BIFURCATION OF A CENTRAL CONFIGURATION

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1. Introduction

This paper presents a relatively simple example of a bifurcation of a central configuration in the four body problem. There have been several indications in the literature that bifurcations can occur, but no one has carried the analysis to completion. Smale (1970) offers a good introduction to the importance of bifurcations of central configurations in the n body problem.

Palmore (1976) gives an example of a degenerate central configuration of the N ($N > 3$) body problem. His example is nested in a one-parameter family of central configurations, but he does not show that the degeneracy gives rise to a bifurcation. Simo (1977) presents a complete numerical study of the central configurations in the 4 body problem. His numerical studies find degeneracies and bifurcations, which he discusses in detail.

Arenstorf (1982) defines a restricted 4 body problem for each central configuration of the 3 body problem. He gives a complete analysis of the critical points of the potential the restricted problems. It is easy to see that a non-degenerate critical point of this potential gives rise to a central configuration of the full 4 body problem with one small mass. He finds that the potential corresponding to the Lagrange triangular configuration has a degenerate critical point for some values of the masses of the primaries. He conjectures that this degenerate critical point can not be continued into the full 4 body problem.

Both Simo (1977) and Arenstorf (1982) found a degenerate central configuration in the restricted 4 body problem. We take an approach between those of the purely numerical approach of Simo and the purely analytic approach of Arenstorf. Our goal is to give a simple proof that the bifurcation that they observe actually occurs in the full 4 body problem with one small mass. First we give a mathematical proof of a theorem that gives conditions when a degenerate critical point of the restricted problem can be continued into the full 4 body problem as a degenerate central configuration. The theorem proves that the degeneracy is actually due to a bifurcation provided other partial derivatives of the potential be non-zero. This is a special case of a theorem in catastrophe theory. Finally we use numerical methods to verify the hypotheses of the theorem in the restricted 4 body problem.

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2. Notation and Background

The general reference for this section is Wintner (1941). Consider the N-body problem in the plane R^2. Thus, the position of the particles is specified by $q=(q_1,\dots,q_N)\in R^{2N}$ where $q_i=(q_i^1,q_i^2)\in R^2$ for $i=1,\dots,N$. Let $\Delta=\{q\in R^{2N}$: for some $i\neq j$, $q_i=q_j\}$. The self-potential of the N-body problem is

$$U_N(q)=\sum_{1\leqslant i<j\leqslant N}\frac{m_i m_j}{\|q_i-q_j\|} \tag{1}$$

where $m_i>0$ for $i=1,\dots,N$. Clearly U_N is a smooth function on $R^{2N}\setminus\Delta$.

A central configuration is a solution $q=\bar{q}$ of the system of equations

$$-\lambda m_i q_i=\frac{\partial U_N}{\partial q_i};\qquad i=1,\dots,N \tag{2}$$

for some constant λ. If $\bar{q}$ is a central configuration corresponding to $\bar{\lambda}$ then $\Sigma m_i\bar{q}_i=0$ and $\bar{\lambda}=U_N(\bar{q})/2I(\bar{q})>0$ where $2I=\Sigma m_i\|q_i\|^2$.

By the Lagrange multiplier theorem, equation (2) can be interpreted as the equations for a critical point of U_N restriced to the set where $I=1$. From the form of U_N it is clear that if $\bar{q}$ is a central configuration and A is a 2×2 rotation matrix then $A\bar{q}=(Aq_1,\dots,Aq_N)$ is a central configuration also.

The concept of a non-degenerate central configuration takes into account all the observations given above. Let $M=\{q\in R^{2N}: \Sigma m_i q_i=0\}$, $S=\{q\in M: I(q)=1\}$ and $\mathscr{S}=S/\sim$ where $\sim$ is the equivalence relation $q\sim q^*$ if $q=Aq^*$ where A is a 2×2 rotation matrix. Let $[q]=\{q^*\in S: q^*\sim q\}$. Since U_N is invariant under rotations, we can define a function $\mathscr{U}'_N:\mathscr{S}'-\Delta\to R$ by $\mathscr{U}'_N[q])=U_N(q)$. It turns out that $\mathscr{S}'$ is a smooth manifold and $\mathscr{U}'_N$ is a smooth function because the graph of the group action is closed (see Abraham and Marsden (1978)). A central configuration $\bar{q}$ is called non-degenerate if the Hessian of $\mathscr{U}'_N$ at $\bar{q}$ is non-singular.

Quotient spaces are awkward to work with in perturbation arguments, so we shall use a slightly more pedestrian but equivalent definition of non-degenerate central configuration. Let $\bar{q}$ be a central configuration. Then $[\bar{q}]$ is the set of all central configurations obtained from q by a rotation. We will select a unique representative from $[q]$ as follows. Since not all the $\bar{q}_i$ are zero, we may assume that $\bar{q}_1\neq 0$. There is a unique rotation matrix A such that $A\bar{q}_1=(\bar{q}_1,0)$, i.e. $A\bar{q}_1$ points along the abscissa. Let $q^*=A\bar{q}$. Define $N=\{q\in R^{2N}: q_1=(q_1^1,0)\}$, $\mathscr{S}=S\cap N$ and $\mathscr{U}_N=U_N|S$. Note that q is a central configuration if and only if q^* is a critical point of $\mathscr{U}_N$. We shall say that $\bar{q}$ is a non-degenerate central configuration if the Hessian of $\mathscr{U}_N$ at q^* is non-singular. The reader should check that the definition is independent of which q_i was chosen to point along the abscissa and that the two definitions of non-degenerate are equivalent.

