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GENERALIZATION OF THE PALEY—WIENER THEOREM IN WEIGHTED SPACES

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1. Introduction

Let X be a linear topological space of complex functions defined on some subset $T \subset \mathbf{R}^n\ (\mathbf{C}^n)$, and assume that a system of functions $e^{\langle t,z\rangle}$, $z \in \Omega$, is complete in this space. Then the generalized Laplace transform, which takes a linear continuous functional S on X to a function $\hat{S}(z) = (S, \exp(\langle t, x\rangle))$, $z \in \Omega$, establishes an isomorphism between the adjoint space X^* and a linear topological space of functions defined on Ω.

Many mathematicians have devoted their work to the problem of describing the adjoint space in terms of generalized Laplace transform. For example in [1] the projective limit of weighted Banach spaces of the form

$$\{f \in H(D)\colon \|f\| = \sup_z [\,|f(z)|/\exp(-\psi(-\ln d(z)))] < \infty\}$$

was considered, where D is a convex, bounded region in $\mathbf{C}^n$, $d(z)$ is the distance from a point z to ∂D and ψ is a convex function, and a complete description was given of the adjoint space in terms of the generalized Laplace transform. In [3, 4] some generalization of the Paley—Wiener theorem for weighted Hilbert spaces.

The present article is devoted to the problem of describing adjoint spaces in terms of the Laplace transform on the space

$$L^2(I, W) = \left\{f \in L_{\mathrm{loc}}(I)\colon \|f\|^2_{L^2(I,W)} \overset{\mathrm{def}}{=} \int_I |f(t)|^2/W(t)\,\mathrm{d}t < \infty\right\},$$

where I is a bounded interval on the real axis and $1/W(t)$ is a measurable function on I.

THEOREM 1. Let $W(t)$ be a function on I bounded from below by a positive constant and bounded from above on each compact subinterval of I. Let $\tilde{h}(x) = \sup_{t\in I}(xt - \ln\sqrt{W(t)})$— Young's conjugate function of the function $\ln\sqrt{W(t)}$, and define $\rho_{\tilde{h}}(x)$ by the condition

$$\int_{x-\rho_{\tilde{h}}(x)}^{x+\rho_{\tilde{h}}(x)} |\tilde{h}'(x) - \tilde{h}'(t)|\,\mathrm{d}t \equiv 1.$$

Then

1. The generalized Laplace transform $\hat{S}(z)$ of the functional S on $L^2(I, W)$ is an entire function satisfying the condition $|\hat{S}|(z)| < C_S \exp(\tilde{h}(x))$,

$$\|\hat{S}\|^2 = \int_{\mathbf{R}}\int_{\mathbf{R}} |\hat{S}(x+iy)|^2 e^{-2\tilde{h}(x)} \rho_{\tilde{h}}(x)\,\mathrm{d}\tilde{h}'(x)\,\mathrm{d}y \leqslant \pi e \|S\|_{L^2(I,W)}.$$

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2. If $\ln W(x)$ is a convex function, then the lower and upper bound

$$(\pi e)^{-1} \| S \|_{L^2(I,W)} \leqslant \| \hat{S} \| \leqslant \pi e \| S \|_{L^2(I,W)} \tag{1}$$

hold. Furthermore, in this case the converse is also true: if $F(z)$ is an entire function of exponential type satisfying the condition $|F(z)| < C_F \exp(\tilde{h}(x))$, $z = x + iy$,

$$\int_{\mathbf{R}}\int_{\mathbf{R}} |F(x+iy)|^2 e^{-2\tilde{h}(x)} \rho_{\tilde{h}}(x)\, d\tilde{h}'(x)\, dy < \infty,$$

then there exists a function S on $L^2(I, W)$ such that

$$\hat{S}(z) \equiv F(z), \quad z \in \mathbf{C}.$$

3. If $W(x) = \int_{-\infty}^{+\infty} \exp(zt)d\mu(t)$, where $\mu(t)$ is a nonnegative measure on $\mathbf{R}$, then $\|S\|^2_{L^2(I,W)} = \int_{\mathbf{R}}\int_{\mathbf{R}} |\hat{S}(x+iy)|^2 d\mu(x)dy$.

Remark. Proofs of these assertions were obtained by the first author. The formulation of the problem and idea of some arguments used here belongs to the second author.

