

APPLICATION OF LIE-SERIES TO REGULARIZED PROBLEMS IN CELESTIAL MECHANICS

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ABSTRACT. In the following paper we tried to apply the Lie-formalism to the regularized restricted three body problem. It will be shown that this algorithm leads to a very simple structure program which is also fast.

1. INTRODUCTION

The application of Lie-series to the n-body problem in celestial mechanics has been first investigated by W. Gröbner (1967), especially the acting of the "Kepler operator" to the coordinate and velocity vector for the two body problem. In two earlier works recursion formulae for the solution of the n-body problem have been derived by means of Lie-series (Hanslmeier 1983; Hanslmeier, Dvorak 1984). With these formulae we have a rapid numerical integration procedure. The structure of the computer program is very simple, so that any modification (e.g. automatic step adaption) can be done without great difficulties.

In this work we will try to show how it is possible and useful from the practical point of view to apply this algorithm to the regularized plane restricted three body problem. It will be seen that in this case too Lie-series provide a fast and very clear method for solving the regularized differential equations. Since regularization theory plays an important role when studying motions of natural and artificial bodies as well as for investigations concerning stability problems of dynamical systems, our investigations may also be of practical interest.

2. SHORT REMARKS ABOUT LIE-SERIES, REGULARIZED TWO BODY PROBLEM

The Lie-operator consists of partial derivatives and holomorphic functions $\theta_i(z)$, z being a complex variable. It is defined according to Gröbner (1967)

$$D = \theta_1(z) \frac{\partial}{\partial z_1} + \theta_2(z) \frac{\partial}{\partial z_2} + \dots + \theta_n(z) \frac{\partial}{\partial z_n} \quad (1)$$

We can act this linear differential operator on any function $f(z)$ which has to be holomorphic within the same domain as are the functions $\theta_i(z)$. Thus we can write (1) as:

$$L(z, t) = \sum_{v=0}^{\infty} \frac{t^v}{v!} D^v f(z) \quad (2)$$

where $D^v f(z) = D^{v-1}(Df(z))$ (2a)
Symbolically eq. (2) can be written in the form

$$e^{tD} f(z) = f(z) + tDf(z) + \frac{t^2 D^2 f(z)}{2!} + \dots \quad (3)$$

The proof of convergence of $L(z, t)$ can be found in Gröbner (1967) where it is also shown that Lie-series are a generalization of Taylor-series. Let us demonstrate how one can solve differential equations by means of Lie-series; as an example we solve the equations of the regularized two body problem (a detailed description of the regularized two body problem is given in Stiefel (1971)).

Let $\vec{x} = (x_1, x_2, x_3)$ be the position vector of mass m_2 with respect to the mass m_1 , r being their mutual distance, t the real time, s the fictitious time. If we look at the ordinary equations of the two body problem, we see that they become singular at the point $r=0$ which corresponds to a collision. Therefore we regularize the equations of motion in two steps:

1) Transformation of time:

$$t = \int r ds$$

2) Transformation of coordinates and velocities according to Kustaanheimo and Stiefel:

$$\vec{x} = L(\vec{u})\vec{u} \quad (4)$$

where $L(\vec{u})$ is the well known K-S matrix.

In the following sections ' denotes differentiation with respect to s . The differential equations for the regularized two body problem are then:

$$u_j'' + \frac{h}{2} u_j = 0 \quad j=1, 2, \dots, 4 \quad (6)$$

h is the "Kepler energy":

$$h = \frac{K^2}{2} - \frac{1}{2} \dot{\vec{x}}^2, \quad K^2 = k^2(m_1 + m_2) \quad (7)$$

The solution of eq. (6) by means of Lie-series is found by determining the functions $\theta_i(z)$:

$$\begin{aligned}\theta_j &= u_j' \\ \theta_j' &= u_j'' = -\frac{h}{2} u_j \quad j = 1, \dots, 4\end{aligned}\quad (8)$$

Now we can write the Lie-operator (1) as:

$$D = \sum_{j=1}^4 \left\{ u_j' \frac{\partial}{\partial u_j} - \frac{h}{2} u_j \frac{\partial}{\partial u_j'} \right\} \quad (9)$$