Now consider the $(N+1)$-body problem with $m_{N+1}=\varepsilon$ and $q_{N+1}=u$. The equations for a central configuration become

$$\begin{aligned} -\lambda m_i q_i &= \frac{\partial U_N}{\partial q_i} + 0(\varepsilon); \qquad i = 1, \dots, N \\ -\lambda u &= \frac{\partial W}{\partial u} \end{aligned} \tag{3}$$

where $W = \Sigma_{j=1}^{N}(m_j/\|u - q_j\|)$. When $\varepsilon = 0$, a solution to (3) is an $(N+1)$-tuple $(\bar{q}_1, \dots, \bar{q}_N, \bar{u})$ where $(\bar{q}_1, \dots, \bar{q}_N)$ is a central configuration of the N-body problem and $\bar{u}$ is a critical point of

$$V(u) = \sum_{j=1}^{N} \frac{m_j}{\|u - \bar{q}_i\|} + \frac{\lambda}{2}\|u\|^2.$$

Note that the value of λ is determined by the condition that the $(\bar{q}_1, \dots, \bar{q}_N)$ are a central configuration of the N-body problem. The function $V(u)$ is called the self-potential of the restricted $(N+1)$-body problem corresponding to the central configuration $(\bar{q}_1, \dots, \bar{q}_N)$.

If $(\bar{q}_1, \dots, \bar{q}_N)$ is a non-degenerate central configuration and $\bar{u}$ is a non-degenerate critical point of V, then a simple application of the implicit function theorem proves that $(\bar{q}_1, \dots, \bar{q}_N, \bar{u})$ can be continued into the $(N+1)$-body problem for small ε. In this case no bifurcation occurs, so we must consider degenerate critical points of V.

Let us assume that we have a one parameter family of central configurations of the N-body problem. Let the parameter be μ, so $\bar{q}_1, \dots, \bar{q}_N$ and λ depend on μ. Let $u = (x, y)$ so $V(u, \mu) = V(x, y, \mu)$.

From catastrophe theory, the simplest bifurcation that can occur is when two critical points collide and disappear as the parameter passes through a critical value. This is called a fold catastrophe or a fold singularity. The amount of catastrophe theory that we shall need for our example is so minimal that we shall reproduce all that is necessary. We shall say that V has a fold for $x = \bar{x}$, $y = \bar{y}$, $\mu = \bar{\mu}$ if

$$\begin{aligned} &V_x = V_y = V_{xy} = V_{yy} = 0 \\ &V_{xx} \neq 0, \quad V_{y\mu} \neq 0, \quad V_{yyy} \neq 0. \end{aligned} \tag{5}$$

This is not the definition of a fold as found in catastrophe theory, but a special case of a fold. One of the main theorems of the theory applied to this example states that there is a change of coordinates such that in the new coordinates $V = x^2 + \mu y + y^3$. Another theorem holds that folds are stable under small peturbations. A special case of this theorem in the context of central configurations is:

THEOREM: *Let $(\bar{q}_1(\mu), \dots, \bar{q}_N(\mu))$ be a smooth one parameter family of central configurations of the N-body problem which is non-degenerate when $\mu = \bar{\mu}$. Let $V(x, y, \mu)$ be the potential of the restricted $(N+1)$-body problem corresponding to this* *central configuration. Let* V *have a fold at* $\bar{u}=(\bar{x},\bar{y})$ *when* $\mu=\bar{\mu}$. *Let* $V_{y\mu}\cdot V_{yyy}<0$ (*resp.* >0).

Then there exist $\varepsilon_0>0$, $\eta>0$, *a neighborhood* θ *of* $(\bar{q}_1(\bar{\mu}),\ldots,\bar{q}_N(\mu),\bar{u})$ *in* R^{2N+2} *and a smooth function* $\mu^*:[0,\varepsilon_0)\to R^1$, $\mu^*(0)=\bar{\mu}$ *such that when* $\varepsilon\cap(0,\varepsilon_0)$ *and* $|\mu-\bar{\mu}|<\eta$,

(i) *The* $(N+1)$*-body problem has a degenerate central configuration in* θ *when* $\mu=\mu^*(\varepsilon)$.

(ii) *The* $(N+1)$*-body problem has no central configuration in* θ *when* $\mu>\mu^*(\varepsilon)$ (*resp.* $\mu<\mu^*(\varepsilon)$).

(iii) *The* $(N+1)$*-body problem has two central configurations in* θ *when* $\mu<\mu^*(\varepsilon)$ (*resp.* $\mu>\mu^*(\varepsilon)$). *Moreover, these two tend to the degenerate one of* (i) *as* $\mu\to\mu^*(\varepsilon)$ *from the left* (*resp. right*).

Proof: The equations to solve are

$$\begin{aligned}
&(a) \qquad -\lambda u=\frac{\partial W}{\partial u}\\
&(b) \qquad -\lambda m_i q_i=\frac{\partial U_N}{\partial q_i}+0(\varepsilon); \qquad i=1,\ldots,N\\
&(c) \qquad 2I=\varepsilon\|u\|^2+\sum_1^N m_i\|q_i\|^2=2 \qquad\qquad (6)\\
&(d) \qquad \varepsilon u+\sum_1^N m_i q_i=0\\
&(e) \qquad q_1^2=0.
\end{aligned}$$

When $\varepsilon=0$, these equations have a solution $u=\bar{u}$ and $q_i=\bar{q}_i$. Moreover, since the central configuration $(\bar{q}_1,\ldots,\bar{q}_N)$ is non-degenerate we can solve the last 4 lines of equations for $q_i=\tilde{q}_i(u,\mu,\varepsilon)$. In order to see this consider the process of constructing local coordinates on $\mathcal{S}$ at $\bar{q}$. The equations (6c, d, e) can be used to express four of the q components in terms of the other qs and u. Having solved for these components the corresponding equations in (6b) must be discarded. For example, (6e) tells us to set $q_1^2=0$ everywhere and discard the equation of the first component of (6b). The important thing to note is that this process can be carried out by solving for q components and eliminating q equations. Therefore, the process yields equations which are well defined when $\varepsilon=0$. Having done this reduction, the fact that $\bar{q}$ is non-degenerate for $\mu=\bar{\mu}$ and $\varepsilon=0$ means that the implicit function theorem can be applied to solve the remaining equations in (6b) for the qs as functions of u, μ and ε. Substituting these values into V yields a function $V=\tilde{V}+0(\varepsilon)$. The equation $\tilde{V}_x=0$ can be solved for x as a function of y,μ and ε since at $\varepsilon=0$ we have $\tilde{V}_{xx}=V_{xx}=0$. Substituting this function into $\tilde{V}$ gives a new function $H(y,\mu,\varepsilon)$ which satisfies $H_y=0$, $H_{yy}=0$, $H_{yyy}\neq 0$ and $H_{\mu y}\neq 0$ when $y=\bar{y}$, $\mu=\bar{\mu}$ and $\varepsilon=0$ by the assumptions in (5).