2. Preparatory Results

Let $u(x)$ be a convex function on $\mathbf{R}$ and $u(x) \not\equiv cx + d$. Define a function $\rho_u(x)$ on $\mathbf{R}$ by the identity $\int_{x-\rho_u(x)}^{x+\rho_u(x)} |u'(t) - u'(x)|dt \equiv 1$ or by the equality $u(x + \rho_u(x)) + u(x - \rho_u(x)) - 2u(x) \equiv 1$. By the continuity of the function $u(x)$ and the second identity we see that the function $\rho_u(x)$ is uniquely defined. The function satisfies the equality

$$\int_0^{\rho_u(x)} (u'(x+t) - u'(x-t))\, dt \equiv 1.$$

LEMMA 1. The function $\rho_u(x)$ has the following properties:

1) $|\rho_u(x) - \rho_u(y)| < |x - y|$ for any $x, y \leq \mathbf{R}$;

2) $\rho_u(x) \leq y + \rho_u(y) + \rho_u(y + \rho_u(y)) - x$, if $y \leq x \leq y + \rho_u(y)$, $\rho_u(x) \leq x - y + \rho_u(y) + \rho_u(y - \rho_u(y))$, if $y - \rho_u(y) \leq x \leq y$;

3) $1 \leq \int_{y-\rho_u(y)}^{y+\rho_u(y)} \rho_u(x)du'(x) \leq 4$.

Proof of Lemma 1. We prove Assertion 1). Let $x \in (y, y + \rho(y))$, $\rho(y) \overset{\text{def}}{=} \rho_u(y)$. Since the function $u'(x)$ is nondecreasing and $x + t \geq y + (t - x + y)$, we have the estimate

$$\int_0^{x+\rho(y)-y} (u'(x+t) - u'(x-t))\, dt \geqslant \int_{x-y}^{x+\rho(y)-y} (u'(y + (t - x + y)) - u'(y - (t - x + y)))\, dt.$$

Hence

$$\int_0^{\rho(y)-y} (u'(x+t) - u'(x-t))\, dt \geqslant \int_0^{\rho(y)} (u'(y+\tau) - u'(y-\tau))\, d\tau = 1.$$

From this and the definition of $\rho(x)$ it follows that $x + \rho(y) - y \geq \rho(x)$ or $\rho(x) - \rho(y) \leq |x - y|$. Similarly from the inequality

$$\int_0^{y+\rho(y)-x} (u'(x+t) - u'(x-t))\, dt \geqslant \int_0^{y+\rho(y)-x} (u'(y + (x + t - y)) - u'(y - (x + t - y)))\, dt$$

we have the estimate $\rho(y) - \rho(x) \leq |x - y|$. By the same proof, if $x \leq x \leq y + \rho(y)$, we have the inequality $|\rho(x) - \rho(y)| \leq |x - y|$. By the symmetry of the definition of $\rho(x)$ this inequality also holds for $y \geq x \geq y - \rho(y)$. Thus Assertion 1) holds for arbitrary $y \in \mathbf{R}$.

Let us prove the first inequality in 2). Introduce the notations $a = y - \rho(y)$, $b = y + \rho(y)$, by the definition of the function $\rho(x)$, the inequality to be proved follows from the inequality

$$\int_0^{b+\rho(b)-x} (u'(x+t)-u'(x-t))\,dt \geqslant 1,$$

which in turn follows from the estimate

$$\int_0^{b+\rho(b)-x} (u'(x+t)-u'(x-t))\,dt \geqslant \int_{b-x}^{b+\rho(b)-x} (u'(b+(x+t-b))-u'(x-t))\,dt \geqslant$$
$$\geqslant \int_{b-x}^{b+\rho(b)-x} (u'(b+(x+t-b))-u'(b-(x+t-b)))\,dt = 1 .$$

The second inequality in 2) can be proved similarly.

We now prove the lower estimate of the integral in 3)

$$A \overset{\text{def}}{=} \int_{y-\rho(y)}^{y+\rho(y)} \rho(x)\,du'(x) = \int_{y-\rho(y)}^{y} \rho(x)\,du'(x) + \int_{y}^{y+\rho(y)} \rho(x)\,du'(x).$$

From 1), which we have proved, we have the estimate $\rho(x) \geq b - x$ if $y \leq x \leq b$ and $\rho(x) \leq x - a$ if $y \geq x \geq a$. From this we have the relation

$$A \geqslant \int_a^y (x-a)\,du'(x) + \int_y^b (b-x)\,du'(x)$$

or

$$A \geqslant -\int_a^y (x-a)\,d(u'(y)-u'(x)) - \int_y^b (x-b)\,d(u'(x)-u'(y)).$$

Integrating by parts on the two integrals, we obtain from the definition of $\rho(x)$

$$A \geqslant \int_{y-\rho(y)}^{y} (u'(y)-u'(x))\,dx + \int_y^{y+\rho(y)} (u'(x)-u'(y))\,dx = 1.$$

