Transformation Groups, Vol. 5, No. 4, 2000, pp. 307–324

GELFAND PAIRS ATTACHED TO REPRESENTATIONS OF COMPACT LIE GROUPS

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Abstract. For each compact Lie algebra $\mathfrak{g}$ and each real representation V of $\mathfrak{g}$ we construct a two-step nilpotent Lie group $N(\mathfrak{g}, V)$, endowed with a natural left-invariant riemannian metric. The main goal of this paper is to show that this construction produces many new Gelfand pairs associated with nilpotent Lie groups. Indeed, we will give a full classification of the manifolds $N(\mathfrak{g}, V)$ which are commutative spaces, using a characterization in terms of multiplicity-free actions.

Introduction

Let N be a simply connected nilpotent Lie group and let K be a compact group of automorphisms of N. We say that (K, N) is a *Gelfand pair* when the convolution algebra $\mathrm{L}^1_K(N)$ of K-invariant integrable functions on N is commutative. If $H = K \ltimes N$, then $\mathrm{L}^1(H/\!/K)$, the convolution algebra of K-bi-invariant integrable functions on H, is isomorphic to $\mathrm{L}^1_K(N)$ (see [L1], for instance). Thus (K, N) is a Gelfand pair precisely when (H, K) is a Gelfand pair in the usual sense (see [GV], p. 36). Gelfand pairs associated with nilpotent Lie groups have been studied in [BJR1, Ki, BJLR, BJR2], for instance, and they are related to the representation theory of groups K, N and H.

A *commutative space* is a connected riemannian homogeneous space M whose algebra of $\mathrm{I}(M)^0$-invariant differential operators is commutative, where $\mathrm{I}(M)^0$ denotes the connected component of the full isometry group $\mathrm{I}(M)$. Commutative spaces have been studied in several articles; see for instance [BTV, KoP, KoPV, KoV, KaR, R, AV].

If $\langle\ ,\ \rangle$ is an inner product on the Lie algebra $\mathfrak{n}$ of N, we endow N with the riemannian left-invariant metric determined by $\langle\ ,\ \rangle$. The isometry group of the resulting riemannian manifold $(N, \langle\ ,\ \rangle)$ is given by $\mathrm{I}(N, \langle\ ,\ \rangle) = K \ltimes N$, where $K = \mathrm{Aut}(\mathfrak{n}) \cap \mathrm{O}(\mathfrak{n}, \langle\ ,\ \rangle)$ is the isotropy subgroup and N acts on itself by left-translations (see [Wi]). In this case, we have the following nice relationship between the commutativity of invariant differential operators and the commutativity of invariant integrable functions (see [H], p. 485):

(K, N) is a Gelfand pair if and only if $(N, \langle\ ,\ \rangle)$ is a commutative space.

In this work, we study Gelfand pairs associated with the following nilpotent Lie groups. Starting from a faithful real representation (π, V) of a compact Lie algebra

*Supported by a fellowship from CONICET and research grants from CONICOR and SeCyT UNC (Argentina).

Received March 18, 1999. Accepted May 19, 1999.

$\mathfrak{g}$, we construct a two-step nilpotent Lie algebra $\mathfrak{n} = \mathfrak{g} \oplus V$ with center $\mathfrak{g}$ and the Lie bracket defined on V by $\langle [v,w], x\rangle = \langle \pi(x)v, w\rangle$ for all $v, w \in V$, $x \in \mathfrak{g}$, where $\langle \ , \ \rangle$ is a fixed $\mathfrak{g}$-invariant inner product on $\mathfrak{n}$. We denote by $N(\mathfrak{g}, V)$ the simply connected Lie group with Lie algebra $\mathfrak{n} = \mathfrak{g} \oplus V$ and we endow $N(\mathfrak{g}, V)$ with the left-invariant metric determined by $\langle \ , \ \rangle$. By a result due to C. Gordon [Go1], the spaces $(N(\mathfrak{g}, V), \langle \ , \ \rangle)$ have a neat geometric characterization within the class of homogeneous nilmanifolds: they are precisely the naturally reductive ones (see [L3]). The isotropy subgroup K of the isometry group of $N(\mathfrak{g}, V)$ is given essentially by $K = G \times U$, where G is the simply connected Lie group with Lie algebra $[\mathfrak{g}, \mathfrak{g}]$ and U is the group of orthogonal intertwining operators of V. The group U acts trivially on the center $\mathfrak{g}$ and each $g \in G$ acts on $\mathfrak{n} = \mathfrak{g} \oplus V$ by $(\mathrm{Ad}(g), \pi(g))$, where we also denote by π the corresponding representation of G on V.

Let T be any maximal torus of G and let $\tilde{V}$ denote a T-invariant complement in V of the zero weight space V_0, regarded naturally as a complex vector space. We give in Section 2 the following necessary condition:

If $(G \times U, N(\mathfrak{g}, V))$ is a Gelfand pair, or equivalently, $N(\mathfrak{g}, V)$ is a commutative space, then the action of $T \times U$ on $\tilde{V}$ is multiplicity-free.

In Section 3 and Section 4 we shall obtain a complete classification of the Gelfand pairs of the form $(G \times U, N(\mathfrak{g}, V))$, determining explicitly the multiplicity free actions given above. This produces many new families of Gelfand pairs associated with nilpotent Lie groups (see Theorem 16). Up to now, relatively few examples were known and in such examples N is one of the following: a product of Heisenberg groups and abelian groups, a free two-step nilpotent Lie group, or an H-type group of a special kind (see [R]).

A connected riemannian manifold M is said to be *weakly symmetric* if for any two points $p, q \in M$, there exists an isometry of M mapping p to q and q to p. These spaces, introduced by A. Selberg in [S], have been studied for instance in [BRV, BTV, BV, KoPV, Z, AV]. It is proved in [S] that any weakly symmetric space is a commutative space (with respect to $\mathrm{I}(M)$-invariance; this coincides with $\mathrm{I}(M)^0$-invariance for homogeneous nilmanifolds [BJR2]). In [S] Selberg remarks that he does not know whether weak symmetry is necessary for the commutativity of a space. It has been proved in [AV] that for homogeneous spaces of reductive algebraic groups, the answer is affirmative. On the other hand, certain modified H-type groups provide counterexamples [L1, L2]; however, none of these is naturally reductive. This motivated the study in [L4] of the weak symmetry condition in the class of manifolds $N(\mathfrak{g}, V)$, obtaining that all the commutative spaces found in the present paper are weakly symmetric as well.

Acknowledgements. I wish to express my deep gratitude to Isabel Dotti, for her invaluable guidance. I am also grateful to Roberto Miatello, Jorge Vargas, Nao Nishihara, Chal Benson and Gail Ratcliff for helpful observations.

1. Preliminaries

We consider a simply connected real nilpotent Lie group N endowed with a left-invariant riemannian metric, denoted by $(N, \langle \ , \ \rangle)$, where $\langle \ , \ \rangle$ is the inner product on the Lie algebra $\mathfrak{n}$ of N determined by the metric. The full group of isometries of

$(N, \langle\ ,\ \rangle)$ is given by

$$\mathrm{I}(N, \langle\ ,\ \rangle) = K \ltimes N \qquad \text{(semidirect product)}, \tag{1}$$

where $K = \mathrm{Aut}(\mathfrak{n}) \cap \mathrm{O}(\mathfrak{n}, \langle\ ,\ \rangle)$ is the isotropy subgroup of the identity and N acts itself by left translations (see [Wi]). Thus, the structure of $\mathrm{I}(N, \langle\ ,\ \rangle)$ is completely determined by K. Note that since we always assume that N is simply connected, we make no distinction between automorphisms of N and $\mathfrak{n}$.

In this section, we shall give some properties of the two-step nilpotent Lie groups constructed as follows. All of the results in this section are proved in [L3].

Definition 1. We say that a triple $(\mathfrak{g}, V, \langle\ ,\ \rangle)$ is a *data set* if

(i) $\mathfrak{g}$ is a compact Lie algebra, i.e., $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \oplus \mathfrak{c}$ where $\mathfrak{c}$ is the center of $\mathfrak{g}$ and $[\mathfrak{g}, \mathfrak{g}]$ is a compact semisimple Lie algebra,

(ii) (π, V) is a real faithful representation of $\mathfrak{g}$ without trivial subrepresentations, i.e., $\bigcap_{x\in\mathfrak{g}} \mathrm{Ker}\, \pi(x) = 0$,

(iii) $\langle\ ,\ \rangle$ is an inner product (positive definite) on $\mathfrak{n} = \mathfrak{g} \oplus V$ satisfying that $\langle\ ,\ \rangle_{\mathfrak{g}} := \langle\ ,\ \rangle|_{\mathfrak{g}\times\mathfrak{g}}$ is $\mathrm{ad}\,\mathfrak{g}$-invariant, $\langle\ ,\ \rangle_V := \langle\ ,\ \rangle|_{V\times V}$ is $\pi(\mathfrak{g})$-invariant and $\langle \mathfrak{g}, V\rangle = 0$. Such an inner product will be called $\mathfrak{g}$-*invariant.*

A data set $(\mathfrak{g}, V, \langle\ ,\ \rangle)$ determines a two-step nilpotent Lie group denoted by $N(\mathfrak{g}, V)$ having Lie algebra $\mathfrak{n} = \mathfrak{g} \oplus V$, with Lie bracket defined by

$$\begin{cases} [\mathfrak{g}, \mathfrak{g}]_{\mathfrak{n}} = [\mathfrak{g}, V]_{\mathfrak{n}} = 0, \qquad [V, V]_{\mathfrak{n}} \subset \mathfrak{g}, \\ \langle [v, w]_{\mathfrak{n}}, x\rangle_{\mathfrak{g}} = \langle \pi(x)v, w\rangle_V \qquad \forall\, x \in \mathfrak{g},\ v, w \in V. \end{cases} \tag{2}$$

Finally, we endow $N(\mathfrak{g}, V)$ with the left-invariant metric determined by $\langle\ ,\ \rangle$.

