HOMOCLINIC AND HETEROCLINIC SOLUTIONS IN THE RESTRICTED THREE-BODY PROBLEM

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Abstract. In this work we have performed a systematic computation of the homoclinic and heteroclinic orbits associated with the triangular equilibrium points of the restricted three-body problem. Some analytical results are given, related to their number when the mass ratio varies.

1. Introduction

From the time of Strömgren [4] it has been known that some families of periodic orbits of the restricted three-body problem end at an 'orbit' formed by a pair of heteroclinic orbits connecting the two triangular equilibrium points. In fact, and for the value of the mass parameter equal to 1/2, Strömgren computed five symmetric heteroclinic orbits, some of whose combinations by pairs are natural endings of well-known families of symmetric periodic orbits (see [5]). Some homoclinic orbits (or asymptotic-periodic orbits, according to the classical nomenclature) were computed by Strömgren too. Families of periodic orbits ending at some of these last ones were given by Danby [1], Szebehely and Nacozy [6], and Szebehely and Van Flandern [7] for the mass ratio $\mu = 0.5$.

In the framework of analytic Hamiltonian systems, Henrard [3] proved Strömgren's conjecture, according to which a class of doubly asymptotic orbits are limit members of families of periodic orbits. Further results of Devaney [2] prove that this phenomenon occurs in both Hamiltonian and reversible systems.

In this work we have done a systematic computation of the homoclinic and heteroclinic orbits associated with the triangular equilibrium points of the restricted three-body problem. For that purpose a preliminar numerical study of the invariant stable and unstable manifolds related to those equilibrium points, has been done. Some

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results about the number and shape of the homoclinic and heteroclinic solutions are obtained when the mass ratio varies.

As a result of the work it is found that the total number of symmetric heteroclinic orbits for $\mu = 0.5$ intersecting the x axis only once perpendicularly is four. This result was known to Strömgren. Two of these orbits are natural terminations of families of symmetric periodic orbits of classes (k) and (l) in Strömgren's terminology (see [5] pp. 484–5).

For a value of the mass ratio in the interval [0.1, 0.2] it has been found that the number of heteroclinic solutions with only one orthogonal cross with the x axis becomes infinite. An explanation of the mechanism that produces this phenomenon, and which is also valid for homoclinic solutions, is given in Proposition 2 and 3.

2. Local Study of the Invariant Manifolds of the Triangular Equilibrium Points

If p is an equilibrium point of a vector field X defined on a certain manifold M, the stable and unstable manifolds of p (which are usually denoted by $W^s(p)$, $W^u(p)$ respectively) are defined by:

$$W^s(p) = \{x \in M,\ \varphi(t,x) \xrightarrow[t \to \infty]{} p\}$$

$$W^u(p) = \{x \in M,\ \varphi(t,x) \xrightarrow[t \to -\infty]{} p\}$$

where $\varphi(t,x)$ denotes the flow associated with the vector field X which passes through x.

If the manifold M, where X is defined, is 4-dimensional the vector field is called R-reversible [2], if R is a diffeomorphism of M satisfying:

(i) $R^2 = \mathrm{Id}$ (R is an involution),
(ii) $\dim \mathrm{Fix}(R) = 2$ (where $\mathrm{Fix}(R) = \{p \in M,\ R(p) = p\}$),
(iii) $DR(X) = -XR$ (Where D stands for the differential).

It is well known (see [5]) that the restricted three-body problem is reversible by

(a) $R_1(q_1, q_2, p_1, p_2) = (q_1, -q_2, -p_1, p_2)$,
(b) for $\mu = 1/2$, $R_2(q_1, q_2, p_1, p_2) = (-q_1, q_2, p_1, -p_2)$

where q_i and p_i stand for the coordinates and momenta of the third body.

Using the above symmetries for the triangular equilibrium points L_4 and L_5, we find that:

$$R_1(W^s(L_4)) = W^u(L_5),$$

$$R_1(W^u(L_4)) = W^s(L_5),$$

$$\text{for } \mu = 1/2, \quad R_2(W^s(L_i)) = W^u(L_i) \quad \text{for} \quad i = 4, 5.$$

These facts will be useful in the computations.

We have used in this work the same system of units and coordinates as in [5]. So if we perform a translation from the origin to L_4 and then a rotation of angle α given by tan $2\alpha = 3^{1/2}(1 - 2\mu)$, where μ is the mass ratio, the equations of motion become:

$$\begin{aligned} \ddot{x} &= 2\dot{y} + \Omega_x, \\ \ddot{y} &= -2\dot{x} + \Omega_y, \end{aligned} \tag{1}$$

where

$$\Omega(x, y) = \frac{3}{2} + \frac{\bar{\lambda}_2}{2}x^2 + \frac{\bar{\lambda}_1}{2}y^2 + O(3)$$

with

$$\bar{\lambda}_1 = \tfrac{3}{2}[1 + (1 - 3\mu(1 - \mu))^{1/2}],$$
$$\bar{\lambda}_2 = \tfrac{3}{2}[1 - (1 - 3\mu(1 - \mu))^{1/2}].$$

The eigenvalues of the linear part of (1) $(DX(L_4))$ at the equilibrium point L_4 are: $\pm(\alpha \pm i\beta)$ with

$$\alpha = \frac{[(27\mu(1-\mu)^{1/2} - 1]^{1/2}}{2} \quad \text{and} \quad \beta = \frac{[(27\mu(1-\mu)^{1/2} + 1]^{1/2}}{2}.$$

Note that, due to the reversibility, $DX(L_5)$ has the same eigenvalues.

So if $\mu > \mu_R$ (where μ_R stands for Routh's mass ratio), the equilibrium is hyperbolic, and the fact that $\beta \neq 0$ implies that the orbits spiral around it.

The solution of the linear part of (1) is:

$$\begin{aligned} x(t) &= e^{\alpha t}(a_1 \cos\beta t + a_2 \sin\beta t) + e^{-\alpha t}(a_3 \cos\beta t + a_4 \sin\beta t) \\ y(t) &= e^{\alpha t}(b_1 \cos\beta t + b_2 \sin\beta t) + e^{-\alpha t}(b_3 \cos\beta t + b_4 \sin\beta t) \end{aligned} \tag{2}$$

with some relations between the integration constants a_i, b_i. In fact the b_i are functions of the a_i so we have only four independent constants. (see [5] pp. 261–264).

It happens that if $a_1 = a_2 = 0$, then $b_1 = b_2 = 0$, so in this case the solution of the linearized equations give an approximation of $W^s(L_4)$ if we are close enough to the equilibrium. Analogously, if $a_3 = a_4 = 0$ then $b_3 = b_4 = 0$ and we get the approximation to $W^u(L_4)$.

3. Numerical Globalization of the Invariant Manifolds

Looking towards a numerical global view of $W^u(L_4)$ and $W^s(L_4)$, we need an adequate set of initial conditions giving us the orbits on these manifolds.

From the preceeding section it follows that the linear approximation to $W^u(L_4)$ is given by:

$$x(t) = e^{\alpha t}(a_1 \cos\beta t + a_2 \sin\beta t),$$
$$y(t) = e^{\alpha t}(b_1 \cos\beta t + b_2 \sin\beta t).$$

At $t=0$ we have, also taking account of the relation between a_i and b_i [5]:

$$\begin{aligned} x(0) &= a_1, \\ y(0) &= b_1 = a_1\alpha' + a_2\beta', \\ \dot{x}(0) &= a_1\alpha + a_2\beta \\ \dot{y}(0) &= a_1(\alpha'\alpha - \beta\beta') + a_2(\alpha\beta' + \alpha'\beta) \end{aligned} \tag{3}$$

where

$$\alpha' = \frac{\alpha}{2|\lambda|^2}(|\lambda|^2 - \bar{\lambda}_2)$$

$$\beta' = \frac{\beta}{2|\lambda|^2}(|\lambda|^2 + \bar{\lambda}_2)$$

$$|\lambda| = \frac{1}{\sqrt{2}}[27\mu(1-\mu)]^{1/4} = |\alpha + i\beta|$$

$$\bar{\lambda}_2 = \tfrac{3}{2}[1-(1-3\mu(1-\mu))^{1/2}].$$

The notation used follows [5] (pp. 261 – 264).

A good set of initial conditions can be obtained for our purposes, taking a_1, a_2 in such a way that when $t=0$ then $(x(0), y(0))$ describes a circle, $(r\cos\theta, r\sin\theta)$, small enough around the equilibrium on the linear part of the manifold. Since the orbits on this manifold spiral from the origin, and the manifold has dimension two, we can be sure that for values of r small enough any orbit going to it has a least one θ associated with it. This can be obtained taking:

$$a_1 = r\cos\theta, \qquad \theta\in[0, 2\pi]$$

$$a_2 = \frac{r\sin\theta - r\alpha'\cos\theta}{\beta'}$$

For the globalizations of $W^s(L_4)$ things are quite similar. After some trials, the computations have been carried out taking $r = 10^{-4}$ and taking a finite set of values of θ in the whole interval $[0, 2\pi]$. Using a Runge–Kutta–Fehlberg (7–8) algorithm for the integration of the equations of motion (1), regularized when needed due to close encounters with one of the primaries, the variations of the value of the Jacobian integral have been less than 10^{-13} from the value of this integral at the equilibrium point, which is equal to 3, and which must be, of course, also the value of the Jacobi constant along the orbits on the invariant manifolds.

Using the method just mentioned, the first intersections of $W^u(L_4)$ and $W^s(L_4)$ with the x axis have been computed for the following values of the mass ratio: $\mu = 0.5$, 0.4, 0.3, 0.2, 0.1. In fact, and due to the symmetries mentioned, for $\mu = 0.5$ only one of the above manifolds must be computed. Those manifolds associated with L_5 are also obtained by symmetry, in all the cases.

The intersections with this axis can be represented in the $(x, \dot{x})$ plane, since $y = 0$ and the value of $\dot{y}$ can be obtained via the Jacobian integral, the graph determines the manifold at the first cut with the x axis. In Figures 1–10 the curves associated with the first intersection are represented for both the stable and the unstable manifold of L_4.

