DYNAMICS OF FLEXIBLE SPACE VEHICLES WITH ACTIVE ATTITUDE CONTROL

PETER C. HUGHES

University of Toronto, Institute for Aerospace Studies, Toronto, Ontario, Canada

(Received 20 December, 1972; in revised form 9 July, 1973)

Abstract. The implications of flexible appendages on the attitude dynamics of a space vehicle are examined in general terms. Two families of natural vibration modes, referred to as 'constrained' and 'unconstrained', are discussed and the relationships between them derived. The incorporation of either set of modes into a simulation of the general attitude motion (under the influence of perturbing torques and control torques) is explained. The influence of rotors on these results is also explored.

1. Introduction

The importance of structural flexibility in the attitude stability and control of spacecraft has been recognized since the earliest artificial earth satellites. Typically, the spacecraft consists of a 'main body', B, and one or more flexible appendages, A. As space hardware has become more sophisticated, the geometrical detail of the appendages has become more complex. And, as attitude control system design has evolved from entirely 'passive' to entirely 'active', the emphasis has shifted accordingly from the influence of flexibility on stability to its significance for control system performance. Specific examples have been documented by NASA (1969) and Likins and Bouvier (1971).

For many purposes the extra degrees of freedom attributable to structural deflections may be considered under two headings: excitations of external origin, and excitations of internal origin. There are instances in which the former are of great importance; however the discussion below will deal with only the latter. The aim of the following development is to present the flexibility/control-system interaction in a quite general way. As will become apparent, there are certain relationships which govern this interaction and which are independent both of the geometrical details of the structure and of the techniques employed by the analyst.

Thus in the ensuing discussion the distributions of mass and elasticity throughout the vehicle are specified in a general way. Referring to Figure 1 (for the present ignoring the modal comments at the base of the figure), let $\mathbf{r}$ represent the components of the position vector as expressed in the main-body-fixed frame. Then the mass density of a volume element $\mathrm{d}v$, at $\mathbf{r}$, can be specified by a scalar function:

$$\text{mass density distribution: } \sigma(\mathbf{r}) \tag{1}$$

where $\mathbf{r}^T = (r_1, r_2, r_3)$. Thus, for example, the mass of all the appendages is found

Celestial Mechanics **9** (1974) 21–39. *All Rights Reserved*

from

$$m_A = \int_A \sigma(\mathbf{r})\,\mathrm{d}v. \tag{2}$$

Next, specify the elasticity distribution by a matrix function:

$$\text{elasticity distribution: } \mathbf{F}(\mathbf{r}, \mathbf{r}_1) \tag{3}$$

which gives the deflection at **r** due to a unit force at $\mathbf{r}_1$. Thus all the flexibility properties

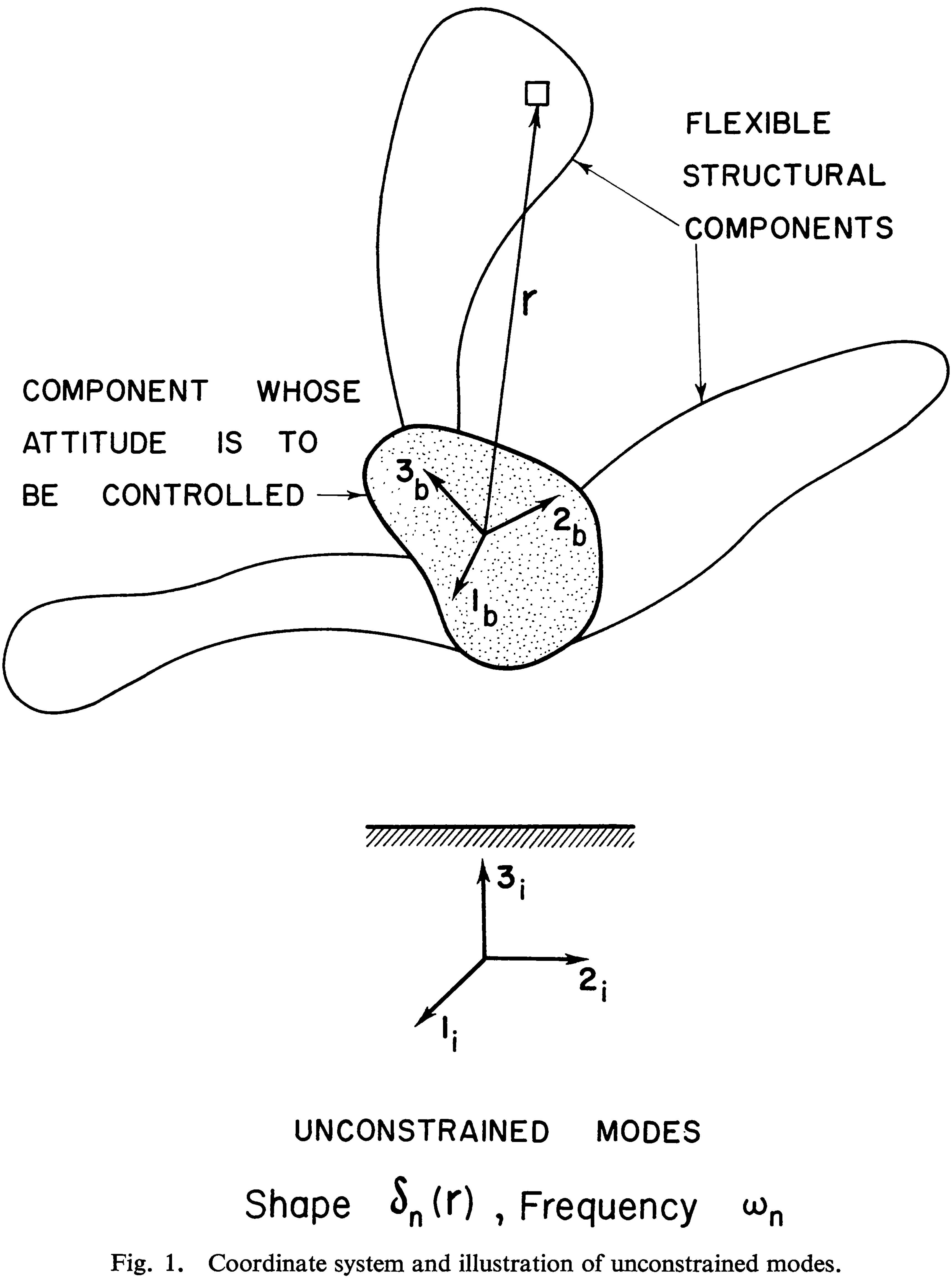

Fig. 1. Coordinate system and illustration of unconstrained modes.

of the spacecraft are implied by $\mathbf{F}$. In particular, for a force/volume distribution $\mathbf{f}(\mathbf{r}_1)$, the distribution of elastic displacement, $\boldsymbol{\delta}(\mathbf{r})$, is given by

$$\boldsymbol{\delta}(\mathbf{r}) = \int_A \mathbf{F}(\mathbf{r}, \mathbf{r}_1)\, \mathbf{f}(\mathbf{r}_1)\, \mathrm{d}v' . \tag{4}$$

The function $\mathbf{F}$ has two important properties, namely

$$\mathbf{F}(\mathbf{r}_1, \mathbf{r}) = \mathbf{F}(\mathbf{r}, \mathbf{r}_1) \tag{5}$$

$$\mathbf{F}^T(\mathbf{r}, \mathbf{r}_1) = \mathbf{F}(\mathbf{r}, \mathbf{r}_1). \tag{6}$$

The integral equation approach to structural analysis used here is convenient because it specifies the linear elastic properties in a general manner, Bolotin (1964). At the same time, details relating to boundary conditions, etc., which are explicitly specified in a differential formulation, are automatically incorporated in $\mathbf{F}$.

It is emphasized that the results below are quite independent of the formulation used to cope with elasticity. Thus, although equations such as Equation (4) imply continuous distributions of force and displacement, completely analogous relations exist for a discretizing procedure, such as the method of finite elements; see, for example, Zienkiewicz (1971). The operation $\int_A (\cdot)\, \mathrm{d}v'$ is simply replaced by a arithmetic sum which has a quite similar appearance if written in matrix notation.