Note that $[\mathfrak{n}, \mathfrak{n}]_{\mathfrak{n}} = \mathfrak{g}$ is the center of $\mathfrak{n}$. The construction of the group $N(\mathfrak{g}, V)$ does not depend on the chosen $\mathfrak{g}$-invariant inner product $\langle\ ,\ \rangle$ (up to a Lie group isomorphism). Moreover, if $\mathfrak{g}$ is a compact Lie algebra, V and V' are representations of $\mathfrak{g}$ as in Definition 1,(ii) and there exist $\phi \in \mathrm{Aut}(\mathfrak{g})$ and $T : V \to V'$ such that $T\pi(x)T^{-1} = \pi'(\phi x)\ \forall\ x \in \mathfrak{g}$, then $N(\mathfrak{g}, V) \simeq N(\mathfrak{g}, V')$ (Lie group isomorphism). In particular, if V is equivalent to V', then $N(\mathfrak{g}, V) \simeq N(\mathfrak{g}, V')$. We shall now describe the isometry group of $N(\mathfrak{g}, V)$. Note that by (1), it suffices to compute the isotropy subgroup K of the isometry group. We first consider the group $U := \{T \in K : T|_{\mathfrak{g}} = I\}$. It is easy to see that $U = \mathrm{End}_{\mathfrak{g}}(V) \cap \mathrm{O}(V, \langle\ ,\ \rangle)$, where $\mathrm{End}_{\mathfrak{g}}(V)$ denotes the algebra of intertwining operators of the representation (π, V) of $\mathfrak{g}$. If $V = V_1^{r_1} \oplus \ldots \oplus V_k^{r_k}$ is the decomposition of V into isotypic components, then $\mathrm{End}_{\mathfrak{g}}(V) = \mathfrak{gl}(r_1, \mathbb{F}_1) \oplus \ldots \oplus \mathfrak{gl}(r_k, \mathbb{F}_k)$, where $\mathbb{F}_l = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ depending on the type of V_l, and $\mathfrak{gl}(r, \mathbb{F})$ denotes the Lie algebra of $(r \times r)$-matrices with coefficients in the ring $\mathbb{F}$ (we refer to [BtD] for definitions and properties of the diferent types of real and complex representations). Each $A = (a_{ij}) \in \mathfrak{gl}(r_l, \mathbb{F}_l)$ acts on $V_l^{r_l}$ by the matrix

$$\begin{bmatrix} a_{11}I_l & \cdots & a_{1r_l}I_l \\ \vdots & \ddots & \vdots \\ a_{r_l1}I_l & \cdots & a_{r_lr_l}I_l \end{bmatrix}, \tag{3}$$

where I_l denotes the identity transformation of V_l. This implies that $U = U_1 \times \ldots \times U_k$, where $U_l = \mathsf{O}(r_l), \mathsf{U}(r_l)$ or $\mathsf{Sp}(r_l)$ depending on the type of V_l.

Theorem 1. [cf. [L3]] *If $N(\mathfrak{g}, V)$ is the two-step nilpotent Lie group corresponding to the data set $(\mathfrak{g}, V, \langle\ ,\ \rangle)$ (see Definition 1), we put $\mathfrak{g} = \overline{\mathfrak{g}} \oplus \mathfrak{c}$ with $\overline{\mathfrak{g}} = [\mathfrak{g}, \mathfrak{g}]$ and $\mathfrak{c}$ the center of $\mathfrak{g}$.*

(i) *The Lie algebra $\mathfrak{k} = \mathrm{Der}(\mathfrak{n}) \cap \mathfrak{so}(\mathfrak{n}, \langle\ ,\ \rangle)$ of the isotropy subgroup K of the isometry group of $(N(\mathfrak{g}, V), \langle\ ,\ \rangle)$ is given by $\mathfrak{k} = \overline{\mathfrak{g}} \oplus \mathfrak{u}$, $[\overline{\mathfrak{g}}, \mathfrak{u}] = 0$, where $\mathfrak{u} = \mathrm{End}_{\mathfrak{g}}(V) \cap \mathfrak{so}(V, \langle\ ,\ \rangle)$ and $\overline{\mathfrak{g}}$ acts on $\mathfrak{n} = \mathfrak{g} \oplus V$ by $(\mathrm{ad}\, x, \pi(x))$ for all $x \in \overline{\mathfrak{g}}$.*

(ii) *The connected component of the identity of K is $K^0 = G \times U^0$, where $U = \mathrm{End}_{\mathfrak{g}}(V) \cap \mathrm{O}(V, \langle\ ,\ \rangle)$, $G = \overline{G}/\mathrm{Ker}\,\pi$ and $\overline{G}$ is the simply connected Lie group with Lie algebra $\overline{\mathfrak{g}}$. The group U acts trivially on $\mathfrak{g}$ and each $g \in G$ acts on $\mathfrak{n} = \mathfrak{g} \oplus V$ by $(\mathrm{Ad}(g), \pi(g))$, where we also denote by π the corresponding representation of G on V.*

(iii) *If $V = V_1^{r_1} \oplus \ldots \oplus V_k^{r_k}$ with V_i irreducible and $V_i \not\simeq V_j$ for all $i \neq j$, then $U = U_1 \times \ldots \times U_k$, where $U_i = \mathrm{O}(r_i), \mathrm{U}(r_i)$ on $\mathrm{Sp}(r_i)$ depending on the type of V_i, and U_i acts on $V_i^{r_i}$ as in (3).*

(iv) *If $\mathrm{Aut}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$, then $K = G \times U$.*

2. Characterization of Gelfand pairs of the form $(G \times U, N(\mathfrak{g}, V))$ via multiplicity-free actions

Let N be a nilpotent Lie group and let K be a compact group of automorphisms. It is shown in [BJR1] that if (K, N) is a Gelfand pair, then N must be two-step nilpotent (or abelian). We will thus assume that N is a two-step nilpotent Lie group. In the following theorem, the relationship between commutativity and Gelfand pairs is given. We shall first recall some preliminary facts and introduce some notation.

If $K \subset \mathrm{Aut}(N) \approx \mathrm{Aut}(\mathfrak{n})$ (we always assume that N is simply connected), we endow $\mathfrak{n}$ with a K-invariant inner product $\langle\ ,\ \rangle$ and we take $\mathfrak{n} = \mathfrak{z} \oplus V$ the orthogonal decomposition, where $\mathfrak{z}$ denotes the center of $\mathfrak{n}$. For each nonzero $x \in \mathfrak{z}$, we consider the Lie algebra $\mathfrak{n}_x = \mathbb{R}x \oplus V_x$, with $V_x = \{v \in V : [v, V] \perp x\}^\perp = (\mathrm{Ker}\, J_x)^\perp$ and defining Lie bracket $[v, w]_x = \langle [v, w], x\rangle x$ for all $v, w \in V_x$, where $J_x : V \longrightarrow V$ is defined by $\langle J_x v, w\rangle = \langle x, [v, w]\rangle$ for all $v, w \in V$, $x \in \mathfrak{z}$. It is clear that the group $N_x = \exp \mathfrak{n}_x$ is isomorphic to a Heisenberg group, unless $J_x = 0$ (i.e., $V_x = 0$), where $N_x \simeq \mathbb{R}$. We have that $K_x := \{k \in K : kx = x\} \subset \mathrm{Aut}(N_x)$. Since $J_x : V_x \to V_x$ is invertible, there exists an orthogonal decomposition $V_x = V_1 \oplus \ldots \oplus V_r$ such that $\dim V_i = 2$ and

$$J_x|_{V_i} = \begin{bmatrix} 0 & -c_i \\ c_i & 0 \end{bmatrix}, \qquad c_i \neq 0, \qquad \forall\, i = 1, \ldots, r. \tag{4}$$

If we take $J : V_x \to V_x$ given by $J|_{V_i} = \frac{1}{c_i} J_x|_{V_i}$, then $J^2 = -I$ and thus J defines a complex structure on V_x. We denote by $\tilde{V}_x$ the corresponding complex vector space (V_x, J). It is easy to see that the elements of K_x commute with J_x and hence they also commute with J; this implies that K_x acts by complex linear transformations on $\tilde{V}_x$.

Finally, for each $v \in V$ we consider the subgroup of K_x given by

$$K_{x,v} = \{k \in K : kx = x, \quad kp_x(v) = p_x(v)\},$$

where $p_x : V \longrightarrow \mathrm{Ker}\, J_x$ is the orthogonal projection.

A complex representation W of a compact Lie group K is said to be *multiplicity free* if the action of K (or equivalently of its complexification $K_{\mathbb{C}}$) on the polynomial ring $\mathbb{C}[W]$ given by $(k.p)(w) = p(k^{-1}w)$ is multiplicity free, i.e., its isotypic components are all irreducible (see [K, Ho] for further information).

