function $\delta : X \to \overset{*}{S}(\sum(X))$ can be defined by $x \to \delta(x)$. It is easy to prove that δ preserves endpoints. By properties (a) and (b) of Definition 3.1, δ is injective. Let's prove that δ is onto: let $P \in \overset{*}{S}(\sum(X))$; if $P = \varnothing$, $P = \delta(p_m)$ and if $P = \sum(X)$, $P = \delta(p_M)$. Then we may suppose that P is a prime lattice filter of $\sum(X)$.

CLAIM 1: $\bigcap\{U \in \sum(X) : U \in P\} \subseteq \bigcup\{V \in \sum(X) : V \notin P\}$.

Suppose, for the contrary, that the inclusion holds. Since $U, V \in \sum(X)$, $U \neq \varnothing$ for all $U \in P$ and $V \neq \varnothing$ for all $V \notin P$. Applying (c) of Definition 3.1, we obtain that $U_1 \cap \cdots \cap U_n \subseteq V_1 \cup \cdots \cup V_m$ for some $n, m \in N$. As $U_1 \cap \cdots \cap U_n \in P$, $V_1 \cup \cdots \cup V_m \in P$ and, since P is a prime filter, $V_i \in P$ for some $1 \leq i \leq m$, which is a contradiction.

Therefore, there exists $x_0 \in \bigcap\{U \in \sum(X) : U \in P\} \setminus \bigcup\{V \in \sum(X) : V \notin P\}$.

CLAIM 2: $\delta(x_0) = P$.

Let $U \in \delta(x_0)$; then $x_0 \in U$; as $x_0 \notin \bigcup\{V \in \sum(X) : V \notin P\}$, $U \in P$. For the opposite inclusion, let $U \in P$; then $x_0 \in U$ and we have $U \in \delta(x_0)$. We have proved that δ is onto.

As the compact and open subsets of X are increasing, it is straightforward to see that δ is order preserving.

Let U be a compact and open subset and suppose first that $U \in \sum(X)$. Note that $x \in \delta^{-1}[\sigma(U)]$ iff $\delta(x) \in \delta(x)$ iff $x \in U$. Then $\delta^{-1}[\sigma(U)] = U$ and $\sigma(U) = \delta(U)$. As $\delta^{-1}[\overset{*}{S}(\sum(X))] = X$ and $\delta^{-1}[\varnothing] = \varnothing$, $\delta(X) = \overset{*}{S}(\sum(X))$ and $\delta(\varnothing) = \varnothing$; we have proved that δ and δ^{-1} are both continuous. □

Let's denote by $\mathscr{D}$ the category of distributive lattices with lattice homorphisms and by $\overset{*}{\mathscr{S}}$ the category of ordered Stone spaces with endpoints, whose morphisms are order and endpoint-preserving strongly continuous functions (recall that $f : X \to Y$ is *strongly continuous* if the inverse image of a compact and open subset of Y is a compact and open subset of X).

THEOREM 3.9: *Duality theorem for distributive lattices*

Let $F : \mathscr{D} \to \overset{*}{\mathscr{S}}$ *be such that for each object L or* $\mathscr{D}$, $F(L) = \langle \overset{*}{S}(L), \overset{*}{\tau}, \subseteq, \varnothing, L \rangle$ *and for each* $f' \in [L, L']_{\mathscr{D}}$, $F(f) : \overset{*}{S}(L') \to \overset{*}{S}(L)$ *is defined by* $F(f)(P') = f^{-1}[P']$ *for each* $P' \in \overset{*}{S}(L')$.

Then F is a contravariant functor providing a duality between $\mathscr{D}$ *and* $\overset{*}{S}$.

Proof. Let $f \in [L, L']_{\mathscr{D}}$; since $f^{-1}[\varnothing] = \varnothing$; $f^{-1}[L'] = L$ and for each prime filter P' of L', $f^{-1}[P']$ is either a prime filter of L or $f^{-1}[P'] \in \{\varnothing, L\}$, $F(f)$ is well defined from $\overset{*}{S}(L')$ onto $\overset{*}{S}(L)$.

As we have that $P' \in F(f)^{-1}[\sigma(a)]$ iff $F(f)(P') \in \sigma(a)$ iff $f^{-1}[P'] \in \sigma(a)$ iff $f(a) \in P'$ iff $P' \in \sigma'(f(a))$, we obtain $F(f)^{-1}[\sigma(a)] = \sigma'(f(a))$ and from this it can be readily seen that $F(f)$ is strongly continuous. Also, it is easy to prove that $F(f)$ is order and endpoint-preserving.

Let now X be an object of $\overset{*}{S}$. From Proposition 3.7, we have that $\sum(X)$ is an object of $\mathscr{D}$ and by Theorem 3.8 $F(\sum(X))$ is isomorphic (in categorical terms) to X. It only remains to prove that if L, L' are objects of $\mathscr{D}$, the function $[L, L']_{\mathscr{D}} \overset{F}{\to} [F(L'), F(L)]_{\overset{*}{\mathscr{S}}}$ is one-one and onto.

Let $f, g \in [L, L']_{\mathscr{D}}$ such that $f \neq g$. Then there is $x \in L$ such that, for example, $f(x) \nleq g(x)$. Let P be a prime filter of L' such that $f(x) \in P$ and $g(x) \notin P$. We have that $F(f)(P) \neq F(g)(P)$.

Now let $g \in [F(L'), F(L)]_{\overset{*}{\mathscr{S}}}$. For each $x \in L$, $\sigma(x)$ is a proper compact and open subset of $\overset{*}{S}(L)$. As g is strongly continuous and order preserving, $g^{-1}[\sigma(x)]$ is also a proper compact and open subset. Then $g^{-1}[\sigma(x)] = \sigma'(y_x)$ with $y_x \in L'$. As y_x is uniquely determined, we can define $f : L \to L'$ by the assignment $x \to y_x$. Let's prove that $f \in [L, L']_{\mathscr{D}}$. As a matter of fact, we can write:

$$\begin{aligned} \sigma'(f(x_1 \vee x_2)) &= \sigma'(y_{x_1 \vee x_2}) = g^{-1}[\sigma(x_1 \vee x_2)] = g^{-1}[\sigma(x_1) \cup \sigma(x_2)] \\ &= g^{-1}[\sigma(x_1)] \cup g^{-1}[\sigma(x_2)] = \sigma'(f(x_1)) \cup \sigma'(f(x_2)), \end{aligned}$$

and from this we obtain $f(x_1 \vee x_2) = f(x_1) \vee f(x_2)$. In a similar way, $f(x_1 \wedge x_2) = f(x_1) \wedge f(x_2)$.

Finally, we prove $F(f) = g$. Let $P \in \overset{*}{S}(L')$; as $x \in g(P)$ iff $g(P) \in \sigma(x)$ iff $P \in g^{-1}[\sigma(x)]$ iff $P \in \sigma'(f(x))$ iff $f(x) \in P$ iff $x \in f^{-1}[P]$ iff $x \in F(f)(P)$, we have that $F(f)(P) = g(P)$. □

In some of the examples stated at Section 1 the lattices considered are not only distributive but bounded. A similar duality can be developed for bounded distributive lattices:

DEFINITION 3.10. We say that an ordered Stone space with endpoints $\langle X, \tau, \leq, p_m, p_M \rangle$ is of *type* 01 if $X \backslash \{p_m\}$ and $\{p_M\}$ are compact and open subsets of X. Let $\mathbf{X} = \langle X, \tau, \leq, p_m, p_M \rangle$ and $\mathbf{X}' = \langle X', \tau', \leq', p'_m, p'_M \rangle$ be spaces of type 01. $f : \mathbf{X} \to \mathbf{X}'$ is a *morphism of type* 01 iff f is a morphism of ordered Stone spaces with endpoints and the following two conditions hold: (0) $f(p) = p'_M$ iff $p = p_M$; (1) $f(p) = p'_m$ iff $p = p_m$.

Let's denote by $\overset{*}{S}_{01}$ the category defined above, and let $\mathcal{D}_{01}$ be the category of bounded distributive lattices. From Proposition 3.2 it can be readily seen that if $L \in \mathcal{D}_{01}$, then $\overset{*}{S}(L)$ is an ordered Stone space with endpoints of type 01. Also, if $X \in \overset{*}{S}_{01}$, $\sum(X) \in \mathcal{D}_{01}$ with $0 = \{p_M\}$ and $1 = X \backslash \{p_m\}$.

One can derive the following:

THEOREM 3.11: *Duality theorem for bounded distributive lattices*
The map F of Theorem 3.9 *establishes a duality between* $\mathcal{D}_{01}$ *and* $\overset{*}{\mathcal{S}}_{01}$.

4. The topological duality for implicative lattices

We are ready now to introduce the topological spaces associated with implicative lattices.

DEFINITION 4.1. $\mathbf{X} = \langle X, \tau, \leq, p_m, p_M, \varphi \rangle$ is an *IL-space* iff:

(a) $\langle X, \tau, \leq, p_m, p_M \rangle$ is an ordered Stone space with endpoints.