2. Motion Equations

The character of the generality sought relates to a general specification of mass and elasticity properties, as by Equations (1) and (3); no attempt is made to represent all conceivable spacecraft subject to the totality of possible torques. The class of vehicle for which the results below are applicable is the non-spinning, three-axes-controlled class. Two other assumptions are made, viz., that the period of all natural structural vibrations is much shorter than the orbital period, and that motions of the vehicle centre-of-mass and attitude motions are uncoupled. The latter assumption is valid if certain geometrical properties are present, and is often satisfied in practice. The former assumption allows the local orbiting reference frame to be taken as inertial in the appendage motion equations. In any case, either assumption can be relaxed without difficulty and the additional terms carried in the equations to follow.

Considering first the appendage motion, the force distribution $\mathbf{f}(\mathbf{r},t)$ is due to 'inertial forces':

$$\mathbf{f}(\mathbf{r}, t) = -\,\sigma(\mathbf{r})\,\{\ddot{\boldsymbol{\delta}}(\mathbf{r}, t) - \mathbf{r}^x \ddot{\boldsymbol{\theta}}\} \tag{7}$$

where $\boldsymbol{\theta}^T = (\theta_1, \theta_2, \theta_3)$ are the three rotations of the main body with respect to the reference axes, and it is noted that $\boldsymbol{\theta}$ is of first order smallness (as is $\boldsymbol{\delta}$) in all of the following. The notation

$$\mathbf{r}^{x} = \begin{pmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{pmatrix} \tag{8}$$

has been adopted because of its kinship to the vector cross-product. The equation governing the flexible motions of the appendage(s) is then obtained by inserting Equation (7) in Equation (4):

$$\boldsymbol{\delta}(\mathbf{r}, t) + \int_A \mathbf{F}(\mathbf{r}, \mathbf{r}_1) \{\ddot{\boldsymbol{\delta}}(\mathbf{r}_1, t) - \mathbf{r}_1^x \ddot{\boldsymbol{\theta}}\} \, \sigma(\mathbf{r}_1) \, dv' = \mathbf{0}. \tag{9}$$

Several remarks are appropriate in connection with Equations (7) and (9). A basic assumption is that the origin in Figure 1 is essentially an inertially fixed point. This is justified in the following stages. First, the attitude motion of the spacecraft is assumed to be uncoupled from its orbital (translational) motion. This assumption is an extremely good one, especially for actively controlled vehicles. A recent examination of this question for uncontrolled (natural) motions has been given in this journal by Mohan *et al.* (1972).

There remains the force fields associated with gravity-gradient and other environmental sources. For normal active attitude control systems however, these fields may be taken as quasi-steady. This latter assumption is perhaps more easily understood in the frequency domain, where the significant frequency content of these external influences are only at the extreme lower end of the controller pass-band. The structural frequencies are, if anything, even higher. Therefore any external force field as may be present leads to a quasi-steady deflection in the flexible appendages. The latter may be superimposed on the dynamic deflections considered below since we are dealing with a linearized analysis. Consistent with this framework, the analysis to follow assumes that the control torques are generated entirely on the rigid part of the spacecraft. If, for example, a gas jet was located out on one of the appendages, other terms would be required in (7) and (9).

A further clarification of (7) and (9) can be made by an amplification of the statement made earlier that the motions of the vehicle center of mass are assumed to be uncoupled from attitude motions. Whether this assumption is tenable or not depends on the existence of certain symmetry properties for the vehicle. Many spacecraft are members of this class; many others are not. Since this paper is focussed on other aspects of the dynamics, we shall avail ourselves of this simplifying restriction on spacecraft configuration. Taken alongside the earlier assumptions, it allows us to treat the vehicle center of mass as an inertially fixed point. If this assumption were removed, as indeed it must be for many vehicles, then several other terms would appear in (7) and (9). An exposition of this more general case has been given by Likins (1970).

As a final comment on Equations (7) and (9), the absence of damping terms should be explained. Dissipative influences are usually treated somewhat heuristically, despite their importance. Thus, a linear rate-dependent damping term, $\dot{\theta}$, is frequently inserted in the equations of motion although it is well known that structural dissipation does not obey such a law over a significant frequency band. Therefore such terms can as well be inserted at a later stage, specifically, in the modal equations to be derived presently.

Turning next to the spacecraft as a whole, the angular momentum, to first order in small quantities, is given by

$$\mathbf{h} = - \int_{A+B} \mathbf{r}^{x}\mathbf{r}^{x}\sigma(\mathbf{r})\,\mathrm{d}v\,\dot{\boldsymbol{\theta}} + \int_{A} \mathbf{r}^{x}\dot{\boldsymbol{\delta}}(\mathbf{r},t)\,\sigma(\mathbf{r})\,\mathrm{d}v\,. \tag{10}$$

The inertia matrices

$$\mathbf{I}_C = - \int_{C} \mathbf{r}^{x}\mathbf{r}^{x}\sigma(\mathbf{r})\,\mathrm{d}v \quad (C = A, B, A + B) \tag{11}$$

are now introduced, whence

$$\mathbf{h} = \mathbf{I}\dot{\boldsymbol{\theta}} + \int_{A} \mathbf{r}^{x}\dot{\boldsymbol{\delta}}(\mathbf{r},t)\,\sigma(\mathbf{r})\,\mathrm{d}v \tag{12}$$

and $\mathbf{I}_{A+B}$ is written $\mathbf{I}$ for simplicity. The motion equation is then found from

$$\dot{\mathbf{h}} + \dot{\boldsymbol{\theta}}^{x}\mathbf{h} = \mathbf{T}(t) \tag{13}$$

wherein the external torques (including control torques) appear on the right side. Noting the the second term on the left side is of second order (and is dropped) the result is obtained:

$$\mathbf{I}\ddot{\boldsymbol{\theta}} + \int_{A} \mathbf{r}^{x}\ddot{\boldsymbol{\delta}}(\mathbf{r},t)\,\sigma(\mathbf{r})\,\mathrm{d}v = \mathbf{T}(t)\,. \tag{14}$$

Certain solutions of these equations are now examined.

3. Natural Motions – Constrained

If no external influences are present ($\mathbf{T} \equiv \mathbf{0}$) the motion may be termed 'natural. Such motions are evidently of interest in themselves, but they take on added distinction inasmuch as the general motion may be viewed as a superposition of natural motions wherein the contributions of the latter to the former vary in time. This approach (i.e., natural modes) is mathematically sound and highly successful because, in practice, an acceptable accuracy can be achieved with only the first few modes.

Often the 'natural' motions are constrained to be those of the appendages when no main body attitude motion is allowed. This is depicted in Figure 2. Since in this case

$$\boldsymbol{\theta}(t) \equiv \mathbf{0} \tag{15}$$

by hypothesis, the spacecraft motion equation, (14), is not needed and only the appendage motion equation, (9), is used. It is well known that the resulting motion is sinusoidal,

$$\boldsymbol{\delta}(\mathbf{r}, t) = \boldsymbol{\Delta}_n(\mathbf{r}) \cos \Omega_n t \tag{16}$$