Since $H_{\mu y}\neq 0$ we can solve $H_y=0$ for μ as a function of y and ε, i.e. there exists

$\phi(y, \varepsilon)$ such that

$$H_y(y, \phi(y, \varepsilon), \varepsilon) \equiv 0, \qquad \phi(\bar{y}, 0) = \bar{\mu}. \tag{7}$$

Differentiate this expression with respect to y to find $\phi_y(\bar{y}, 0) = 0$. Differentiate again to get $\phi_{yy}(\bar{y}, 0) = -H_{yyy}/H_{\mu y} \neq 0$. Assume $\phi_{yy}(\bar{y}, 0) > 0$.

Thus for $\varepsilon = 0$, the function $\phi(y, 0)$ has a non-degenerate minimum at y. Since $\phi_{yy}(\bar{y}, 0) \neq 0$ we can solve $\phi_y = 0$ for y, i.e. there is a function $y^*(\varepsilon)$ such that $\phi_y(y^*(\varepsilon), \varepsilon) \equiv 0$. Define $\mu^*(\varepsilon) = \phi(y^*(\varepsilon), \varepsilon)$.

For fixed small ε, the function $\phi(y, \varepsilon)$ has a local minimum at $y^*(\varepsilon)$ with minimum value $\mu^*(\varepsilon)$. Moreover, this is a non-degenerate minimum since $\phi_{yy}(y^*(\varepsilon), \varepsilon)$ is positive for small ε. Thus if $\mu > \mu^*(\varepsilon)$ but close, then there are two values y say y_1 and y_2 such that $\mu = \phi(y_i, \varepsilon)$, $i = 1, 2$ (see Figure 1). Thus

$$H_y(y_i, \mu, \varepsilon) = H_y(y_i, \phi(y_i, \varepsilon), \varepsilon) = 0.$$

When $\mu = \mu^*(\varepsilon)$ there is precisely one nearby y namely $y = y^*(\varepsilon)$ such that $\mu = \phi(y, \varepsilon)$. When $\mu < \mu^*(\varepsilon)$ there are *no* nearby y such that $\mu = \phi(y, \varepsilon)$. ■

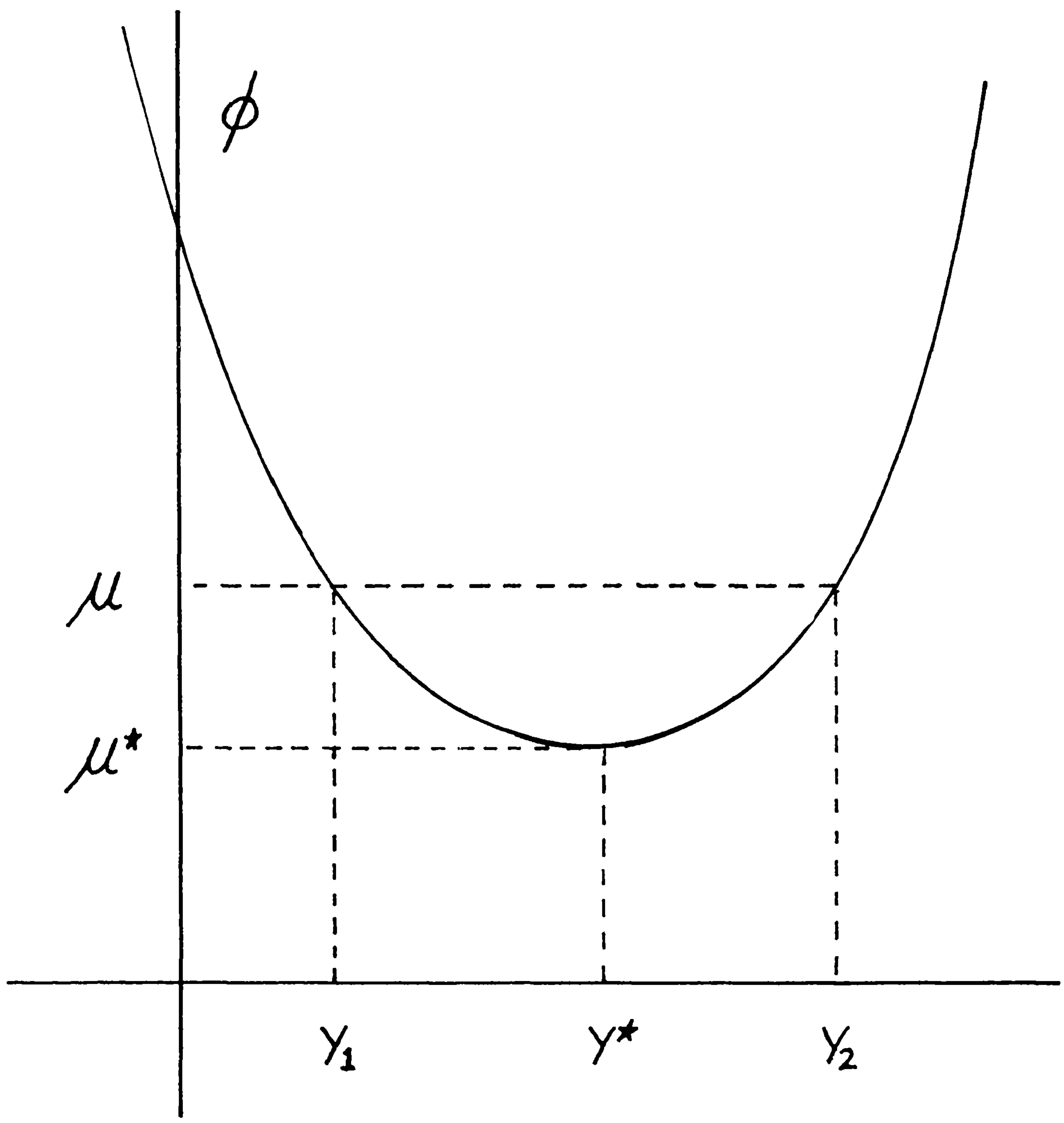

Fig. 1.