(b) φ is a continuous function from the product space $X \times X$ to X that is order-preserving in each variable with $\varphi(p, p_M) = p_M$ for $p \neq p_m$ and $\varphi(p_M, p_m) = \varphi(p_m, p_M) = p_m$.

(c) For each proper compact and open subset $U \subseteq X$ and for each $p \in X$, there exists p_U, the greatest q (respect to $\leq$) such that $\varphi(q, p) \notin U$.

(d) If U and U' are proper compact and open subsets of X and $p \in X$, then either $p_U \leq p_{U'}$ or $p_{U'} \leq p_U$.

(e) If U, V are proper compact and open subsets of X, $\bigcap_{p \in U} \varphi_p^{-1}[V]$ is a compact subset (where for each $p \in X$, $\varphi_p : X \to X$ is defined by $\varphi_p(q) = \varphi(p, q)$).

PROPOSITION 4.2. *Let* $\mathbf{A} = \langle A, \vee, \wedge, \to \rangle$ *be an implicative lattice and* Φ *the binary function* (*) *defined at Proposition* 2.1. *Then* $IL(A) = \langle \overset{*}{S}(A), \overset{*}{\tau}, \subseteq, \varnothing, A, \Phi \rangle$ *is an IL-space.*

Proof. Let's check conditions (a)–(e) of Definition 4.1. From Proposition 3.2, $\langle \overset{*}{S}(A), \overset{*}{\tau}, \subseteq, \varnothing, A \rangle$ is an ordered Stone space with endpoints and, from Observation 2.2, $\Phi(P, A) = A$ if $P \neq \varnothing$, $\Phi(A, \varnothing) = \Phi(\varnothing, A) = \varnothing$ and Φ is order-preserving in each variable.

To prove the continuity of Φ, let V be a compact and open subset of $IL(A)$. As $\Phi^{-1}[\varnothing] = \varnothing$ and $\Phi^{-1}[\overset{*}{S}(A)] = \overset{*}{S}(A) \times \overset{*}{S}(A)$, we may suppose that V is proper. Then $V = \sigma(b)$ with $b \in A$. Let $(P, Q) \in \Phi^{-1}[\sigma(b)]$; as $\Phi(P, Q) \in \sigma(b)$, $b \in \Phi(P, Q)$ and there is $a \in b$ such that $a \to b \in Q$. Then $(P, Q) \in \sigma(a) \times \sigma(a \to b)$, which is an open subset of the product space contained in $\Phi^{-1}[\sigma(b)]$.

Conditions (c) and (d) follow from (ii) and (iii) of Lemma 2.4 (in the sequel we will use without explanation the identity $P_a = P_{\sigma(a)}$). Condition (e) follows from Theorem 2.5. □

PROPOSITION 4.3. *Let* $\mathbf{X} = \langle X, \tau, \leq, p_m, p_M, \varphi \rangle$ *be an IL-space and let* $\sum(X)$ *be the family of proper compact and open subsets of X. We define for* $U, V \in \sum(X)$,

$$U \Rightarrow V = \bigcap_{p \in U} \varphi_p^{-1}[V]. \tag{***}$$

Then $\langle \sum(X), \cup, \cap, \Rightarrow \rangle$ *is an implicative lattice.*

Proof. Let's first prove that for $U, V \in \sum(X)$, $U \Rightarrow V \in \sum(X)$. As U and V are proper subsets, $U \neq \varnothing, X$ and $V \neq \varnothing, X$; as U and V are increasing, $p_m \notin U, V$ and $p_M \in U, V$. From Definition 4.1, condition (b), $\varphi(p, p_M) = p_M$ for all $p \neq p_m$; then $p_M \in U \Rightarrow V$ and $U \Rightarrow V \neq \varnothing$. Now suppose that $p_m \in U \Rightarrow V$. Then, for all $p \in U$ we have $\varphi(p, p_m) \in V$. But $p_M \in U$ and $\varphi(p_M, p_m) = p_m \notin V$. Thus, $p_m \notin U \Rightarrow V$ and so $U \Rightarrow V \neq X$. We have proved that $U \Rightarrow V$ is a proper subset of X. From Definition 4.1, condition (e), $U \Rightarrow V$ is compact; it remains to prove that $U \Rightarrow V$ is open. Let $q \in \bigcap_{p \in U} \varphi_p^{-1}[V]$. From the continuity of φ (condition b), we have, for each $p \in U$, two compact and open subsets $U_q(p)$, $V_p(q)$ such that $p \in U_q(P)$, $q \in V_p(q)$ and $U_q(p) \times V_p(q) \subseteq \varphi^{-1}[V]$. Since $U \subseteq \bigcup_{p \in U} U_q(P)$ and U is a compact subset, there is a finite subset $F \subseteq U$ such that $U \subseteq \bigcup_{p \in F} U_q(p)$. Let's consider $V_0(q) = \bigcap_{p \in F} V_p(q)$. Note that $q \in V_0(q)$ and, as X is a Stone space, $V_0(q)$ is a compact and open subset. Let's show that $V_0(q) \subseteq \bigcap_{p \in U} \varphi_p^{-1}[V]$. Let $q' \in V_0(q)$ and $p \in U$. Choose $p' \in F$ such that $p \in U_q(p')$. As $q' \in \bigcap_{p \in F} V_p(q)$, $q' \in V_{p'}(q)$. It follows that $(p, q') \in U_q(p') \times V_{p'}(q) \subseteq \varphi^{-1}[V]$; then we have that $\varphi_p(q') = \varphi(p, q') \in V$.

Conditions $IL(1)$ and $IL(2)$ of Definition 1.1 are satisfied from definition (***) by the properties of the inverse image of a function with respect to unions and intersections. Also it can be readily seen that for $U, V, V' \in \sum(X)$, $(U \Rightarrow V) \cup (U \Rightarrow V') \subseteq U \Rightarrow (V \cup V')$ and $(U \Rightarrow V) \cup (U' \Rightarrow V) \subseteq (U \cap U') \Rightarrow V$.

Let's now prove $U \Rightarrow (V \cup V') \subseteq (U \Rightarrow V) \cup (U \Rightarrow V')$. Choose $q \in U \Rightarrow (V \cup V')$. If $q \notin (U \Rightarrow V) \cup (U \Rightarrow V') = (\bigcap_{P \in U} \Phi_P^{-1}[V]) \cup (\bigcap_{P \in U} \Phi_P^{-1}[V'])$ then there exist $p, p' \in U$ such that $\Phi(p, q) \notin V$ and $\Phi(p', q) \notin V'$.

Let's consider the elements q_V and $q_{V'}$ given by (c) of Definition 4.1. As $p, p' \in U$, $q_V, q_{V'} \in U$. From condition (d) $q_V \leq q_{V'}$ or $q_{V'} \leq q_V$. Suppose (for example) $q_V \leq q_{V'}$. As $\varphi(q_{V'}, q) \notin V'$ and φ is order preserving in the first variable, $\varphi(q_V, q) \notin V'$. From this $\varphi(q_V, q) \notin V$ and $\varphi(q_V, q) \notin V'$, i.e., $\varphi(q_V, q) \notin V \cup V'$. Since $q_V \in U$, we obtain a contradiction.

We prove finally that $(U \cap U') \Rightarrow V \subseteq (U \Rightarrow V) \cup (U' \Rightarrow V)$. Choose $q \in (U \cap U') \Rightarrow V$. If $q \notin (U \Rightarrow V) \cup (U' \Rightarrow V) = (\bigcap_{p \in U} \varphi_p^{-1}[V]) \cup (\bigcap_{p \in U'} \varphi_p^{-1}[V])$, then there exist $p \in U$ and $p' \in U'$ such that $\varphi(p, q) \notin V$ and $\varphi(p', q) \notin V$. Let's consider q_V or condition (c). As $p, p' \leq q_V$, $q_V \in U \cap U'$. But $\varphi(q_V, q) \notin V$, a contradiction. □

OBSERVATION 4.4. We have proved that for each U, V proper compact and open subsets of an *IL-space* **X**, $\bigcap_{p \in U} \varphi_p^{-1}[V]$ is also open. Then we can replace condition (e) of Definition 4.1 by the following:

(e′) For any two proper compact and open subsets U, V, the greatest open subset W such that $U \times W \subseteq \varphi^{-1}[V]$ is a compact subset.

Note now that if $\langle X, \tau, \leq, p_m, p_M, \varphi \rangle$ is an *IL* space, the family $\overset{*}{\mathcal{O}}(X)$ of the proper open subsets of X is a lattice under union and intersection. If we consider the set $\overset{*}{\mathcal{O}}(X \times X)$ of proper open subsets of the product space $X \times X$, we have that φ induces a function $\psi : \overset{*}{\mathcal{O}}(X) \to \overset{*}{\mathcal{O}}(X \times X)$ defined by $\psi(U) = \varphi^{-1}[U]$. From the properties of the inverse image of a function, ψ is a lattice homomorphism. Let's denote $\langle \overset{*}{\mathcal{O}}(X), \psi \rangle$ the lattice $\overset{*}{\mathcal{O}}(X)$ endowed with the lattice homomorphism ψ.