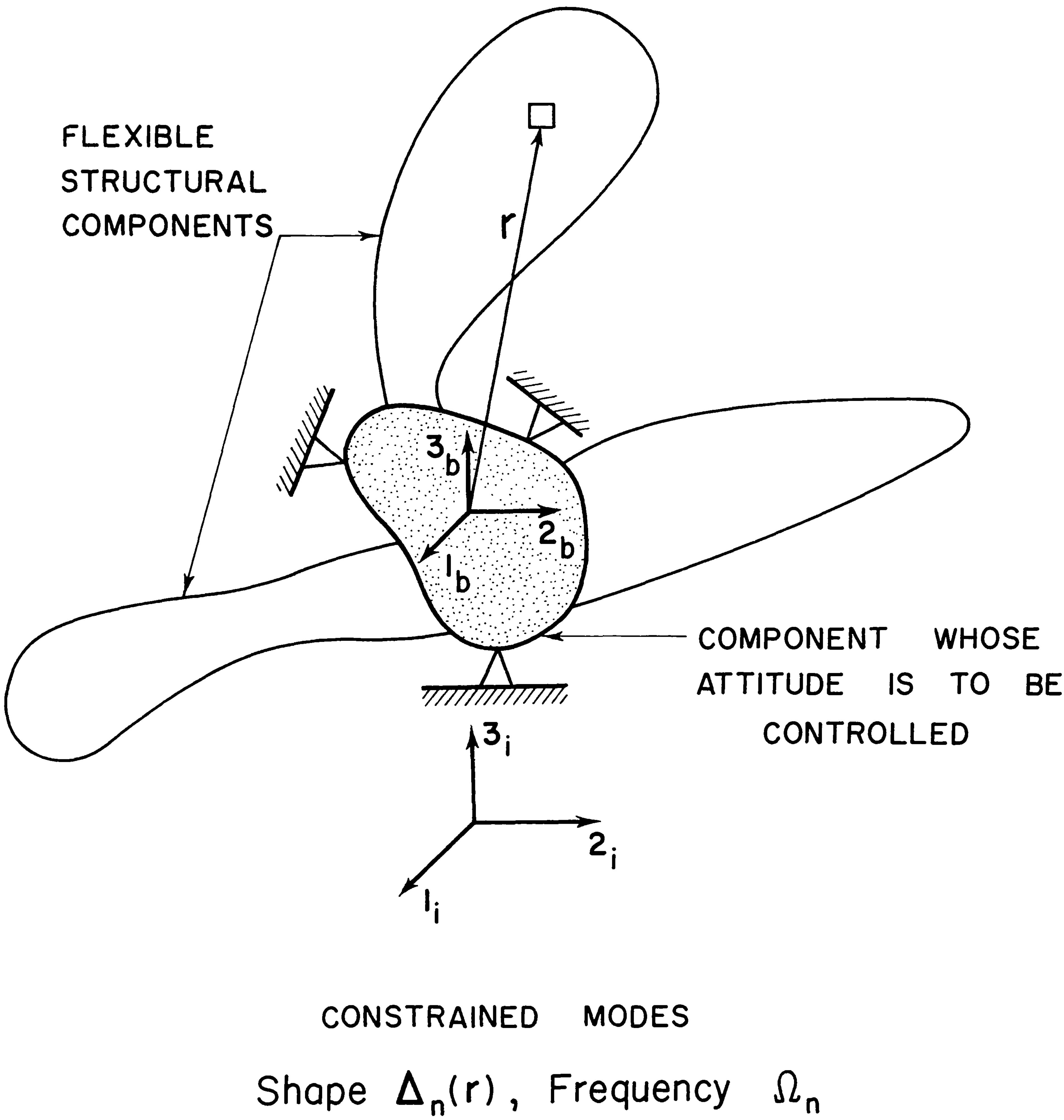

Fig. 2. Illustration of constrained modes.

and that the nth mode shape, $\boldsymbol{\Delta}_n(\mathbf{r})$, and frequency, Ω_n, are the solution of an eigenvalue problem; in the present case, the problem is

$$\boldsymbol{\Delta}_n(\mathbf{r}) = \Omega_n^2 \int_A \mathbf{F}(\mathbf{r}, \mathbf{r}_1)\, \boldsymbol{\Delta}_n(\mathbf{r}_1)\, \sigma(\mathbf{r}_1)\, \mathrm{d}v' . \tag{17}$$

These mode shapes are orthogonal. The proof follows well-trodden paths and is omitted for brevity; Equations (5) and (6) are summoned during the proof and the outcome is

$$\int_A \boldsymbol{\Delta}_n^T(\mathbf{r})\, \boldsymbol{\Delta}_m(\mathbf{r})\, \sigma(\mathbf{r})\, \mathrm{d}v = 0 \quad (\Omega_n \neq \Omega_m). \tag{18}$$

The modes are therefore specified to within a constant multiplier. A unique definition is secured by selecting a normality condition. The one chosen here is

$$\int_A \boldsymbol{\Delta}_n^T(\mathbf{r})\, \boldsymbol{\Delta}_n(\mathbf{r})\, \sigma(\mathbf{r})\, \mathrm{d}v = I_a \tag{19}$$

where I_a is characteristic of the appendages, and has dimensions corresponding to moment of inertia.

This section is concluded by the observation that

$$\mathbf{F}(\mathbf{r}, \mathbf{r}_1) = \sum_{m=1}^{\infty} \frac{\boldsymbol{\Delta}_m(\mathbf{r})\, \boldsymbol{\Delta}_m^T(\mathbf{r}_1)}{I_a \Omega_m^2} \tag{20}$$

expressing the flexibility kernel in terms of the constrained natural modes.

4. Natural Motions – Unconstrained

If the spacecraft as a whole is allowed to oscillate, Equation (15) no longer applies and only external influences are prohibited:

$$\mathbf{T}(t) \equiv \mathbf{0}. \tag{21}$$

The situation corresponds to Figure 1. In addition to Equation (9), the motion equation for the spacecraft, (14), is also needed. The solution is constructed from

$$\begin{aligned} \boldsymbol{\delta}(\mathbf{r}, t) &= \boldsymbol{\delta}_n(\mathbf{r}) \cos \omega_n t \\ \boldsymbol{\theta}(t) &= \boldsymbol{\theta}_n \cos \omega_n t \end{aligned} \tag{22}$$

which, when inserted, generates the following eigenvalue problem:

$$\begin{aligned} &\boldsymbol{\delta}_n(\mathbf{r}) - \omega_n^2 \int_A \mathbf{F}(\mathbf{r}, \mathbf{r}_1)\, \boldsymbol{\delta}_n(\mathbf{r}_1)\, \sigma(\mathbf{r}_1)\, \mathrm{d}v' = - \omega_n^2 \int_A \mathbf{F}(\mathbf{r}, \mathbf{r}_1)\, \mathbf{r}_1^{\times} \sigma(\mathbf{r}_1)\, \mathrm{d}v'\, \boldsymbol{\theta}_n \\ &\mathbf{I}\boldsymbol{\theta}_n = \mathbf{a}_n \end{aligned} \tag{24}$$

where the definition

$$\mathbf{a}_n = - \int_A \mathbf{r}^{\times} \boldsymbol{\delta}_n(\mathbf{r})\, \sigma(\mathbf{r})\, \mathrm{d}v \tag{25}$$

is convenient. The solution may be expressed in terms of an auxiliary matrix function $\mathbf{X}(\mathbf{r};\,\omega^2)$ defined as the solution to the integral equation:

$$\mathbf{X}(\mathbf{r};\,\omega^2) - \omega^2 \int\limits_A \mathbf{F}(\mathbf{r},\mathbf{r}_1)\,\mathbf{X}(\mathbf{r}_1;\,\omega^2)\,\sigma(\mathbf{r}_1)\,\mathrm{d}v' = = -\,\omega^2 \int\limits_A \mathbf{F}(\mathbf{r},\mathbf{r}_1)\,\mathbf{r}_1^{\times}\sigma(\mathbf{r}_1)\,\mathrm{d}v'\,. \tag{26}$$

Then

$$\boldsymbol{\delta}_n(\mathbf{r}) = \mathbf{X}_n(\mathbf{r})\,\boldsymbol{\theta}_n \tag{27}$$

where the notation $\mathbf{X}_n(\mathbf{r}) = \mathbf{X}(\mathbf{r};\omega_n^2)$ has been adopted.

Placing Equation (27) in Equation (25), Equation (24) becomes

$$[\mathbf{I} + \mathbf{B}(\omega_n^2)]\,\boldsymbol{\theta}_n = \mathbf{0} \tag{28}$$

where

$$\mathbf{B}(\omega^2) = \int\limits_A \mathbf{r}^{\times}\mathbf{X}(\mathbf{r};\,\omega^2)\,\sigma(\mathbf{r})\,\mathrm{d}v\,. \tag{29}$$

This shows that the natural frequencies are the zeros of the equation

$$\mathrm{d}(\omega^2) \equiv \det[\mathbf{I} + \mathbf{B}(\omega^2)] = 0 \tag{30}$$

and the participation of the main body in the nth mode, $\boldsymbol{\theta}_n$, is also obtained from Equation (28). The mode shape is finally obtained from Equation (27).