Theorem 2. *If N is a two-step nilpotent Lie group, K is a compact subgroup of* $\mathrm{Aut}(N)$ *and $H = K \ltimes N$, then the following conditions are equivalent.*

(i) *The algebra of H^0-invariant differential operators on N is commutative. In particular, if K is the isotropy subgroup of the isometry group of $(N, \langle\ ,\ \rangle)$, this means that $(N, \langle\ ,\ \rangle)$ is a commutative space.*

(ii) *(K^0, N) is a Gelfand pair.*

(iii) *(K, N) is a Gelfand pair.*

(iv) *$(K_{x,v}, N_x)$ is a Gelfand pair for all $x \in \mathfrak{z}$, $v \in V$.*

(v) *The action of $K_{x,v}$ (or $K^0_{x,v}$) on the complex vector space $\tilde{V}_x$ defined in (4) is multiplicity free for all $x \in \mathfrak{z}$, $v \in V$.*

It is well known that (i) is equivalent to the commutativity of the algebra $\mathrm{L}^1(H^0/\!\!/K^0)$ (see [H], p. 486); thus the equivalence of (i) and (ii) follows from the isomorphism $\mathrm{L}^1(H^0/\!\!/K^0) \simeq \mathrm{L}^1_{K^0}(N)$. It is proved that (ii) and (iii) are equivalent in [BJLR] and [BJR2]. The equivalence of (iii) and (iv) is called *localization*, and it has been proved in [Ki] and [BJR2]. The description of the localization procedure in terms of the operators J_x considered here is proved in [N]. Finally, conditions (iv) and (v) are equivalent by [BJR1].

Definition 2. $(N, \langle\ ,\ \rangle)$ is said to be an *almost-commutative* space if (K_x, N_x) is a Gelfand pair for all $x \in \mathfrak{z}$, where $K = \mathrm{Aut}(\mathfrak{n}) \cap \mathrm{O}(\mathfrak{n}, \langle\ ,\ \rangle)$ is the isotropy subgroup.

We consider this weaker notion of commutativity (i.e., $v = 0$ in (iv) of the above theorem), just because in the particular case $N = N(\mathfrak{g}, V)$ (see Definition 1) and $K = G \times U$ (see Theorem 1), we have the following neat characterization in terms of multiplicity free actions.

Theorem 3. [cf. [L3]] *A group $N(\mathfrak{g}, V)$ with $\mathfrak{g}$ semisimple is an almost-commutative space if and only if the action of $e^{\pi(\mathfrak{t})} \times U^0$ on $\tilde{V}$ is multiplicity free, where $\mathfrak{t}$ is any maximal torus of $\mathfrak{g}$ and $\tilde{V}$ is the complex vector space $\tilde{V}_h$ defined in (4) for any $h \in \mathfrak{t}$ satisfying $\lambda(h) \neq 0$ for all nonzero weights λ of V. Note that $V = \tilde{V} \oplus V_0$, where V_0 denotes the zero weight space of the representation V with respect to $\mathfrak{t}$.*

We note that if $\mathfrak{t}$ denotes a maximal torus of $\mathfrak{g}$ (maximal abelian subalgebra), then $\lambda \in \mathfrak{t}^*$ is called a *weight* of a real representation (π, V) of $\mathfrak{g}$ if there exist $v, w \in V$ such that $\pi(h')v = \lambda(h')w$ and $\pi(h')w = -\lambda(h')v$ for all $h' \in \mathfrak{t}$. We shall now give, using the characterization in the theorem above, two families of examples of groups $N(\mathfrak{g}, V)$ that are almost-commutative spaces. We first need to recall a well known lemma about multiplicity free actions of a torus, which is proved in [L3] for instance.

Lemma 4. *Let $\mathbb{C}^*$ denote the multiplicative group $\mathbb{C} - \{0\}$. A complex representation W of an n-dimensional torus T^n is multiplicity free if and only if the set of weights $P(W) \subset \mathfrak{t}^*$ of W is $\mathbb{R}$-linearly independent. In particular, if W is multiplicity free, then $\dim_{\mathbb{C}} W \leq n$.*

Example 1. Consider the group $N(\mathfrak{su}(n), \mathbb{C}^n)$, $n \geq 2$, where $\mathbb{C}^n$ is the standard representation of $\mathfrak{su}(n)$ regarded as a real representation. The subspace $\mathfrak{t}$ of $\mathfrak{su}(n)$ given by diagonal matrices is a maximal torus of $\mathfrak{su}(n)$. The representation $\mathbb{C}^n$ is of complex type; thus $e^{\pi(\mathfrak{t})} \times U^0 = e^{\pi(\mathfrak{t})} \times \mathrm{S}^1$. Furthermore, since $(\mathbb{C}^n)_0 = 0$, we have that $\tilde{V} = \mathbb{C}^n$. The Lie algebra of $e^{\pi(\mathfrak{t})} \times \mathrm{S}^1$ can be identified with $\mathfrak{t} \oplus \mathbb{R}$, and thus the weights of $\mathbb{C}^n$

are given by $P(\mathbb{C}^n) = \{\lambda_1 + \lambda, \dots, \lambda_n + \lambda\}$, where $\lambda_j(H,r) = ih_j$ and $\lambda(H,r) = ir$ for all $H \in \mathfrak{t}$, $r \in \mathbb{R}$. Since $P(\mathbb{C}^n)$ is a linearly independent subset of $(\mathfrak{t} \oplus \mathbb{R})^*$, we obtain from Lemma 4 that the action of $e^{\pi(\mathfrak{t})} \times S^1$ on $\mathbb{C}^n$ is multiplicity free, and hence $N(\mathfrak{su}(n), \mathbb{C}^n)$ is an almost-commutative space by Theorem 3. We shall prove in Section 4 that $N(\mathfrak{su}(n), \mathbb{C}^n)$ is actually a commutative space.

Example 2. We consider the group $N(\mathfrak{so}(n), \mathbb{R}^n)$, $n \geq 2$, where $\mathbb{R}^n$ denotes the standard representation of $\mathfrak{so}(n)$. In this case $e^{\pi(\mathfrak{t})} \times U^0 = e^{\pi(\mathfrak{t})}$, since $\mathbb{R}^n$ is of real type. If $n = 2k+1$ or $n = 2k$, we choose the standard maximal torus of $\mathfrak{so}(n)$ (see [Kn], p. 63). It is clear that in both cases we have to analyze the action of $e^{\pi(\mathfrak{t})}$ on $\tilde{V} = \mathbb{C}^k$ given by $e^{\pi(H)}.(c_1, \dots, c_k) = (ih_1c_1, \dots, ih_kc_k)$ (see (4)). The Lie algebra of $e^{\pi(\mathfrak{t})}$ is $\mathfrak{t}$ and $P(\mathbb{C}^k) = \{\lambda_1, \dots, \lambda_k\}$, where $\lambda_j(H) = ih_j$; thus $P(\mathbb{C}^k)$ is a linearly independent subset of $\mathfrak{t}^*$. By Lemma 4 we have that the action of $e^{\pi(\mathfrak{t})}$ on $\mathbb{C}^k$ is multiplicity free and thus $N(\mathfrak{so}(n), \mathbb{R}^n)$ is an almost-commutative space (see Theorem 3). We shall prove in Section 4 that $N(\mathfrak{so}(n), \mathbb{R}^n)$ is actually commutative.

Remark 1. It is easy to see that the group $N(\mathfrak{so}(n), \mathbb{R}^n)$ is precisely the so-called *free two-step nilpotent Lie group on n generators.* The almost-commutativity of these groups has been proved in [BJR1]. Moreover, it was also proved in [BJR1] that the only Gelfand pair of the form $(K, N(\mathfrak{so}(n), \mathbb{R}^n))$ is $(\mathsf{SO}(n), N(\mathfrak{so}(n), \mathbb{R}^n))$. The weak symmetry of these groups has been proved in [Z].

3. Classification of the groups $N(\mathfrak{g}, V)$ which are almost-commutative spaces

In this section, we shall give an explicit classification of the almost-commutative two-step nilpotent Lie groups $N(\mathfrak{g}, V)$ (see Definition 2), using the characterization in terms of multiplicity free actions given in Theorem 3 (see Theorems 14, 15).

If V is a real representation of $\mathfrak{g}$, we denote by $V_{\mathbb{C}}$ its complexification $\mathbb{C} \otimes_{\mathbb{R}} V$, which is naturally a complex representation of $\mathfrak{g}$. If W is irreducible and $W = V_{\mathbb{C}}$, we shall put $W_{\mathbb{R}} = V$ and sometimes we shall regard a complex representation W as a real representation denoted also by W.

Theorem 5. *Let $\mathfrak{g}$ be a compact Lie algebra and let V be a representation of $\mathfrak{g}$ as in Definition* 1, (ii). *If $\overline{\mathfrak{g}} = [\mathfrak{g}, \mathfrak{g}]$ and $N(\mathfrak{g}, V)$ is an almost-commutative space then $N(\overline{\mathfrak{g}}, V)$ is so, where we also denote by V the restriction of the representation V to $\overline{\mathfrak{g}}$.*

Proof. Suppose that $N(\mathfrak{g}, V)$ is almost-commutative. If $h \in \overline{\mathfrak{g}}$, then the action of K_h^0 on $\tilde{V}_h$ is multiplicity free by Theorem 2. The group K preserves $\overline{\mathfrak{g}}$ and it is clear that $K|_{\overline{\mathfrak{g}} \oplus V} \subset \overline{K}$, where $\overline{K}$ denotes the isotropy subgroup of $N(\overline{\mathfrak{g}}, V)$. Thus the action of $\overline{K}_h^0$ on $\tilde{V}$ is also multiplicity free, as it has to be shown. □

In view of the result above, we first analyze the groups $N(\mathfrak{g}, V)$ with $\mathfrak{g}$ semisimple. We shall now deduce necessary conditions for the almost-commutativity of $N(\mathfrak{g}, V)$ with V irreducible, depending on the type of V. In the quaternionic case, the condition obtained is also sufficient (we refer to [BtD] for definitions and properties of the diferent types of real and complex representations). We first give a lemma about multiplicity free representations, which will be useful in all the classification.