From these remarks and Proposition 4.3 we establish to which extent implicative lattices can be represented by means of the set-theoretical operations:

THEOREM 4.5. *Let $\langle A, \vee, \wedge, \to \rangle$ be an implicative lattice. Then A is isomorphic to a sublattice $\mathscr{C}$ of the lattice $\langle \overset{*}{\mathcal{O}}(X), \psi \rangle$ of an IL-space X satisfying: for all $U, V \in \mathscr{C}$, the greatest $W \in \overset{*}{\mathcal{O}}(X)$ such that $U \times W \subseteq \psi(V)$ belongs to $\mathscr{C}$.*

We now prove that there is a duality between implicative lattices and *IL*-spaces.

DEFINITION 4.6. Let $\langle X, \tau, \leq, p_m, p_M, \varphi \rangle$, $\langle X', \tau', \leq', p'_m, p'_M, \varphi' \rangle$ be *IL*-spaces. A function $f: X \to X'$ is said to be a *morphism of IL-spaces* iff f is a morphism of ordered Stone spaces with endpoints satisfying:

(i) $\varphi'(f(p), f(q)) \leq' f(\varphi(p, q))$.

(ii) For each $q \in X$ and V' proper compact and open subset of X', $(f(q))_{V'} = f(q_{f^{-1}[V']})$.

PROPOSITION 4.7.

(1) *If* X *and* X' *are IL-spaces and* $f: X \to X'$ *is a morphism of IL-spaces,* $f^{-1}[\bigcap_{p' \in U'} \varphi'^{-1}_{p'}[V']] = \bigcap_{p \in f^{-1}[U']} \varphi_p^{-1}[f^{-1}[V']]$.

(2) *If* A, A' *are implicative lattices and* $h: A \to A'$ *is an homomorphism of implicative lattices, then* $F(h): \overset{*}{S}(A') \to \overset{*}{S}(A)$ *such that* $F(h)(P') = h^{-1}[P']$ *is a morphism of IL-spaces.*

Proof. To prove (1) let $q \in X$ be such that $f(q) \in \bigcap_{p' \in U'} \varphi'^{-1}_{p'}[V']$ and let $p \in f^{-1}[U']$. Since $f(p) \in U'$, $\varphi'_{f(p)} f(q) \in V'$. From (i) of Definition 4.6, as V' is increasing, $f(\varphi(p, q)) \in V'$, and so $\varphi_p(q) \in f^{-1}[V']$. For the opposite inclusion, let $q \in \bigcap_{p \in f^{-1}[U']} \varphi_p^{-1}[f^{-1}[V']]$ and suppose $f(q) \notin \bigcap_{p' \in U'} \varphi'^{-1}_{p'}[V']$. Let $p' \in U'$ such that $\varphi'_{p'} f(q) \notin V'$. Since U' is increasing $(f(q))_{V'} \in U'$. From Definition 4.6 (ii), $f(q_{f^{-1}[V']}) \in U'$, so we get $q_{f^{-1}[V']} \in f^{-1}[U']$; thus $\varphi(q_{f^{-1}[V']}, q) \in f^{-1}[V']$, which is a contradiction.

From Theorem 3.9, to prove (2) we have only to check

(i) $\Phi(F(h)(P'), F(h)(Q')) \subseteq F(h)(\Phi'(P', Q'))$.

(ii) $(F(h)(Q'))_{\sigma(b)} = F(h)(Q_{(F(h))^{-1}[\sigma(b)]})$.

From the definition of $F(h)$, (i) is equivalent to $\Phi(h^{-1}[P'], h^{-1}[Q']) \subseteq h^{-1}[\Phi'(P', Q')]$.

Let $y \in \Phi(h^{-1}[P'], h^{-1}[Q'])$ and let $x \in h^{-1}[P']$ be such that $x \to y \in h^{-1}[Q']$. Then $h(x \to y) = h(x) \to' h(y) \in Q'$. Since $h(x) \in P'$, $h(y) \in \Phi'(P', Q')$ and $y \in h^{-1}[\Phi'(P', Q')]$. (ii) is equivalent to the identity $(h^{-1}[Q']) = h^{-1}[Q'_{\sigma(h(b))}]$. Now, $x \in (h^{-1}[Q'])$ iff $x \to b \notin h^{-1}[Q']$ iff $h(x \to b) \notin Q'$ iff $h(x) \to' h(b) \notin Q'$ iff $h(x) \in Q'_{\sigma(h(b))}$ iff $x \notin h^{-1}[Q'_{\sigma(h(b))}]$. □

Let $\mathcal{IL}$ be the category whose objects are implicative lattices and whose morphisms are lattice homomorphisms which preserve the implication. Let IL be the category whose objects are IL-spaces and whose morphisms are the morphisms of IL-spaces.

THEOREM 4.8: *Duality theorem for implicative lattices*

The map $F: \mathcal{IL} \to IL$ *such that for each implicative lattice* $\mathbf{A}$, $F(\mathbf{A}) = IL(\mathbf{A})$ *and for each morphism of implicative lattices* $f: \mathbf{A} \to \mathbf{A}'$, $F(f): IL(\mathbf{A}') \to IL(\mathbf{A})$ *is defined by* $F(f)(P') = f^{-1}[P']$, *establishes a duality between* $\mathcal{IL}$ *and IL.*

Proof. By Proposition 4.2 and Proposition 4.7(2), F is a contravariant functor from $\mathcal{IL}$ to IL. Note also that for each object $\mathbf{X}$ of IL, we have from Proposition 4.3 that $\sum(\mathbf{X}) = \langle \sum(X), \cup, \cap, \Rightarrow \rangle$ is an object of $\mathcal{IL}$.

Let's prove that the map $\delta: X \to F(\sum(X))$ such that $\delta(p) = \{U \in \sum(X) : p \in U\}$ is an isomorphism of IL-spaces. From Theorem 3.9 we already know that

δ is an isomorphism of ordered Stone spaces with endpoints. In order to prove that δ is a morphism of IL-spaces let's check conditions (i) and (ii) of Definition 4.6.

For (i) $\Phi(\delta(p), \delta(q)) \subseteq \delta(\varphi(p, q))$, let $V \in \Phi(\delta(p), \delta(q))$ and let $U \in \delta(p)$ be such that $U \Rightarrow V \in \delta(q)$. As $U \Rightarrow V = \bigcap_{p \in U} \varphi_p^{-1}[V]$ and $p \in U$, $\varphi(p, q) \in V$; then $V \in \delta(\varphi(p, q))$.

To prove (ii) $(\delta(q))_{\sigma(V)} = \delta(q_{\delta^{-1}[\sigma(V)]})$, note first that $\delta^{-1}[\sigma(V)] = V : q \in \delta^{-1}[\sigma(V)]$ iff $\delta(q) \in \sigma(V)$ iff $V \in \delta(q)$ iff $q \in V$. Then, we will prove $(\delta(q))_{\sigma(V)} = \delta(q_V)$. Let $U \in \sum(X)$ be such that $U \Rightarrow V \notin \delta(q)$; then $q \notin U \Rightarrow V = \bigcap_{p \in U} \varphi_p^{-1}[V]$. Let $p \in U$ be such that $\varphi(p, q) \notin V$. As $p \leq q_V$ and U is increasing, $q_V \in U$ and we obtain $U \in \delta(q_V)$. For the opposite inclusion, let $U \in \sum(X)$ be such that $q_V \in U$ and suppose $U \Rightarrow V \in \delta(q)$. Then $q \in U \Rightarrow V$. As $q_V \in U$, we would have $\varphi(q_V, q) \in V$, which is a contradiction. Then $U \Rightarrow V \notin \delta(q)$ and we have $U \in (\delta(q))_{\sigma(V)}$.