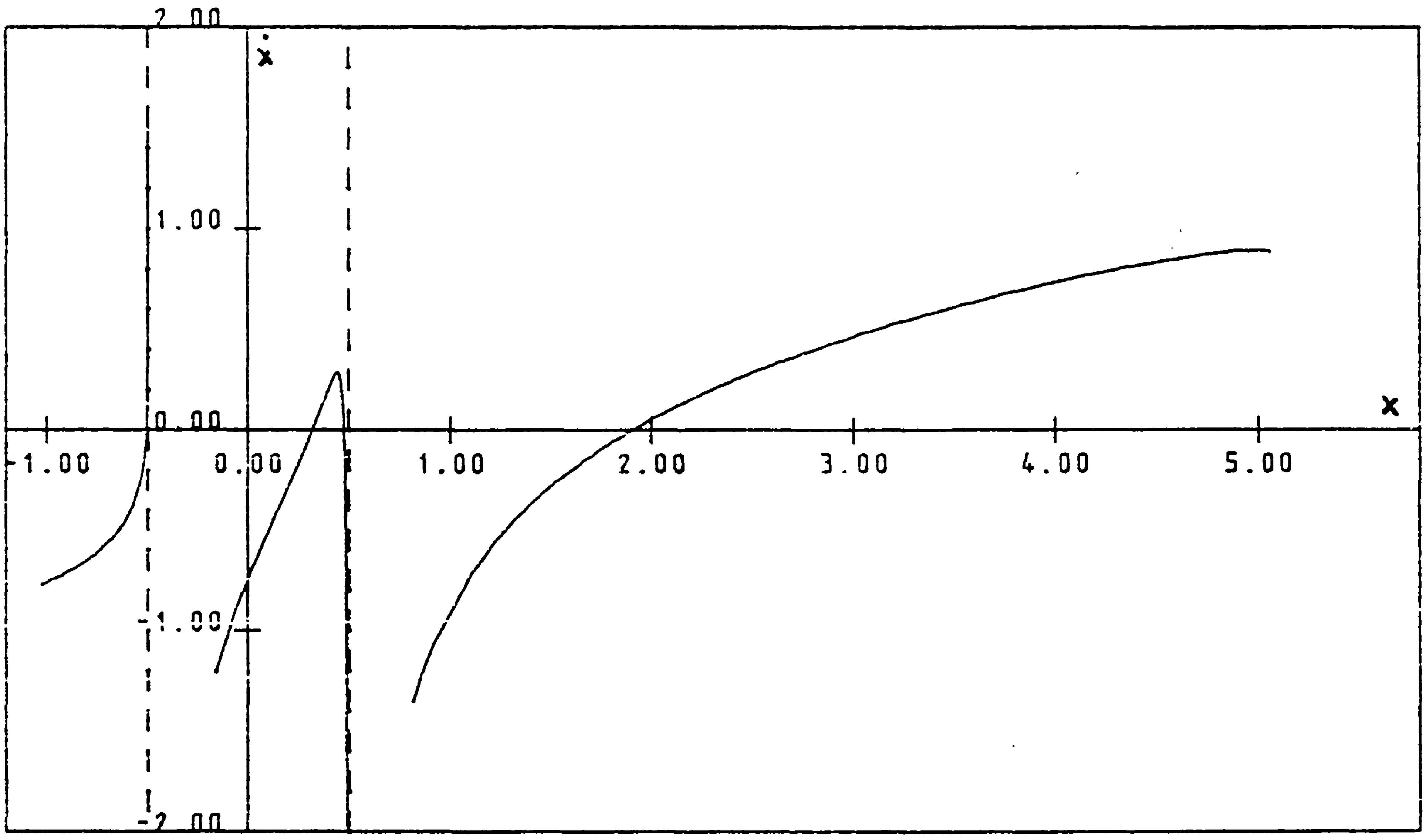

Fig. 1. $W^u(L_4)$ at the first cut for $\mu = 0.5$

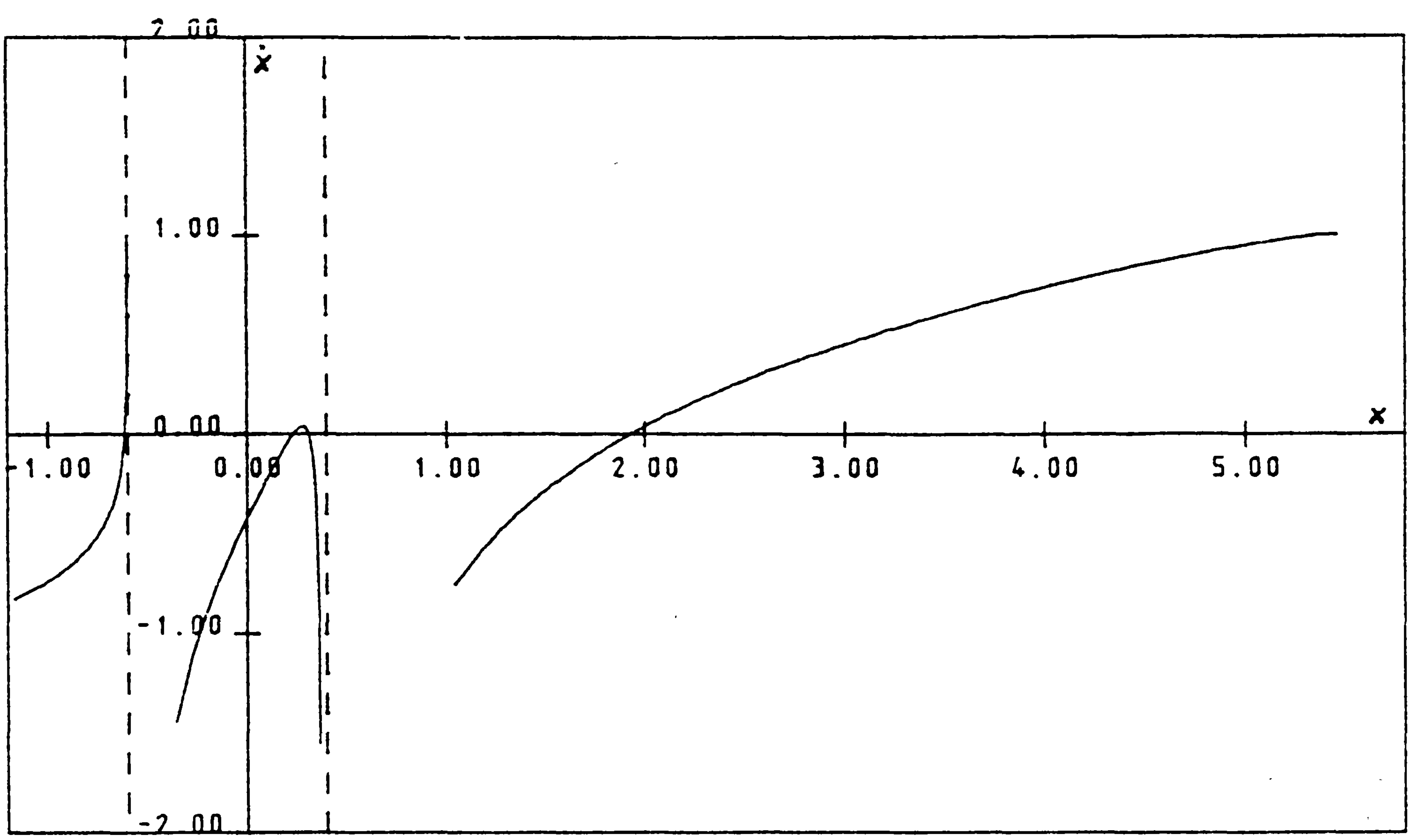

Fig. 2. $W^u(L_4)$ at the first cut for $\mu = 0.4$.

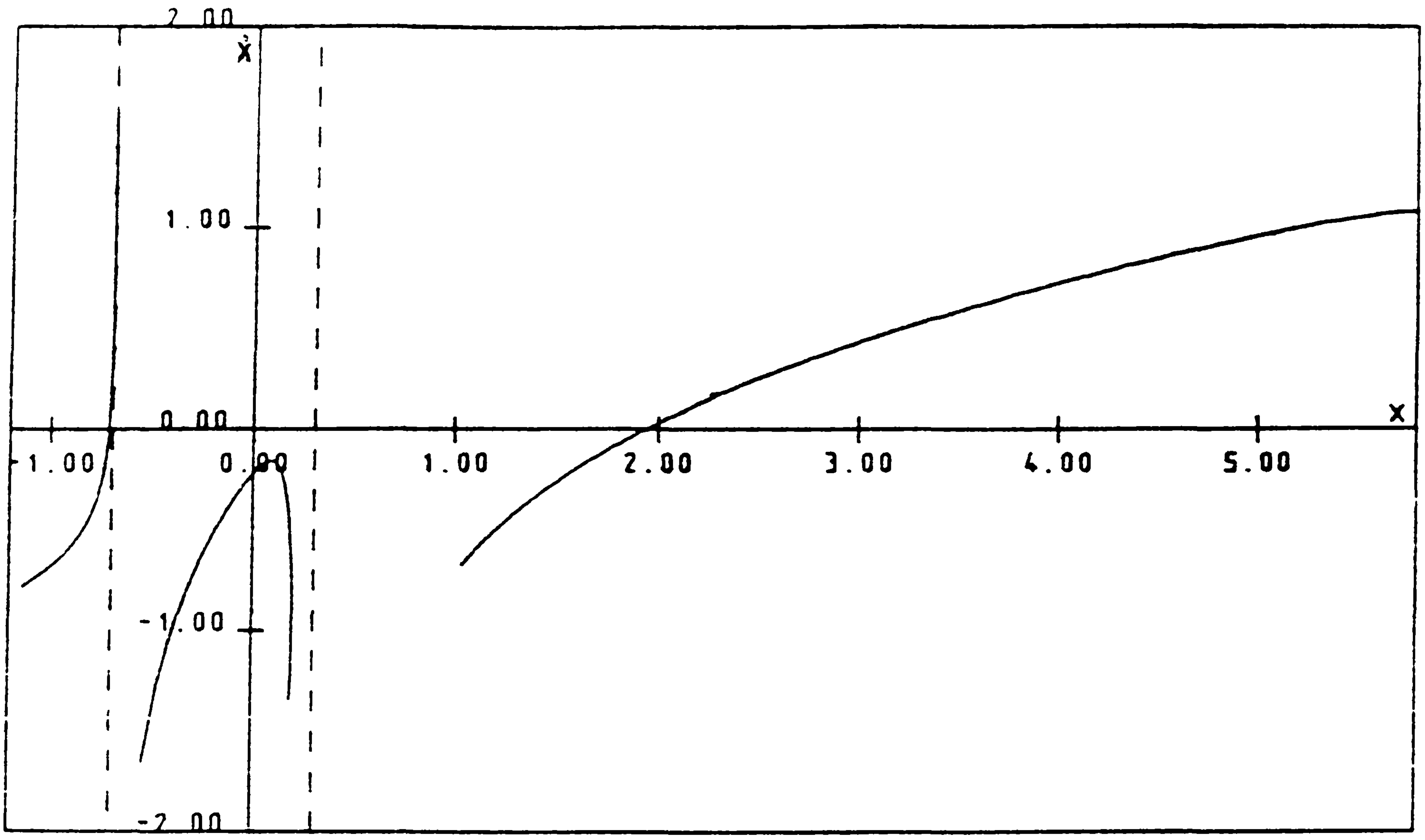

Fig. 3. $W^u(L_4)$ at the first cut for $\mu = 0.3$.

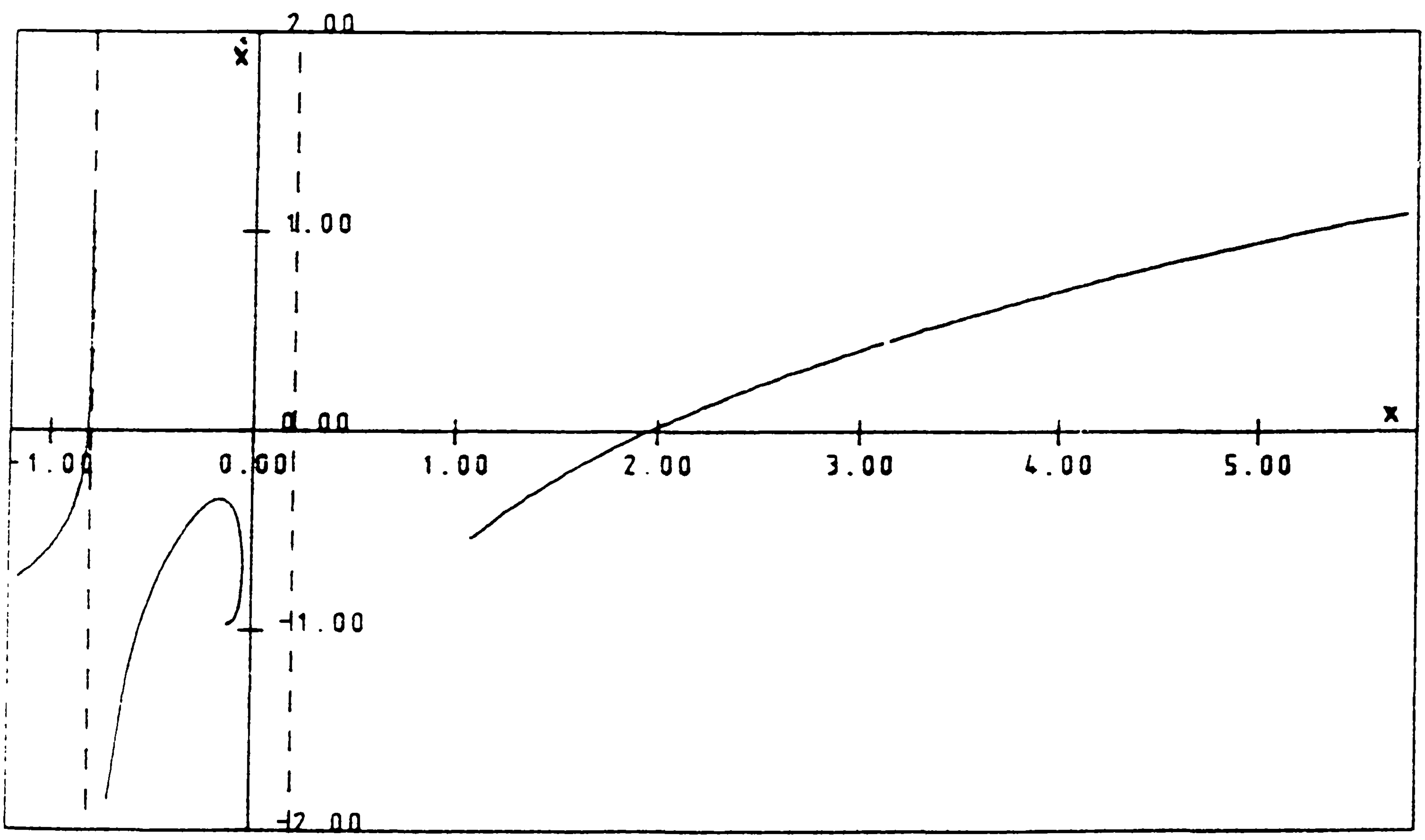

Fig. 4. $W^u(L_4)$ at the first cut for $\mu = 0.2$.

It must be noted that the curves appearing in the figures have two kinds of discontinuities, one near the primaries due to the possible collision orbits, and a second one due to the non-global character of $y = 0$ as a surface of section. Due to this last fact any tangent orbit produces a discontinuity, on the representation of the manifold, with

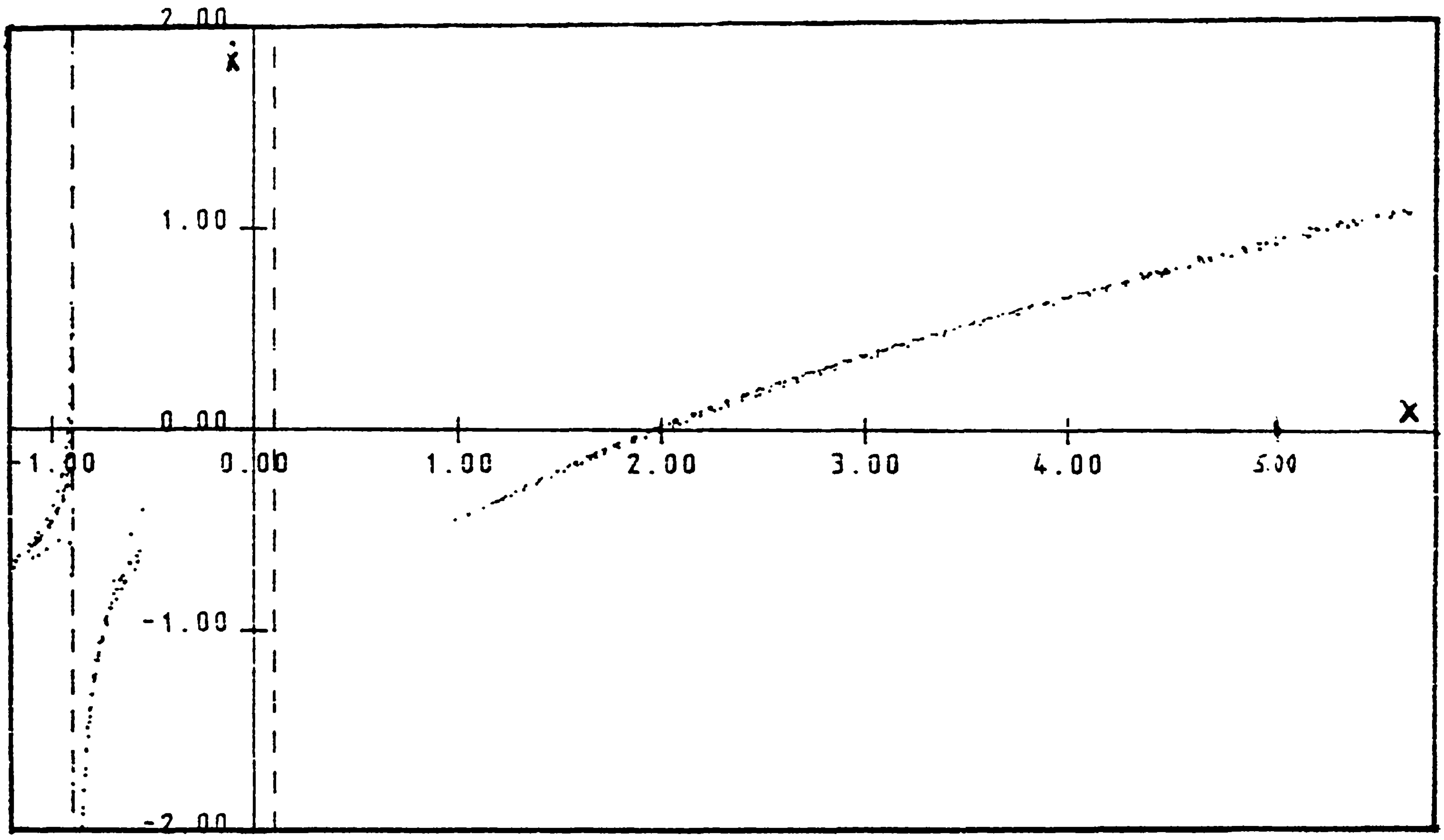

Fig. 5. $W^u(L_4)$ at the first cut for $\mu = 0.1$.