Lemma 6. (i) *Let G be one of the classical groups $\mathsf{SO}(k, \mathbb{C})$, $\mathsf{Sl}(k, \mathbb{C})$, $\mathsf{Gl}(k, \mathbb{C})$ or $\mathsf{Sp}(\frac{k}{2}, \mathbb{C})$, where $k \geq 2$ and in the last case k is even. If $\mathbb{C}^k$ denotes the standard* *representation of G, we consider the action of $(\mathbb{C}^*)^n \times G$ on $(\mathbb{C}^k)^n = \mathbb{C}^k \oplus \ldots \oplus \mathbb{C}^k$ given by*

$$\rho(a_1, \ldots, a_n, A)(v_1, \ldots, v_n) = (a_1 A v_1, \ldots, a_n A v_n) \ \forall\, a_i \in \mathbb{C}^*,\ A \in G.$$

If $G = \mathsf{Sl}(k,\mathbb{C})$, $\mathsf{Gl}(k,\mathbb{C})$ or $\mathsf{Sp}(\frac{k}{2},\mathbb{C})$, then this action is multiplicity free if and only if $n = 1, 2$. For $G = \mathsf{SO}(k,\mathbb{C})$ the action is multiplicity free if and only if $n = 1$.

(ii) *The action of $\mathbb{C}^* \times \mathbb{C}^* \times \mathsf{Sp}(k,\mathbb{C})$ on $\mathbb{C}^2 \oplus \mathbb{C}^{2k} \oplus \mathbb{C}^{2k}$ given by*

$$\rho(a,b,A)(v_1, v_2, v, w) = (av_1, bv_2, aAv, bAw),\ a, b \in \mathbb{C}^*,\ A \in \mathsf{Sp}(k,\mathbb{C}),$$

is not multiplicity free for all $k \geq 1$.

(iii) *The action of $\mathbb{C}^* \times \mathsf{SO}(k,\mathbb{C}) \times \mathsf{Gl}(n_1,\mathbb{C}) \times \ldots \times \mathsf{Gl}(n_r,\mathbb{C})$ on $\mathbb{C}^k \oplus (\mathbb{C}^{n_1} \oplus \mathbb{C}^{n_1}) \oplus \ldots \oplus (\mathbb{C}^{n_r} \oplus \mathbb{C}^{n_r})$ given by*

$$\rho(a, A, B_1, \ldots, B_r)(v, v_1, w_1, \ldots, v_r, w_r) = (aAv, aB_1v_1, a^{-1}B_1w_1, \ldots, aB_rv_r, a^{-1}B_rw_r)$$

for all $a \in \mathbb{C}^*$, $A \in \mathsf{SO}(k,\mathbb{C})$, $B_i \in \mathsf{Gl}(n_i,\mathbb{C})$, is multiplicity free if and only if $k = 0$ and $r = 1$.

Part (i) follows directly from [BR], Theorem 2 or [Le], Theorem 2.5. It is easy to prove parts (ii) and (iii) using [BR], Theorem 7. □

Case V of quaternionic type. We take the complex representation W of $\mathfrak{g}$ such that $V = W$, regarded as a real representation. We have that $e^{\pi(\mathfrak{t})} \times U^0 = e^{\pi(\mathfrak{t})} \times \mathsf{Sp}(1)$. If $V = \tilde{V} \oplus V_0$ as in Theorem 3, then there exists a real basis of $\tilde{V}$ denoted by

$$\tilde{V} = \{v_1, iv_1, jv_1, -ijv_1, \ldots, v_n, iv_n, jv_n, -ijv_n\}_{\mathbb{R}},$$

where $i, j : V \to V$ are the complex and quaternionic structures of W respectively, such that

$$\pi(h)|_{\{v_k, iv_k, jv_k, -ijv_k\}_{\mathbb{R}}} = \begin{bmatrix} 0 & -\lambda_k(h) & & \\ \lambda_k(h) & 0 & & \\ & & 0 & -\lambda_k(h) \\ & & \lambda_k(h) & 0 \end{bmatrix} \quad \forall\, h \in \mathfrak{t}. \tag{5}$$

Thus, $\tilde{V}$ as a complex vector space has a basis $\tilde{V} = \{v_1, jv_1, \ldots, v_n, jv_n\}_{\mathbb{C}}$ and each $\pi(h)$ with $h \in \mathfrak{t}$ acts on $\tilde{V}$ by $\pi(h)|_{\{v_k, jv_k\}_{\mathbb{C}}} = i\lambda_k(h)I$. If $T \in \mathsf{Sp}(1)$, then T acts diagonally on $\tilde{V} = \mathbb{C}^2 \oplus \ldots \oplus \mathbb{C}^2$ (n copies) and in the standard way on each copy $\mathbb{C}^2 = \{v_k, jv_k\}_{\mathbb{C}}$. By Theorem 3 we have that if $N(\mathfrak{g}, V)$ is almost-commutative, then the action of the complexification $(e^{\pi(\mathfrak{t})} \times \mathsf{Sp}(1))_{\mathbb{C}} = (\mathbb{C}^*)^{\dim \mathfrak{t}} \times \mathsf{Sl}(2,\mathbb{C})$ on $\tilde{V} = (\mathbb{C}^2)^n$ is multiplicity free. This implies that the action given in Lemma 6,(i), must be multiplicity free and thus $\dim \mathfrak{t} = 1$ or $\dim \mathfrak{t} = 2$.

If $\dim \mathfrak{t} = 1$, then $\mathfrak{g} = \mathfrak{su}(2)$ and thus $V = \mathbb{C}^2$, obtaining the group $N(\mathfrak{su}(2), \mathbb{C}^2)$. If $\dim \mathfrak{t} = 2$, we have that $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$ or $\mathfrak{g} = \mathfrak{sp}(2)$. It follows from $\dim W = 4 + \dim W_0$ that the only possibility is $\mathfrak{g} = \mathfrak{sp}(2)$ and $V = \mathbb{C}^4$, the standard representation of $\mathfrak{sp}(2)$.

We then obtain the following result.

Theorem 7. *If* $\mathfrak{g}$ *is semisimple and* V *is irreducible of quaternionic type, then* $N(\mathfrak{g}, V)$ *is an almost-commutative space if and only if* $\mathfrak{g} = \mathfrak{su}(2)$ *and* $V = \mathbb{C}^2$ *or* $\mathfrak{g} = \mathfrak{sp}(2)$ *and* $V = \mathbb{C}^4$*, where* $\mathbb{C}^2$ *and* $\mathbb{C}^4$ *denote the corresponding standard representations.*

Case V of real type. In this case $e^{\pi(\mathfrak{t})} \times U^0 = e^{\pi(\mathfrak{t})}$. If $V = \tilde{V} \oplus V_0$ as in Theorem 3, we take a real basis $\tilde{V} = \{v_1, w_1, \dots, v_n, w_n\}_{\mathbb{R}}$ of $\tilde{V}$ such that $\pi(h)|_{\{v_j, w_j\}_{\mathbb{R}}} = \begin{bmatrix} 0 & -\lambda_j(h) \\ \lambda_j(h) & 0 \end{bmatrix}$ $\forall\ h \in \mathfrak{t}$. Thus $\tilde{V} = \{v_1, \dots, v_n\}_{\mathbb{C}}$ and the action is given by $\pi(h)v_j = i\lambda_j(h)v_j$ for all $h \in \mathfrak{t}$. Suppose that $N(\mathfrak{g}, V)$ is almost-commutative. Since the action of $e^{\pi(\mathfrak{t})}$ on $\tilde{V}$ is multiplicity free, we obtain from Lemma 4 that $n \leq \dim \mathfrak{t}$ and $\{\lambda_1, \dots, \lambda_n\}$ is a linearly independent subset of $\mathfrak{t}^*$. If $W = V_{\mathbb{C}}$, then $\dim_{\mathbb{C}} W - \dim_{\mathbb{C}} W_0 = \dim V - \dim V_0 = \dim_{\mathbb{R}} \tilde{V} = 2n \leq 2 \dim \mathfrak{t}$. We thus obtain that the complex representation W of $\mathfrak{g}$ of real type satisfies

$$\begin{aligned} &\dim W \leq 2\,\mathrm{rank}(\mathfrak{g}) + \dim W_0, \\ &\dim W_\lambda = 1 \quad \forall\, \lambda \in P(W) - \{0\}, \end{aligned} \tag{6}$$

where W_λ denotes the λ-weight space of W.

Case V of complex type. Let W be the complex representation such that $V = W$, regarded as a real representation. We have that $e^{\pi(\mathfrak{t})} \times U^0 = e^{\pi(\mathfrak{t})} \times S^1$. Consider, as in the cases above, a real basis $\tilde{V} = \{v_1, iv_1, \dots, v_n, iv_n\}_{\mathbb{R}}$ of $\tilde{V}$, where $i : W \to W$ is the complex structure of W such that $\pi(h)|_{\{v_j, iv_j\}_{\mathbb{R}}} = \begin{bmatrix} 0 & -\lambda_j(h) \\ \lambda_j(h) & 0 \end{bmatrix}$ $\forall\, h \in \mathfrak{t}$. We then have that $\tilde{V} = \{v_1, \dots, v_n\}_{\mathbb{C}}$ and $\pi(h)v_j = i\lambda_j(h)v_j$ for all $h \in \mathfrak{t}$. Analogous to the previous case we obtain that if $N(\mathfrak{g}, V)$ is almost-commutative, then $n \leq \dim \mathfrak{t} + 1$ and $\lambda_i \neq \lambda_j$ for all $i \neq j$. This implies that $\dim_{\mathbb{C}} W - \dim_{\mathbb{C}} W_0 = \frac{1}{2} \dim V - \frac{1}{2} \dim V_0 = \frac{1}{2} \dim_{\mathbb{R}} \tilde{V} = n \leq \dim \mathfrak{t} + 1$. Thus we obtain that the complex representation W of $\mathfrak{g}$ of complex type satisfies

$$\begin{aligned} &\dim W \leq \mathrm{rank}(\mathfrak{g}) + 1 + \dim W_0, \\ &\dim W_\lambda = 1 \quad \forall\, \lambda \in P(W) - \{0\}. \end{aligned} \tag{7}$$

Lemma 8. *Let* W *be a complex representation of* $\mathfrak{g}$ *such that* $\dim W_\lambda = 1$ *for all* $\lambda \in P(W) - \{0\}$. *Then* $\dim W_0 \leq \mathrm{rank}(\mathfrak{g})$.