Note that δ also satisfies $\delta(\varphi(p, q)) \subseteq \Phi(\delta(p), \delta(q))$: let $V \in \sum(X)$ such that $\varphi(p, q) \in V$. Then $(p, q) \in \varphi^{-1}[V]$. Since φ is a continuous function from the product space $X \times X$ to X and the family of compact and open subsets of X is a basis, there are compact and open subsets $U(p)$, $U'(q)$, such that $(p, q) \in U(p) \times U'(q) \subseteq \varphi^{-1}[V]$. Moreover, $U(p)$ is proper as we now show. As $p \in U(p)$ we have $U(p) \neq \emptyset$. Also, $U(p) \neq X$ because if $p_m \in U(p)$, $(p_m, p_M) \in U(p) \times U'(q)$, but from Definition 4.1(b), $\varphi(p_m, p_M) = p_m$, so we would have that $p_m \in V$, which is a contradiction, because V is proper. Using $\varphi(p_M, p_m) = p_m$ it follows in the same way that $U'(q)$ is proper, too. Then, from Proposition 4.3 and the fact that $\varphi(U(p) \times U'(q)) \subseteq V$ we get $q \in U'(q) \subseteq U(p) \Rightarrow V$. Further, $U(p) \Rightarrow V \in \sum(X)$. Thus, $U(p) \Rightarrow V \in \delta(q)$ and $V \in \varphi(\delta(p), \delta(q))$. We can now prove that $\delta^{-1} : F(\sum(X)) \to X$ is a morphism of IL-spaces: as a matter of fact, from the above identity $\delta(\varphi(\delta^{-1}(P), \delta^{-1}(Q))) = \Phi(\delta(\delta^{-1}(P)), \delta(\delta^{-1}(Q)))$, we obtain $\varphi(\delta^{-1}(P), \delta^{-1}(Q)) = \delta^{-1}(\Phi(P, Q))$; then, condition (i) of Definition 4.6 is satisfied. Let's prove (ii) $(\delta^{-1}(Q))_V = \delta^{-1}(Q_{(\delta^{-1})^{-1}[V]})$. As $(\delta^{-1})^{-1}[V] = \delta(V) = \{\delta(q) : q \in V\}$ and δ is a bijection, it suffices to show that $\delta((\delta^{-1}(Q))_V) = Q_{\delta(V)}$. Since $V = \delta^{-1}[\delta(V)]$ and δ is a morphism of IL-spaces, we can write $\delta((\delta^{-1}(Q))_V) = \delta((\delta^{-1}(Q))_{\delta^{-1}[\delta(V)]}) = (\delta(\delta^{-1}(Q))_{\delta(V)}) = Q_{\delta(V)}$.

We have proved that δ is an isomorphism of IL-spaces between X and $F(\sum(X))$. It remains to prove that if $\mathbf{A}$, $\mathbf{A}'$ are implicative lattices, the function $[\mathbf{A}, \mathbf{A}']_{\mathscr{I}\mathscr{L}} \xrightarrow{F} [IL(\mathbf{A}'), IL(\mathbf{A})]_{IL}$ is one-one and onto.

The fact that F is one-one follows exactly as in Theorem 3.9. To prove that F is onto, let $g \in [IL(\mathbf{A}'), IL(\mathbf{A})]_{\mathscr{I}\mathscr{L}}$. As g is in particular a morphism of ordered Stone spaces with endpoints, we can define a function $f : \mathbf{A} \to \mathbf{A}'$ with the same assignment of Theorem 3.9: $x \to y_x$ (where $g^{-1}[\sigma(x)] = \sigma'(y_x)$). We already know that $f \in [\mathbf{A}, \mathbf{A}']_{\mathscr{D}}$; let's prove that f also preserves the implication: $\sigma'(f(x_1 \to x_2)) = \sigma'(y_{x_1 \to x_2}) = g^{-1}[\sigma(x_1 \to x_2)] = g^{-1}[\bigcap_{P \in \sigma(x_1)} \Phi_p^{-1}[\sigma(x_2)]]$. The last equality follows from Theorem 2.5 (1). Now, from Proposition 4.7 (i),

$$g^{-1}\left[\bigcap_{P \in \sigma(x_1)} \Phi_p^{-1}[\sigma(x_2)]\right] = \bigcap_{P' \in g^{-1}[\sigma(x_1)]} \Phi'^{-1}_{P'}[g^{-1}[\sigma(x_2)]].$$

Then,

$$\sigma'(f(x_1 \to x_2)) = \sigma'(g^{-1}[\sigma(x_1)] \to' g^{-1}[\sigma(x_2)]) = \sigma'(f(x_1) \to' f(x_2)).$$

Thus $f(x_1 \to x_2) = f(x_1) \to' f(x_2)$. □

We conclude this section giving a topological reformulation of some algebraic conditions that appear frequently in dealing with algebras coming from logic and also in lattice ordered groups.

PROPOSITION 4.9. *Let $\mathbf{A} = \langle A, \vee, \wedge, \to \rangle$ be an implicative lattice and let $\mathbf{X} = \langle X, \tau, \leq, p_m, p_M, \varphi \rangle$ be an IL-space. For each one of the following conditions on the left, the condition on the right is the corresponding topological translation (in the following sense: if* $\mathbf{A}$ *satisfies (a_i). IL*(**A**) *satisfies (a_i') and, if X satisfies (a_i'), $\sum(X)$ satisfies (a_i))*

(a_1) $x \leq (x \to y) \to y$; (a_1') $\varphi(p, q) = \varphi(q, p)$

(a_2) $x \to (y \to z) = y \to (x \to z)$; (a_2') $\varphi(p, \varphi(p', q)) = \varphi(p', \varphi(p, q))$.

(a_1) and (a_2) hold together if and only if φ is associative and commutative.

Proof. Let $\langle \overset{*}{S}(A), \overset{*}{\tau}, \subseteq, \varnothing, A, \Phi \rangle$ be the IL-space corresponding to A and $\langle \sum(X), \cup, \cap, \Rightarrow \rangle$ be the implicative lattice corresponding to X. Suppose that A satisfies (a_1) and let $y \in \Phi(P, Q)$. Then, there is $x \in P$ such that $x \to y \in Q$. From (a_1), $x \leq (x \to y) \to y$. Then $(x \to y) \to y \in P$. Thus we have $y \in \Phi(Q, P)$ and $\Phi(P, Q) \subseteq \Phi(Q, P)$. In a similar way, $\Phi(Q, P) \subseteq \Phi(P, Q)$. Hence, $IL(A)$ satisfies (a_1').

Suppose now that φ satisfies (a_1'); let's prove $U \subseteq (U \Rightarrow V) \Rightarrow V$. Let $q \in U$ and suppose $q \notin \bigcap_{p \in U \Rightarrow V} \varphi_p^{-1}[V]$. Then, there is $p \in U \Rightarrow V$ such that $\varphi(p, q) \notin V$. As $p \in U \Rightarrow V$ and $q \in U$, $\varphi(q, p) \in V$. From this we get $\varphi(p, q) \neq \varphi(q, p)$ which is a contradiction.

To prove (a_2') let $z \in \Phi(P, \Phi(P', Q))$; then, there is $y \in P$ such that $y \to z \in \Phi(P', Q)$ and $x \in P'$ such that $x \to (y \to z) \in Q$. From (a_2) it follows that $y \to (x \to z) \in Q$. Then $x \to z \in \Phi(P, Q)$ and $z \in \Phi(P', \Phi(P, Q))$. We have proved $\Phi(P, \Phi(P', Q)) \subseteq \Phi(P', \Phi(P, Q))$. The opposite inclusion follows in a similar way.

Suppose now that φ satisfies (a_2'). Let's prove $U \Rightarrow (V \Rightarrow W) = V \Rightarrow (U \Rightarrow W)$. Let $q \in U \Rightarrow (V \Rightarrow W)$ and suppose $q \notin V \Rightarrow (U \Rightarrow W)$. Then, there is $p \in V$ such that $\varphi(p, q) \notin U \Rightarrow W$ and $p' \in U$ such that $\varphi(p', \varphi(p, q)) \notin W$. From (a_2') it follows that $\varphi(p, \varphi(p', q)) \notin W$. As $p \in V$, $\varphi(p', q) \notin V \Rightarrow W$ and, as $p' \in U$, $q \notin U \Rightarrow (V \Rightarrow W)$, a contradiction. The opposite inclusion follows similarly.

If (a_1) and (a_2) hold, we can write $\varphi(p, \varphi(p', q)) = \varphi(p, \varphi(q, p')) = \varphi(q, \varphi(p, p')) = \varphi(\varphi(p, p')q)$.

Then φ is associative. The converse also follows easily. □

OBSERVATION 4.10. From Proposition 4.9 above, if A is an implicative lattice satisfying (a_1) and (a_2), $\overset{*}{S}(A)$ is a compact ordered abelian topological semigroup (possibly non Hausdorff). Thus, a surprising connection arises with another branch of Mathematics: the well studied theory of topological semigroups.

5. The duality for ℓ-groups and for abelian ℓ-groups

In order to apply our duality theory to ℓ-groups and to abelian ℓ-groups, we characterize these groups as implicative lattices with a distinguished element.

PROPOSITION 5.1. *Let* $\mathbf{G} = \langle G, \vee, \wedge, \cdot, ^{-1}, e\rangle$ *be an* ℓ*-group and define* $x \to y = x^{-1}y$. *Then* $\langle G, \vee, \wedge, \to\rangle$ *is an implicative lattice with a distinguished element e satisfying*:

(ℓ_1) $(x \to e) \to e = x$

(ℓ_2) $x \to x = e$

(ℓ_3) $x \to ((y \to e) \to z) = ((x \to y) \to e) \to z.$

We call it the implicative lattice of $\mathbf{G}$ *and we denote it by* $\vec{\mathbf{G}}$.

Conversely, let $\mathbf{A} = \langle A, \vee, \wedge, \to\rangle$ *be an implicative lattice with a distinguished element* $e \in A$ *satisfying* (ℓ_1), (ℓ_2) *and* (ℓ_3) *above. Then defining* $x \cdot y = (x \to e) \to y$ *and* $x^{-1} = x \to e$, $\langle A, \vee, \wedge, \cdot, ^{-1}, e\rangle$ *is an* ℓ*-group. We call it the* ℓ-group of $\mathbf{A}$ *and we denote it by* $\dot{\mathbf{A}}$.