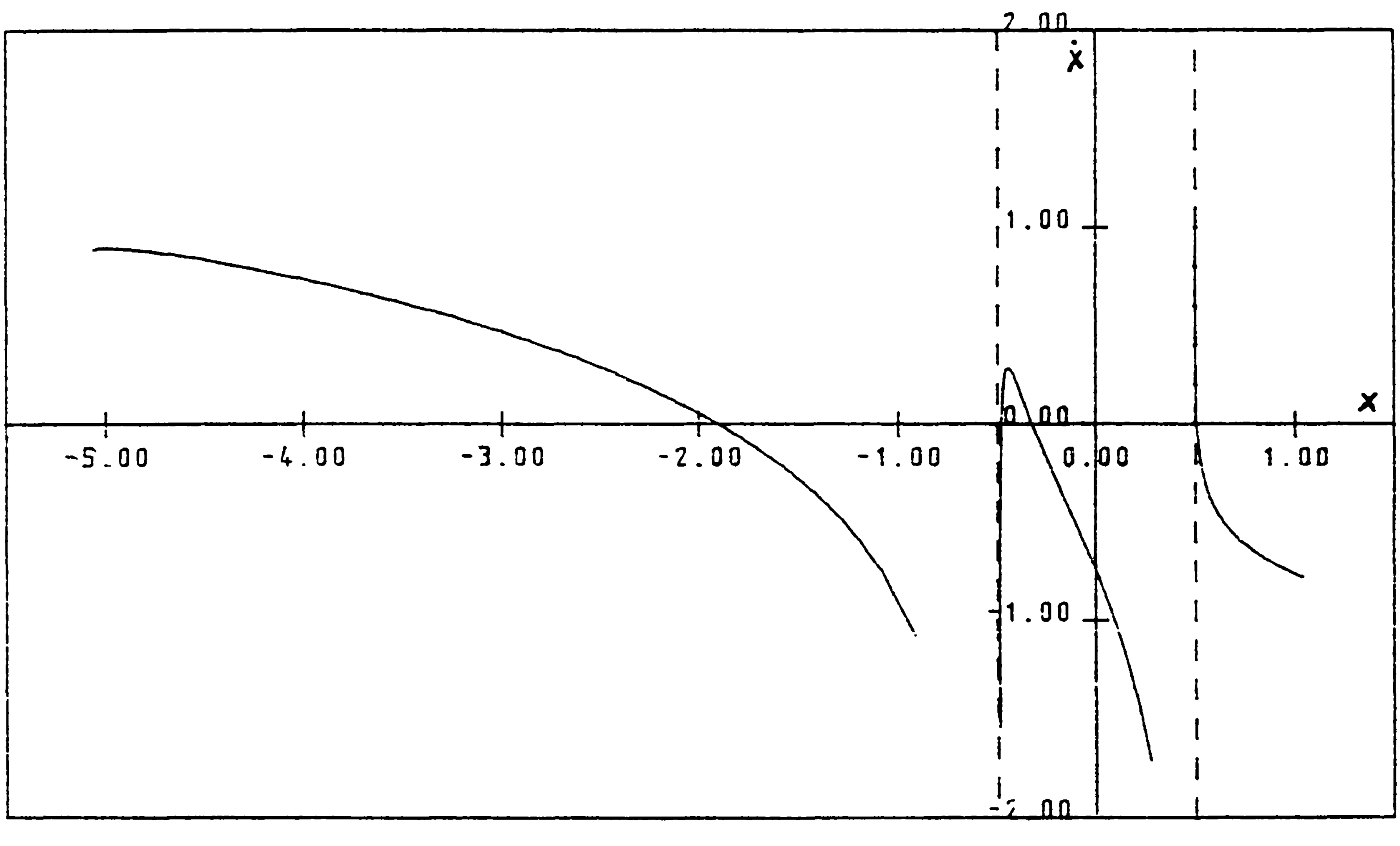

Fig. 6. $W^s(L_4)$ at the first cut for $\mu = 0.5$

regard to nearby initial conditions, as can be seen in Figure 11. Anyhow, both these facts do not affect our purpose at all.

It must also be said that in the Figures corresponding to the mass parameters ranging from 0.5 to 0.2 the graph of the invariant manifolds has three continuous

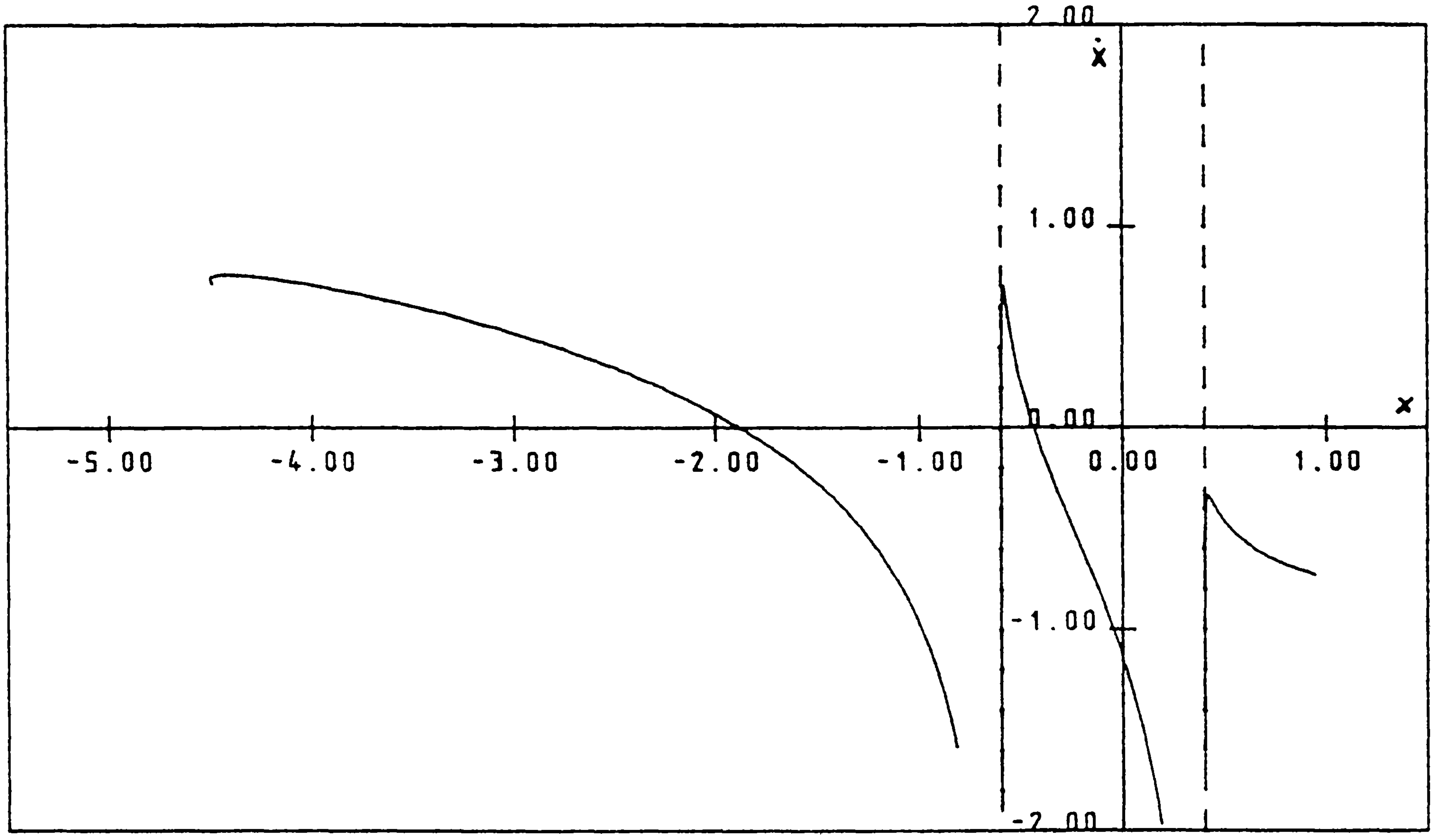

Fig. 7. $W^s(L_4)$ at the first cut for $\mu = 0.4$.

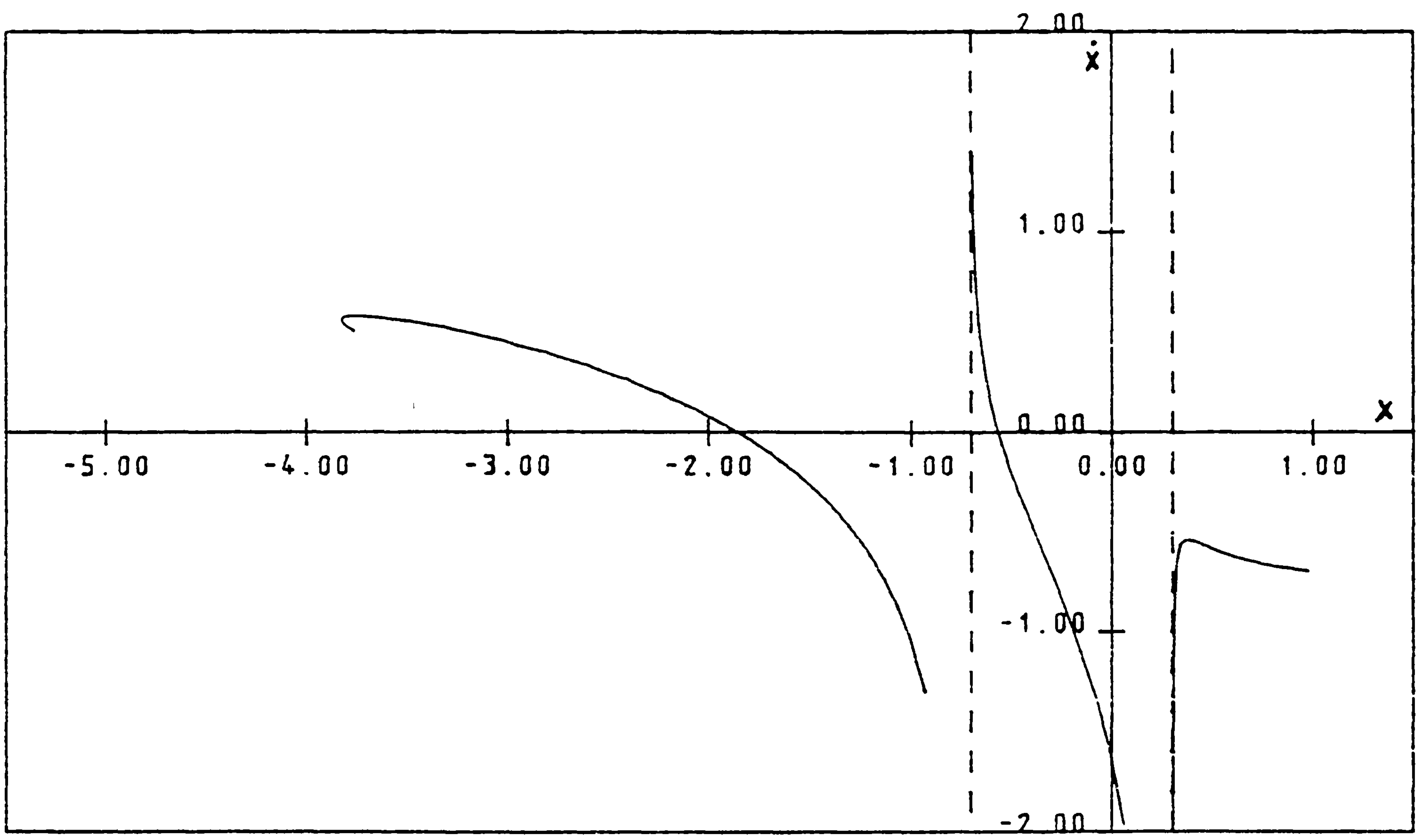

Fig. 8. $W^s(L_4)$ at the first cut for $\mu = 0.3$.

branches. For $\mu = 0.1$ (Figures 5 and 10) the apparent diffusion that can be observed in the representation of the manifolds must be understood as an accumulation of the different branches of them. It is difficult to give a good picture representing these manifolds if we join the computed points by smooth curves, so we have prefered to represent them discontinuously. An explanation of the behaviour of these manifolds is given in the next section.