Proof. If $r = \mathrm{rank}(\mathfrak{g})$, we take $\Delta = \{\alpha_1, \dots, \alpha_r\}$ the set of simple roots of $\mathfrak{g}$. Denote by $\lambda_1 \in P(W)$ the maximal weight of W and let $w_1 \in W_{\lambda_1} - \{0\}$. If $x_{-\alpha_{i_1}} \dots x_{-\alpha_{i_t}} w_1 \in W_0$ with $x_{-\alpha_{i_j}} \in \mathfrak{g}_{-\alpha_{i_j}}$, then $x_{-\alpha_{i_2}} \dots x_{-\alpha_{i_t}} w_1 \in W_{\alpha_{i_1}}$, and thus $x_{-\alpha_{i_1}} \dots x_{-\alpha_{i_t}} w_1 \in x_{-\alpha_{i_1}} W_{\alpha_{i_1}}$. Since W_0 is $\mathbb{C}$-linearly generated by the elements of the form $x_{-\alpha_{i_1}} \dots x_{-\alpha_{i_t}} w_1$ we have that $W_0 = \langle x_{-\alpha_1} W_{\alpha_1} \cup \dots \cup x_{-\alpha_r} W_{\alpha_r} \rangle_{\mathbb{C}}$. Now, using that $\dim W_{\alpha_i} \leq 1$ for all i we obtain that $\dim W_0 \leq r$. □

Case $\mathfrak{g}$ *simple and* V *irreducible.* In view of (6),(7) and Lemma 8, we should compute all the irreducible complex representations W of any simple Lie algebra $\mathfrak{g}$ satisfying $\dim W \leq 3\,\mathrm{rank}(\mathfrak{g})$. Using tables of representations of simple Lie algebras (see for instance [MPR]), one can check that such representations are precisely the following:

$$
\begin{array}{llrl}
\mathfrak{su}(2): & \mathbb{C}^2, \mathbb{C}^3, & \mathfrak{so}(5): & \mathbb{C}^5, \Delta_5, \\
\mathfrak{su}(3): & \mathbb{C}^3, \mathbb{C}^6, \overline{\mathbb{C}^6}, \overline{\mathbb{C}^3}, & \mathfrak{so}(7): & \mathbb{C}^7, \Delta_7, \\
\mathfrak{su}(4): & \mathbb{C}^4, \Lambda^2\mathbb{C}^4, \overline{\mathbb{C}^4}, & \mathfrak{so}(2n+1): & \mathbb{C}^{2n+1},\ n \geq 4, \\
\mathfrak{su}(5): & \mathbb{C}^5, \Lambda^2\mathbb{C}^5, \Lambda^3\mathbb{C}^5, \overline{\mathbb{C}^5}, & \mathfrak{sp}(n): & \mathbb{C}^{2n},\ n \geq 3, \\
\mathfrak{su}(6): & \mathbb{C}^6, \Lambda^2\mathbb{C}^6, \Lambda^4\mathbb{C}^6, \overline{\mathbb{C}^6}, & \mathfrak{so}(8): & \mathbb{C}^8, \Delta_+^4, \Delta_-^4, \\
\mathfrak{su}(n): & \mathbb{C}^n, \overline{\mathbb{C}^n}, \quad n \geq 7, & \mathfrak{so}(2n): & \mathbb{C}^{2n},\ n \geq 5,
\end{array}
\tag{8}
$$

where $\mathbb{C}^3$ denotes the only 3-dimensional representation of $\mathfrak{su}(2)$ and $\mathbb{C}^6, \overline{\mathbb{C}^6}$ are the representations of $\mathfrak{su}(3)$ corresponding to the dominant weights $(2,0)$ and $(0,2)$ respectively. The representations denoted with Δ are spin representations and bar denotes the dual representation.

The representation $\mathbb{C}^3$ of $\mathfrak{su}(2)$ is of real type and it corresponds to the group $N(\mathfrak{so}(3), \mathbb{R}^3)$, which is commutative by Example 2. Since $\mathbb{C}^6$ and $\overline{\mathbb{C}^6}$ are of complex type, we should have by (7) that $6 \leq 2\,\mathrm{rank}(\mathfrak{g}) + 1 = 5$, which is a contradiction. Similarly, we can disregard the representations $\Lambda^2\mathbb{C}^5, \Lambda^3\mathbb{C}^5$ of $\mathfrak{su}(5)$ and $\Lambda^2\mathbb{C}^6, \Lambda^4\mathbb{C}^6$ of $\mathfrak{su}(6)$.

It is easy to see that $\Lambda^2\mathbb{C}^4 = (\mathbb{R}^6)_{\mathbb{C}}$, where $\mathbb{R}^6$ denotes the standard representation of $\mathfrak{so}(6) = \mathfrak{su}(4)$; thus $N(\mathfrak{su}(4), (\Lambda^2\mathbb{C}^4)_{\mathbb{R}}) = N(\mathfrak{so}(6), \mathbb{R}^6)$ is an almost-commutative space by Example 2.

Since $\Delta_5 = \mathbb{C}^4$, the standard representation of $\mathfrak{sp}(2) = \mathfrak{so}(5)$, we have by Theorem 7 that $N(\mathfrak{so}(5), \mathbb{C}^4) = N(\mathfrak{sp}(2), \mathbb{C}^4)$ is an almost-commutative space. The representation Δ_7 of $\mathfrak{so}(7)$ is of real type and satisfies $(\Delta_7)_0 = 0$ (see [BtD], p. 280), by (6) we should have $\dim_{\mathbb{C}} \Delta_7 = 8 \leq 2\,\mathrm{rank}(\mathfrak{g}) + \dim W_0 = 6$, which is a contradiction.

The representation $\mathbb{C}^{2n}$ of $\mathfrak{sp}(n)$ is of quaternionic type for all $n \geq 1$; thus $N(\mathfrak{sp}(n), \mathbb{C}^{2n})$ is an almost-commutative space if and only if $n = 1, 2$ (see Theorem 7).

Finally, the groups $N(\mathfrak{so}(8), \mathbb{R}^8)$, $N(\mathfrak{so}(8), (\Delta_+^4)_{\mathbb{R}})$ and $N(\mathfrak{so}(8), (\Delta_-^4)_{\mathbb{R}})$ are pairwise isomorphic, since Δ_+^4 and Δ_-^4 can be obtained from $\mathbb{C}^8$ composing with an outer automorphism ϕ of $\mathfrak{so}(8)$ (see Section 1). Furthermore, it is easy to see that the representations $\mathbb{C}^n$ and $\overline{\mathbb{C}^n}$ of $\mathfrak{su}(n)$ are equivalent, regarded as real representations, thus $N(\mathfrak{su}(n), \mathbb{C}^n)$ and $N(\mathfrak{su}(n), \overline{\mathbb{C}^n})$ are isomorphic (see Section 1).

We have obtained the following classification.

Theorem 9. *The groups $N(\mathfrak{g}, V)$ with $\mathfrak{g}$ simple and V irreducible that are almost-commutative spaces are*

(i) $N(\mathfrak{su}(n), \mathbb{C}^n)$, $n \geq 2$,
(ii) $N(\mathfrak{so}(n), \mathbb{R}^n)$, $n \geq 3$, $n \neq 4$,
(iii) $N(\mathfrak{sp}(2), \mathbb{C}^4)$.

Case $\mathfrak{g}$ semisimple, nonsimple and V irreducible. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_k$ $(k \geq 2)$ be the decomposition into simple ideals of $\mathfrak{g}$. As in the above case, we shall first compute all the complex representations W of $\mathfrak{g}$ satisfying (6) or (7). There exist representations W_i of $\mathfrak{g}_i$ such that $W = W_1 \otimes \ldots \otimes W_k$ (see [BtD], p. 82); thus it follows from Lemma 8 that

$$\dim W_1 \ldots \dim W_k \leq 3(r_1 + \ldots + r_k), \tag{9}$$

where $r_i = \mathrm{rank}(\mathfrak{g}_i)$. Since $\dim W_i \geq r_i + 1$ for all i (see (8)), we obtain from (9) that

$$(r_1 + 1) \ldots (r_k + 1) \leq 3(r_1 + \ldots + r_k). \tag{10}$$

It is easy to see that if (10) holds, then $k = 2, 3$, and furthermore, the only 3-tuples (r_1, r_2, r_3) with $r_1 \geq r_2 \geq r_3$ satisfying (10) are $(1,1,1)$ and $(2,1,1)$. For $(1,1,1)$ we have that $\mathfrak{g} = \mathfrak{su}(2)\oplus\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ and $\dim W \leq 9$, thus $W = \mathbb{C}^2\otimes\mathbb{C}^2\otimes\mathbb{C}^2$. But $W_0 = 0$ in this case, and this implies that if (6) or (7) hold, then $8 = \dim W \leq 2\,\mathrm{rank}(\mathfrak{g}) = 6$, which is a contradiction. In a similar fashion we can disregard the 3-tuple $(2,1,1)$.