Proof. From Example 1.4 we already know that $\vec{\mathbf{G}}$ is an implicative lattice. Condition (ℓ_1) follows from the fact that $(x^{-1})^{-1} = x$ and (ℓ_2) from the identity $x^{-1}x = e$; we obtain (ℓ_3) from these equalities:

$$x \to ((y \to e) \to z) = x^{-1}((y \to e)^{-1}z) = x^{-1}((y^{-1})^{-1}z) = x^{-1}(yz)$$
$$= (x^{-1}y)z = ((x^{-1}y)^{-1})^{-1}z = ((x \to y) \to e) \to z.$$

For the converse, note first that (ℓ_3) guarantees the associativity of:

$$x(yz) = (x \to e) \to ((y \to e) \to z) = (((x \to e) \to y) \to e) \to z = (xy)z.$$

As we have from (ℓ_1) that $xe = (x \to e) \to e = x$, e is a right unit for $\cdot$; from (ℓ_2) $x(x \to e) = (x \to e) \to (x \to e) = e$. Then, for all $x \in A$, $x \to e$ is a right inverse for x. It is well known that the conditions above suffice to prove that $\langle A, \cdot, {}^{-1}, e\rangle$ is a group. The equations $c(a \wedge b) = ca \wedge cb$; $(a \wedge b)c = ac \wedge bc$; $c(a \vee b) = ca \vee cb$; $(a \vee b)c = ac \vee bc$, follow respectively from $IL(1)$, $IL(2)$, $IL(3)$ and $IL(4)$ of Definition 1.1. □

PROPOSITION 5.2. *Let* $\mathbf{G} = \langle G, \vee, \wedge, \cdot, {}^{-1}, e\rangle$ *be an abelian* ℓ*-group. Then* $\vec{\mathbf{G}}$ *satisfies conditions* (ℓ_1) *and* (ℓ_2) *of Proposition* 5.1 *and the following*:

(ℓ_3') $\quad x \to (y \to z) = y \to (x \to z)$

(ℓ_4') $\quad (y \to e) \to (x \to e) = x \to y.$

Conversely, if $\mathbf{A} = \langle A, \vee, \wedge, \to\rangle$ *is an implicative lattice with a distinguished element e satisfying* (ℓ_1), (ℓ_2), (ℓ_3') *and* (ℓ_4'), *then* $\dot{\mathbf{A}}$ *is an abelian* ℓ*-group.*

Proof. Let $\mathcal{G}$ be an abelian ℓ-group. Using associativity and commutativity of $\cdot$, we can write

$$x \to (y \to z) = x^{-1}(y^{-1}z) = (x^{-1}y^{-1})z = (y^{-1}x^{-1})z$$
$$= y^{-1}(x^{-1}z) = y \to (x \to z)$$

and we obtain (ℓ_3').

(ℓ_4') also follows at once using $yx^{-1} = x^{-1}y$.

For the converse, commutativity follows using (ℓ_1) and (ℓ_3') from these equalities:

$$xy = (x \to e) \to y = (x \to e) \to ((y \to e) \to e) = (y \to e) \to ((x \to e) \to e)$$
$$= (y \to e) \to x = yx.$$

Using both (ℓ'_3) and commutativity we obtain that $\cdot$ is associative from the equalities:

$$x(yz) = (x \to e) \to ((y \to e) \to z) = (x \to e) \to ((z \to e) \to y)$$
$$= (z \to e) \to ((x \to e) \to y) = (((x \to e) \to y)) \to e) \to z = (xy)z.$$

The remaining conditions follow as in Proposition 5.1. □

Let $\mathscr{G}$ be the category of ℓ-groups, where the morphisms are all homomorphisms and let's denote $\mathscr{I}$ the category whose objects are the implicative lattices with distinguished element which satisfy equations (ℓ_1)–(ℓ_3) of Proposition 5.1 and whose morphisms are the homomorphisms of implicative lattices preserving the distinguished element.

Note that for each object $\mathbf{A}$ of $\mathscr{I}$, $\vec{\mathbf{A}} = \mathbf{A}$; also, it is clear that if $f : \mathbf{G} \to \mathbf{G}'$ is a morphism of ℓ-groups, $\vec{f} : \vec{\mathbf{G}} \to \vec{\mathbf{G}}'$ such that $\vec{f}(x) = f(x)$ for all $x \in G$, is a morphism of $\mathscr{I}$. In fact, one can state the following:

PROPOSITION 5.3. *There is a categorical equivalence between* $\mathscr{G}$ *and* $\mathscr{I}$.

In a similar way, let's denote by $\mathscr{G}a$ the category of abelian ℓ-groups and by $\mathscr{I}a$ the category of implicative lattices with distinguished element which satisfy the equations of Proposition 5.2. Again, one can state the following:

PROPOSITION 5.4. *There is a categorical equivalence between* $\mathscr{G}a$ *and* $\mathscr{I}a$.

Let's now show how to translate conditions (ℓ_1), (ℓ_2) and (ℓ_3) of Proposition 5.1 in our topological spaces.

PROPOSITION 5.5. *Let* $\mathbf{A} = \langle A, \vee, \wedge, \to \rangle$ *be an implicative lattice with a distinguished element* $e \in A$ *satisfying* (ℓ_1), (ℓ_2) *and* (ℓ_3) *of Proposition* 5.1. *Let* $IL(\mathbf{A}) = \langle \overset{*}{S}(A), \overset{*}{\tau}, \subseteq, \emptyset, A, \Phi \rangle$ *be the IL-space of* $\mathbf{A}$. *Then*:

(1) $\sigma(e)$ *satisfies* $P \in \sigma(e)$ *iff* $P_{\sigma(e)} \notin \sigma(e)$ *for all* $P \in \overset{*}{S}(A)$ *and it is the unique proper compact and open subset of* $\overset{*}{S}(A)$ *with this property.*
(2) $P \subseteq Q$ *implies* $Q_e \subseteq P_e$ *for all* $P, Q \in \overset{*}{S}(A)$.
(3) $P = (P_e)_e$ *for all* $P \in \overset{*}{S}(A)$.
(4) $P \subseteq P_e$ *or* $P_e \subseteq P$ *for all* $P \in \overset{*}{S}(A)$.
(5) *If* $P \notin \sigma(e)$, $P_a \in \sigma(a)$ *for all* $P \in \overset{*}{S}(A)$ *and for all* $a \in A$.
(6) *If* $P \in \sigma(e)$, $Q \subseteq \Phi(Q, P)$ *for all* $P, Q \in \overset{*}{S}(A)$.
[(7) $(\Phi(P, Q))_a = (\Phi(P, (Q_a)_e))_e$ *for all* $P, Q \in \overset{*}{S}(A)$ *and for all* $a \in A$.]

Proof. Let **A** be an implicative lattice with $e \in A$ satisfying (ℓ_1), (ℓ_2) and (ℓ_3). As we have from (ℓ_2) that $e \to e = e$, it follows, for all $P \in \overset{*}{S}(A)$, that $e \in P$ iff $e \to e \in P$ iff $e \notin P_e$. Recalling that $P_a = P_{\sigma(a)}$ for all $a \in A$ we obtain $P \in \sigma(e)$ iff $P_{\sigma(e)} \notin \sigma(e)$ for all $P \in \overset{*}{S}(A)$.

Now let $U \subseteq \overset{*}{S}(A)$ be a proper compact and open subset satisfying $P \in U$ iff $P_U \notin U$ for all $P \in \overset{*}{S}(A)$. Let $a \in A$ such that $U = \sigma(a)$; let's prove that $\sigma(a) = \sigma(e)$. If $P \in \sigma(a)$, from the hypothesis we have $P_{\sigma(a)} \notin \sigma(a)$. As $P_{\sigma(a)} = P_a$ we obtain $a \notin P_a$; then $a \to a \in P$. But from (ℓ_2) $a \to a = e$. Then, if $P \in \sigma(a)$, $P \in \sigma(e)$. For the other inclusion let $P \in \sigma(e)$ and suppose $P \notin \sigma(a)$. Then $P_{\sigma(a)} \in \sigma(a)$. Using again $P_{\sigma(a)} = P_a$, we obtain $a \in P_a$ and from this we derive $a \to a = e \notin P$, which is a contradiction.

For (2) let $P, Q \in \overset{*}{S}(A)$ be such that $P \subseteq Q$ and let $y \in Q_e$. As $y \to e \notin Q$, $y \to e \notin P$. Then $y \in P_e$.

For (3) note that $x \in (P_e)_e$ iff $x \to e \notin P_e$ iff $(x \to e) \to e \in P$. As we have from (ℓ_1) that $(x \to e) \to e = x$, we conclude $(P_e)_e = P$.