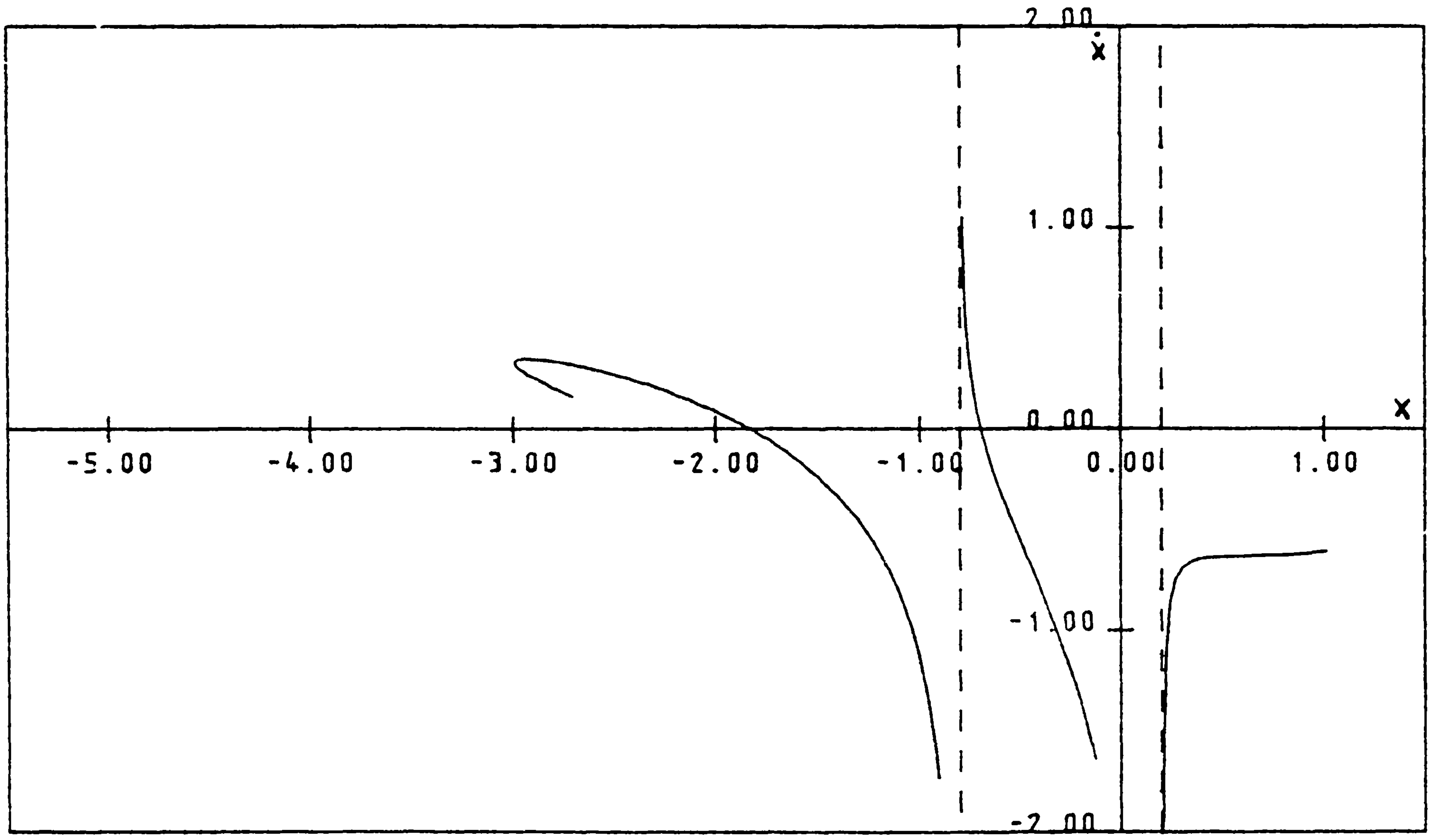

Fig. 9. $W^s(L_4)$ at the first cut for $\mu = 0.2$.

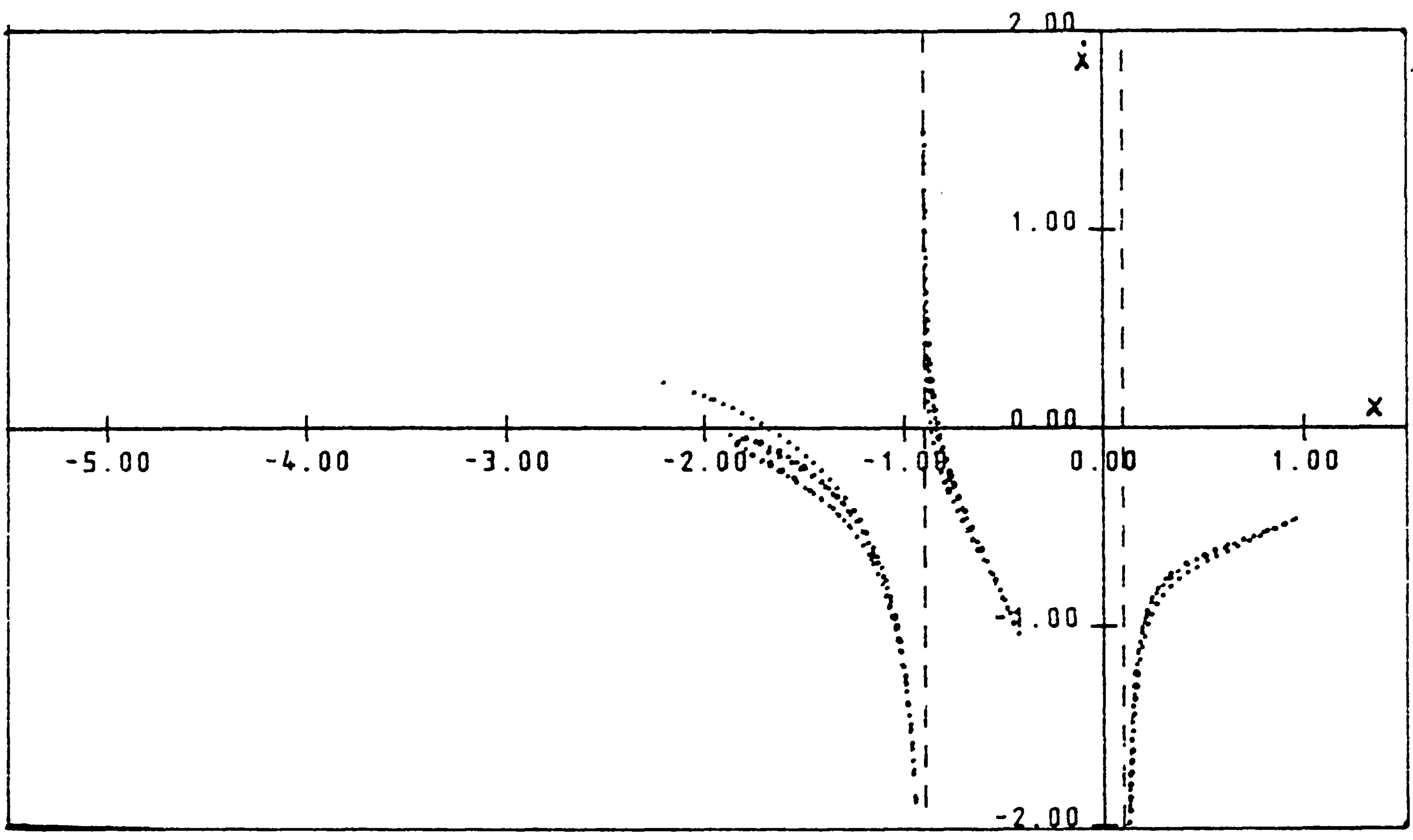

Fig. 10. $W^s(L_4)$ at the first cut for $\mu = 0.1$.

4. Homoclinic and Heteroclinic Orbits

Let p, q be two hyperbolic epuilibrium points of a vector field X as before. If $x \in W^s(p) \cap W^u(p)$ then the orbit that passes through x is called homoclinic. If $x \in W^u(p) \cap W^s(q)$ then it is called heteroclinic from p to q. A homoclinic orbit, γ, is said

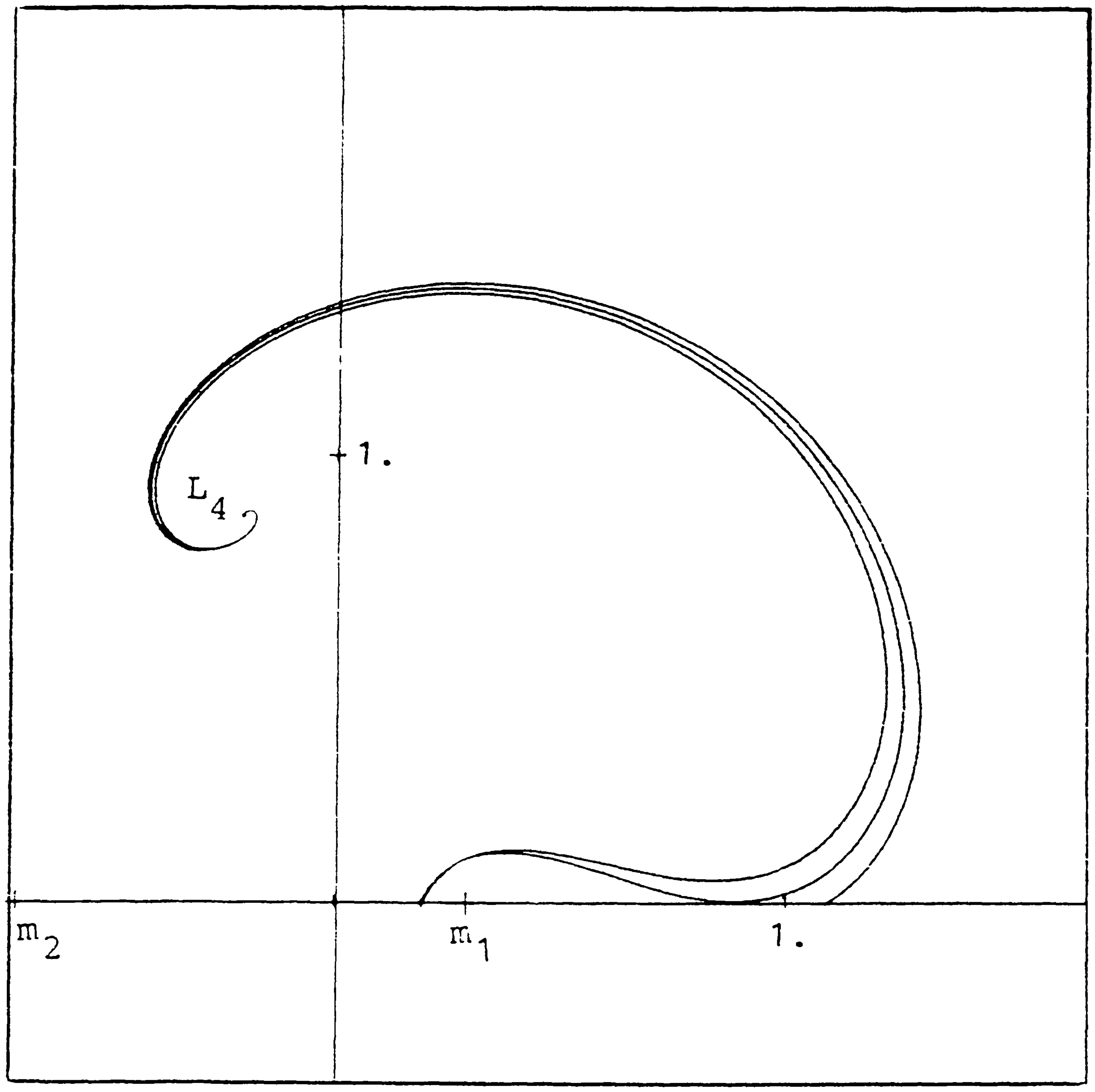

Fig. 11. Behaviour of the orbits near a tangent one which produces a discontinuity on the representation, in the $(x, \dot{x})$ plane, of the invariant manifolds.

to be non-degenerate if

$$\dim(TW^s(p) \cap TW^u(p)) = 1$$

where $TW^s(p)$ and $TW^u(p)$ are the tangent spaces to $W^s(p)$ and $W^u(p)$ on the points of γ. An analogous definition can be used for a non-degenerate heteroclinic orbit, (see [3, 2] for their properties).

We have classified the homoclinic and heteroclinic orbits attending to their number of intersections with the x axis. In this way, given $x \in W^u(L_i) \cap W^s(L_i)$, $i \in \{4, 5\}$, we say that the orbit γ that passes through x is a $2k$-homoclinic one, if its (x, y)-projection has $2k$-crosses with the x axis, $k = 0, 1, \ldots$. If the orbit has some tangency then the multiplicity is taken into account. Analogously we can define $(2k + 1)$-heteroclinic orbits if we look to $W^u(L_i) \cap W^s(L_j)$, $i \neq j$.

If X is R-reversible and p is an equilibrium point of X, then p is said to be symmetric if $p \in \mathrm{Fix}(R)$. An orbit γ of X is symmetric if $R(\gamma) = \gamma$.

In the restricted problem due to the symmetry mentioned in Section 2, we have that $W^s(L_5) = R_1(W^u(L_4))$ and $W^u(L_5) = R_1(W^s(L_4))$, so if we want to compute all the 1-heteroclinic orbits from L_4 to L_5 (resp. from L_5 to L_4) we have to compute all the intersections between the graph corresponding to $W^u(L_4)$ (resp. $W^s(L_4)$) and its symmetry with respect to the x axis ($\dot{x} = 0$). So a point $(x, \dot{x})$ on the graph of the manifold corresponds to a 1-heteroclinic orbit if the point $(x, -\dot{x})$ also belongs to the graph.

The next proposition summarizes the numerical results obtained for the 1-heteroclinic orbits.

PROPOSITION 1. *For the mass ratios studied the number of* 1-*heteoclinic orbits is given in the following table.*

μ	$W^u(L_5) \cap W^s(L_4)$	$W^s(L_5) \cap W^u(L_4)$
0.5	4	4
0.4	3	4
0.3	2	2
0.2	2	2
0.1	?	?

Remark 1. For the value of $\mu = 0.1$ infinitely many 1-heteroclinic orbits apparently appear. An explanation of the number of these orbits, for values of μ less than 0.2, is given later.

Remark 2. The orbits found for the case $\mu = 0.5$ were well known to Strömgren [4].

Remark 3. For the values of the mass parameters ranging from 0.5 to 0.2 the intersection of the manifolds happens when $y = \dot{x} = 0$ so all the 1-heteroclinic orbits are symmetric. For other values of μ (i.e. $\mu = 0.14$) some non-symmetric 1-heteroclinic $L_5 \to L_4$ orbits appear.

In Figures 12–19 the heteroclinic orbits computed have been represented for the first four mass parameters. In Figures 20–23 some typical cases of heteroclinic orbits corresponding to $W^u(L_4)$ when $\mu = 0.1$ have been plotted, as well as a 0-homoclinic orbit.

The numerical study for $\mu = 0.5$ of the unstable manifold around L_4 at the second cut (Figure 24) presents two parts that are well differentiated. A 'proper' one which has been represented by continuous curves, and a second one which is an accumulation (in the sense previously mentioned at the end of Section 3) around the unstable manifold of L_5 at the first cut (symmetrical to Figure 6). In this way, for the 2-homoclinic orbits $L_4 \to L_5$ obtained by the intersection $W^u_2(L_4) \cap W^s_1(L_5)$, leaving aside some isolated proper ones, there appears an infinite number of them which 'remember' essentially the shape of a pair of 1-heteroclinic orbits. This fact gives us the next Proposition.