3. The Example

Consider the restricted 4 body problem corresponding to the Lagrange equilateral triangular central configuration of the three body problem (Figure 2). Assume that two of the primaries have equal mass say $1-\mu$, $0<\mu<1$, and that the third has mass 2μ so that the total mass of the system is 2. Let the y-axis be the line of symmetry. Specifically, let

$$\begin{array}{lll} m_1 = 1-\mu, & \bar{q}_1 = (1, -\sqrt{3}\mu), & \|\bar{q}_1\|^2 = 1+3\mu^2 \\ m_2 = 1-\mu, & \bar{q}_2 = (-1, -\sqrt{3}\mu), & \|\bar{q}_2\|^2 = 1+3\mu^2 \\ m_3 = 2\mu, & \bar{q}_3 = (0, \sqrt{3}(1-\mu)), & \|\bar{q}_3\|^2 = 3(1-\mu)^2 \\ \|q_i - q_j\| = 2, & \lambda = 1/4. & \end{array} \tag{8}$$

The potential is

$$V = \frac{1-\mu}{\sqrt{(x-1)^2 + (y+\sqrt{3}\mu)^2}} + \frac{1-\mu}{\sqrt{(x+1)^2 + (y+\sqrt{3}\mu)^2}} + \frac{2\mu}{\sqrt{x^2 + (y-\sqrt{3}(1-\mu))^2}} + \tfrac{1}{8}(x^2+y^2). \tag{9}$$

Since V is even in x, we have $V_x(0,y) = V_{xy}(0,y) = 0$, therefore we shall search for

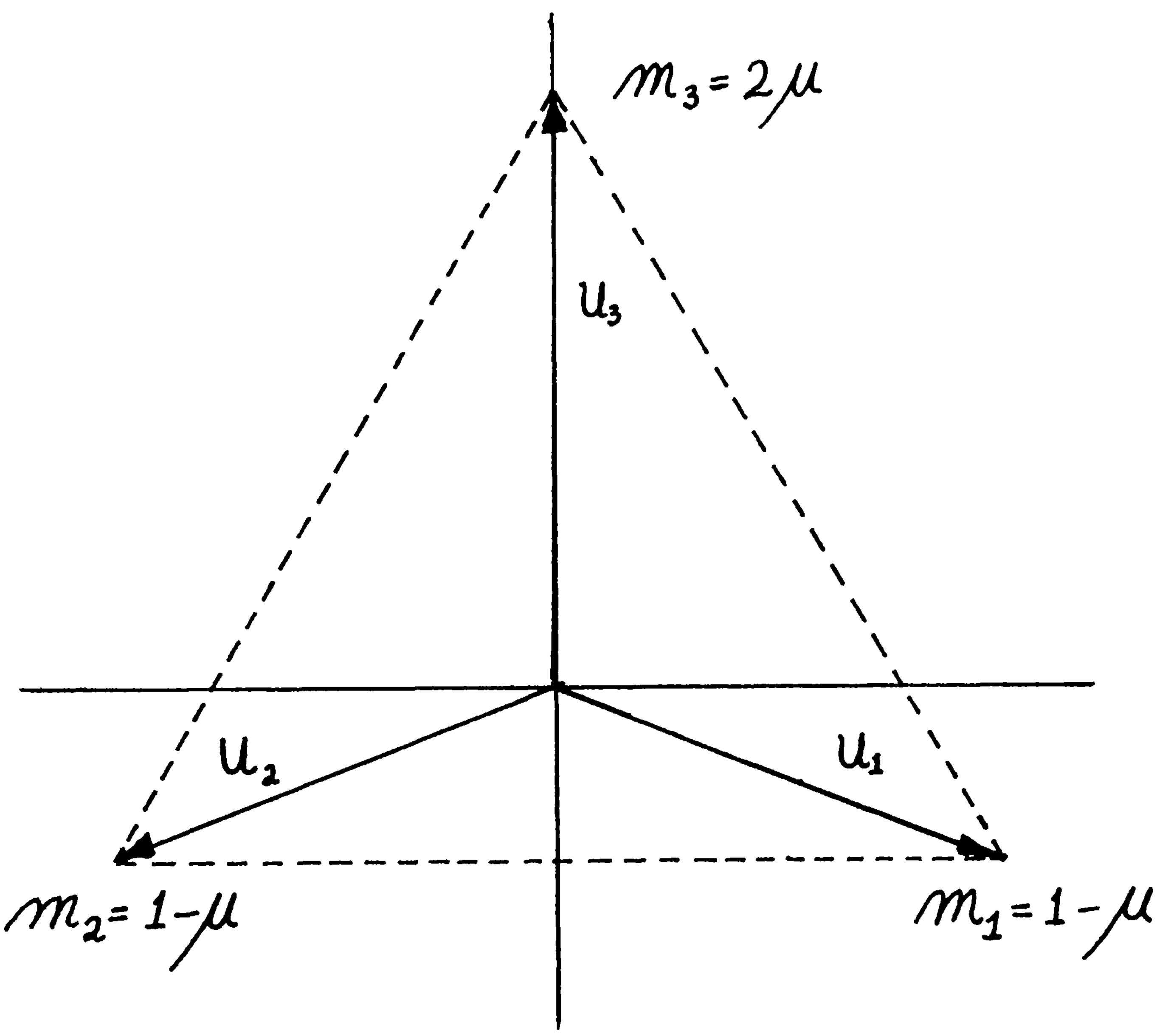

Fig. 2.