For (4) let's recall first from Proposition 5.1 that defining $x \cdot y = (x \to e) \to y$ and $x^{-1} = x \to e$, the structure $\langle A, \vee, \wedge, ^{-1}, \cdot, e\rangle$ becomes an ℓ-group. Then, from a well known result ([2], Chapter III, §4, Theorem 7) we have that $e \leq y \vee y^{-1}$ for all $y \in A$. As $x \wedge x^{-1} = (x^{-1} \vee x)^{-1}$ we also have from all $x \in A$ that $x \wedge x^{-1} \leq e$. Then for all $x, y \in A$, $x \wedge x^{-1} \leq y \vee y^{-1}$, or, in terms of the implication $\to$, $(*)$ $x \wedge (x \to e) \leq y \vee (y \to e)$. Suppose now that there is $P \in \overset{*}{S}(A)$ such that $P \nsubseteq P_e$ and $P_e \nsubseteq P$; choose $x \in P$ such that $x \notin P_e$ and choose $y \in P_e$ such that $y \notin P$. As $x \in P$ and $(x \to e) \in P$, $x \wedge (x \to e) \in P$. From $(*)$ $y \vee (y \to e) \in P$. As $P \in \overset{*}{S}(A)$, $y \in P$ or $y \to e \in P$. As $y \in P_e$, $y \to e \notin P$. Then, $y \in P$, which is a contradiction.

(5) and (6) follow at once from (ℓ_2). [To prove (7) let $y \in (\Phi(P, Q))_a$ and suppose that $y \notin (\Phi(P, (Q_a)_e))_e$. Then $y \to e \in \Phi(P, (Q_a)_e)$. Let $x \in P$ such that $x \to (y \to e) \in (Q_a)_e$. Then $(x \to (y \to e)) \to e \notin Q_a$ and we obtain $((x \to (y \to e)) \to e) \to a \in Q$. From (ℓ_3) $x \to (((y \to e) \to e) \to a) \in Q$ and from (ℓ_1) we get $x \to (y \to a) \in Q$. Then, as $x \in P$, $y \to a \in \Phi(P, Q)$, which contradicts $y \in (\Phi(P, Q))_a$.

For the other inclusion, let $y \in (\Phi(P, (Q_a)_e)_e$ and suppose $y \notin (\Phi(P, Q))_a$; then $y \to a \in \Phi(P, Q)$. Let $x \in P$ such that $x \to (y \to a) \in Q$. From (ℓ_1) and (ℓ_3) we can write $x \to (y \to a) = x \to (((y \to e) \to e) \to a) = ((x \to (y \to e)) \to e) \to a$. Then, $x \to (y \to e) \in (Q_a)_e$. As $x \in P$, $y \to e \in \Phi(P, (Q_a)_e)$; then $y \notin (\Phi(P, (Q_a)_e))_e$ which is a contradiction.] □

OBSERVATION 5.6. Recall that a function g from an ordered set $\langle X, \leq\rangle$ to $\langle X, \leq\rangle$ is said to be an *involution* iff $g^2 = Id$ and $x \leq y$ implies $g(y) \leq g(x)$ for all $x \in X$.

By (2) and (3) of Proposition 5.5, we can define an involution $G : \langle \overset{*}{S}(A), \subseteq \rangle \to \langle \overset{*}{S}(A), \subseteq \rangle$ by the stipulation $G(P) = P_e$ (note that $G(\emptyset) = A$ and $G(A) = \emptyset$).

We are now in a position to define the topological spaces associated with ℓ-groups.

DEFINITION 5.7. Let $\mathbf{X} = \langle X, \tau, \leq, p_m, p_M, \varphi \rangle$ be an IL-space. We'll say that $\mathbf{X}$ is an *ℓ-space* iff:

(i) There is one and only one proper compact and open subset $U \subseteq X$ such that $p \in U$ iff $p_U \notin U$. We call it U_e.

(ii) The function $g : (X, \leq) \to (X, \leq)$ defined by $g(p) = p_{U_e}$ is an involution satisfying $g(p) \leq p$ or $p \leq g(p)$ for all $p \in X$.

(iii) If $p \notin U_e$, $p_U \in U$ for all proper compact and open subsets $U \subseteq X$.

(iv) If $p \in U_e$, $q \leq \varphi(q, p)$ for all $q \in X$.

[(v) $(\varphi(p, q))_U = g(\varphi(p, g(q_U)))$ for all $p, q \in X$ and for all proper compact and open subsets $U \subseteq X$.]

Of course, if $\mathbf{A}$ is an implicative lattice with a distinguished element e satisfying (ℓ_1), (ℓ_2) and (ℓ_3) of Proposition 5.1, it follows from Proposition 5.5 and Observation 5.6 that $IL(\mathbf{A})$ is an ℓ-space: condition (i) follows from (1) of Proposition 5.5. As $G(P) = P_{\sigma(e)}$ and $P_{\sigma(e)} = P_e$, from (2), (3) and (4) of Proposition 5.5 we obtain (ii); (iii) follows from (5) and the fact that $P_a = P_{\sigma(a)}$; (iv) follows at once from (6). [(v) follows from (7) in this way: let U be a proper compact and open subset of $\overset{*}{S}(A)$; then, there is $a \in A$ such that $U = \sigma(a)$ and we can write the equalities:

$$(\Phi(P, Q))_U = (\Phi(P, Q))_{\sigma(a)} = (\Phi(P, Q))_a = (\Phi(P, (Q_a)_e)_e = (\Phi(P, (Q_{\sigma(a)})_{\sigma(e)}))_{\sigma(e)}$$
$$= G(\Phi(P, G(Q_{\sigma(a)}))) = G(\Phi(P, G(Q_U))).]$$

In the sequel we shall need the following:

LEMMA 5.8. *Let $\mathbf{X} = \langle X, \tau, \leq, p_m, p_M, \varphi \rangle$ be an ℓ-space and let U be a proper compact and open subset of X. Then, for all $q \in X$, $q \in U \Rightarrow U_e$ iff $g(q) \notin U$.*

Proof. In fact, if $q \in U \Rightarrow U_e = \bigcap_{p \in U} \varphi_p^{-1}[U_e]$ and $g(q) = q_{U_e} \in U$, we would obtain $\varphi(q_{U_e}, q) \notin U_e$ which contradicts the definition of q_{U_e}.

For the converse, suppose $g(q) = q_{U_e} \notin U$ and $q \notin U \Rightarrow U_e$. Then, there is $p \in U$ such that $\varphi(p, q) \notin U_e$. From Definition 4.1 (c) $p \leq q_{U_e}$; as U is increasing and $p \in U$, we would obtain $q_{U_e} \in U$, which is a contradiction. □

THEOREM 5.9. *Let* $\mathbf{X} = \langle X, \tau, \leq, p_m, p_M, \varphi \rangle$ *be an* ℓ*-space. Then* $\sum(X) = \langle \sum(X), \cup, \cap, \Rightarrow \rangle$, *the implicative lattice of* $\mathbf{X}$, *with* U_e *as the distinguished element, satisfies* (ℓ_1), (ℓ_2) *and* (ℓ_3) *of Proposition* 5.1.

Proof. To prove (ℓ_1) $(U \Rightarrow U_e) \Rightarrow U_e = U$, let $q \in (U \Rightarrow U_e) \Rightarrow U_e$ and suppose $q \notin U$. Then, from Lemma 5.8, $g(q) = q_{U_e} \in U \Rightarrow U_e$. As $q \in (U \Rightarrow U_e) \Rightarrow U_e = \bigcap_{p \in (U \Rightarrow U_e)} \varphi_p^{-1}[U_e]$ and $q_{U_e} \in U \Rightarrow U_e$, we derive $\varphi(q_{U_e}, q) \in U_e$, which contradicts Definition 4.1 (c).

For the other inclusion, let $q \in U$ and suppose $q \notin (U \Rightarrow U_e) \Rightarrow U_e$; then there is $p \in U \Rightarrow U_e$ such that $\varphi(p, q) \notin U_e$; from Definition 4.1 (c), $p \leq q_{U_e}$. Using Definition 5.7 (ii) we get $g(q_{U_e}) \leq g(p)$, i.e. $(q_{U_e})_{U_e} \leq p_{U_e}$. But $(q_{U_e})_{U_e} = g(g(q)) = q$. As $q \in U$ and U is increasing, $p_{U_e} \in U$. As $p \in U \Rightarrow U_e$ and $p_{U_e} \in U$, we would have $\varphi(p_{U_e}, p) \in U_e$, which contradicts Definition 4.1 (c) again.

To prove (ℓ_2) $U \Rightarrow U = U_e$, let $q \in U \Rightarrow U$ and suppose that $q \notin U_e$. Then, from (iii) of Definition 5.7, $q_U \in U$. As $U \Rightarrow U = \bigcap_{p \in U} \varphi_p^{-1}[U]$, we would obtain $\varphi(q_U, q) \in U$, which is a contradiction. For the opposite inclusion let $q \in U_e$; then, from (iv) of Definition 5.7, $p \leq \varphi(p, q)$ for all $p \in X$. As U is an increasing set, $\varphi(p, q) \in U$ for all $p \in U$, i.e., $q \in \bigcap_{p \in U} \varphi_p^{-1}[U] = U \Rightarrow U$.