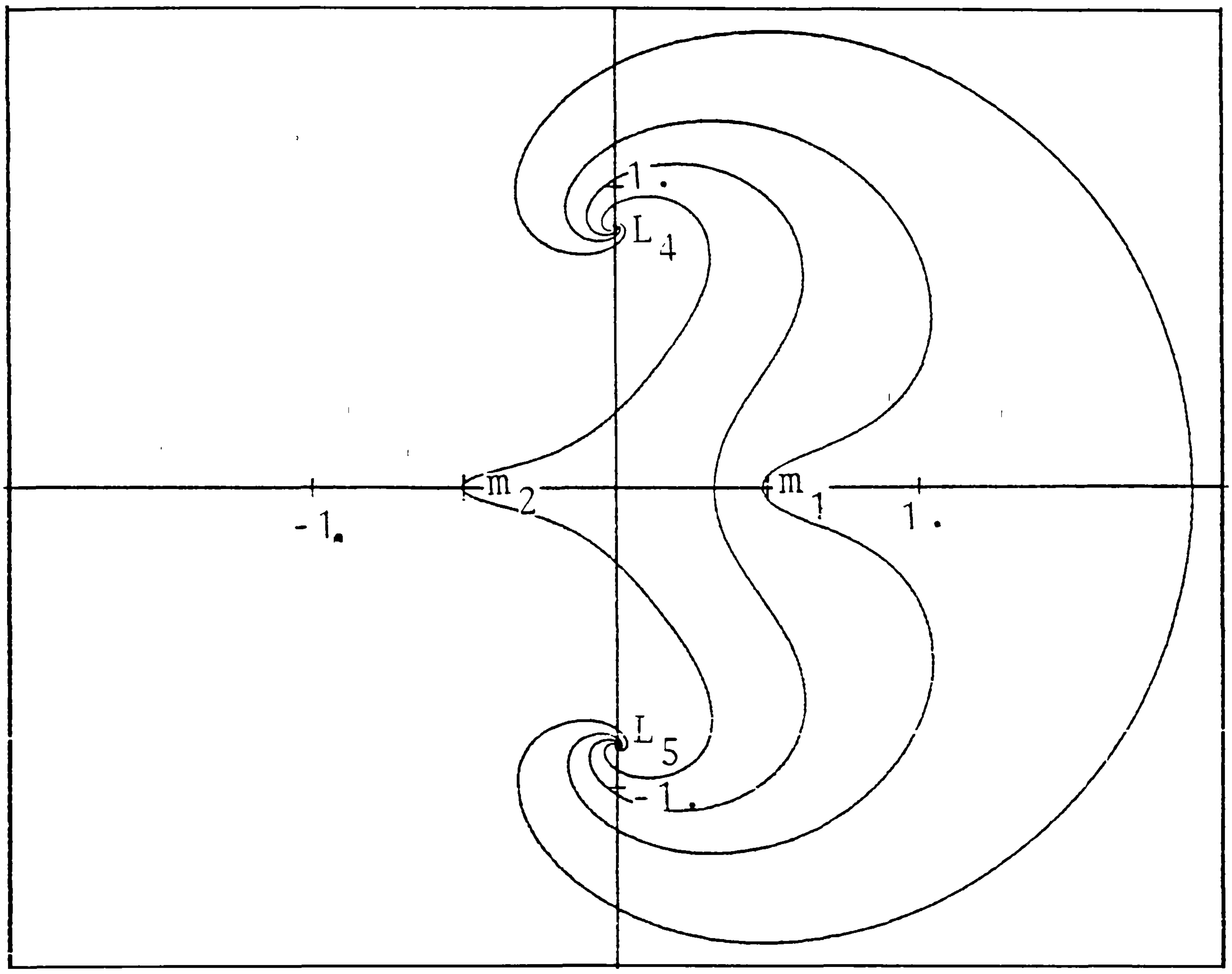

Fig. 12. 1-heteroclinic orbits $L_4 \to L_5$ for $\mu = 0.5$.

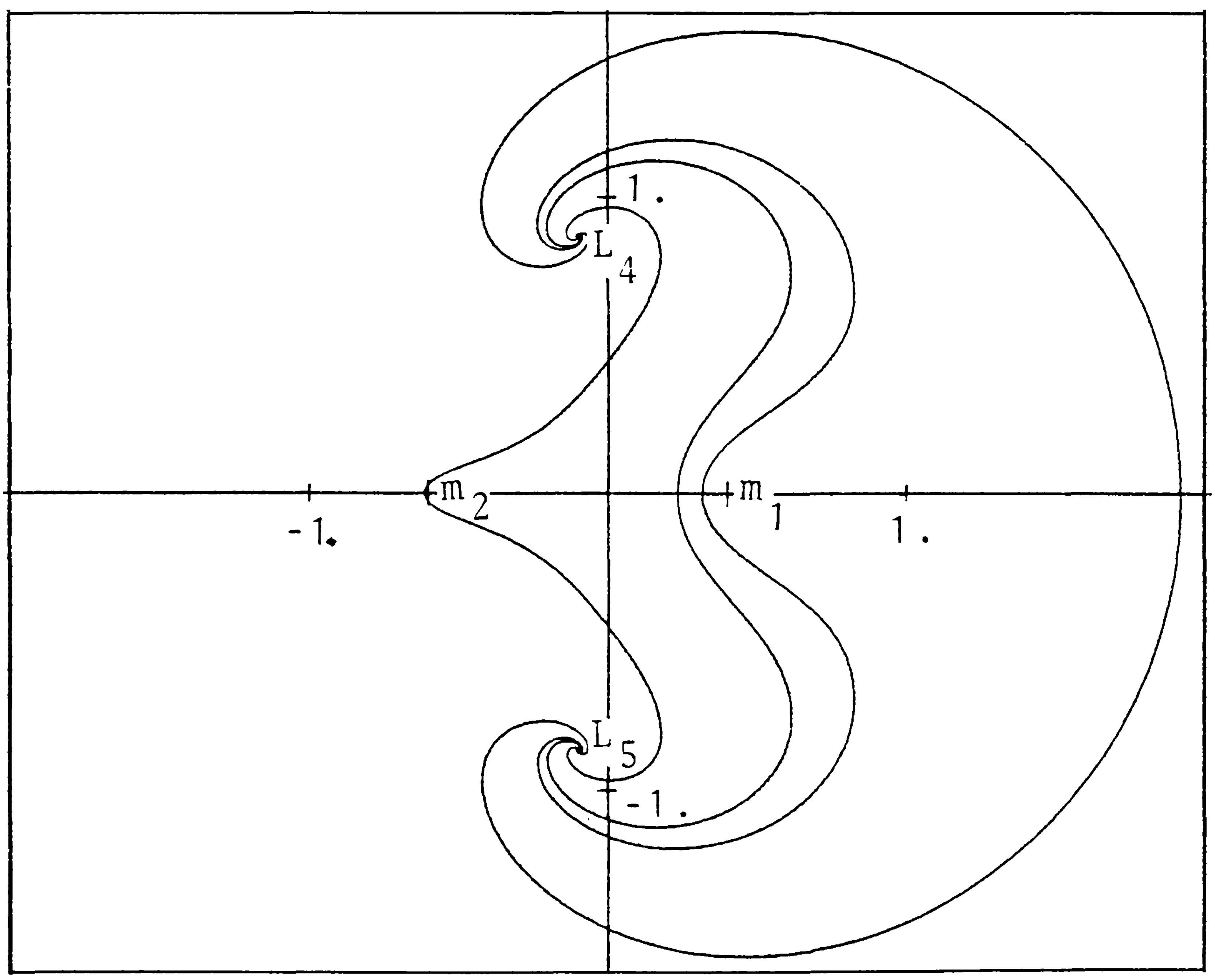

Fig. 13. 1-heteroclinic orbits $L_4 \to L_5$ for $\mu = 0.4$.

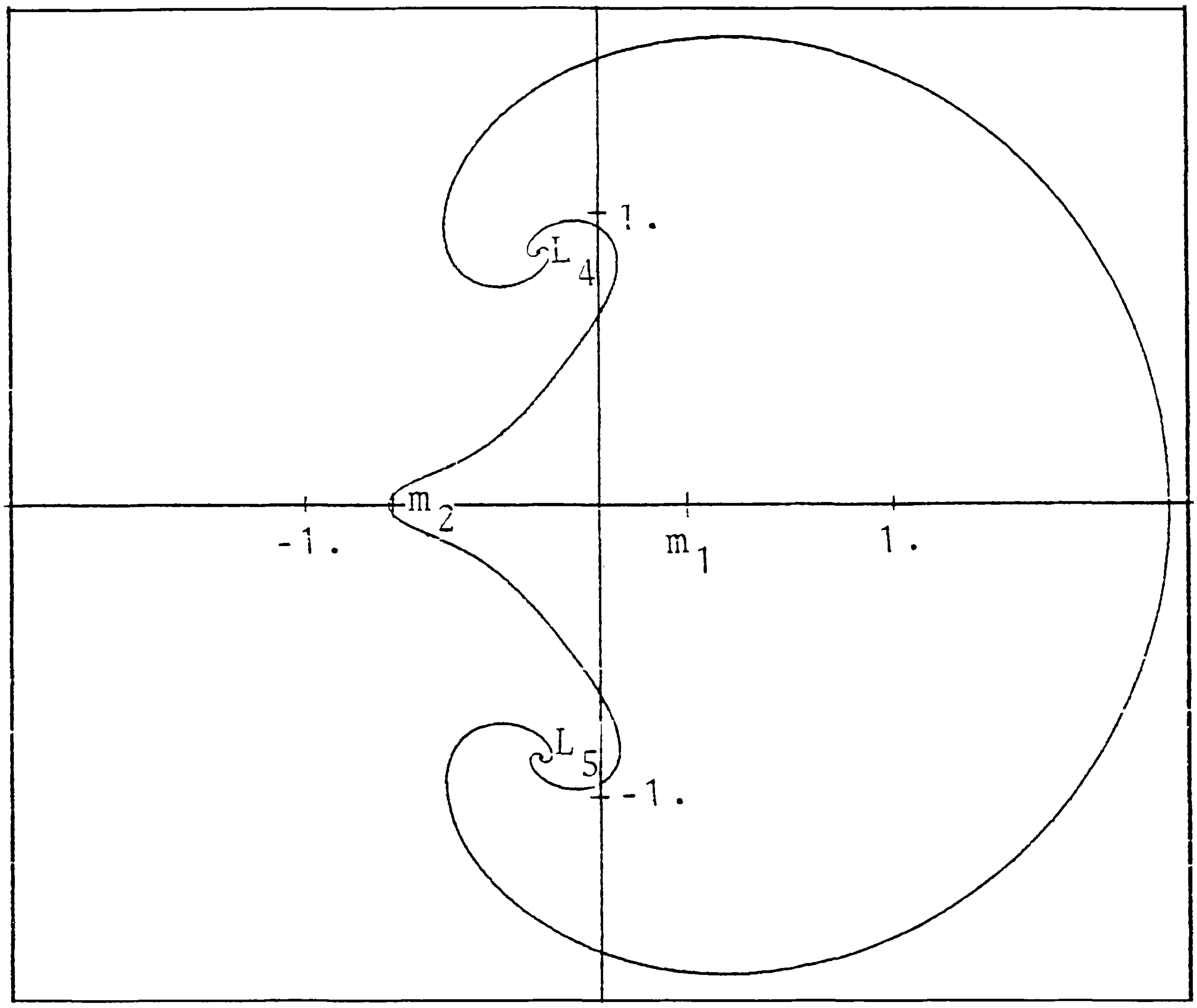

Fig. 14. 1-heteroclinic orbits $L_4 \rightarrow L_5$ for $\mu = 0.3$.

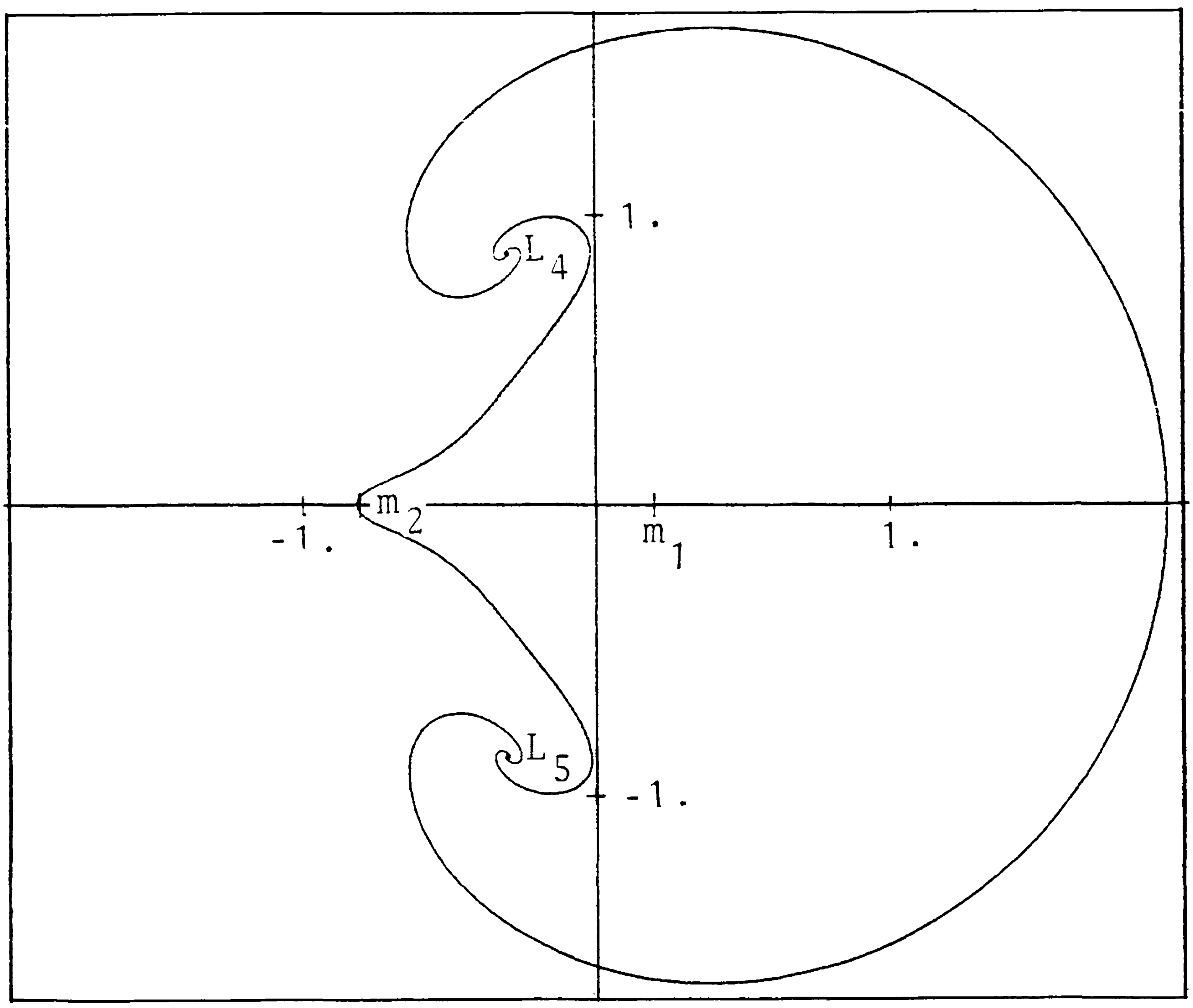

Fig. 15. 1-heteroclinic orbits $L_4 \rightarrow L_5$ for $\mu = 0.2$.