We now consider the case $k = 2$. It is easy to prove that under the assumption $(W_1)_0 = 0$ or $(W_2)_0 = 0$ then $(W_1 \otimes W_2)_0 = 0$; thus (6) or (7) imply that

$$\dim W_1 \dim W_2 \leq 2(r_1 + r_2). \tag{11}$$

Since $\dim W_i \geq r_i + 1$ we have that $(r_1+1)(r_2+1) \leq 2(r_1+r_2), r_1r_2-(r_1+r_2)+1 \leq 0$, $(1-r_1)(1-r_2) \leq 0$, and if we suppose $r_1 \geq r_2$, then $r_2 = 1$. Moreover, it follows from (11) that $\dim W_1 = r_1 + 1$ and $\dim W_2 = 2$, and thus the only possibility is $\mathfrak{g} = \mathfrak{su}(n) \oplus \mathfrak{su}(2)$ and $W = \mathbb{C}^n \otimes \mathbb{C}^2$, with $n \geq 2$. If $n \geq 3$, since $\mathbb{C}^n \not\simeq \overline{\mathbb{C}^n}$ and $\mathbb{C}^2 \simeq \overline{\mathbb{C}^2}$, then $W \not\simeq \overline{W}$, i.e., W is of complex type and thus (7) must hold. Therefore $2n = \dim W \leq \mathrm{rank}(\mathfrak{g}) + 1 + \dim W_0 = n + 1$, which is a contradiction. Thus $n = 2$, $\mathfrak{g} = \mathfrak{su}(2) \oplus \mathfrak{su}(2) = \mathfrak{so}(4)$ and $W = \mathbb{C}^2 \otimes \mathbb{C}^2$, which is of real type. We have that $(\mathbb{C}^2 \otimes \mathbb{C}^2)_{\mathbb{R}} = \mathbb{R}^4$, the standard representation of $\mathfrak{so}(4)$, and thus we have obtained the group $N(\mathfrak{su}(2) \oplus \mathfrak{su}(2), \mathbb{R}^4)$, which is an almost-commutative space by Example 2.

Finally, we suppose that $(W_1)_0 \neq 0$ and $(W_2)_0 \neq 0$. It follows from (8) that $\dim W_i \geq 2r_i$; thus we obtain from (9) that

$$4r_1r_2 \leq 3(r_1 + r_2). \tag{12}$$

Assuming $r_1 \geq r_2$ it is easy to see that (12) implies $r_2 = 1$, and thus $\mathfrak{g}_2 = \mathfrak{su}(2)$ and $\dim W_2 \geq 3$. Using (9) again we have $(\dim W_1)3 \leq 3(r_1 + 1)$; thus $\dim W_1 \leq r_1 + 1$, which is a contradiction since $(W_1)_0 \neq 0$.

We then obtain in this case the following result.

Theorem 10. *The only group $N(\mathfrak{g}, V)$ with $\mathfrak{g}$ semisimple, nonsimple and V irreducible which is an almost-commutative space is $N(\mathfrak{su}(2) \oplus \mathfrak{su}(2), \mathbb{R}^4)$, where $\mathbb{R}^4$ denotes the standard representation of $\mathfrak{so}(4) = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$.*

Case $\mathfrak{g}$ semisimple and V isotypic nonirreducible. Suppose that $V = V_1 \oplus \ldots \oplus V_k$ $(k \geq 2)$ with $V_i \simeq V_j$ for all $i, j = 1, \ldots, k$. We assume that the corresponding group $N(\mathfrak{g}, V)$ is an almost-commutative space and we shall use constantly the characterization given in Theorem 3.

For each $j = 1, \ldots, k$ consider the decomposition

$$V_j = \{v_1^j, w_1^j, \ldots, v_n^j, w_n^j\}_{\mathbb{R}} \oplus (V_j)_0 \tag{13}$$

such that $\pi(h)|_{\{v_i^j, w_i^j\}_{\mathbb{R}}} = \begin{bmatrix} 0 & -\lambda_i(h) \\ \lambda_i(h) & 0 \end{bmatrix}$ $\forall\, h \in \mathfrak{t}$. Note that the nonzero weights $\{\lambda_1, \ldots, \lambda_n\} \subset \mathfrak{t}^*$ of V_j do not depend on j. The complex vector space $\tilde{V}$ in Theorem 3 is given by $\tilde{V} = \{v_1^1, \ldots, v_n^1\}_{\mathbb{C}} \oplus \ldots \oplus \{v_1^k, \ldots, v_n^k\}_{\mathbb{C}}$, and $\pi(h)v_i^j = \sqrt{-1}\lambda_i(h)v_i^j$ for all $h \in \mathfrak{t}$, $1 \leq i \leq n$, $1 \leq j \leq k$. If we define $W_i = \{v_i^1, v_i^2, \ldots, v_i^k\}_{\mathbb{C}}$ for all $i = 1, \ldots, n$, then the decomposition of $\tilde{V}$ into irreducibles as a representation of $e^{\pi(\mathfrak{t})} \times U^0$ is $\tilde{V} = W_1 \oplus \ldots \oplus W_n$, assuming that V is of real or complex type. In the case when V is of quaternionic type, we have that n is even and furthermore, we can choose the vectors v_i^j, w_i^j in (13) such that $\lambda_1 = \lambda_2, \ldots, \lambda_{n-1} = \lambda_n$ (see (5)). Thus the decomposition into irreducibles is $\tilde{V} = (W_1 \oplus W_2) \oplus \ldots \oplus (W_{n-1} \oplus W_n)$.

Note that in any case the group $U^0 = \mathsf{SO}(k), \mathsf{U}(k), \mathsf{Sp}(k)$ acts diagonally on $\tilde{V}$ and via the standard representation on each irreducible component. Moreover, $\pi(h)|_{W_i} = \sqrt{-1}\lambda_i(h)I$ for all $h \in \mathfrak{t}$ and $i = 1, \ldots, n$.

If V is of real type, then the action of the complexification $(e^{\pi(\mathfrak{t})} \times \mathsf{SO}(k))_{\mathbb{C}} = (\mathbb{C}^*)^{\dim \mathfrak{t}} \times \mathsf{SO}(k, \mathbb{C})$ on $\tilde{V} = (\mathbb{C}^k)^n$ must be multiplicity free. This implies that the action of $(\mathbb{C}^*)^n \times \mathsf{SO}(k, \mathbb{C})$ on $(\mathbb{C}^k)^n$ given in Lemma 6,(i) is multiplicity free and hence $n = 1$. This implies that $\mathfrak{g} = \mathfrak{su}(2)$ and $V_j = \mathbb{R}^3$, obtaining the two-step group $N(\mathfrak{su}(2), (\mathbb{R}^3)^k)$.

We now suppose that V is of complex type. Thus the action of the complexification $(e^{\pi(\mathfrak{t})} \times \mathsf{U}(k))_{\mathbb{C}} = (\mathbb{C}^*)^{\dim \mathfrak{t}} \times \mathsf{Gl}(k, \mathbb{C})$ on $\tilde{V} = (\mathbb{C}^k)^n$ must be multiplicity free. It follows from this that the action of $(\mathbb{C}^*)^n \times \mathsf{Gl}(k, \mathbb{C})$ on $(\mathbb{C}^k)^n$ given in Lemma 6,(i) is multiplicity free, therefore $n = 1$ or $n = 2$. Since (π, V) is faithful, we have that $\dim \mathfrak{t} = 1$ or $\dim \mathfrak{t} = 2$, and it is easy to check that there are no representations of complex type satisfying any of the above conditions.

Finally, if V is of quaternionic type, then the action of the complexification $(e^{\pi(\mathfrak{t})} \times \mathsf{Sp}(k))_{\mathbb{C}} = (\mathbb{C}^*)^{\dim \mathfrak{t}} \times \mathsf{Sp}(k, \mathbb{C})$ on $\tilde{V} = (\mathbb{C}^{2k})^{\frac{n}{2}}$ must be multiplicity free, and this implies that the action of $(\mathbb{C}^*)^{\frac{n}{2}} \times \mathsf{Sp}(k, \mathbb{C})$ on $(\mathbb{C}^{2k})^{\frac{n}{2}}$ given in Lemma 6,(i) is multiplicity free, and thus $n = 2$ or $n = 4$. We then obtain that $\dim \mathfrak{t} = 1$, i.e., $\mathfrak{g} = \mathfrak{su}(2)$ and $V_j = \mathbb{C}^2$, or $\dim \mathfrak{t} = 2$. It is easy to see that the only possibility in the last case is $\mathfrak{g} = \mathfrak{sp}(2)$ and $V_j = \mathbb{C}^4$, obtaining the group $N(\mathfrak{sp}(2), (\mathbb{C}^4)^k)$.

We have the following result in this case.

Theorem 11. *A group $N(\mathfrak{g}, V)$ with $\mathfrak{g}$ semisimple and V isotypic, nonirreducible is an almost-commutative space if and only if it is one of the following groups:*

(i) $N(\mathfrak{su}(2), (\mathbb{R}^3)^k)$, $k \geq 2$,
(ii) $N(\mathfrak{su}(2), (\mathbb{C}^2)^k)$, $k \geq 2$,
(iii) $N(\mathfrak{sp}(2), (\mathbb{C}^4)^k)$, $k \geq 2$.