By using customary procedures the orthogonality condition

$$\int\limits_A \boldsymbol{\delta}_n^T(\mathbf{r})\,\boldsymbol{\delta}_m(\mathbf{r})\,\sigma(\mathbf{r})\,\mathrm{d}v = \boldsymbol{\theta}_n^T\mathbf{I}\boldsymbol{\theta}_m \quad (\omega_n \neq \omega_m) \tag{31}$$

may be verified. It may be also expressed in the equivalent form

$$\int\limits_{A+B} [\boldsymbol{\delta}_n(\mathbf{r}) - \mathbf{r}^{\times}\boldsymbol{\theta}_n]^T\,[\boldsymbol{\delta}_m(\mathbf{r}) - \mathbf{r}^{\times}\boldsymbol{\theta}_m]\,\sigma(\mathbf{r})\,\mathrm{d}v = 0 \quad (\omega_n \neq \omega_m) \tag{32}$$

in which its role as an orthogonality condition is more clearly seen ($\boldsymbol{\delta} \equiv \mathbf{0}$ over B). The modes are rendered unique by the aid of the normality condition

$$\int\limits_A \boldsymbol{\delta}_n^T(\mathbf{r})\,\boldsymbol{\delta}_n(\mathbf{r})\,\sigma(\mathbf{r})\,\mathrm{d}v = I_a \tag{33}$$

where I_a is a moment of inertia characteristic of the appendages. This latter condition also may be expressed in an equivalent form:

$$\int\limits_{A+B} [\boldsymbol{\delta}_n(\mathbf{r}) - \mathbf{r}^{\times}\boldsymbol{\theta}_n]^T\,[\boldsymbol{\delta}_n(\mathbf{r}) - \mathbf{r}^{\times}\boldsymbol{\theta}_n]\,\sigma(\mathbf{r})\,\mathrm{d}v = I_a - \boldsymbol{\theta}_n^T\mathbf{I}\boldsymbol{\theta}_n\,. \tag{34}$$

This section is concluded by noting that

$$\mathbf{F}(\mathbf{r}, \mathbf{r}_1) = \sum_{m=1}^{\infty} \frac{[\boldsymbol{\delta}_m(\mathbf{r}) - \mathbf{r}^{\times}\boldsymbol{\theta}_m][\boldsymbol{\delta}_m(\mathbf{r}_1) - \mathbf{r}_1^{\times}\boldsymbol{\theta}_m]^T}{\omega_m^2 (I_a - \boldsymbol{\theta}_m^T \mathbf{I} \boldsymbol{\theta}_m)} \tag{35}$$

which is interesting to compare with Equation (20).

5. Natural Frequencies

Natural frequencies of both the constrained and unconstrained modes were defined above. They were denoted Ω_n and ω_n, respectively, and the corresponding eigenvalue problems formulated. Certain relationships between these frequencies exist and these are now derived. First, expand $\mathbf{r}^{\times}$ in terms of constrained modes,

$$\mathbf{r}^{\times} = \sum_{m=1}^{\infty} \boldsymbol{\Delta}_m(\mathbf{r})\, \mathbf{b}_m^T \quad (\mathbf{r} \in A) \tag{36}$$

where, using Equations (18) and (19), the constants $\mathbf{b}_n$ can be calculated fıom

$$\mathbf{b}_n = -\frac{1}{I_a} \int_A \mathbf{r}^{\times} \boldsymbol{\Delta}_n(\mathbf{r})\, \sigma(\mathbf{r})\, \mathrm{d}v\,. \tag{37}$$

Next, expand $\mathbf{X}$ also in terms of constrained modes

$$\mathbf{X}(\mathbf{r}; \omega^2) = \sum_{m=1}^{\infty} \boldsymbol{\Delta}_m(\mathbf{r})\, \mathbf{c}_m^T(\omega^2) \tag{38}$$

and substitute both Equations (36) and (38) into Equation (26). After calling upon the definition of $\boldsymbol{\Delta}_m$, Equation (17), and the orthogonality and normality conditions, Equations (18) and (19), $\mathbf{c}_n$ is found in terms of $\mathbf{b}_n$,

$$\mathbf{c}_n = \left(\frac{\omega^2}{\omega^2 - \Omega_n^2}\right) \mathbf{b}_n\,. \tag{39}$$

A more explicit form for $\mathbf{B}(\omega^2)$ is now facilitated. If Equations (36) and (38) are inserted into the definition of $\mathbf{B}$, Equation (29), there follows:

$$\begin{aligned}
\mathbf{B}^T &= -\int_A \mathbf{X}^T \mathbf{r}^{\times} \sigma(\mathbf{r})\, \mathrm{d}v = \\
&= -\sum_{m=1}^{\infty} \sum_{k=1}^{\infty} \mathbf{c}_m \int_A \boldsymbol{\Delta}_m^T(\mathbf{r})\, \boldsymbol{\Delta}_k(\mathbf{r})\, \sigma(\mathbf{r})\, \mathrm{d}v\, \mathbf{b}_k^T = \\
&= I_a \sum_{m=1}^{\infty} \left(\frac{\omega^2}{\Omega_m^2 - \omega_2^2}\right) \mathbf{b}_m \mathbf{b}_m^T\,.
\end{aligned} \tag{40}$$

Since the last expression is clearly a symmetric matrix, it is also an expression for **B**. It is now plain that the determinant in Equation (30), whose zeros are the (unconstrained) natural frequencies, will have the general character sketched in Figure 3, where

$$\Omega_1 \leqslant \omega_1 \leqslant \Omega_2 \leqslant \omega_2 \leqslant \cdots \leqslant \Omega_n \leqslant \omega_n \leqslant \cdots . \tag{41}$$

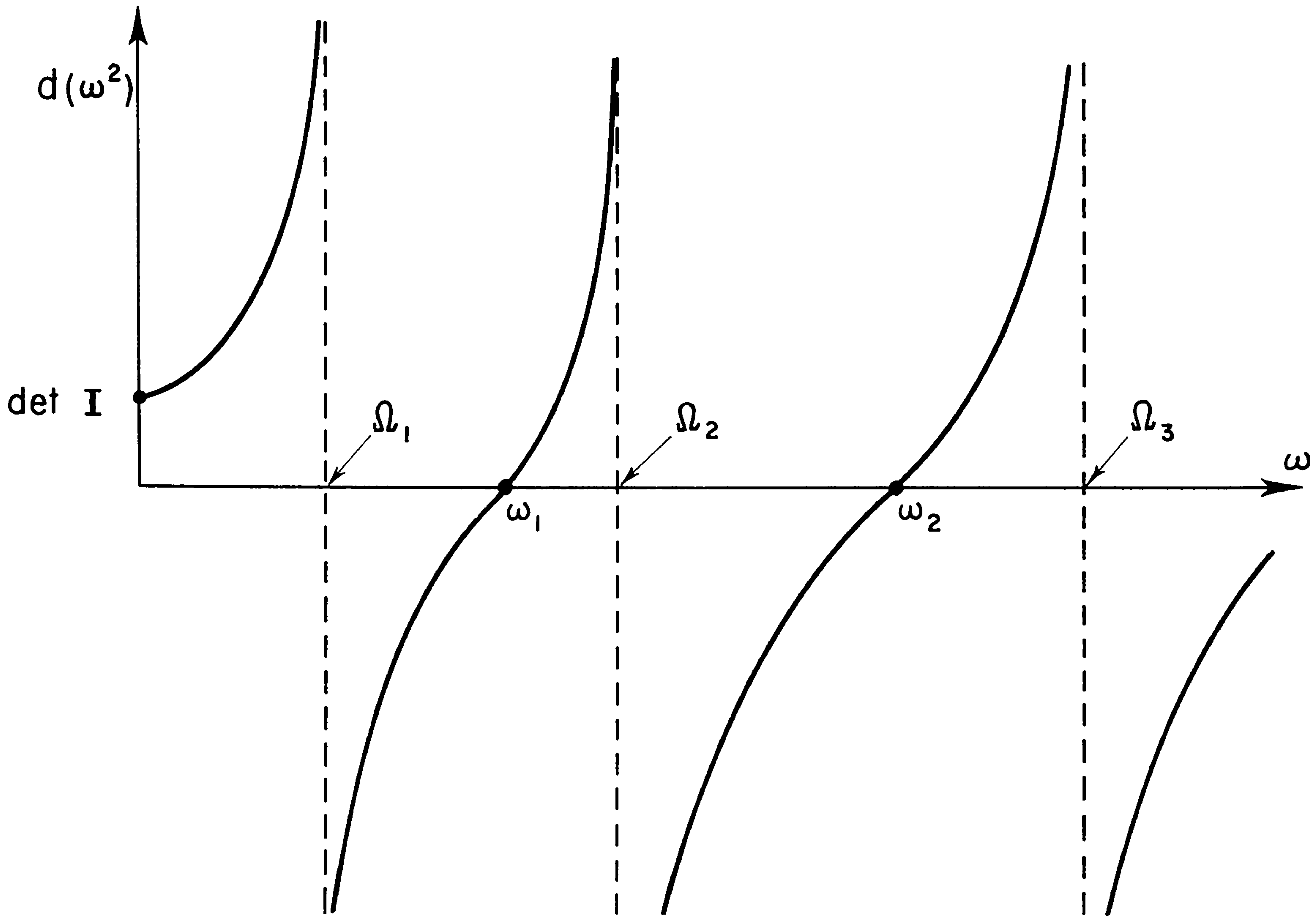

Fig. 3. Plot of the determinant $d(\omega^2)$.