We now prove the upper estimate of the integral A. We use the relations in 2). We have

$$A \leqslant -\int_a^y (x-a+\rho(a))\,d(u'(y)-u'(x)) - \int_y^b (x-b-\rho(b))\,d(u'(x)-u'(y)).$$

Integration by parts gives the estimate

$$A \leqslant \rho(a)[u'(y)-u'(a)] + \rho(b)[u'(b)-u'(y)] + \int_{y-\rho(y)}^{y+\rho(y)} |u'(y)-u'(x)|\,dx =$$
$$= \rho(a)[u'(y)-u'(a)] + \rho(b)[u'(b)-u'(y)] + 1. \tag{2}$$

Let us estimate the first summand on the right side of the last inequality:

$$\rho(a)[u'(y)-u'(a)] \leqslant \int_a^{\max(y,\,a+\rho(a))} [u'(y)-u'(a)]\,dx + \int_{\max(y,\,a+\rho(a))}^{a+\rho(a)} [u'(y)-u'(a)]\,dx.$$

Since if $x \in [\max(y, a+\rho(a)), a+\rho(a)]$, $u'(x) \geq u'(y)$, we have

$$\rho(a)[u'(y)-u'(a)] \leqslant \int_a^{\max(y,a+\rho(a))} (u'(y)-u'(x))\,dx +$$
$$+ \int_a^{\max(y,a+\rho(a))} (u'(x)-u'(a))\,dx + \int_{\max(y,a+\rho(a))}^{a+\rho(a)} (u'(x)-u'(a))\,dx.$$

The sum of the last two integrals in this inequality does not exceed 1 by the definition of $\rho(a)$. On the other hand, if $x \in [y, \max(y, a+\rho(a))]$, the function to be integrated in the first integral is nonpositive. Therefore, we have the estimate

$$\rho(a)[u'(y)-u'(a)] \leqslant 1 + \int_a^y (u'(y)-u'(x))\,dx.$$

Similarly, we can obtain an estimate for the second summand in (2)

$$\rho(b)[u'(b)-u'(y)] \leqslant 1+\int_y^b (u'(x)-u'(y))\,dx.$$

From the last two inequalities and (2) and the definition of $\rho(y)$ we obtain the desired inequality $A \le 4$.

LEMMA 2. Let $u(x)$ be a convex function defined on $\mathbf{R}$, $u(x) \not\equiv cx + d$, $\tilde{u}(t)$ is Young's conjugate function of $u(x)$. Then for t such that $\tilde{u}(t) < \infty$, we have the inequality

$$e^{-2} \leqslant \exp(-2\tilde{u}(t))\int_{-\infty}^{+\infty} \exp(2tx-2u(x))\rho(x)\,du'(x) \leqslant \frac{\pi}{2}e^2.$$

Proof of Lemma 2. Without loss of generality we may assume that t is an exterior point of the integral on which $\tilde{u}(t) < \infty$. Then we can find at least one point x_0 satisfying the equality $\tilde{u}(t) = tx_0 - u(x_0)$.

By the choice of the point t the equation $zt - u(z) - \tilde{u}(t) = -n$ for any $n > 0$ has exactly two solutions $z = x_n$ and $z = x_{-n}$, and $x_{-n-1} < x_{-n} < x_0 < x_n < x_{n+1}$. By the definition of the sequence x_n, $n > 0$, we have the equality

$$\int_{x_n}^{x_{n+1}} (u'(x)-t)\,dx = 1 \qquad (n=1,2,\ldots).$$

Let $x_n^* = (x_n + x_{n+1})/2$. Because for $n > 0$, $u'(x_n) \ge t$, we have

$$1 \geqslant \int_{x_n^*}^{x_{n+1}} (u'(x)-t)\,dx \geqslant \int_{x_n^*}^{x_{n+1}} (u'(x)-u'(x_n^*))\,dx + \int_{x_n^*}^{x_{n+1}} (u'(x_n^*)-u'(x_n))\,dx.$$

Using the definition of $\rho(x_n^*)$ and the monotonicity of $u'(x)$, we obtain the inequality

$$1 \geqslant \int_{x_n^*}^{x_{n+1}} (u'(x)-u'(x_n^*))\,dx + \int_{x_n}^{x_n^*} (u'(x_n^*)-u'(x))\,dx.$$

From this and the definition of $\rho(x_n^*)$ we have

$$\rho(x_n^*) \geqslant x_n^* - x_n = x_{n+1} - x_n^*.$$

Consequently, by the assertion of 3) of Lemma 1 we have the estimate

$$\int_{x_n}^{x_{n+1}} \rho(x)\,du'(x) \leqslant 4.$$

On the interval (x_n, x_{n+1}) the function $tx - u(x) - \tilde{u}(t)$ does not exceed $-n$. Thus the inequality

$$\int_{-\infty}^{x_0} \exp(2tx-2u(x)-2\tilde{u}(t))\rho(x)\,du'(x) \leqslant 4\sum_{n=0}^{\infty} \exp(-2n) \leqslant \frac{\pi}{4}e^2$$

holds.