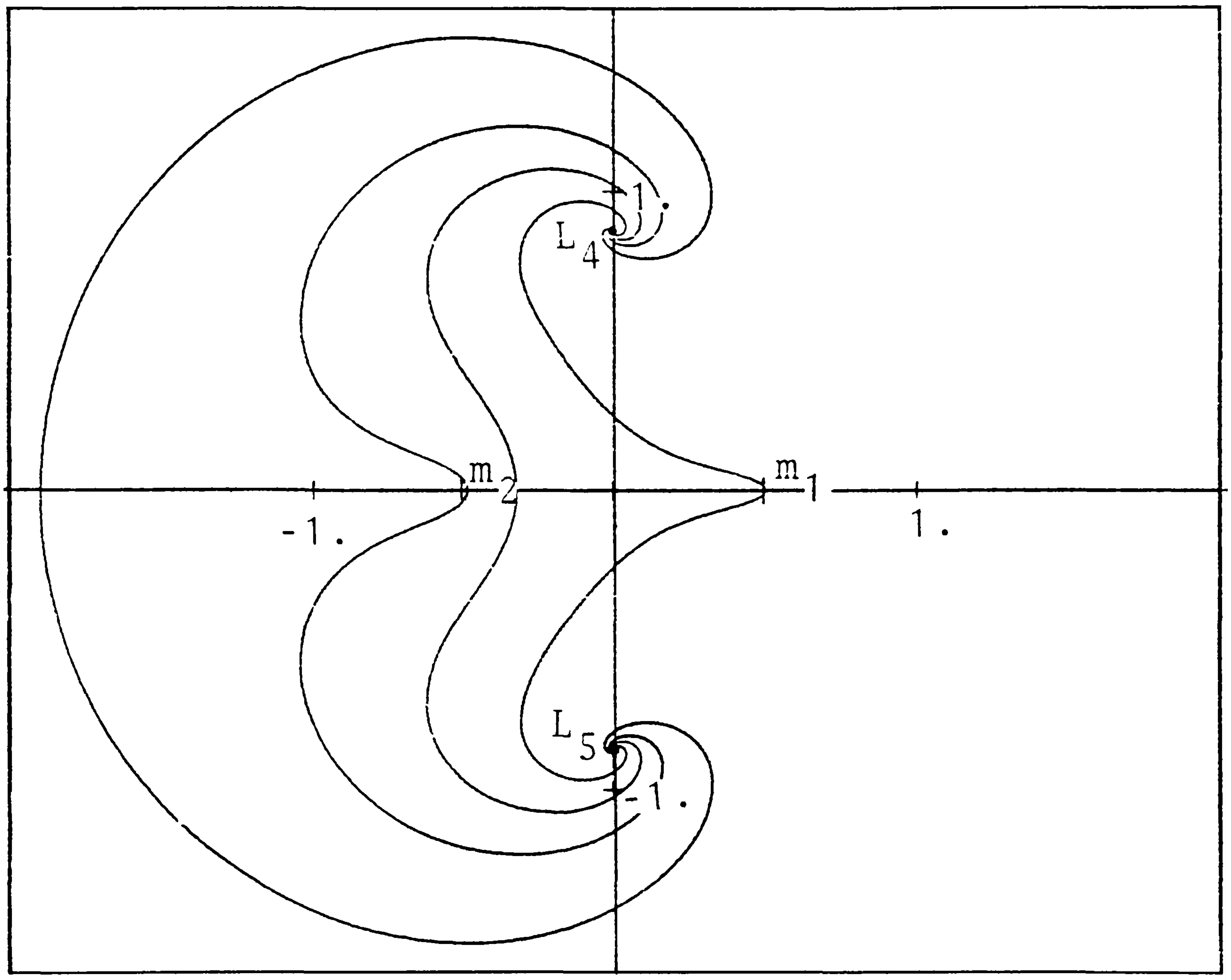

Fig. 16. 1-heteroclinic orbits $L_5 \rightarrow L_4$ for $\mu = 0.5$.

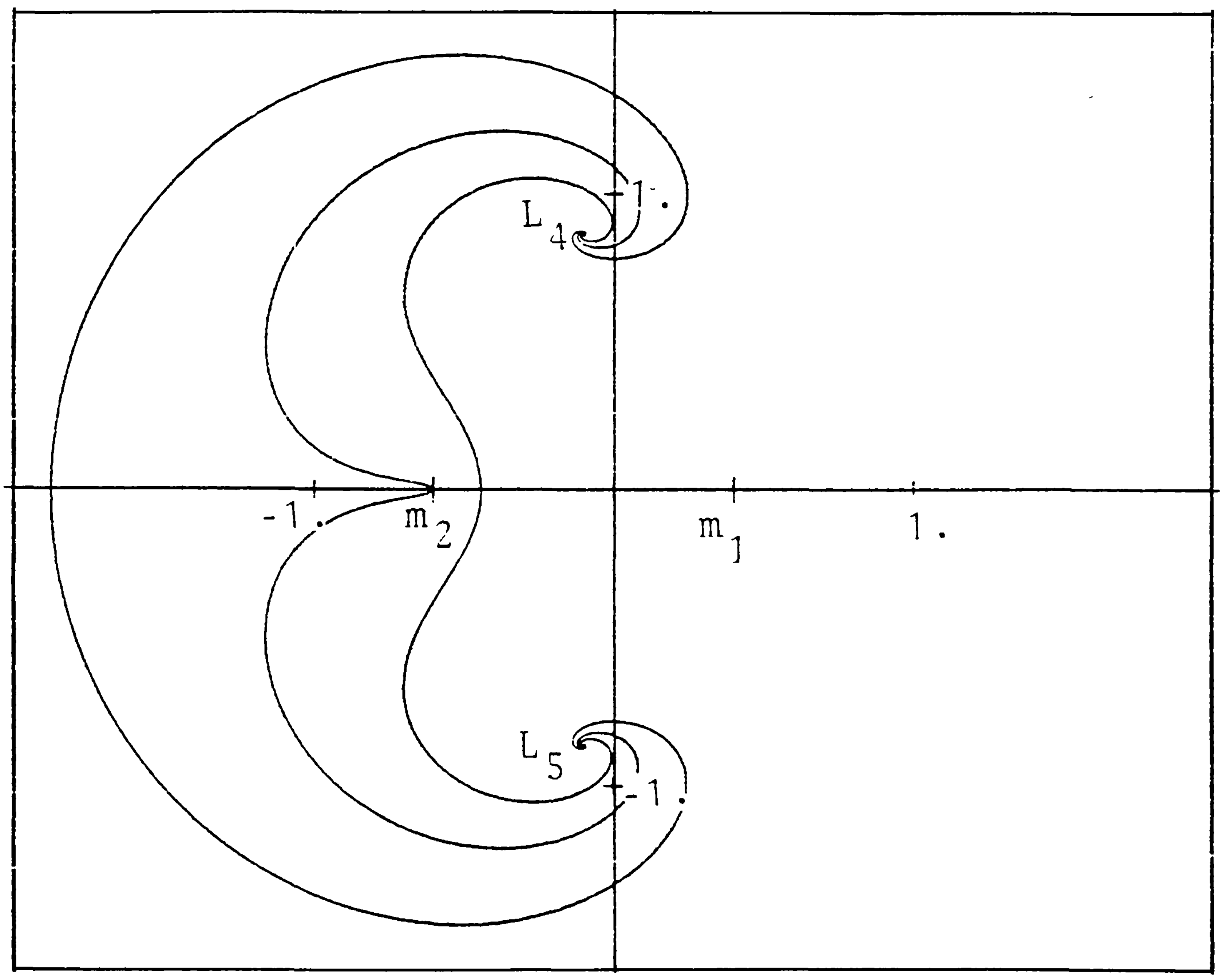

Fig. 17. 1-heteroclinic orbits $L_5 \rightarrow L_4$ for $\mu = 0.4$.

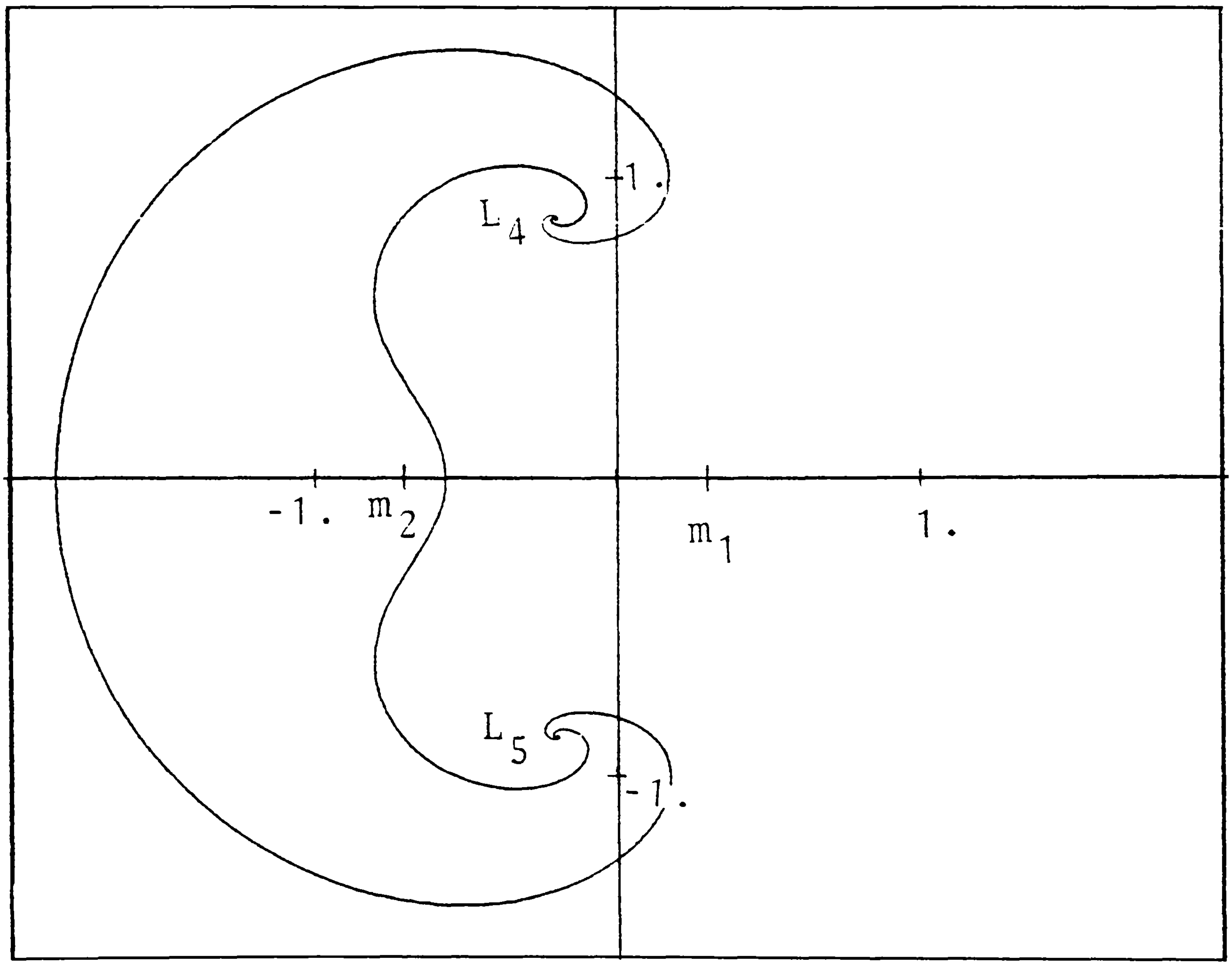

Fig. 18. 1-heteroclinic orbits $L_5 \to L_4$ for $\mu = 0.3$.

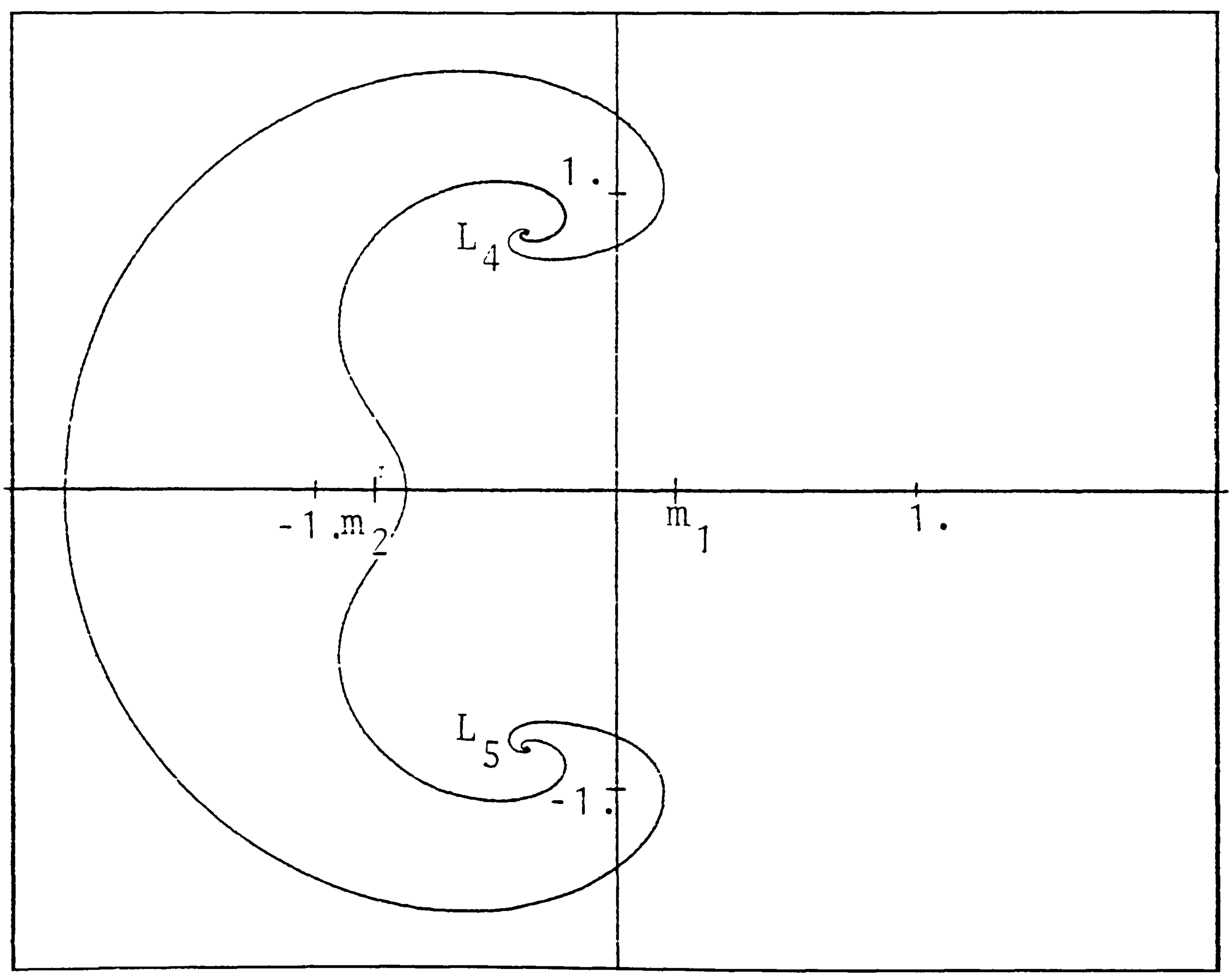

Fig. 19. 1-heteroclinic orbits $L_5 \to L_4$ for $\mu = 0.2$.