The physical significance of $\mathbf{B}(\omega^2)$ is seen by impressing a sinusoidal torque on the spacecraft:

$$\mathbf{T}(t) = \mathbf{T}_0 \cos \omega t . \tag{42}$$

This gives rise to the steady-state response

$$\begin{aligned} \boldsymbol{\delta}(\mathbf{r}, t) &= \boldsymbol{\delta}_0(\mathbf{r}) \cos \omega t \\ \boldsymbol{\theta}(t) &= \boldsymbol{\theta}_0 \cos \omega t \end{aligned} \tag{43}$$

where

$$-\omega^2 \boldsymbol{\theta}_0 = [\mathbf{I} + \mathbf{B}(\omega^2)]^{-1} \mathbf{T}_0 . \tag{44}$$

Thus $\mathbf{I}+\mathbf{B}(\omega^2)$ may be interpreted as the 'effective' inertia at frequency ω. In particular, if $\omega \to \omega_n$ then $\boldsymbol{\theta}_0 \to \infty$ (resonance) and if $\omega \to \Omega_n$ then $\boldsymbol{\theta}_0 \to \mathbf{0}$ (the appendages combine to act as a vibration adsorber).

6. General Motion in Terms of Constrained Modes

A foundation has been laid for an examination of the more general spacecraft motion in response both to perturbing external torques, $\mathbf{T}_e$, and to the compensating control torques, $\mathbf{T}_c$. The use of constrained natural modes will be illustrated in general terms. To this end, form the expansion

$$\boldsymbol{\delta}(\mathbf{r}, t) = \sum_{m=1}^{\infty} \boldsymbol{\Delta}_m(\mathbf{r})\, Q_m(t) \tag{45}$$

where the Q_m assume the role of generalized coordinates characterizing the additional degrees of freedom attributable to structural flexibility. This expansion is now substituted in the motion equations, (14) and (9). The former becomes

$$\mathbf{I}\ddot{\boldsymbol{\theta}} + \sum_{m=1}^{\infty} \int_A \mathbf{r}^{\times} \boldsymbol{\Delta}_m(\mathbf{r})\, \sigma(\mathbf{r})\, \mathrm{d}v\, \ddot{Q}_m = \mathbf{T}(t) \tag{46}$$

and, noting that $\mathbf{r}^{\times}$ is skew-symmetric, Equation (36) together with Equation (18) and (19), this reduces to

$$\mathbf{I}\ddot{\boldsymbol{\theta}} - I_a \sum_{m=1}^{\infty} \mathbf{b}_m \ddot{Q}_m = \mathbf{T}(t). \tag{47}$$

The appendage equation of motion, Equation (9), when subjected to a similar series of operations yield

$$\sum_{m=1}^{\infty} (\ddot{Q}_m + \Omega_m^2 Q_m)\, \boldsymbol{\Delta}_m(\mathbf{r}) = \mathbf{r}^{\times}\ddot{\boldsymbol{\theta}}. \tag{48}$$

A set of uncoupled differential equations for the 'elastic' degrees of freedom is obtained by performing the operation $\int_A \boldsymbol{\Delta}_n^T(\mathbf{r})\{\cdot\}\sigma(\mathbf{r})\,\mathrm{d}v$. This leads to

$$\ddot{Q}_n + \Omega_n^2 Q_n = \mathbf{b}_n^T \ddot{\boldsymbol{\theta}} \quad (n = 1, 2, \ldots). \tag{49}$$

These results form the basis of an explicit relationship between torque and main-body attitude response. Such a relationship is particularly attractive from an attitude control standpoint. A convenient means of expressing this relationship is in terms of Laplace transformed variables; these will be identified by an overbar ($^-$). The entries corresponding to initial conditions will be omitted in the interests of succinctness. The relationship, which follows from Equations (47) and (49), is

$$\bar{\mathbf{I}}_e(s)\, s^2 \bar{\boldsymbol{\theta}} = \bar{\mathbf{T}}(s) \tag{50}$$

where the 'effective' inertia matrix with flexibility present, $\bar{\mathbf{I}}_e$, has been defined as ($\mathbf{1}$ is the unit matrix):

$$\bar{\mathbf{I}}_e(s) = \left\{\mathbf{1} - \sum_{m=1}^{\infty} \frac{s^2 \mathbf{K}_m}{s^2 + \Omega_m^2}\right\} \mathbf{I}. \tag{51}$$

The contraction in notation

$$\mathbf{K}_n = I_a \mathbf{b}_n \mathbf{b}_n^T \mathbf{I}^{-1} \tag{52}$$

has been employed.

Equations (50) and (52) are equivalent to Equations (298) and (9) given by Likins in (1970) and (1971), respectively.

The manifestation of Equation (51) in a block diagram is shown in Figure 4. Dissipative influences may be incorporated at this stage by replacing the denominators $(s^2+\Omega_n^2)$ by $(s^2+2\zeta_n\Omega_n s+\Omega_n^2)$. It can be seen that the interaction between the attitude control system and structural flexibility will be negligible if the following relationships hold:

$$\|\mathbf{K}_n\| \left(\frac{\omega_{\mathrm{BW}}}{\Omega_n}\right)^2 \ll 1\,; \quad n = 1, 2, \ldots \tag{53a}$$