critical points on the y axis. Let $U(y,\mu)=V(0,y,\mu)$ so

$$U=\frac{2(1-\mu)}{\sqrt{1+(y+\sqrt{3}\mu)^2}}-\frac{2\mu}{\sqrt{y-\sqrt{3}(1-\mu)}}+\tfrac{1}{8}y^2 \tag{10}$$

for $y<\sqrt{3}(1-\mu)$. Note that the terms in V and U are positive and so the second term in U must have a negative sign when y is in the range $y<\sqrt{3}(1-\mu)$. Now $U\to\infty$ as $y\to-\infty$ and as $y\to\sqrt{3}(1-\mu)^-$, so U has at least one minimum in $(-\infty,\sqrt{3}(1-\mu))$. In order to find a degenerate critical point, we solve $U_y=0$, $U_{yy}=0$ for y and μ by Broyden's method (see Dennis and Schnabel (1983)). The calculations were carried out at double precision until the absolute values of U_y and U_{yy} were less than 10^{-15}. These values of y and μ where substituted into the remaining derivatives to yield:

$$V_x=0,\quad V_{xy}=0,\qquad V_y\approx 0.00\,000,\qquad V_{yy}\approx 0.00\,000$$

$$V_{xx}\approx 1.80\,616,\qquad V_{yyy}\approx 3.26\,104,\qquad V_{yy}\approx 1.01\,715$$

when $\bar{x}=0$, $\bar{y}\approx -0.45\,286\,3$ and $\bar{\mu}\approx 0.42\,344\,8$. Thus numerically, the hypothesis of the theorem of the previous section have been verified.

The three-dimensional plots (3a) for $\mu=0.2$ and (3b) for $\mu=0.5$ illustrate the

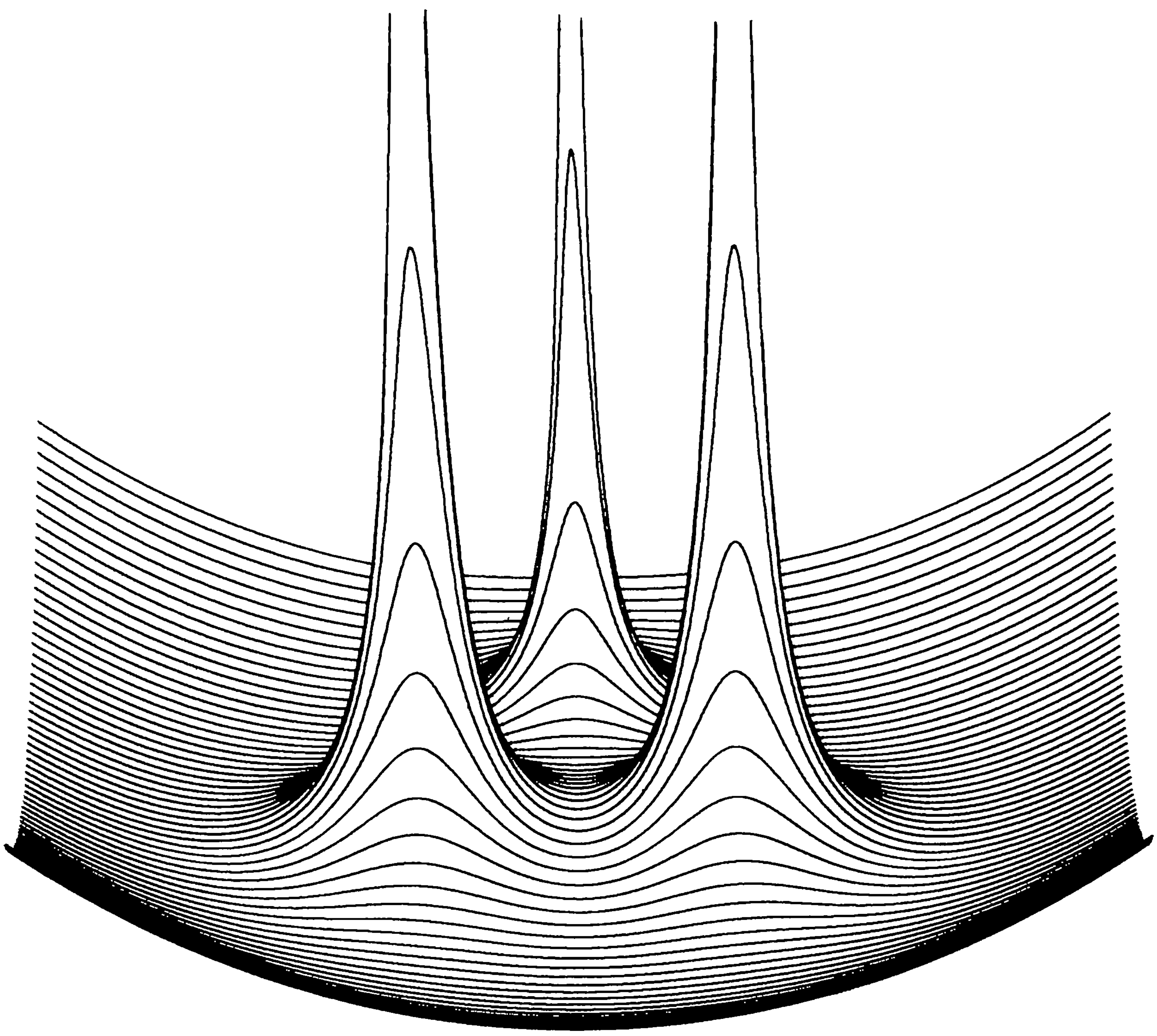

Fig. 3a. $\mu=0.2$.