To prove (ℓ_3) $U \Rightarrow ((V \Rightarrow U_e) \Rightarrow W) = ((U \Rightarrow V) \Rightarrow U_e) \Rightarrow W$, let $q \in U \Rightarrow ((V \Rightarrow U_e) \Rightarrow W)$ and suppose $q \notin ((U \Rightarrow V) \Rightarrow U_e) \Rightarrow W$. Then there is $p \in (U \Rightarrow V) \Rightarrow U_e$ such that $\varphi(p, q) \notin W$. As $p \in (U \Rightarrow V) \Rightarrow U_e$, from Lemma 5.8, we have that $g(p) \notin U \Rightarrow V$. Then there exists $p' \in U$ such that $\varphi(p', g(p)) \notin V$. As g is an involution we can write $g(g(\varphi(p', g(p)))) \notin V$ to obtain $g(\varphi(p', g(p))) \in V \Rightarrow U_e$ using again Lemma 5.8. As $p' \in U$ and $q \in U \Rightarrow ((V \Rightarrow U_e) \Rightarrow W)$, $\varphi(p, q) \in (V \Rightarrow U_e) \Rightarrow W$ and, as $g(\varphi(p', g(p))) \in V \Rightarrow U_e$, we obtain $(*)$ $\varphi(g(\varphi(p', g(p))), \varphi(p', q)) \in W$. Now, as $\varphi(p, q) \notin W$, $p \leq q_W$. As g is an involution $g(q_W) \leq g(p)$. Then, using the fact that φ is order-preserving in each variable, $\varphi(p', g(q_W)) \leq \varphi(p', g(p))$. Then $g(\varphi(p', g(p))) \leq g(\varphi(p', g(q_W)))$. Using (v) of Definition 5.7. we obtain $g(\varphi(p', g(p))) \leq (\varphi(p', q))_W$. Now, as $\varphi((\varphi(p', q))_W, \varphi(p', q)) \notin W$ and φ is order-preserving in the first variable, $\varphi(g(\varphi(p', g(p))), \varphi(p', q)) \notin W$. But this contradicts $(*)$.

For the other inclusion let $q \in ((U \Rightarrow V) \Rightarrow U_e) \Rightarrow W$ and suppose $q \notin U \Rightarrow ((V \Rightarrow U_e) \Rightarrow W)$. Then there is $p \in U$ such that $\varphi(p, q) \notin (V \Rightarrow U_e) \Rightarrow W$. Then, there is $p' \in V \Rightarrow U_e$ such that $\varphi(p', \varphi(p, q)) \notin W$. As $p' \leq (\varphi(p, q))_W$ and $p' \in V \Rightarrow U_e$, which is an increasing set, we have $(\varphi(p, q))_W = g(\varphi(p, g(q_W))) \in V \Rightarrow U_e$. Using Lemma 5.8. and the fact that g is an involution, we obtain $\varphi(p, g(q_W)) \notin V$. As $p \in U$, $g(q_W) \notin \bigcap_{p \in U} \varphi_p^{-1}[V] = U \Rightarrow V$. Then, from Lemma 5.8, $q_W \in (U \Rightarrow V) \Rightarrow U_e$. As $q \in ((U \Rightarrow V) \Rightarrow U_e) \Rightarrow W$ and $q_W \in (U \Rightarrow V) \Rightarrow U_e$, we obtain $\varphi(q_W, q) \in W$, which is a contradiction. □

Let $\mathscr{L}$ be the category whose objects are ℓ-spaces and whose morphisms are defined as follows:

DEFINITION 5.10. $f:\mathbf{X}\to\mathbf{X}'$ is a *morphism of ℓ-spaces* iff f is a morphism of IL-spaces which also satisfies:

$$f(g(p)) = g'(f(p)) \qquad \text{for all } p \in X \tag{$*$}$$

(where, of course, g and g' are the involutions introduced in Definition 5.7).

THEOREM 5.11: *Duality theorem for ℓ-groups*
Consider the map F such that for each object $\mathbf{A}$ of $\mathscr{I}$, $F(\mathbf{A}) = Il(\mathbf{A})$ and for each morphism $f:\mathbf{A}\to\mathbf{A}'$, $F(f):F(\mathbf{A}')\to F(\mathbf{A})$ is defined by $F(f)(P') = f^{-1}[P']$ for all $P'\in F(\mathbf{A}')$. Then F is a duality between the category $\mathscr{I}$ and the category $\mathscr{L}$ of ℓ-spaces. Thus, by Proposition 5.3, there is a duality between $\mathscr{G}$ and $\mathscr{L}$.

Proof. For each ℓ-space $\mathbf{X}$ of $\mathscr{L}$, let's consider the implicative lattice $\sum(\mathbf{X}) = \langle \sum(X), \cup, \cap, \Rightarrow\rangle$. From Theorem 5.9 and Proposition 5.1 $\sum(\mathbf{X})$ with U_e as the distinguished element is an object of $\mathscr{I}$. Let's consider again, as in Theorem 4.8, the function $\delta:\mathbf{X}\to IL(\sum(\mathbf{X}))$ defined by $\delta(p) = \{U\in\sum(\mathbf{X}) : p\in U\}$. We already know that δ is an isomorphism of IL-spaces. Let's prove that δ is also an isomorphism of ℓ-spaces. To prove this we'll show that δ and δ^{-1} are both morphisms of ℓ-spaces. As δ and δ^{-1} are morphisms of IL-spaces, from Definition 5.10 it suffices to prove: (i) $\delta(g(q)) = G(\delta(q))$ for all $q\in X$ and (ii) $\delta^{-1}(G(Q)) = g(\delta^{-1}(Q))$ for all $Q\in IL(\sum(\mathbf{X}))$.

To prove (i) recall that $G(\delta(q)) = (\delta(q))_{U_e}$. Now $U\in\delta(g(q))$ iff $g(q)\in U$. From Lemma 5.8 $g(q)\in U$ iff $q\notin U\Rightarrow U_e$; then $U\in\delta(g(q))$ iff $U\Rightarrow U_e\notin\delta(q)$ iff $U\in(\delta(q))_{U_e}$.

To prove (ii) let $Q\in IL(\sum(\mathbf{X}))$; recall that δ is a bijection function and take $q, q'\in X$ such that $\delta(q) = Q$ and $\delta(q') = G(Q)$. We have proved in (i) that $\delta(g(q)) = G(Q)$. Then $q' = g(q)$ and we obtain $\delta^{-1}(G(Q)) = \delta^{-1}(\delta(q')) = q' = g(q) = g(\delta^{-1}(\delta(q))) = g(\delta^{-1}(Q))$.

At this point we have proved that $\mathbf{X}$ and $F(\sum(\mathbf{X}))$ are isomorphic as ℓ-spaces. It remains to prove that if $\mathbf{A}, \mathbf{A}'$ are objects of $\mathscr{I}$, the function $[\mathbf{A},\mathbf{A}']_{\mathscr{I}} \overset{F}{\to} [F(\mathbf{A}'), F(\mathbf{A})]_{\mathscr{L}}$ induced by the functor F is one-one and onto. From Theorem 4.8, we already know that if f and f' are distinct morphisms, then $F(f)\neq F(f')$.

Now, let $h: F(\mathbf{A}')\to F(\mathbf{A})$ be a morphism of ℓ-spaces and consider the function $f:\mathbf{A}\to\mathbf{A}'$ of Theorem 4.8, defined by the assignment $x\to y_x$ (where $h^{-1}[\sigma(x)] = \sigma'(y_x)$). We already know that f is a morphism of implicative lattices such that $F(f) = h$.

Let's show that $f(e) = e'$. For this, we'll need the following:

CLAIM. *Let* **X** *and* **X'** *be* ℓ*-spaces; if* $h : \mathbf{X} \to \mathbf{X}'$ *is a morphism of* ℓ*-spaces then* $h^{-1}[U_{e'}] = U_e$.

To prove this claim let $p \in h^{-1}[U_e']$ and suppose $p \notin U_e$. From (i) of Definition 5.7, $p_{U_e} \in U_e$. From (ii) of the same definition, $p \le p_{U_e}$ or $p_{U_e} \le p$. As U_e is increasing, $p_{U_e} \in U_e$ and $p \notin U_e$, we conclude $p \le p_{U_e} = g(p)$. As $p \in h^{-1}[U_{e'}]$, $h(p) \in U_{e'}$. From (i) and (ii) of Definition 5.7 $(h(p))_{U_{e'}} = g'(h(p)) \notin U_{e'}$. As h is a morphism of ℓ-spaces, from (*) of Definition 5.10, we obtain $h(g(p)) \notin U_{e'}$. Then $g(p) \notin h^{-1}[U_{e'}]$. As $p \le g(p)$ and $h^{-1}[U_{e'}]$ is an increasing set, we derive $p \notin h^{-1}[U_{e'}]$, which is a contradiction.