Similarly we obtain the estimate

$$\int_{-\infty}^{x_0} \exp(2tx-2u'(x)-2u'(t))\rho(x)\,du'(x) \leqslant 4\sum_{n=0}^{\infty} \exp(-2n) \leqslant \frac{\pi}{4}e^2.$$

Therefore, the right inequality in Lemma 2 is proved.

We now prove the lower estimate

$$\int_{-\infty}^{+\infty} \exp(2tx-\tilde{u}(t)-u(x))\rho(x)\,du'(x) \geqslant \int_{x_{-1}}^{x_1} \exp(2tx-\tilde{u}(t)-u(x))\rho(x)\,du'(x) \geqslant e^{-2}\int_{x_{-1}}^{x_1} \rho(x)\,du'(x). \quad (3)$$

Suppose that $x_1 - x_0 \le x_0 - x_{-1}$ and $x' = x_0 - (x_1 - x_0)$. Then by the definitions of the points x_1 and x' we have the inequality

$$1 \leqslant \int_{x_0}^{x_1} (u'(x)-t)\,dx + \int_{x'}^{x_0} (t-u'(x))\,dx$$

or

$$1\leqslant\int_{x_0}^{x_1}(u'(x)-u'(x_0))\,dx+\int_{x'}^{x_0}(u'(x_0)-u'(x))\,dx.$$

By the same token, from the definition of $\rho(x_0)$ we have $\rho(x_0) \le x_1 - x_0 = x_0 - x' \le x_0 - x_{-1}$. From this and (3) we have the inequality

$$\int_{-\infty}^{+\infty}\exp(2tx-\tilde{u}(t)-u(x))\rho(x)\,du'(x)\geqslant e^{-2}\int_{x_0-\rho(x_0)}^{x_0+\rho(x_0)}\rho(x)\,du'(x),$$

which together with the assertion of 3) of Lemma 1 gives the desired lower estimate.

3. Proof of Theorem 1. Let us prove the first assertion of Theorem 1. Let S be a functional on $L^2(I, W)$, generated by the function $f \in L^2(I, W)$: $\hat{S}(z) = \int_I \overline{f(t)}\exp(zt)/W(t)dt$.

The analyticity of $\hat{S}(z)$ on $\mathbf{C}$ follows from the Paley–Wiener theorem, since $\hat{S}(z)$ is the classical Fourier transform of the function $\overline{f(t)}/W(t)$. From the Cauchy–Bunyakovski inequality we have $|\hat{S}(z)| \le \|f\|_{L^2(I,W)}\exp(\tilde{h}(x))\sqrt{|I|}$. By the Plancherel–Parseval formula

$$\int_{-\infty}^{+\infty}|\hat{S}(x+iy)|^2\,dy=2\pi\int_I|f(t)|^2\exp(2xt)/W^2(t)\,dt.$$

From this we obtain the inequality

$$\|\hat{S}\|^2\overset{\mathrm{def}}{=}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|\hat{S}(x+iy)|^2\exp(-2\tilde{h}(x))\rho_{\tilde{h}}(x)\,dy\,d\tilde{h}'(x)=2\pi\int_I|f(t)|^2/W^2(t)\int_{-\infty}^{+\infty}\exp(2xt-2\tilde{h}(x))\rho_{\tilde{h}}(x)\,d\tilde{h}'(x)\,dt.\quad(4)$$

From this, by applying the assertion of Lemma 2 to the function $u(x) = 2\tilde{h}(x)$ and the point $2t$, we have the estimate

$$\|\hat{S}\|^2\leqslant(\pi e)^2\int_I|f(t)|^2\exp(2\tilde{\tilde{h}}(t))/W(t)\,dt,$$

where $\tilde{\tilde{h}}(t)$ is Young's conjugate function of $\tilde{h}(x)$. Since $\tilde{\tilde{h}}(t) \le \ln\sqrt{W(t)}$, the last inequality gives Assertion 1) of Theorem 1.