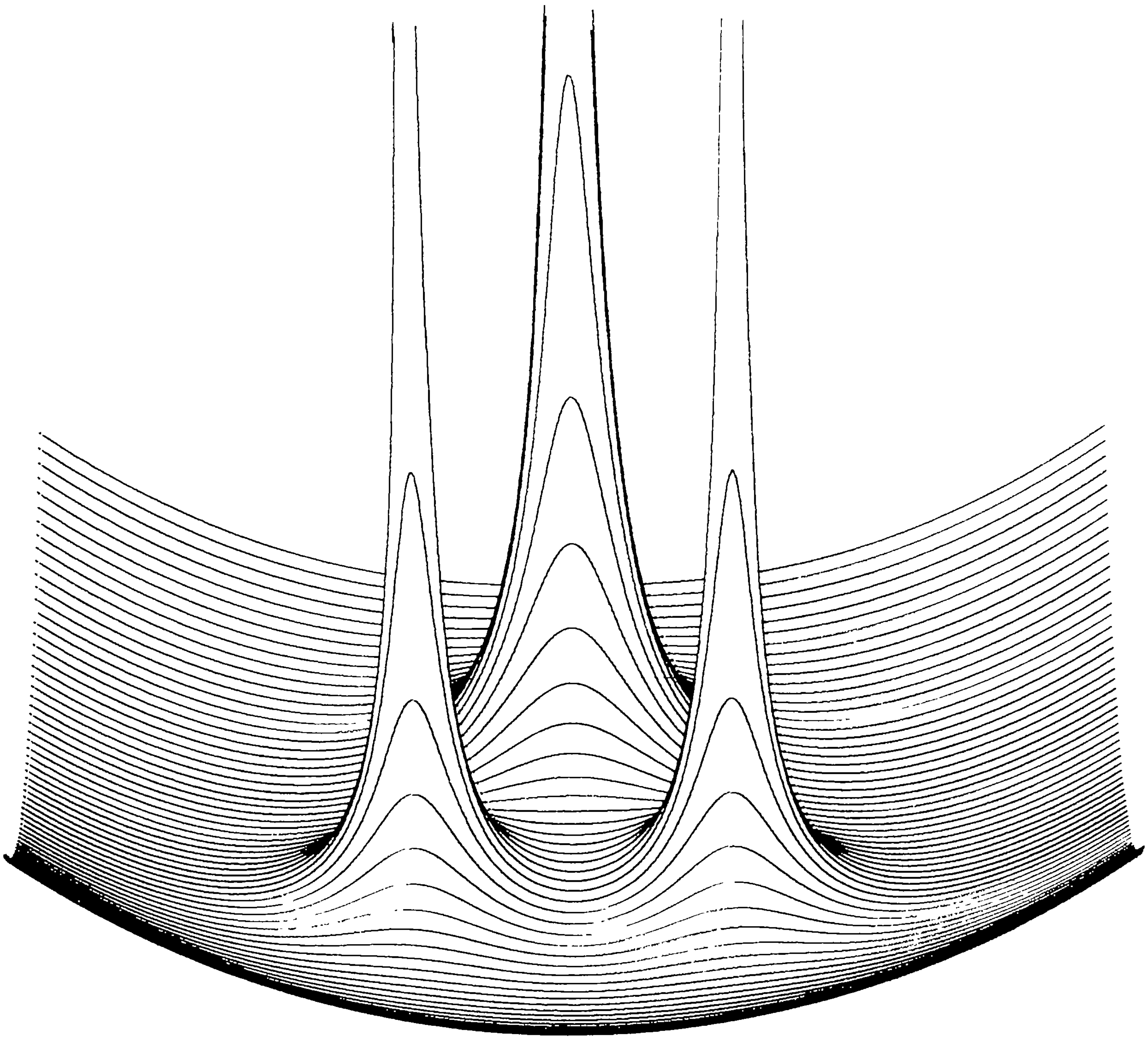

Fig. 3b. $\mu = 0.5$.

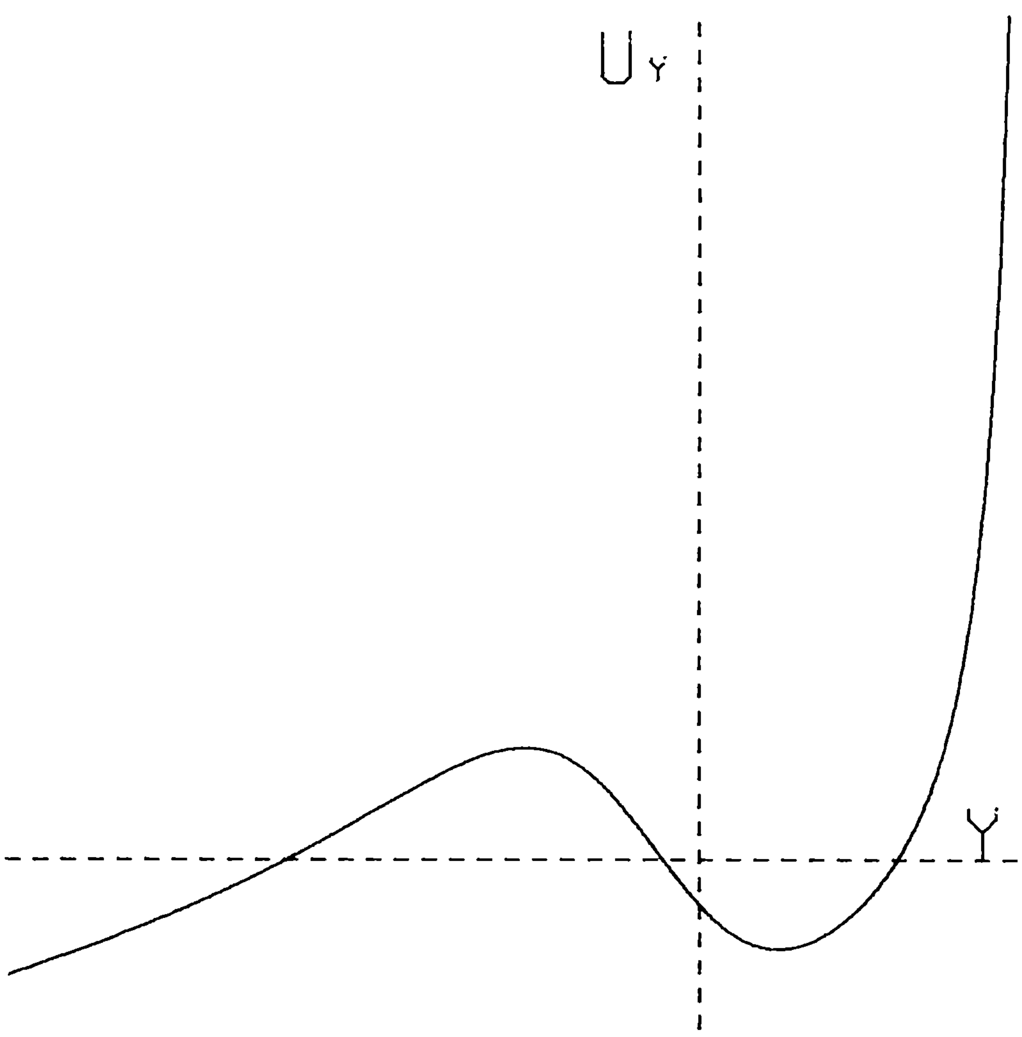

Fig. 4a. $\mu = 0.1$.