For the other inclusion let $p \in U_e$ and suppose $p \notin h^{-1}[U_e']$. As $p \in U_e$, $p_{U_e} \notin U_e$. As U_e is increasing, $p \nleq p_{U_e}$. From (ii) of Definition 5.7, $p_{U_e} \le p$, i.e., $g(p) \le p$. As $h(p) \notin U_{e'}$, $(h(p))_{U_{e'}} = g'(h(p)) \in U_{e'}$. As h is a morphism of ℓ-spaces $h(g(p)) \in U_{e'}$; then $g(p) \in h^{-1}[U_{e'}]$. As $g(p) \le p$ and $h^{-1}[U_{e'}]$ is increasing, $p \in h^{-1}[U_{e'}]$, which is a contradiction.

Using this claim for $\mathbf{X} = F(\mathbf{A}')$ and $\mathbf{X}' = F(\mathbf{A})$ and recalling that $U_e = \sigma(e)$ and $U_{e'} = \sigma(e')$, we can write $\sigma'(y_e) = h^{-1}[\sigma(e)] = h^{-1}[U_e] = U_{e'} = \sigma'(e')$. As σ' is a bijection, $y_e = f(e) = e'$.

Thus, we have proved that $f \in [\mathbf{A}, \mathbf{A}']_{\mathscr{I}}$; since $F(f) = h$, the function $[\mathbf{A}, \mathbf{A}']_{\mathscr{I}} \overset{F}{\to} [F(\mathbf{A}'), F(\mathbf{A})]_{\mathscr{S}}$ induced by the functor F is onto. This ends our proof. □

The dual space is especially well behaved in the abelian case.

DEFINITION 5.12. Let $\mathbf{X} = \langle X, \tau, \le, p_m, p_M, \varphi \rangle$ be an IL-space. We'll say that **X** is an *abelian* ℓ*-space*, iff **X** satisfies:

(i) There is one and only one proper compact and open subset $U \subseteq X$ such that $p \in U$ iff $p_U \notin U$. We call it U_e.

(ii) The function $g : (X, \le) \to (X, \le)$ defined by $g(p) = p_{U_e}$ is an involution satisfying $g(p) \le p$ or $p \le g(p)$ for all $p \in X$.

(iii) If $p \notin U_e$, then $p_U \in U$ for all proper compact and open subsets $U \subseteq X$.

(iv) If $p \in U_e$, $q \le \varphi(q, p)$ for all $q \in X$.

(v) φ is associative and commutative.

(vi) $\varphi(g(\varphi(p, q)), q) \le g(p)$.

Note that conditions (i)–(iv) are the same of Definition 5.7; from (v) and the fact that φ is continuous and order-preserving in each variable, it follows that abelian ℓ-spaces are compact, ordered, abelian, topological semigroups.

PROPOSITION 5.13. *Let* **A** *be an object of* $\mathcal{I}a$. *Then* $IL(\mathbf{A})$ *is an abelian* ℓ*-space.*

Proof. As $IL(\mathbf{A})$ is an ℓ-space, (i)–(iv) are verified. To prove (v) it suffices to show that **A** satisfies both (a) $x \leq (x \to y) \to y$ and (b) $x \to (y \to z) = y \to (x \to z)$ of Proposition 4.9.

As $(x \to y) \to y = (x^{-1}y)^{-1}y = (y^{-1}x)y$ and from commutativity $y^{-1}x = xy^{-1}$, we obtain $(x \to y) \to y = x$; this proves (a); (b) also follows from the commutativity by the equalities $x \to (y \to z) = x^{-1}(y^{-1}z) = y^{-1}(x^{-1}z) = y \to (x \to z)$.

To prove (vi) $\Phi(G(\Phi(P, Q)), Q) \subseteq G(P)$, let $y \in \Phi(G(\Phi(P, Q)), Q)$ and suppose $y \notin G(P) = Pe$; then $y \to e \in P$. Let $x \in G(\Phi(P, Q))$ such that $x \to y \in Q$. As **A** is abelian, $x \to y = x^{-1}y = yx^{-1} = (y \to e) \to (x \to e)$. Then $(y \to e) \to (x \to e) \in Q$. As $x \in G(\Phi(P, Q)) = (\Phi(P, Q))_e$, $x \to e \notin \Phi(P, Q)$. But $y \to e \in P$ and $(y \to e) \to (x \to e) \in Q$; then $x \to e \in \Phi(P, Q)$, which is a contradiction. □

THEOREM 5.14. *Let* $\mathbf{X} = \langle X, \tau, \leq, p_m, p_M, \varphi \rangle$ *be an abelian* ℓ*-space. Then the implicative lattice of* $\mathbf{X}$, $\sum(\mathbf{X}) = \langle \sum(X), \cup, \cap, \Rightarrow \rangle$ *with* U_e *as the distinguished element, satisfies* (ℓ_1), (ℓ_2), (ℓ'_3) *and* (ℓ'_4) *of Proposition* 5.2.

Proof. Conditions (ℓ_1) and (ℓ_2) follow as in Theorem 5.7. (To prove these conditions only (i)–(iv) of Definition 5.7 were used and we also have (i)–(iv) in the definition of abelian ℓ-spaces.) (ℓ'_3) follows from (v) of Definition 5.12 and from Proposition 4.9. Let's prove (ℓ'_4) $(V \Rightarrow U_e) \Rightarrow (U \Rightarrow U_e) = U \Rightarrow V$.

Let $q \in (V \Rightarrow U_e) \Rightarrow (U \Rightarrow U_e)$ and suppose $q \notin U \Rightarrow V = \bigcap_{p \in U} \varphi_p^{-1}[V]$; then there is $p \in U$ such that $\varphi(p, q) \notin V$. Then, from Lemma 5.8, $g(\varphi(p, q)) \in V \Rightarrow U_e$. As $q \in (V \Rightarrow U_e) \Rightarrow (U \Rightarrow U_e)$ and $g(\varphi(p, q)) \in V \Rightarrow U_e$, $\varphi(g(\varphi(p, q)), q) \in U \Rightarrow U_e$. From (vi) of Definition 5.10 and the fact that $U \Rightarrow U_e$ is an increasing set, we obtain $g(p) \in U \Rightarrow U_e$. As $p \in U$, $\varphi(p, g(p)) \in U_e$. But φ is commutative; then $\varphi(g(p), p) \in U_e$. As $g(p) = p_{U_e}$ we derive $\varphi(p_{U_e}, p) \in U_e$, which is a contradiction.

For the other inclusion, let $q \in U \Rightarrow V$ and suppose that $q \notin (V \Rightarrow U_e) \Rightarrow (U \Rightarrow U_e)$, then there is $p \in V \Rightarrow U_e$ such that $\varphi(p, q) \notin U \Rightarrow U_e$; from Lemma 5.8 $g(\varphi(p, q)) \in U$. As $q \in U \Rightarrow V = \bigcap_{p \in U} \varphi_p^{-1}[V]$, we obtain $\varphi(g(\varphi(p, q)), q) \in V$. From (vi) of Definition 5.12, and the fact that V is an increasing set, $g(p) \in V$. As $p \in V \Rightarrow U_e$ and $g(p) \in V$, we obtain $\varphi(g(p), p) \in U_e$. But $g(p) = p_{U_e}$, then $\varphi(p_{U_e}, p) \in U_e$, which is a contradiction. □

Let's now consider the category $\mathcal{L}a$ of abelian ℓ-spaces, whose objects are the abelian ℓ-spaces and whose morphisms are the morphisms of IL-spaces which satisfy $(*)$ of Definition 5.10. It can be derived, with the same arguments used in Theorem 5.11, the following:

THEOREM 5.15: *Duality theorem for abelian ℓ-groups*

Let F be the map such that for each object $\mathbf{A}$ *of* $\mathscr{I}a$, $F(\mathbf{A}) = IL(\mathbf{A})$ *and for each morphism* $f : \mathbf{A} \to \mathbf{A}'$, $F(f) : F(\mathbf{A}') \to F(\mathbf{A})$ *is defined by* $F(f)(P') = f^{-1}[P']$. *Then F is a duality between the category* $\mathscr{I}a$ *and the category of abelian ℓ-spaces. By Proposition* 5.4, *there is a duality between* $\mathscr{G}a$ *and* $\mathscr{L}a$.

In fact, for each abelian ℓ-space $\mathbf{X}$, we have by Theorem 5.14 that $\sum(\mathbf{X})$ is an object of $\mathscr{I}a$; thus, defining δ as in Theorem 5.11, as δ is a morphism of abelian ℓ-spaces, it follows that $\mathbf{X}$ and $F(\sum(\mathbf{X}))$ are isomorphic as abelian ℓ-spaces. Finally, the fact that the function $[\mathbf{A}, \mathbf{A}']_{\mathscr{I}a} \overset{F}{\to} [F(\mathbf{A}'), F(\mathbf{A})]_{\mathscr{L}a}$ induced by the functor F is one–one and onto follows exactly as in Theorem 5.11. □

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Universidad de Buenos Aires
Buenos Aires, Argentina