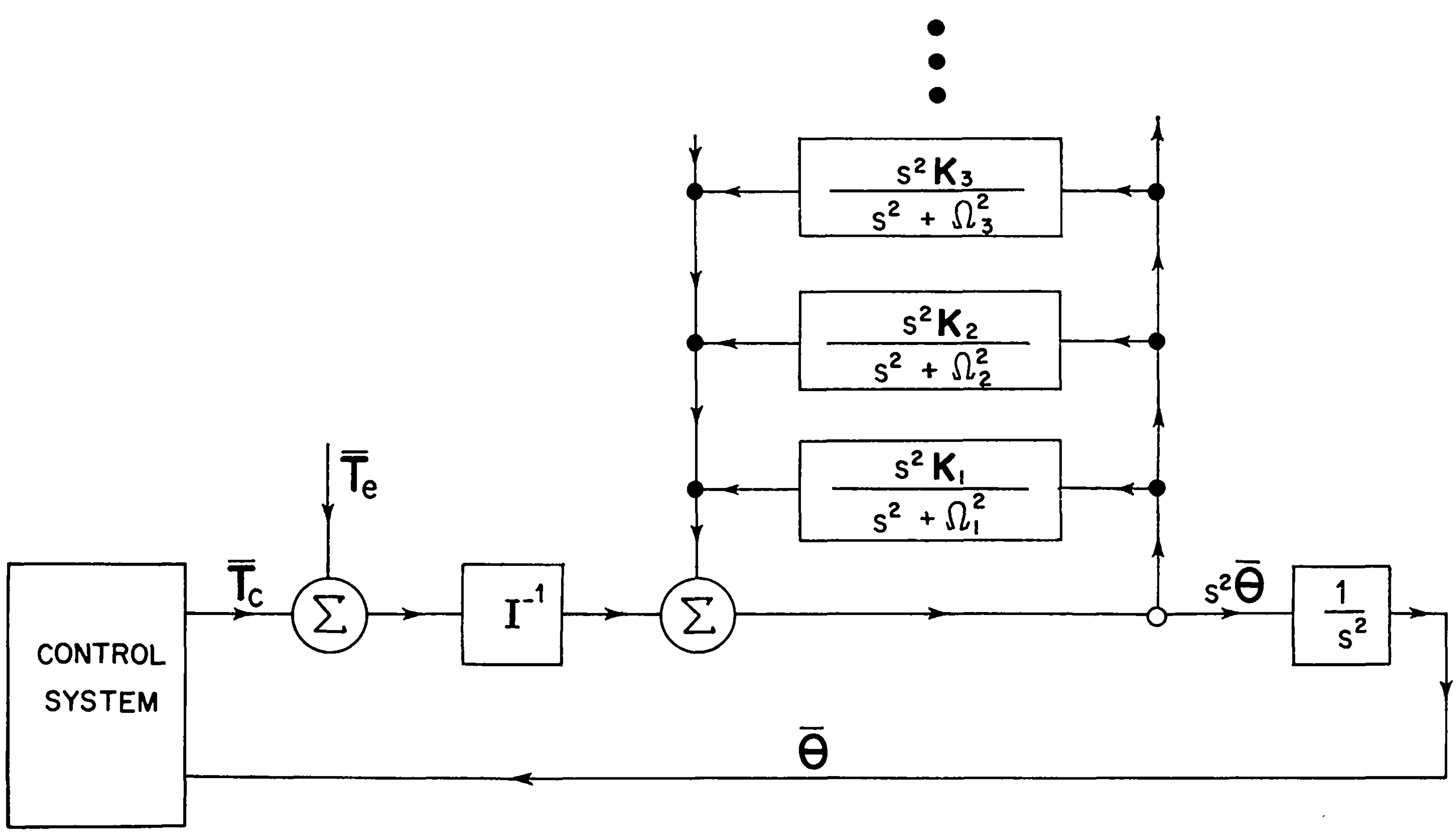

Fig. 4. Block diagram using constrained modes.

where ω_{BW} is the band-width of the controller. Equation (53a) combines a limitation on the degree of structural flexibility (it normally need be applied only for $n=1$) with a limitation on the ratio of the flexible appendage inertia to the vehicle inertia. This last statement is substantiated by Equation (63) below. If any of the conditions in Equation (53a) are violated then structure/controller interactions are likely to be significant.

7. General Motion in Terms of Unconstrained Modes

The general motion of the spacecraft may also be expressed in terms of the unconstrained modes. More precisely, the following expansions are appropriate:

$$\begin{aligned}\boldsymbol{\delta}(\mathbf{r}, t) &= \sum_{m=1}^{\infty} \boldsymbol{\delta}_m(\mathbf{r})\, q_m(t) \\ \boldsymbol{\theta}(t) &= \sum_{m=1}^{\infty} \boldsymbol{\theta}_m q_m(t) + \boldsymbol{\Theta}(t).\end{aligned} \tag{54}$$

The spacecraft motion equation, (14), then becomes

$$\mathbf{I}\ddot{\boldsymbol{\Theta}} = \mathbf{T}(t) \tag{55}$$

after Equations (24) and (25) have been recognized. Thus $\boldsymbol{\Theta}(t)$ is simply the response the spacecraft would have if it were rigid. The appendage motion equation, (9), under the influence of these expansions becomes

$$\sum_{m=1}^{\infty} (\ddot{q}_m + \omega_m^2 q_m)\,[\boldsymbol{\delta}_m(\mathbf{r}) - \mathbf{r}^x\boldsymbol{\theta}_m] = \mathbf{r}^x\ddot{\boldsymbol{\Theta}} \quad (r \in A). \tag{56}$$

To decouple these differential equations for the 'elastic' degrees of freedom, perform the operation $\int_A [\boldsymbol{\delta}_n(\mathbf{r}) - \mathbf{r}^x\boldsymbol{\theta}_n]^T\{\cdot\}\,\sigma(\mathbf{r})\,\mathrm{d}v$ on Equation (56), and bear in mind Equations (31), (33), (25) and (24). The result is

$$\ddot{q}_n + \omega_n^2 q_n = \frac{\boldsymbol{\theta}_n^T\mathbf{I}}{I_a - \boldsymbol{\theta}_n^T\mathbf{I}\boldsymbol{\theta}_n}\,\ddot{\boldsymbol{\Theta}}. \tag{57}$$

An explicit result again can be written between torque and main-body attitude response. The Laplace-transformed versions of Equations (54), (55) and (57) may be combined to verify the result:

$$\bar{\mathbf{I}}_e(s)\, s^2\bar{\boldsymbol{\theta}} = \bar{\mathbf{T}} \tag{58}$$

where the 'effective' inertia matrix with flexibility piesent, $\bar{\mathbf{I}}_e$, has been defined as

$$\bar{\mathbf{I}}_e(s) = \left(\mathbf{1} + \sum_{m=1}^{\infty} \frac{s^2\mathbf{k}_m}{s^2 + \omega_m^2}\right)^{-1} \mathbf{I} \tag{59}$$

and the definition

$$\mathbf{k}_n = \frac{\mathbf{I}\boldsymbol{\theta}_n\boldsymbol{\theta}_n^T}{I_a - \boldsymbol{\theta}_n^T\mathbf{I}\boldsymbol{\theta}_n} \tag{60}$$

has been introduced. That the denominator in this equation is always positive may be seen from Equation (34).

A block diagram corresponding to Equation (59) is shown in Figure 5. It is interesting to compare this diagram, in which the modal contributions are feed-forward loops, with Figure 4, where we have feed-back loops. Damping terms, $2\zeta_n\omega_n s$, may easily be inserted in the denominators of these loops. Conditions for negligible structure/controller interaction are seen to be

$$\|\mathbf{k}_n\| \left(\frac{\omega_{\mathrm{BW}}}{\omega_n}\right)^2 \ll 1; \quad n = 1, 2, \ldots \tag{53b}$$

which are comparable to those in Equation (53a). The squared factor is a limitation on the degree of structural flexibility, while $\|\mathbf{k}_n\|$ is directly related to the ratio of the flexible appendage inertia to the rigid body inertia. The latter assertion is justified by Equation (66) below.

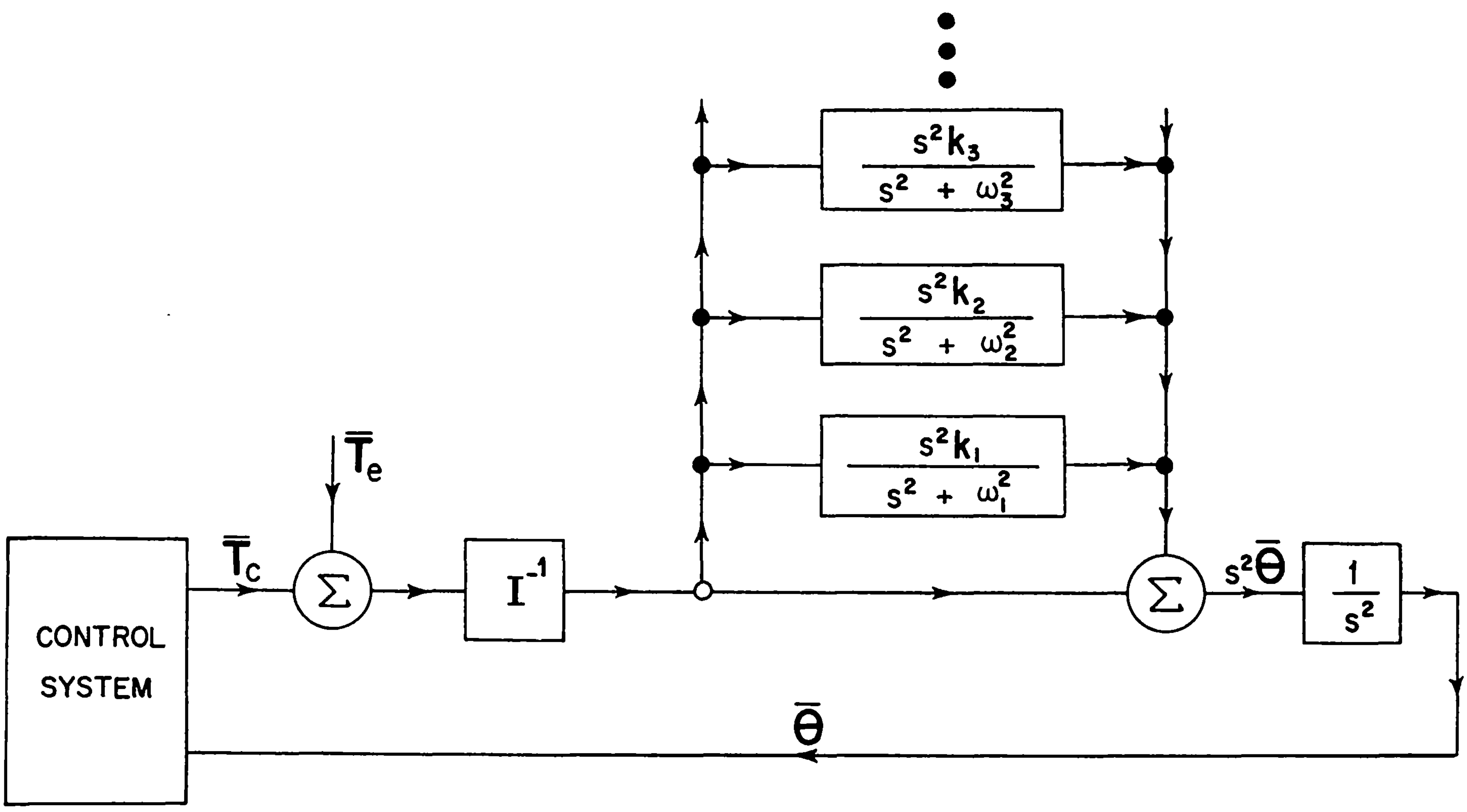

Fig. 5. Block diagram using unconstrained modes.