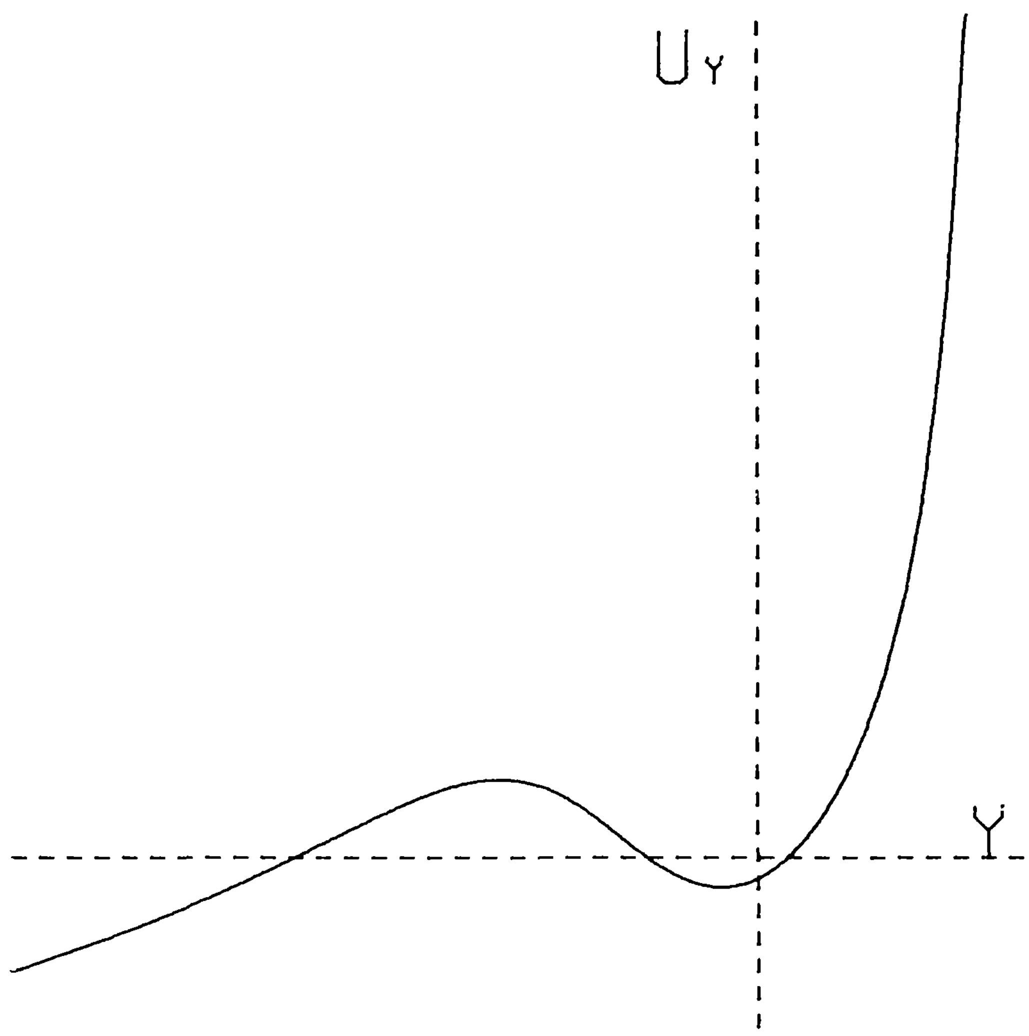

Fig. 4b. $\mu = 0.3$.

bifurcation. The viewer is situated above the negative y-axis at a point that is on a line that makes a 60° angle withe the V or z-axis. That is, the spherical angular coordinates of the viewer are $\theta = -90°$, $\phi = 60°$. There are several critical points in these figures, but the ones of interest can be seen in (3a). Looking down the line of symmetry one can see a minimum in the foreground, then a saddle approximately between the two symmetric singularities, then a minimum and finally the other