Case $\mathfrak{g}$ simple and V nonisotypic. We suppose that $V = V_1^{k_1} \oplus \ldots \oplus V_r^{k_r}$ $(r \geq 2)$, with V_i irreducible representations of $\mathfrak{g}$ and $V_i \not\simeq V_j$ for all $i \neq j$. If $N(\mathfrak{g}, V)$ is an almost-commutative space, then the groups $N(\mathfrak{g}, V_j^{k_j})$ are as well, since the isotypic components $V_j^{k_j} = V_j \oplus \ldots \oplus V_j$ are $(e^{\pi(\mathfrak{t})} \times U^0)$-invariant (see Theorem 3). It then follows from Theorems 9,11 that we are in one of the following cases:

(i) $\mathfrak{g} = \mathfrak{su}(2) = \mathfrak{so}(3)$, $V = (\mathbb{R}^3)^k \oplus (\mathbb{C}^2)^n$, $k, n \geq 1$,
(ii) $\mathfrak{g} = \mathfrak{su}(4) = \mathfrak{so}(6)$, $V = \mathbb{C}^4 \oplus \mathbb{R}^6$,
(iii) $\mathfrak{g} = \mathfrak{sp}(2) = \mathfrak{so}(5)$, $V = \mathbb{R}^5 \oplus (\mathbb{C}^4)^k$, $k \geq 1$.

Case (i). Since $e^{\pi(\mathfrak{t})} \times U^0 = \mathsf{S}^1 \times \mathsf{SO}(k) \times \mathsf{Sp}(n)$ and $\tilde{V} = \mathbb{C}^k \oplus \mathbb{C}^{2n}$, we have that the action ρ of the complexification $(e^{\pi(\mathfrak{t})} \times U^0)_{\mathbb{C}} = \mathbb{C}^* \times \mathsf{SO}(k, \mathbb{C}) \times \mathsf{Sp}(n, \mathbb{C})$ on $\tilde{V}$ is given by $\rho(a, A, B)(v_1, v_2) = (aAv_1, aBv_2)$, and so it is multiplicity free (see [BR], Theorem 7). We then obtain from Theorem 3 that the groups $N(\mathfrak{su}(2), (\mathbb{R}^3)^k \oplus (\mathbb{C}^2)^n)$ are almost-commutative spaces.

Case (ii). In this case $e^{\pi(\mathfrak{t})} \times U^0$ is a 4-dimensional torus and $\tilde{V} = \mathbb{C}^4 \oplus \mathbb{C}^3$. This implies that the corresponding action cannot be multiplicity free (see Lemma 4).

Case (iii). We have that the action of $(e^{\pi(\mathfrak{t})} \times U^0)_{\mathbb{C}} = \mathbb{C}^* \times \mathbb{C}^* \times \mathsf{Sp}(k,\mathbb{C})$ on $\tilde{V} = \mathbb{C}^2 \oplus \mathbb{C}^{2k} \oplus \mathbb{C}^{2k}$ is the given in Lemma 6, (ii) and thus it is not multiplicity free. Hence, this group is not almost-commutative.

We have obtained in this case the following result.

Theorem 12. *A group $N(\mathfrak{g}, V)$ with $\mathfrak{g}$ simple and V nonisotypic is commutative if and only if $\mathfrak{g} = \mathfrak{su}(2)$ and $V = (\mathbb{R}^3)^k \oplus (\mathbb{C}^2)^n$, with $k, n \geq 1$.*

Case $\mathfrak{g}$ semisimple, nonsimple and V nonisotypic. Let $\mathfrak{g} = \mathfrak{g}_1 \oplus \ldots \oplus \mathfrak{g}_m$ $(m \geq 2)$ be the decomposition of $\mathfrak{g}$ into simple ideals and suppose that $V = V_1^{k_1} \oplus \ldots \oplus V_r^{k_r}$ $(r \geq 2)$ is the decomposition of V into isotypic components $V_i^{k_i}$.

Definition 3. We say that a group $N(\mathfrak{g}, V)$ is *indecomposable* if there are no nonzero ideals $\mathfrak{h}_1, \mathfrak{h}_2$ of $\mathfrak{g}$ and nonzero subspaces V_1, V_2 of V such that $\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2$, $V = V_1 \oplus V_2$ and $\pi(\mathfrak{h}_1)|_{V_2} = \pi(\mathfrak{h}_2)|_{V_1} \equiv 0$. Otherwise, we will say that $N(\mathfrak{g}, V)$ is *decomposable.*

Note that if $N(\mathfrak{g}, V)$ is decomposable, then

$$N(\mathfrak{g}, V) = N(\mathfrak{h}_1, V_1) \times N(\mathfrak{h}_2, V_2) \tag{14}$$

is a direct product of Lie groups, since $\mathfrak{n}_1 = \mathfrak{h}_1 \oplus V_1$ and $\mathfrak{n}_2 = \mathfrak{h}_2 \oplus V_2$ are Lie subalgebras of $\mathfrak{n}$ such that $\mathfrak{n} = \mathfrak{n}_1 \oplus \mathfrak{n}_2$ and $[\mathfrak{n}_1, \mathfrak{n}_2] = 0$ (see Definition 1). Moreover, $\mathfrak{n}_1$ and $\mathfrak{n}_2$ are orthogonal subspaces with respect to any $\mathfrak{g}$-invariant inner product on $\mathfrak{n} = \mathfrak{g} \oplus V$; thus (14) is also a direct product of riemannian manifolds. This implies that $N(\mathfrak{g}, V)$ will be an almost-commutative space if and only if both $N(\mathfrak{h}_1, V_1)$ and $N(\mathfrak{h}_2, V_2)$ are as well. We then assume that $N(\mathfrak{g}, V)$ is an indecomposable almost-commutative space.

If $\mathfrak{h} \subset \mathfrak{g}$ is an ideal and $W \subset V$ is an $\mathfrak{h}$-invariant subspace of V where $\mathfrak{h}$ acts faithfully and without trivial subrepresentations on W, then $N(\mathfrak{h}, W)$ is also an almost-commutative space by Theorem 3. In fact, $\tilde{W} = \tilde{V} \cap W$ and

$$e^{\pi(\mathfrak{t})} \times U^0|_{\tilde{W}} \subset e^{\pi(\mathfrak{t} \cap \mathfrak{h})} \times U^0_{\mathfrak{h},W} : \tilde{W} \to \tilde{W}, \tag{15}$$

where $U_{\mathfrak{h},W} = \mathrm{End}_{\mathfrak{h}}(W) \cap \mathsf{O}(W)$ and $\mathfrak{t}$ is a cartan subalgebra of $\mathfrak{g}$ such that $\mathfrak{t} \cap \mathfrak{h}$ is a cartan subalgebra of $\mathfrak{h}$, obtaining that the action of $e^{\pi(\mathfrak{t}\cap\mathfrak{h})} \times U^0_{\mathfrak{h},W}$ on $\tilde{W}$ is also multiplicity free.

We fix a simple ideal $\mathfrak{g}_i$ of $\mathfrak{g}$. If W_i denotes the subspace of V where $\mathfrak{g}_i$ acts without trivial subrepresentations, then W_i is a sum of certain isotypic components of V. From (15) we have that the two-step groups $N(\mathfrak{g}_i, W_i)$ and $N(\mathfrak{h}_i, W_i)$ are almost-commutative spaces, where $\mathfrak{h}_i$ is the maximal ideal of $\mathfrak{g}$ acting faithfully on W_i.

Suppose that $\mathfrak{g}_i \neq \mathfrak{su}(2), \mathfrak{sp}(2)$. Thus it follows from Theorems 11 and 12 that W_i is $\mathfrak{g}_i$-irreducible. Since $\mathfrak{g}_i \subset \mathfrak{h}_i$ we have that W_i is $\mathfrak{h}_i$-irreducible too, and thus $\mathfrak{g}_i = \mathfrak{h}_i$ by Theorem 10. This implies that $\mathfrak{g}_i^\perp$ acts trivially on W_i, and since $\mathfrak{g}_i$ acts trivially on $W_i^\perp$, we can decompose $N(\mathfrak{g}, V) = N(\mathfrak{g}_i, W_i) \times N(\mathfrak{g}_i^\perp, W_i^\perp)$ (see Definition 3), which is a contradiction.

Thus $\mathfrak{g}_i = \mathfrak{su}(2)$ or $\mathfrak{sp}(2)$ for all $i = 1, \ldots, m$. Fix an isotypic component $V_j^{k_j}$ of V and let $\mathfrak{h}_j$ denote the maximal ideal of $\mathfrak{g}$ acting faithfully on $V_j^{k_j}$. Since $N(\mathfrak{h}_j, V_j^{k_j})$ is an almost-commutative space (see (15)), we obtain from Theorems 9, 10, 11 that one of the following conditions must hold:

(i) $\mathfrak{h}_j = \mathfrak{su}(2)$, $V_j = \mathbb{R}^3$,

(ii) $\mathfrak{h}_j = \mathfrak{su}(2)$, $V_j = \mathbb{C}^2$,
(iii) $\mathfrak{h}_j = \mathfrak{su}(2) \oplus \mathfrak{su}(2)$, $V_j = (\mathbb{C}^2 \otimes \mathbb{C}^2)_{\mathbb{R}} = \mathbb{R}^4$, $k_j = 1$,
(iv) $\mathfrak{h}_j = \mathfrak{sp}(2)$, $V_j = \mathbb{C}^4$.