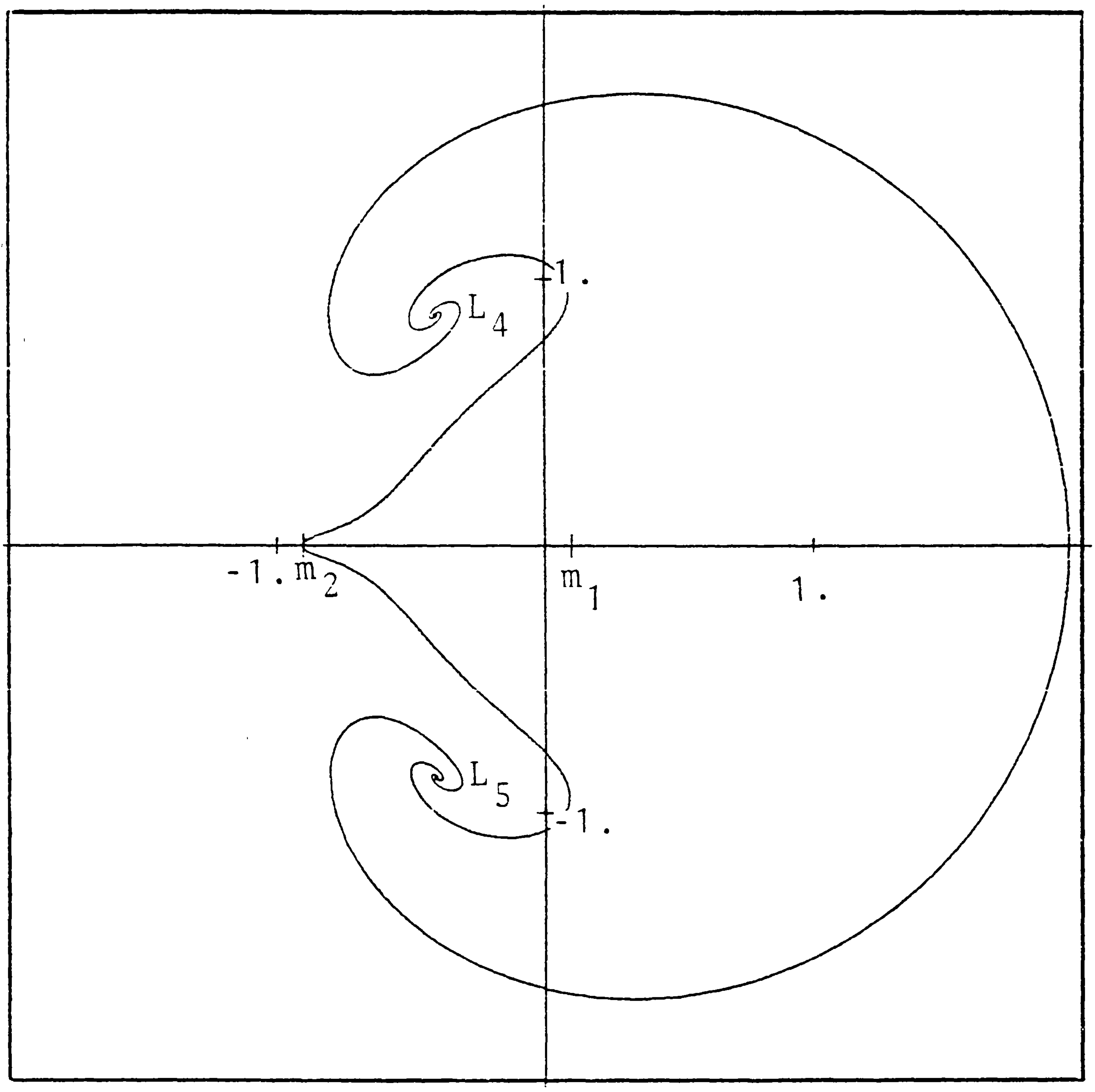

Fig. 20. The simplest 1-heteroclinic orbits $L_4 \to L_5$ for $\mu = 0.1$

PROPOSITION 2. *For a fixed value of μ, the existence of a pair of non-degenerate 1-heteroclinic orbits, one from L_5 to L_4 and the other from L_4 to L_5, implies the existence of an infinite number of 2k-homoclinic (L_4 to L_4 and L_5 to L_5) and (2k + 1)-heteroclinic (L_4 to L_5 and L_5 to L_4) orbits for $k = 1, 2, 3, \ldots$.*

Proof. Let C_1 and C_2 be two small circles of radius r around the equilibrium points, L_4, L_5 respectively, parametrized by the angles θ^1 and θ^2 respectively.

Let γ be some 1-heteroclinic orbit from L_4 to L_5 and η some 1-heteroclinic orbit from L_5 to L_4. Assume that the first intersection between γ and C_1 takes place for a value of the parameter equal to θ^1_γ. Consider $(\theta^1_\gamma - \varepsilon, \theta^1_\gamma + \varepsilon)$, we know that $\dim W^u(L_i) = \dim W^s(L_i) = 2, i = 4, 5$. By virtue of the continuity with respect to the initial conditions, and due to the fact that γ spirals infinitely around L_5, we can take an orbit, γ_n, on $W^u(L_4)$ and close to γ in such a way that, inside C_2 and before leaving it, it turns clockwise around L_5 exactly n times. If we denote $\theta^1_{\gamma_n}$ the value of the parameter

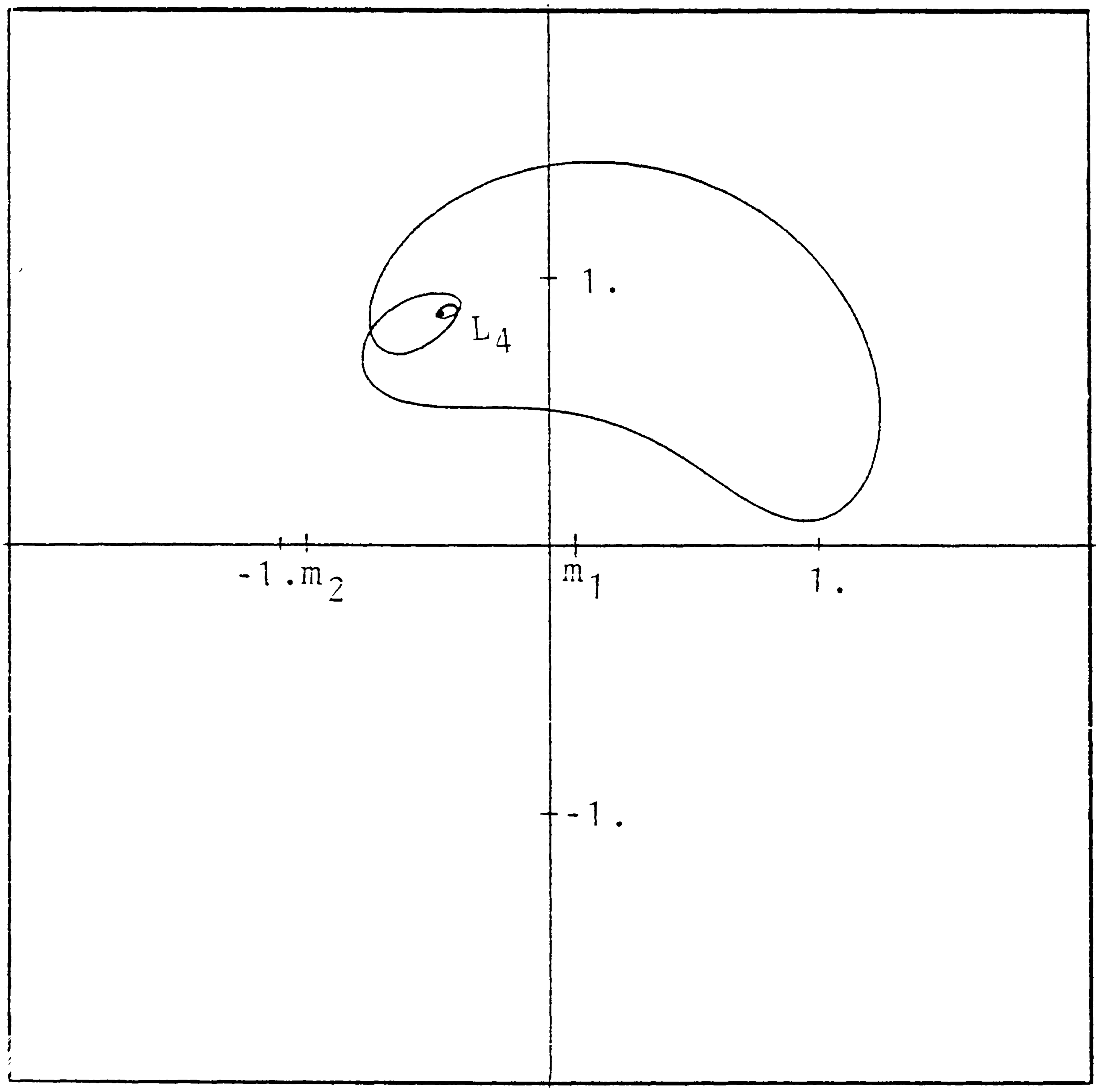

Fig. 21. 0-homoclinic orbit $L_4 \to L_4$ for $\mu = 0.1$.

corresponding to the first intersection of γ_n with C_1, we can assume that for an adequate ε we have

$$-\varepsilon + \theta_\gamma^1 < \theta_{\gamma_n}^1 < \theta_\gamma^1$$

In the same way we can say that there exists an orbit γ_{n+1} on $W^u(L_4)$ that inside C_2 turns exactly $n+1$ times clockwise around L_5 and with $\theta_{\gamma_n}^1 < \theta_{\gamma_{n+1}}^1 < \theta_\gamma^1$.

If n is large enough, say $n \geqslant n_0$, the orbits of $W^u(L_4)$ that leave L_4 between $\theta_{\gamma_n}^1$ and $\theta_{\gamma_{n+1}}^1$, after going outside C_2, fall in a neighbourhood of $W^u(L_5)$ and so they 'reproduce' $W^u(L_5)$ in the sense that these orbits form a two-dimensional manifold that, when going outside C_2, can be considered as a small perturbation of $W^u(L_5)$, say $W^{u,n}(L_5)$ (see Figure 25).

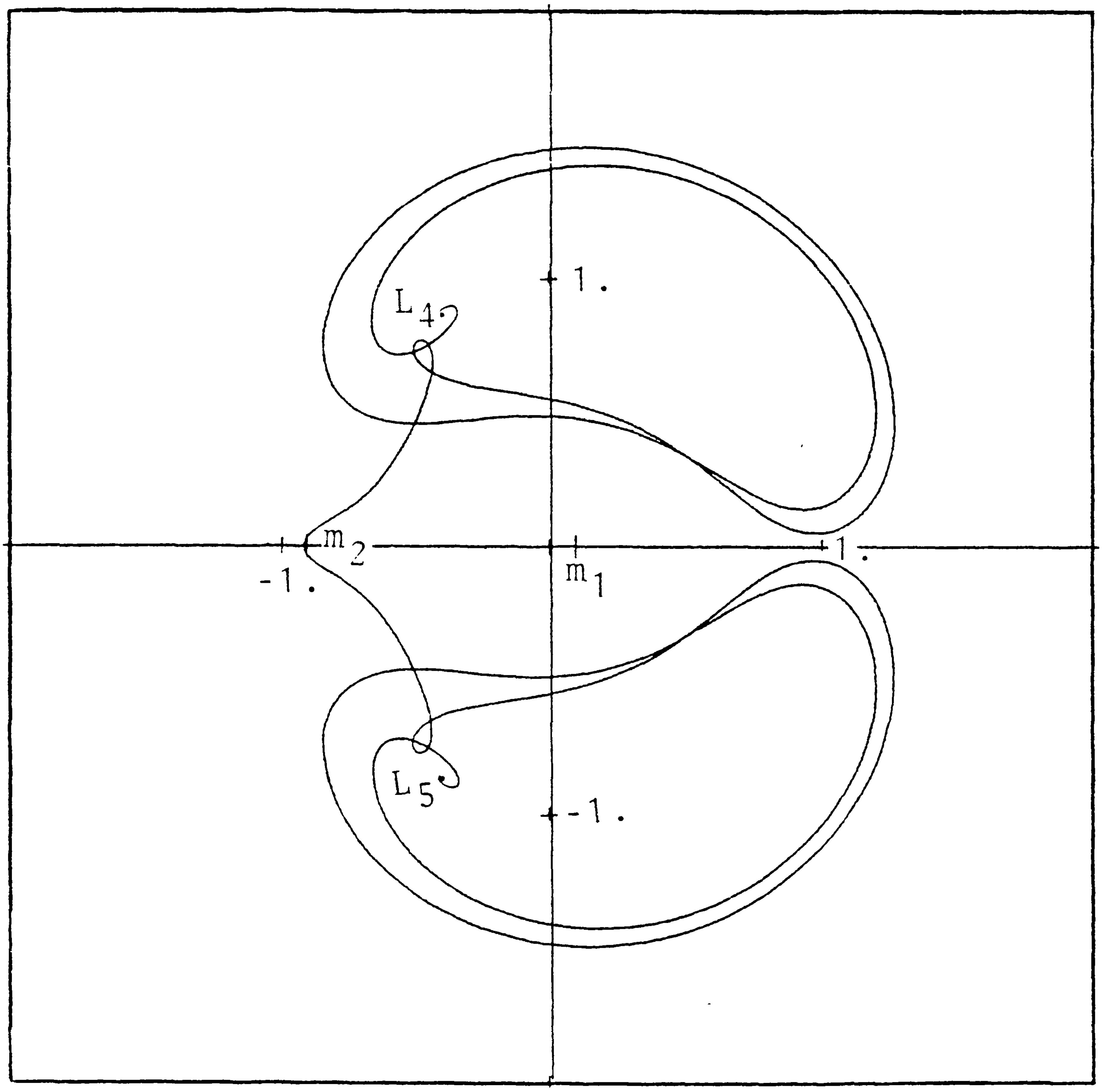

Fig. 22. Example of a 1-heteroclinic orbit $L_4 \to L_5$ for $\mu = 0.1$ involving the shape of the 0-homoclinic orbit.