8. Relationships Concerning K_n and $\mathbf{k}_n$

A formula for the sum of the $\mathbf{K}_n$ is useful since the remainder after N terms can quickly be estimated. To derive such a formula, select Equation (48), and perform the operation $\int_A \mathbf{r}^\times\{\cdot\}\,\sigma(\mathbf{r})\,dv$. Using Equations (11) and (36), an intermediate result is found:

$$I_a \sum_{m=1}^{\infty} (\ddot{Q}_m + \Omega_m^2 Q_m)\,\mathbf{b}_m = \mathbf{I}_A \ddot{\boldsymbol{\theta}}. \tag{61}$$

However since $\ddot{Q}_m + \Omega_m^2 Q_m$ is known from Equation (49), we have

$$I_a \sum_{m=1}^{\infty} \mathbf{b}_m \mathbf{b}_m^T \ddot{\boldsymbol{\theta}} = \mathbf{I}_A \ddot{\boldsymbol{\theta}}. \tag{62}$$

Recognizing Equation (52), and that $\ddot{\boldsymbol{\theta}}$ is not in general zero, one has

$$\sum_{m=1}^{\infty} \mathbf{K}_m = \mathbf{I}_A \mathbf{I}^{-1}. \tag{63}$$

A similar formula may be found for the sum of the $\mathbf{k}_n$. The derivation is initiated by performing the operation $\int_A \mathbf{r}^\times\{\cdot\}\sigma(\mathbf{r})\,dv$ on Equation (56). Using Equations (11), (24) and (25), an intermediate result in found:

$$(\mathbf{I}-\mathbf{I}_A)\sum_{m=1}^{\infty}(\ddot{q}_m+\omega_m^2 q_m)\,\boldsymbol{\theta}_m=\mathbf{I}_A\ddot{\boldsymbol{\Theta}}. \tag{64}$$

However, since $\ddot{q}_m+\omega_m{}^2 q_m$ is known from Equation (57), we have

$$(\mathbf{I}-\mathbf{I}_A)\sum_{m=1}^{\infty}\left(\frac{\boldsymbol{\theta}_m\boldsymbol{\theta}_m^T\mathbf{I}}{I_a-\boldsymbol{\theta}_m^T\mathbf{I}\boldsymbol{\theta}_m}\right)\ddot{\boldsymbol{\Theta}}=\mathbf{I}_A\ddot{\boldsymbol{\Theta}}. \tag{65}$$

Recognizing Equation (60), and that $\ddot{\boldsymbol{\Theta}}$ is not in general zero, one has

$$\sum_{m=1}^{\infty}\mathbf{k}_m=(\mathbf{1}-\mathbf{I}_A\mathbf{I}^{-1})^{-1}\,\mathbf{I}_A\mathbf{I}^{-1}=(\mathbf{I}\mathbf{I}_A^{-1}-\mathbf{1})^{-1}=\mathbf{I}_A\mathbf{I}_B^{-1}. \tag{66}$$

Assuming that $\mathbf{K}_m$ (or $\mathbf{k}_m$) ($m=1, 2, \ldots, M$) have been calculated, Equation (63) (or 66) provides a useful upper bound for $\mathbf{K}_{M+1}$ (or $\mathbf{k}_{M+1}$).

Further relationships may be discovered by recognizing the equivalence of the two expansions for the 'effective' inertia, $\bar{\mathbf{I}}_e$, as given by Equations (51) and (59). This equivalence may be written as

$$\left\{\mathbf{1}-\sum_{m=1}^{\infty}\frac{s^2\mathbf{K}_m}{s^2+\Omega_m^2}\right\}\left\{\mathbf{1}+\sum_{m=1}^{\infty}\frac{s^2\mathbf{k}_m}{s^2+\omega_m^2}\right\}=\mathbf{1}. \tag{67}$$

If one evaluates Equation (67) at $s=i\Omega_n$ (where $i^2=-1$) it is learned that

$$\det\left\{\mathbf{1}-\sum_{m=1}^{\infty}\frac{\Omega_n^2\mathbf{k}_m}{\omega_m^2-\Omega_n^2}\right\}=0\quad(n=1,2,\ldots). \tag{68}$$

If one evaluates Equation (67) at $s=i\omega_n$, on the other hand, one obtains

$$\det\left\{\mathbf{1}+\sum_{m=1}^{\infty}\frac{\omega_n^2\mathbf{K}_m}{\Omega_m^2-\omega_n^2}\right\}=0\quad(n=1,2,\ldots) \tag{69}$$

which is a recovery of Equation (30). Thus, as has been pointed out earlier, if the appendages are characterized (i.e., $\mathbf{K}_m$ and Ω_m are known) the system frequencies ω_n can be found from Equation (69). Information also is available on the parameters $\mathbf{k}_n$ from Equation (68); in fact if planar rotation is under consideration, Equation (68) may be regarded as linear equations in the unknown scalars k_n.

9. Stored Angular Momentum – Constrained Modes

There are many directions in which the preceeding results can be extended. The remainder of this paper will be confined to one such extension – a treatment of flexible spacecraft motions when there is within the main body, a stored angular momentum, $\mathbf{h}_s$. The physical origin of this angular momentum might be a momentum wheel, control moment gyros, etc., or possibly other internal or external rotors. The column matrix $\mathbf{h}_s$, whose elements are components along body axes, is taken to be essentially constant over a time interval of interest from a structural vibration standpoint.

This additional ingredient does not affect the appendage motion equation, (9), nor the constrained mode shapes and frequencies. However the spacecraft motion equation, (14), must be re-examined. The expression for total system angular momentum, given previously by Equation (12) must now be augmented thus:

$$\mathbf{h} = \mathbf{I}\dot{\boldsymbol{\theta}} + \int_A \mathbf{r}^x \dot{\boldsymbol{\delta}}(\mathbf{r}, t)\, \sigma(\mathbf{r})\, dv + \mathbf{h}_s \tag{70}$$

whence the new spacecraft equation of motion becomes

$$\mathbf{I}\ddot{\boldsymbol{\theta}} + \int_A \mathbf{r}^x \ddot{\boldsymbol{\delta}}(\mathbf{r}, t)\, \sigma(\mathbf{r})\, dv + \dot{\boldsymbol{\theta}}^x \mathbf{h}_s = \mathbf{T}(t). \tag{71}$$

If it is desirable to use constrained modal amplitudes as generalized coordinates, the substitution of Equation (45) is made into Equations (9) and (71), leading to

$$\ddot{Q}_n + \Omega_n^2 Q_n = \mathbf{b}_n^T \ddot{\boldsymbol{\theta}} \quad (n = 1, 2, \ldots) \tag{72}$$

$$\mathbf{I}\ddot{\boldsymbol{\theta}} - I_a \sum_{m=1}^{\infty} \mathbf{b}_m \ddot{Q}_m + \dot{\boldsymbol{\theta}}^x \mathbf{h}_s = \mathbf{T}(t) \tag{73}$$

which is identical to the earlier situation ($\mathbf{h}_s = \mathbf{0}$) except for the new torque $-\dot{\boldsymbol{\theta}}^x \mathbf{h}_s$. The transfer function from torque to attitude is then found from

$$\{s^2 \bar{\mathbf{I}}_e(s) - s\mathbf{h}_s^x\}\, \bar{\boldsymbol{\theta}} = \bar{\mathbf{T}}(s) \tag{74}$$

where $\bar{\mathbf{I}}_e$ is the expansion in Equation (51).