The solutions of eq. (6) with Lie-series are then:

$$\begin{aligned}u_j &= (e^{tD} u_j) (0) \\ D(u_j) &= u_j' \\ D^2(u_j) &= -\frac{h}{2} u_j \\ D^3(u_j) &= -\frac{h}{2} D(u_j) \\ &\vdots \\ D^n(u_j) &= -\frac{h}{2} D^{n-2}(u_j) \quad n = 4, 5, \dots, k \\ &\quad j = 1, \dots, 4\end{aligned}\quad (10)$$

The CPU-time in this simple case is for numerical integration about 1/3 than that needed for the ordinary two body problem and can be compared with the CPU-time needed for the integration with Stumpff- functions (Lichtenegger 1984).

Of course no automatic step adaption process is required when integrating very excentric orbits. In the next section we try to apply this formalism to the plane restricted three body problem.

3. REGULARIZED PLANE RESTRICTED THREE BODY PROBLEM

We consider three masses m_1, m_2, m_3 , where m_3 does not have any gravitational influence on the two other masses. We further use the well known rotating coordinate system where the distance $\overline{m_1 m_2}$ remains fix. The units used are then:

unit of mass: $m_1 + m_2, m_2 = \mu, m_1 = 1 - \mu$

unit of length: distance $\overline{m_1 m_2}$

unit of time: 1 period of revolution of m_2 about m_1 divided by

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(11)

With these units chosen the value of the gravitational constant will be 1. Before applying the Lie-series let us briefly recall the Hamiltonian formalism.

Let H be the Hamiltonian function, x and y be the coordinates, p_x, p_y the momenta. The equations of motion are then:

$$\begin{aligned} \dot{x} &= \frac{\partial H}{\partial p_x} & \dot{p}_x &= -\frac{\partial H}{\partial x} \\ \dot{y} &= \frac{\partial H}{\partial p_y} & \dot{p}_y &= -\frac{\partial H}{\partial y} \end{aligned} \quad (12)$$

In this case, the Lie-operator (1) can be written in the following form (Giacaglia, 1981):

$$D = \sum_i \left\{ \frac{\partial H}{\partial p_i} \frac{\partial}{\partial q_i} - \frac{\partial H}{\partial q_i} \frac{\partial}{\partial p_i} \right\} = [\cdot, H] \quad (13)$$

q_i ... generalized coordinates

p_i ... generalized momenta

$[\cdot]$... Poisson brackets

If we act D on a function $f(z)$, we get:

$$\begin{aligned} D(f) &= [f, H] \\ D^2(f) &= [[f, H], H] \\ D^3(f) &= [[[f, H], H], H] \end{aligned} \quad (14)$$

We follow the regularization with parabolic coordinates as has been given by A. Deprit and R. Broucke (1964).

Let x, y be the coordinates of the massless body m_3 , ρ_1 be the distance to m_1 , ρ_2 be the distance of m_3 to m_2 .

$$\begin{aligned} \rho_1^2 &= \left(x + \frac{1}{2} \right)^2 + y^2 \\ \rho_2^2 &= \left(x - \frac{1}{2} \right)^2 + y^2 \end{aligned} \quad (15)$$

We finally derive that the Hamiltonian function H is equal to the expression:

$$\begin{aligned}
 H = & \frac{1}{2} (p_x^2 + p_y^2) + (y p_x - x p_y) - (1 - \mu) / \rho_1 - \\
 & - (\frac{1}{2} - \mu) x - \mu / \rho_2
 \end{aligned}
 \tag{16}$$

From (13) we can deduce the equations of motion for the plane restricted three body problem.

It is obvious that we have two singular points now at m_1 and m_2 . The regularization is a conformal mapping of the complex plane $z = x + i y$ to the complex plane $\zeta = \xi + i \eta$.