We turn to the proof of the second assertion of the theorem. If $\ln W(t)$ is a convex function, then $\tilde{\tilde{h}}(t) \equiv \ln\sqrt{W(t)}$. Therefore, Ineq. (1) follows from Eq. (4) and Lemma 2 applied to the function $2\tilde{h}(x)$ and the point $2t$.

Suppose now that $F(z)$ is an entire function of exponential type and satisfying the conditions of Theorem 1. From the convergence of the integral

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}|F(x+iy)|^2\exp(-2\tilde{h}(x))\rho_{\tilde{h}}(x)\,dy\,d\tilde{h}'(x)$$

we see that there exists a pont x_0 such that

$$\int_{-\infty}^{+\infty}|F(x_0+iy)|^2\,dy<\infty.$$

By the Paley–Wiener theorem [2] and the uniform estimate on $|F(z)|$ we can find a function $g(t) \in L^2(I)$ such that

$$F(x_0+iy)=\int_I\overline{g(t)}\exp[(x_0+iy)t]\,dt.$$

The function $F_1(x + iy) \overset{\mathrm{def}}{=} \int_I \overline{g(t)}\exp[(x_0 + iy)t]dt$ is an entire function and $F_1(x_0 + iy) \equiv F(x_0 + iy)$ $\forall y \in \mathbf{R}$,; therefore $F(z) = \int_I \overline{g(t)}\exp(zt)dt$. Let $f(t) = g(t)W(t)$. Then

$$F(z)=\int_I\overline{f(t)}\exp(zt)/W(t)\,dt,$$

and it remains to be shown that $f(t) \in L^2(I, W)$. By the Plancherel–Parserval formula

$$\int_{-\infty}^{+\infty}|F(x+iy)|^2\,dy=2\pi\int_I|f(t)|^2\exp(2xt)/W^2(t)\,dt,$$

therefore

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |F(x+iy)|^2 \exp(-2\tilde{h}(x))\rho_{\tilde{h}}(x)\,dy\,d\tilde{h}'(x) = 2\pi\int_I |f(t)|^2/W^2(t)\int_{-\infty}^{+\infty}\exp(2xt-2\tilde{h}(x))\rho_{\tilde{h}}(x)\,d\tilde{h}'(x)\,dt.$$

From this and the lower estimate in Lemma 2 we have

$$(\pi e)^{-2}\int_I |f(t)|^2/W(t)\,dt \leqslant \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} |F(x+iy)|^2\exp(-2\tilde{h}(x))\rho_{\tilde{h}}(x)\,dy\,d\tilde{h}'(x).$$

This proves the second assertion.

The third assertion follows from Eq. (4), in which we should let $\exp(-2\tilde{h}(x))\rho_{\tilde{h}}(x)) \times d\tilde{h}'(x) = d\mu(x)$. The proof of Theorem 1 is completed.

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COMPLETENESS OF A SYSTEM OF EXPONENTIALS ON THE HALFLINE

A. M. Sedletskii

1. The problem of determination of conditions for the completeness of the systems of exponentials

$$(\exp(-\lambda_n t))_{n=1}^{\infty}, \quad \operatorname{Re}\lambda_n > 0, \tag{1}$$

in the spaces $L^p = L^p(0, \infty)$ $(1 \le p < \infty)$ is equivalent to the classical Müntz–Szasz problem of determination of conditions for the completeness of the systems of powers $(x^{\mu_n})_{n=1}^{\infty}$, $\operatorname{Re}\mu_n > -1/p$, in the spaces $L^p(0, 1)$ [under the substitution $x = \exp(-t)$]. By the Szasz theorem [1] (see also [2, p. 283]), the criterion for the completeness of system (1) in L^2 is that

$$\sum_{n=1}^{\infty}(\operatorname{Re}\lambda_n)/(1+|\lambda_n|^2) = \infty. \tag{2}$$

For the completeness of system (1) in L^p, condition (2) is sufficient for $p \ge 2$ and is necessary for $1 \le p \le 2$ [3]. This proposition reverts if

$$\int_{-\infty}^{\infty}(1+y^2)^{-1}\log\operatorname{dist}(iy, \Lambda)\,dy > -\infty, \quad \Lambda = (\lambda_n)$$

[4, 5], and then, consequently, system (1) is either complete in all the space L^p, $1 \le p < \infty$, or incomplete in all of them. In this situation we say that the property of completeness of system (1) in L^p does not divide the exponents $p \in [1, \infty)$. However, for an arbitrary disposition of the points λ_n in the right halfplane, conditions (2) is not sufficient for the completeness of system (1) in L^p, $1 \le p < 2$ [3, 6]. The search for sufficient cond-

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