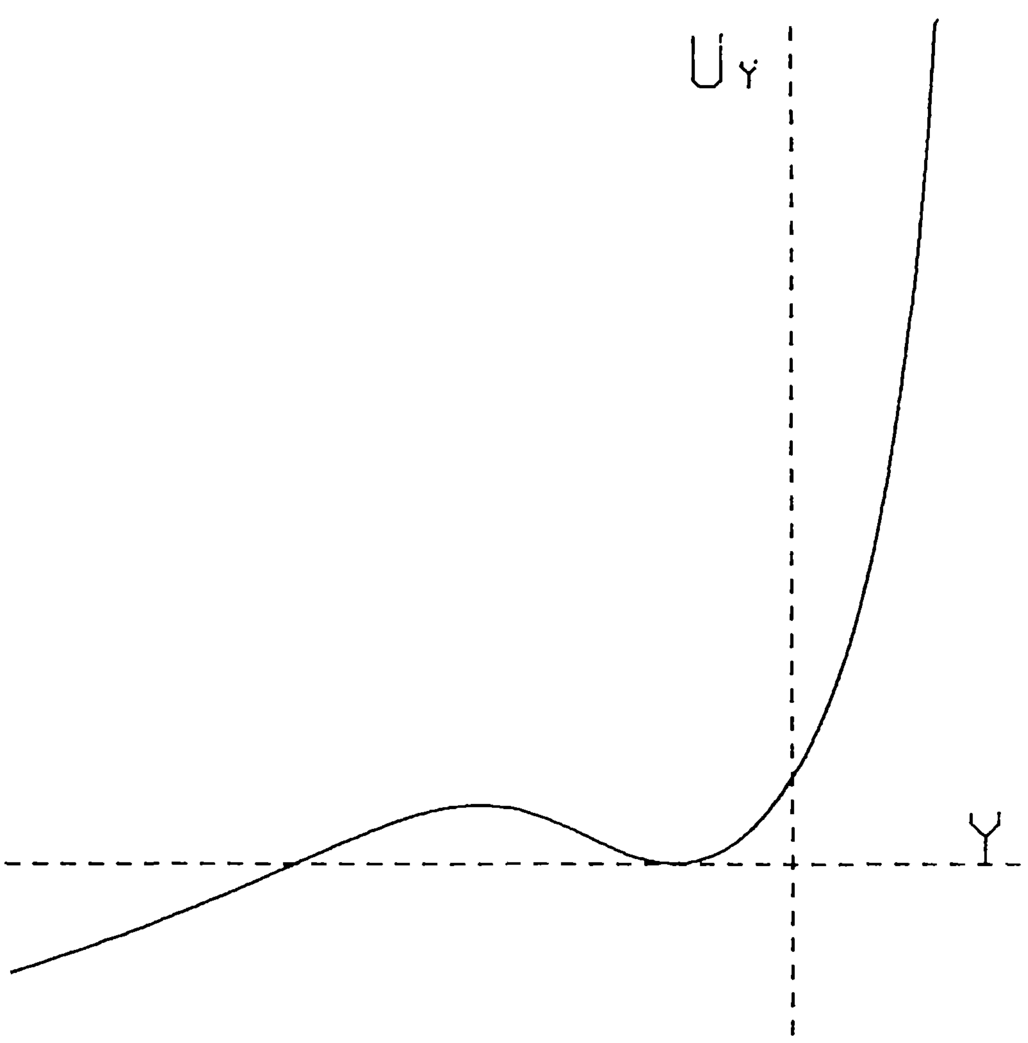

Fig. 4c. $\mu = 0.423$.

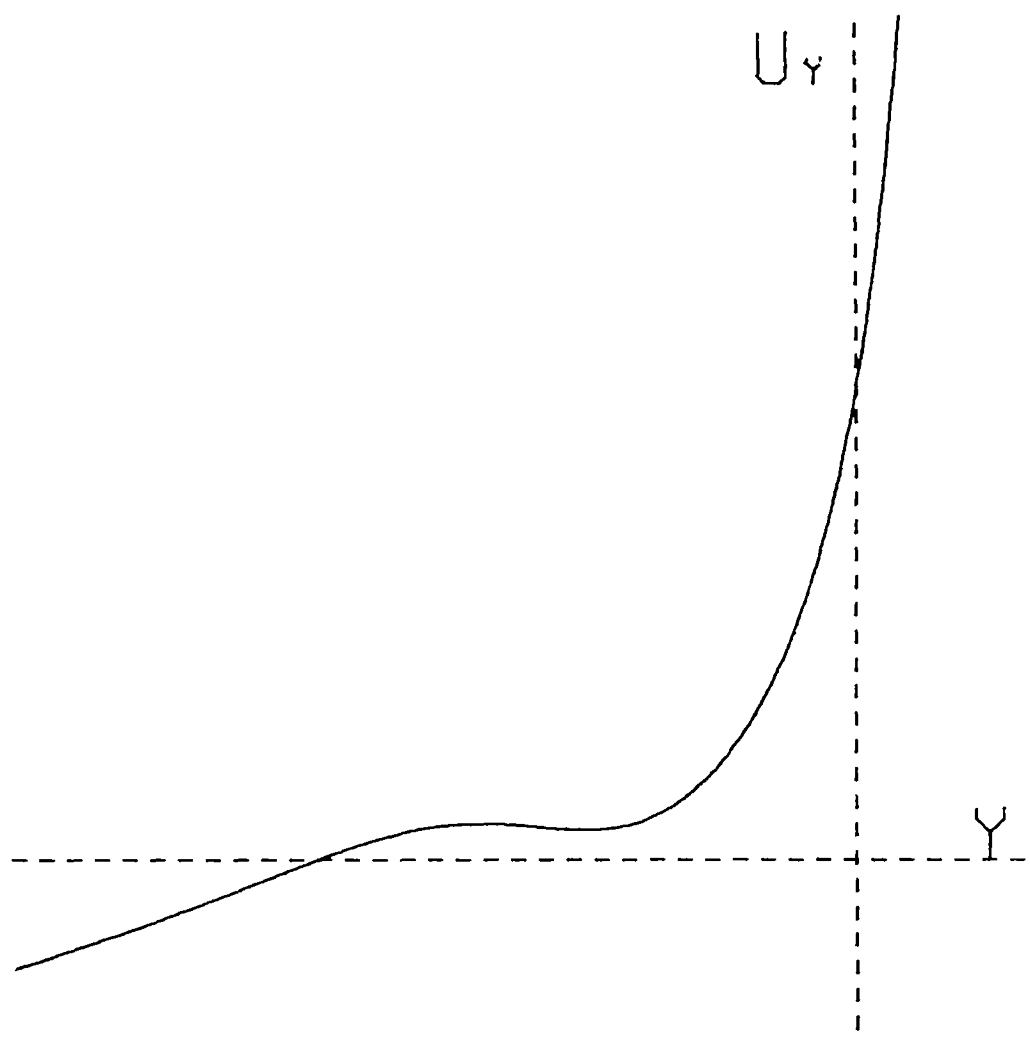

Fig. 4d. $\mu = 0.6$.

singularity. In figure (3b) one sees that the first minimum remains but the saddle and the other minimum have disappeared.

The graphs in figure (4) are of the function $V_y(y, \mu) = V_y(0, y, u)$ for different values of μ. These plots are self-explanatory.

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