If we consider regularization with respect to m_1 , using parabolic coordinates, the conformal mapping has the form:

$$z = -\frac{1}{2} + \zeta_1^2 \tag{17}$$

whereas if we regularize with respect to m_2 , we have to use the mapping

$$z = \frac{1}{2} + \zeta_2^2 \tag{18}$$

We can treat both cases if we introduce at the beginning of the program the following parameters:

Table 1: List of parameters

Parameter	at m_1	at m_2
ϵ	-1	1
ρ	ρ_1	ρ_2
σ	ρ_2	ρ_1
α	$1-\mu$	μ

We have the following relations:

$$\begin{aligned}
 x &= \frac{1}{2} \epsilon + \xi^2 - \eta^2 & y &= 2\xi\eta \\
 2\rho p_x &= \xi p_\xi - \eta p_\eta & 2\rho p_y &= \eta p_\xi + \xi p_\eta \\
 \rho &= \xi^2 + \eta^2 \\
 \sigma &= (\rho^2 - 2(\xi^2 - \eta^2) + 1)^{1/2}
 \end{aligned}
 \tag{19}$$

The Hamiltonian function is then:

$$\begin{aligned}
 H = & \frac{1}{2} (p_\xi^2 + p_\eta^2) + 2 \left\{ (\rho - \frac{1}{2}\epsilon)\eta p_\xi - (\rho + \frac{1}{2}\epsilon) \xi p_\eta \right\} \\
 & - 4\alpha - 4(1 - \alpha) (\rho/\sigma) - 4 \left(\frac{1}{2} - \mu \right) \left(\frac{1}{2}\epsilon + \xi^2 - \eta^2 \right) \rho \\
 & - 4h\rho
 \end{aligned} \tag{20}$$

The Lie-operator is finally found according to (13):

$$\begin{aligned}
 D = & [p_\xi + (2\rho - \epsilon)\eta] \frac{\partial}{\partial \xi} - [4\xi\eta p_\xi - 4\xi^2 p_\eta - 2(\rho + \frac{1}{2}\epsilon) p_\eta \\
 & - A(2\xi\sigma^{-1}(1 - \rho^2\sigma^{-2} + \rho\sigma^{-2})) - M(4\xi^3 + \epsilon\xi) - 8h\xi] \frac{\partial}{\partial p_\xi} \\
 & + [p_\eta - (2\rho + \epsilon)\xi] \frac{\partial}{\partial \eta} - [4\eta^2 p_\xi - 4\eta\xi p_\eta + 2(\rho - \frac{1}{2}\epsilon) p_\xi \\
 & - A(2\eta\sigma^{-1}(1 - \rho^2\sigma^{-2} - \sigma^{-2}\rho)) - M(\epsilon\eta - 4\eta^3) - 8h\eta] \frac{\partial}{\partial p_\eta}
 \end{aligned} \tag{21}$$

Here we introduce the abbreviations:

$$\begin{aligned}
 A &= 4(1 - \alpha) \\
 M &= 4 \left(\frac{1}{2} - \mu \right)
 \end{aligned} \tag{22}$$

The solutions are of the form:

$$\xi = e^{tD} \xi_{(0)} \quad \eta = e^{tD} \eta_{(0)} \tag{23}$$

The first term of the Lie-series can be found very easily.

$$\begin{aligned}
 D\xi &= p_\xi + (2\rho - \epsilon)\eta \\
 D\eta &= p_\eta - (2\rho + \epsilon)\xi
 \end{aligned} \tag{24}$$

Since the equations are similar for the x and the y coordinates, we consider here only the Lie-series for x!

We also can evaluate the second Lie-term:

$$D^2\xi = Dp_\xi + D\eta(2\rho - \epsilon) + \eta 2D\rho \tag{25}$$

We see that:

$$Dp_\xi = \theta_2 \tag{26}$$

At this point, we are able to give some kind of recursion relation for the $D^{n+2}\xi$, $n=1,2,\dots$:

$$\begin{aligned}
D^{n+2}\xi = & - \left\{ 4 \sum_{v=0}^n \binom{n}{v} D^v \delta D^{n-v} P_\xi - 4 \sum_{v=0}^n \binom{n}{v} D^v \xi^2 D^{n-v} P_\eta - \right. \\
& - 2 \sum_{v=0}^n \binom{n}{v} D^v \left(\rho + \frac{1}{2}\epsilon \right) D^{n-v} P_\eta - 2A \left[\sum_{v=0}^n \binom{n}{v} D^v \xi D^{n-v} \sigma^{-1} - \right. \\
& - \sum_{v=0}^n \binom{n}{v} D^v \rho^2 D^{n-v} \sigma_\xi + \sum_{v=0}^n \binom{n}{v} D^v \rho D^{n-v} \sigma_\xi \left. \right] - M (4D^n \xi^3 + \\
& + D^n \xi) - 8hD^n \xi \left. \right\} + \sum_{v=0}^n \binom{n}{v} D^v (2\rho - \epsilon) D^{n-v} \eta \quad (27)
\end{aligned}$$