The natural frequencies of the overall spacecraft are now found from setting $\bar{\mathbf{T}} \equiv \mathbf{0}$, $s = i\omega$ in Equation (74). The characteristic equation, whose zeros are the natural frequencies ω_m, $m = 1, 2, \ldots$ is thus shown to be

$$\det\{\omega \mathbf{I} + \omega \mathbf{B}(\omega^2) + i\mathbf{h}_s^x\} = 0. \tag{75}$$

This result has a more explicit form which does not contain i:

$$\omega^2 \det\{\mathbf{I} + \mathbf{B}(\omega^2)\} - \mathbf{h}_s^T \{\mathbf{I} + \mathbf{B}(\omega^2)\}\, \mathbf{h}_s = 0. \tag{76}$$

It should be borne in mind that effects attributable to flexibility enter exclusively via $\mathbf{B}$. If the spacecraft were entirely rigid, $\mathbf{B} \equiv \mathbf{0}$, then there would exist only one natural frequency, namely that corresponding to precession,

$$\omega_1 = \omega_P = \sqrt{\frac{\mathbf{h}_s^T \mathbf{I} \mathbf{h}_s}{\det \mathbf{I}}} \tag{77}$$

and ω_P will accordingly be called the rigid precession frequency.

To ascertain the interaction between stored angular momentum and structural flexibility requires, in general, the solution of Equation (76). It can be said, however, that when the appendages are sufficiently rigid that

$$\Omega_1 \gg \omega_P \tag{78}$$

then $\mathbf{B}(\omega_P^2)$ will make an insignificant correction to $\mathbf{I}$, and the first natural frequency will still be given by Equation (77).

10. Stored Angular Momentum – Unconstrained Modes

Since the unconstrained modes are, by definition, modes of the main-body/appendage system, their modal frequencies ω_n and parameters k_n will be dependent on $\mathbf{h}_s$. The last two paragraphs contained in effect, a discussion of $\omega_n(\mathbf{h}_s)$. It is of interest to investigate the mode shapes defined in this way. The equation governing such shapes are found by making the substitutions

$$\boldsymbol{\delta}(\mathbf{r}, t) = \sum_{m=1}^{\infty} \boldsymbol{\delta}_m(\mathbf{r}) \exp(i\omega_m t) \tag{79}$$

$$\boldsymbol{\theta}(t) = \sum_{m=1}^{\infty} \boldsymbol{\theta}_m \exp(i\omega_m t) \tag{80}$$

into Equations (9) and (71), and putting $\mathbf{T}(t) \equiv \mathbf{0}$. The results are

$$\boldsymbol{\delta}_n(\mathbf{r}) - \omega_n^2 \int_A \mathbf{F}(\mathbf{r}, \mathbf{r}_1)\, \boldsymbol{\delta}_n(\mathbf{r}_1)\, \sigma(\mathbf{r}_1)\, dv' = -\omega_n^2 \int_A \mathbf{F}(\mathbf{r}, \mathbf{r}_1)\, \mathbf{r}_1^x \sigma(\mathbf{r}_1)\, dv'\, \boldsymbol{\theta}_n \tag{81}$$

(as before) and

$$\omega_n^2 \mathbf{I} \boldsymbol{\theta}_n + \omega_n^2 \int_A \mathbf{r}^x \boldsymbol{\delta}_n(\mathbf{r})\, \sigma(\mathbf{r})\, dv + i\omega_n \mathbf{h}_s^x \theta_n = 0\,. \tag{82}$$

Then following the development of Section 4,

$$\boldsymbol{\delta}_n(\mathbf{r}) = \mathbf{X}_n(\mathbf{r})\, \boldsymbol{\theta}_n \tag{83}$$

and

$$\{\omega_n \mathbf{I} + \omega_n \mathbf{B}(\omega_n^2) + i\mathbf{h}_s^x\}\, \boldsymbol{\theta}_n = \mathbf{0}\,. \tag{84}$$

Here $\mathbf{B}$ and $\mathbf{X}$ are real matrices but $\boldsymbol{\delta}_n$ and $\boldsymbol{\theta}_n$ are complex in general. Equation (24) is replaced by

$$(\mathbf{I} + i\omega_n^{-1} \mathbf{h}_s^x)\, \boldsymbol{\theta}_n = \mathbf{a}_n \tag{85}$$

which, unfortunately, also changes the property of orthogonality, and Equation (32) now becomes

$$\int_{A+B} [\boldsymbol{\delta}_n(\mathbf{r}) - \mathbf{r}^x \boldsymbol{\theta}_n]^T [\boldsymbol{\delta}_m(\mathbf{r}) - \mathbf{r}^x \boldsymbol{\theta}_m]^* \sigma(\mathbf{r})\, dv = i\, \frac{\boldsymbol{\theta}_n^T \mathbf{h}_s^x \boldsymbol{\theta}_m^*}{\omega_n + \omega_m} \quad (\omega_n \neq \omega_n) \tag{86}$$

where ( )* denotes the complex conjugate. Although Equation (56) is still valid, the decoupling into separate differential equations for the modal coordinates as was done in Equation (57) is not facilitated by Equation (86) unless $\boldsymbol{\theta}_n^T \mathbf{h}_s^\times \boldsymbol{\theta}_m^* = 0$ which is true only if the $\boldsymbol{\theta}_m$ and $\mathbf{h}_s$ are all coplanar; in particular it is true if the flexibility is about a single axis. In the general case, the modal coordinates satisfy a coupled set of differential equations of the form

$$\mathbf{M}\ddot{\mathbf{q}} + i\mathbf{G}\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \mathbf{Y}^T \ddot{\boldsymbol{\Theta}}$$

where **M** and **K** are symmetric and **G** is a skew-symmetric matrix.

11. Concluding Remarks

The representation of elastic flexibility in the context of attitude dynamics and control has been discussed in considerable generality. In fact, two approaches were considered in parallel and many of the interrelationships between them were noted. The question naturally arises as to which is the best representation. Actually both the 'constrained' and the 'unconstrained' modal expansions have points in their favour. Both are mathematically equivalent if an infinite number of terms are taken in each of the two expansions. In practice, however, only a finite number of terms are included in the sum and therefore the nature of the approximation thus made is of interest. If it is important that the dynamic response be accurately modelled in frequency bands of large response, then M unconstrained modes will be more accurate than M constrained modes.

Accuracy in frequency bands of small response will favour the constrained expansion. These observations are true whether one is simulating 'flexible' attitude dynamics prior to flight, or analyzing flight data.

However the constrained modes have advantages also. If ground tests are done on the appendages alone then it is the constrained modal information which is learned. The use of constrained modes is also much more straightforward and makes fewer demands on the analyst. In more complicated situations where the appendages are not rigidly fixed to the main body but are, for example, articulated, unconstrained modes may not even be defined. Finally, it was seen that when rotors are included on the main body constrained modes are much easier to use especially since the uncoupling of unconstrained modal equations is not straightforward.

Acknowledgements

For the financial support of this investigation, the author wishes to thank Spar Aerospace Products Ltd. (Toronto), in connection with Sections 1–7, and the Federal Department of Communications Research Centre (Ottawa), in connection with Sections 8–10.

The reviewers selected by Professor Roberson for this paper were Professor Peter Likins and Dr Ralph Pringle; their comments were incorporated into the paper and are acknowledged with thanks.

References

Bolotin, V. V.: 1964, *The Dynamic Stability of Elastic Systems,* Holden-Day.
Likins, P. W.: 1970, 'Dynamics and Control of Flexible Space Vehicles', JPL Tech. Report 32-1329, Rev. 1.
Likins, P. W.: 1971, *J. Spacecraft Rockets* **8**, 264–273.
Likins, P. W. and Bouvier, H. K.: 1971, *Astronaut. Aeronaut.* 00, 64–71.
NASA: 1969, 'Effects of Structural Flexibility on Spacecraft Control Systems', NASA SP-8016.
Zienkiewicz, O. C.: 1971, *The Finite Element Method in Engineering Science*, McGraw-Hill.