Since η is non-degenerate $W^u(L_5) \cap W^s(L_4)$ is transversal, and so will $W^{u,n}(L_5) \cap W^s(L_4)$ be for $n > n_0$, since as $n \to \infty$ then $W^{u,n}(L_5)$ approaches $W^u(L_5)$.

In this way we have a set of 2-homoclinic non-degenerate orbits, γ_n^2, from $L_4 \to (L_5) \to L_4$ for $n > n_0$.

Taking any of these 2-homoclinic orbits, γ_k^2, we can repeat the arguments considering the value of the parameter $\theta^2_{\gamma_k^2}$ at the first intersection of γ_k^2 with C_2, when γ_k^2 spirals clockwise around L_5. Between $(\theta^2_{\gamma_k^2} - \varepsilon,\ \theta^2_{\gamma_k^2} + \varepsilon)$ we can find an infinite number of 3-heteroclinic orbits $L_4 \to (L_5 \to L_4) \to L_5$ in the same way.

Starting with a 1-heteroclinic orbit $L_5 \to L_4$ we should get the families of $2k$-homoclinic $(L_5 \to L_5)$ and $2k+1$-heteroclinic $(L_5 \to L_4)$ orbits.

Using the same kind of arguments we can prove

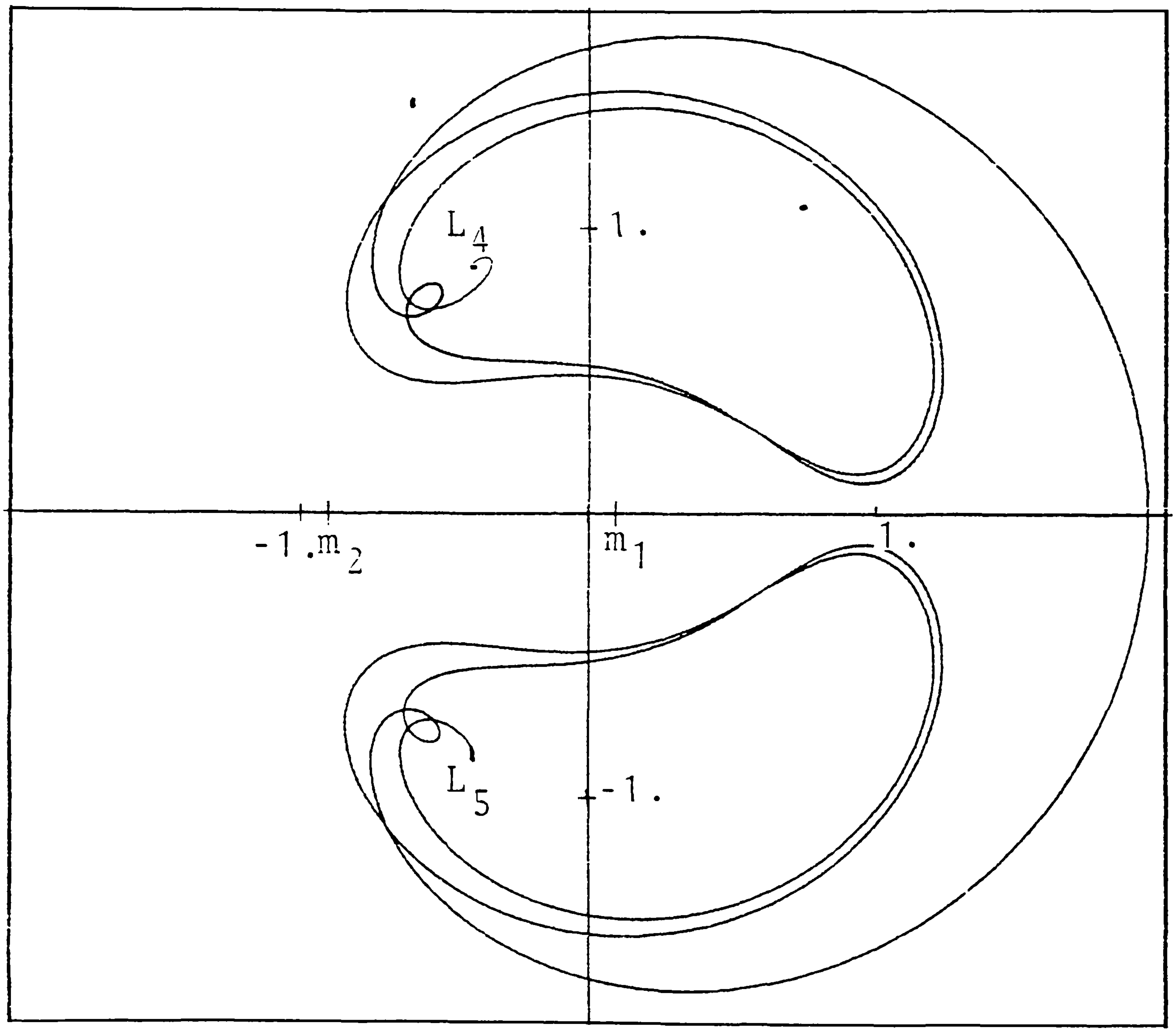

Fig. 23. Example of a 1-heteroclinic orbit $L_4 \to L_5$ for $\mu = 0.1$ involving the shape of the 0-homoclinic orbit.

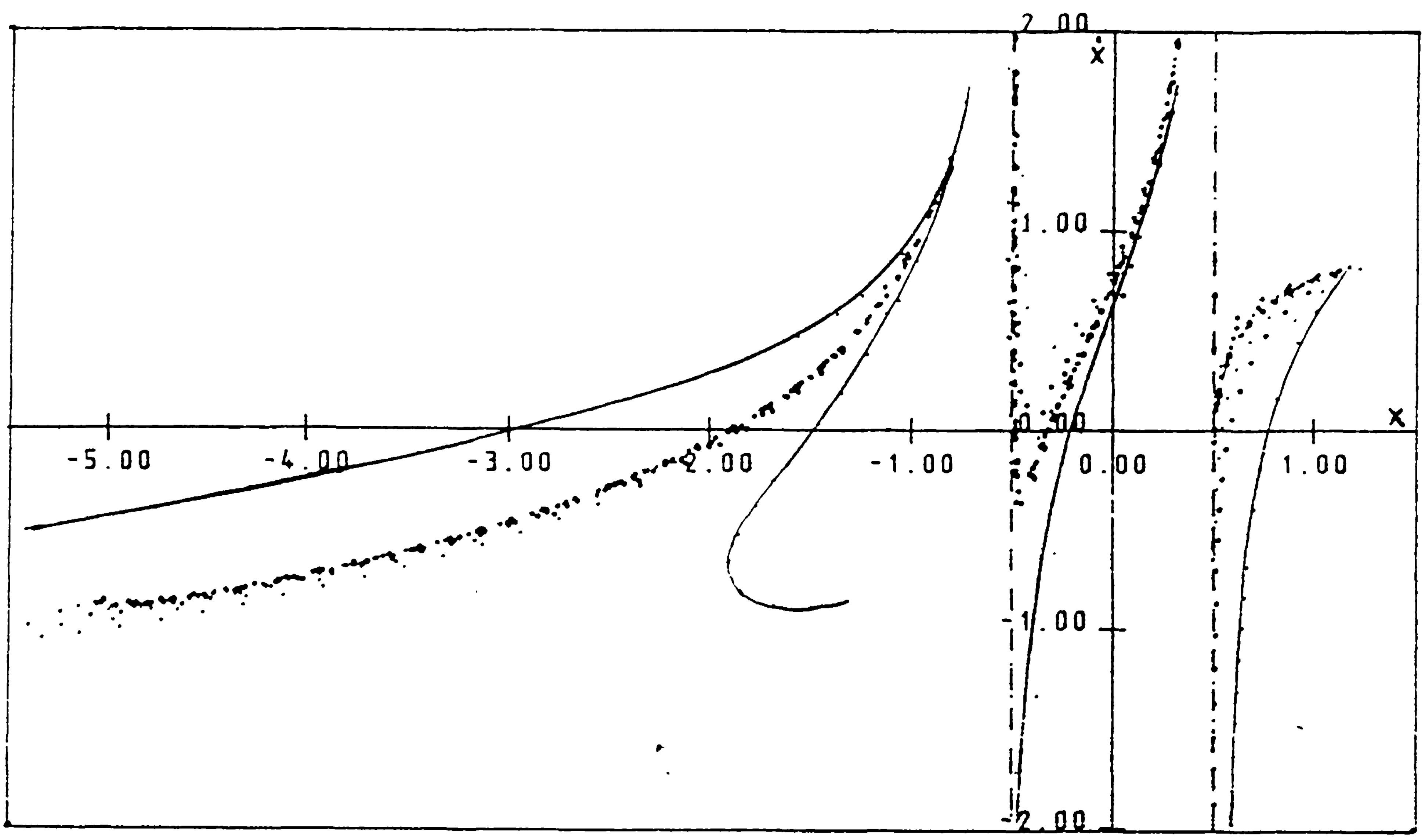

Fig. 24. $W^u(L_4)$ at the second cut for $\mu = 0.5$.

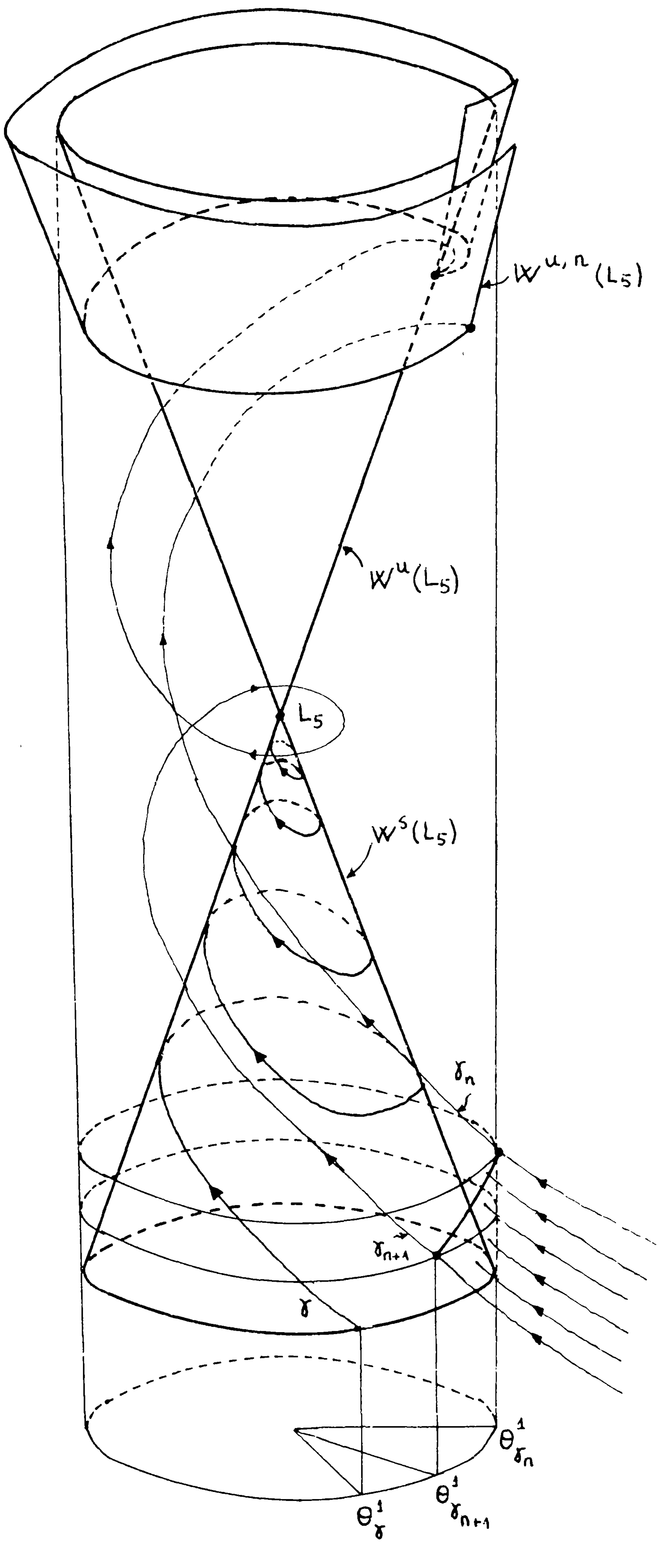

Fig. 25. Process of generation of $W^{u,n}(L_5)$.

PROPOSITION 3. *Under the same hypothesis as Proposition* 2, *the existence of a* 0-*homoclinic non-degenerate orbit* ($L_4 \to L_4$ *or* $L_5 \to L_5$) *implies the existence of an infinite number of* 0-*homoclinic orbits* $L_4 \to L_4$ *or* $L_5 \to L_5$ *and an infinite number of* 1-*heteroclinic orbits* ($L_4 \to L_5$ *or* $L_5 \to L_4$ *respectively*).

Remark 1. This explains the situation observed numerically for $\mu = 0.1$.

Remark 2. It has been computed numerically that the first non-degenerate 0-homoclinic orbit appears for a value of μ in the interval (0.1108, 0.1109). For values of the mass parameter less than this critical one and greater than the Routh's value ($\mu = 0.03852\ldots$), there are, according to the last proposition, an infinite number of 0-homoclinic orbits and 1-heteroclinic orbits.

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