For brevity we introduced:

$$S = \sum_{v=0}^n \binom{n}{v}$$

The expression (27) contains the following "derivatives" for which recursion relations can be found:

$$D^n \delta = D^n (\xi \eta) = S D^v \xi D^{n-v} \eta \quad (28)$$

$$D^{n+1} \xi^2 = 2 S D^v \xi D^{n-v+1} \xi \quad (29)$$

$$\begin{aligned}
D^{n+1} P_\xi = & - \left\{ 4 S D^v \delta D^{n-v} P_\xi - 4 S D^v \xi^2 D^{n-v} P_\eta - \right. \\
& - 2 S D^v \left(\rho + \frac{1}{2}\epsilon \right) D^{n-v} P_\eta - 2 A \left[S D^v \xi D^{n-v} \sigma^{-1} - \right. \\
& - S D^v \rho^2 D^{n-v} \sigma_\xi + S D^v \rho D^{n-v} \sigma_\xi \left. \right] - M (4D^n \xi^3 + \epsilon D^n \xi) - \\
& - 8hD^n \xi \left. \right\} \quad (30)
\end{aligned}$$

$$D^{n+1} \left(\rho + \frac{1}{2}\epsilon \right) = 2 (S D^v \xi D^{n-v+1} \xi + S D^v \eta D^{n-v+1} \eta) \quad (31)$$

$$D^{n+1} \rho^2 = 2 S D^v \rho D^{n-v+1} \rho \quad (32)$$

The only remaining problem is to find a recursion relation for the $D^n \sigma$ which seems to be impossible. But it is not difficult to

evaluate the terms:

$$\begin{aligned}
 \sigma &= (\rho^2 - 2(\xi^2 + \eta^2) + 1)^{1/2} \\
 D\sigma &= \sigma^{-1} (\rho D\rho - 2\xi D\xi - 2\eta D\eta) \\
 D^2\sigma &= -\sigma^{-2} D\sigma (\rho D\rho - 2\xi D\xi - 2\eta D\eta) + \sigma^{-1} D(\rho D\rho - 2\xi D\xi - 2\eta D\eta) \\
 &= \sigma^{-1} (-(D\sigma)^2 + D(\rho D\rho - 2\xi D\xi - 2\eta D\eta)) \\
 D^3\sigma &= \sigma^{-1} (-3D\sigma D^2\sigma + D^2(\rho D\rho - 2\xi D\xi - 2\eta D\eta)) \tag{33}
 \end{aligned}$$

It is evident, how the higher derivatives are evaluated. The terms $D^n \sigma^{-3}$ and $D^n \sigma^{-1}$ can be derived and the term appearing in (27) stands for:

$$\begin{aligned}
 \sigma_\xi &= \sigma^{-3} \xi \\
 D^n \sigma_\xi &= S D^\nu \sigma^{-3} D^{n-\nu} \xi \tag{34}
 \end{aligned}$$

With these relations, we can in principle calculate all Lie-terms up to the n-th order. In the next section we show that for practical uses it will be sufficient to work with 5 terms.

4. DISCUSSION

If we use the appropriate unit system (see section 3.), than the numerical values of $\xi, \eta, p_\xi, p_\eta, \rho$ for a typical Trojan orbit are less or equal 1, σ being greater than 1. So the convergence of the Lie-series depends mainly on

$$\frac{s^n}{n!}$$

where $dt = 4 \rho ds$.

It will be sufficient to work with 5 Lie-terms, where the computing-time needed to evaluate the terms will not be great. The advantage of this solution is that we avoided trigonometric series and that the solution itself is very simple in structure. The program works very fast and can be changed easily. A detailed description of the regularization of the restricted three body problem has been given by V.Szebehely (1967). In a future work we plan to give some numerical tests and we want to compare this method to other numerical integration techniques.

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