If (iv) holds and $\mathfrak{h}_j$ acts nontrivially on some $V_l^{k_l}$, then $\mathfrak{h}_j^\perp$ must act trivially on $V_l^{k_l}$ by Theorem 11 and thus $j = l$. This implies that $\mathfrak{h}_j$ acts trivially on $(V_j^{k_j})^\perp$ and hence we can decompose $N(\mathfrak{g}, V) = N(\mathfrak{h}_j, V_j^{k_j}) \times N(\mathfrak{h}_j^\perp, (V_j^{k_j})^\perp)$, which is a contradiction. Thus $\mathfrak{g}_i = \mathfrak{su}(2)$ for all $i = 1, \dots, m$. We can deal with the case (i) similarly to (iv); therefore

$$V = \mathbb{R}^4 \oplus \dots \oplus \mathbb{R}^4 \oplus (\mathbb{C}^2)^{k_1} \oplus \dots \oplus (\mathbb{C}^2)^{k_{r-r'}} \tag{16}$$

with r' copies of $\mathbb{R}^4$. If $\mathfrak{g}_i$ acts trivially on all isotypic components $\mathbb{R}^4$, then $\mathfrak{g}_i^\perp$ must act trivially on $W_i = (\bigcap_{x \in \mathfrak{g}_i} \operatorname{Ker} \pi(x))^\perp$. This implies, as before, that we can decompose and this is a contradiction, obtaining that $\mathfrak{g}$ acts faithfully on $\mathbb{R}^4 \oplus \dots \oplus \mathbb{R}^4$ (r' copies). Hence $N(\mathfrak{g}, \mathbb{R}^4 \oplus \dots \oplus \mathbb{R}^4)$ is a commutative space (see (15)) and thus the action of $e^{\pi(\mathfrak{t})} = T^m$ on $(\mathbb{C}^2)^{r'}$ is multiplicity free. We then obtain from Lemma 4 that $2r' \le m$, and since at most two ideals $\mathfrak{g}_i$ can act nontrivially on each copy of $\mathbb{R}^4$, we also have $m \le r' + 1$. Henceforth $r' = 1$ and $m = 2$.

Thus we have obtained in this case that the only indecomposable possibility is $N(\mathfrak{su}(2) \oplus \mathfrak{su}(2), (\mathbb{C}^2)_1^{k_1} \oplus \mathbb{R}^4 \oplus (\mathbb{C}^2)_1^{k_2})$, where the first copy of $\mathfrak{su}(2)$ acts only on $(\mathbb{C}^2)^{k_1}$ and the second one only on $(\mathbb{C}^2)^{k_2}$. This group is almost-commutative since the action ρ of $(e^{\pi(\mathfrak{t})} \times U^0)_{\mathbb{C}} = \mathbb{C}^* \times \mathbb{C}^* \times \mathsf{Sp}(k_1) \times \mathsf{Sp}(k_2)$ on $\tilde{V} = \mathbb{C}^{2k_1} \oplus \mathbb{C}^2 \oplus \mathbb{C}^{2k_2}$ is given by $\rho(a, b, A, B)(v, v_1, v_2, w) = (aAv, av_1, bv_2, bBw)$ is multiplicity free (see [BR], Theorem 7).

Theorem 13. *A group $N(\mathfrak{g}, V)$ with $\mathfrak{g}$ semisimple, nonsimple and V nonisotypic is an almost-commutative space if and only if it is*

(i) *$N(\mathfrak{su}(2) \oplus \mathfrak{su}(2), (\mathbb{C}^2)^{k_1} \oplus \mathbb{R}^4 \oplus (\mathbb{C}^2)^{k_2})$, where the first copy of $\mathfrak{su}(2)$ acts only on $(\mathbb{C}^2)^{k_1}$ and the second one acts only on $(\mathbb{C}^2)^{k_2}$,*
(ii) *a direct product of some of the groups listed in Theorems* 9, 10, 11, 12 *and part* (i).

It would be appropriate now to summarize the results obtained in the cases above, see Theorems 9, 10, 11, 12, 13.

Theorem 14. *The two-step nilpotent Lie groups $N(\mathfrak{g}, V)$ with $\mathfrak{g}$ semisimple that are almost-commutative spaces are*

(i) *$N(\mathfrak{su}(2), (\mathbb{R}^3)^k \oplus (\mathbb{C}^2)^n)$, $k, n \ge 0$,*
(ii) *$N(\mathfrak{su}(2) \oplus \mathfrak{su}(2), (\mathbb{C}^2)^{k_1} \oplus \mathbb{R}^4 \oplus (\mathbb{C}^2)^{k_2})$, $k_1, k_2 \ge 0$, where the first copy of $\mathfrak{su}(2)$ acts only on $(\mathbb{C}^2)^{k_1}$ and the second one only on $(\mathbb{C}^2)^{k_2}$,*
(iii) *$N(\mathfrak{sp}(2), (\mathbb{C}^4)^k)$, $k \ge 1$,*
(iv) *$N(\mathfrak{su}(n), \mathbb{C}^n)$, $n \ge 3$,*
(v) *$N(\mathfrak{so}(n), \mathbb{R}^n)$, $n \ge 5$,*
(vi) *a direct product of some of the two-step nilpotent Lie groups above.*

Case $\mathfrak{g}$ nonsemisimple. By Theorem 5, we have that if $N(\mathfrak{g}, V)$ is an almost-commutative space, then $N(\overline{\mathfrak{g}}, V)$ is so, where $\overline{\mathfrak{g}} = [\mathfrak{g}, \mathfrak{g}]$. Thus $N(\overline{\mathfrak{g}}, V)$ must be one of the groups listed in Theorem 14,(i)–(vi), unless $\overline{\mathfrak{g}} = 0$. Each one of these cases will be analysed separately.

Case (i). Suppose that $N(\mathfrak{su}(2)\oplus\mathfrak{c},(\mathbb{R}^3)^k\oplus(\mathbb{C}^2)^n)$ is an almost-commutative space with center $\mathfrak{c}\neq 0$. Since $\mathbb{R}^3$ is of real type, the center $\mathfrak{c}$ must act trivially on $(\mathbb{R}^3)^k$. We consider the decomposition $(\mathbb{C}^2)^n = (\mathbb{C}^2)^{n_1}\oplus\ldots\oplus(\mathbb{C}^2)^{n_r}\oplus(\mathbb{C}^2)^{n'}$ such that $\pi(c)|_{(\mathbb{C}^2)^{n_i}}=\lambda_i(c)J_i$ for all $c\in\mathfrak{c}$ ($\lambda_i\in\mathfrak{c}^*-\{0\}$), where J_i is a skew-symmetric transformation satisfying $J_i^2=-I$, and $\pi(\mathfrak{c})|_{(\mathbb{C}^2)^{n'}}\equiv 0$. We have that $\mathrm{End}_{\mathfrak{u}(2)}((\mathbb{C}^2)^{n_i})=\mathfrak{gl}(n_i,\mathbb{C})$, since $\mathbb{C}^2$ is of complex type as a representation of $\mathfrak{u}(2)$. Henceforth, if $h=\begin{bmatrix}\sqrt{-1} & 0\\ 0 & -\sqrt{-1}\end{bmatrix}\in\mathfrak{su}(2)$, then $K_h^0=e^{\pi(\mathbb{R}h)}\times\mathsf{SO}(k)\times\mathsf{U}(n_1)\times\ldots\times\mathsf{U}(n_r)\times\mathsf{Sp}(n')$, (see Theorem 1), and it follows from Theorem 2 that the action of $(K_h^0)_{\mathbb{C}}=\mathbb{C}^*\times\mathsf{SO}(k,\mathbb{C})\times\mathsf{Gl}(n_1,\mathbb{C})\times\ldots\times\mathsf{Gl}(n_r,\mathbb{C})\times\mathsf{Sp}(n',\mathbb{C})$ on $\tilde{V}_h=\mathbb{C}^k\oplus(\mathbb{C}^2)^{n_1}\oplus\ldots\oplus(\mathbb{C}^2)^{n_r}\oplus\mathbb{C}^{2n'}$ is multiplicity free. This implies that the action of $\mathbb{C}^*\times\mathsf{SO}(k,\mathbb{C})\times\mathsf{Gl}(n_1,\mathbb{C})\times\ldots\times\mathsf{Gl}(n_r,\mathbb{C})$ on $\mathbb{C}^k\oplus(\mathbb{C}^{n_1})^2\oplus\ldots\oplus(\mathbb{C}^{n_r})^2$ given in Lemma 6,(iii) is multiplicity free as well, hence $r=1$, $k=0$. We then conclude that the only possibility in this case is a group $N(\mathfrak{su}(2)\oplus\mathbb{R},(\mathbb{C}^2)^{n_1}\oplus(\mathbb{C}^2)^n)$, where $\mathbb{R}$ acts only on $(\mathbb{C}^2)^{n_1}$. These groups are almost-commutative spaces for all $n_1,n\geq 0$ since for all $h\in\mathfrak{su}(2)\oplus\mathbb{R}$ there exists a compact subgroup $e^{\pi(\mathbb{R}h_1)}\times\mathsf{U}(n_1)\times\mathsf{Sp}(n)\subset K_h^0|_{\tilde{V}}$ whose complexification $\mathbb{C}^*\times\mathsf{Gl}(n_1,\mathbb{C})\times\mathsf{Sp}(n,\mathbb{C})$ acts on $\mathbb{C}^{n_1}\oplus\mathbb{C}^{n_1}\oplus\mathbb{C}^{2n}$ by $\rho(a,A,B)(v_1,v_2,v_3)=(aAv_1,a^{-1}Av_2,aBv_3)$ and such action is multiplicity free (see [BR], Theorem 7).

Case (ii). We now suppose that a group $N(\mathfrak{su}(2)\oplus\mathfrak{su}(2)\oplus\mathfrak{c},(\mathbb{C}^2)^k\oplus\mathbb{R}^4\oplus(\mathbb{C}^2)^n)$ is almost-commutative. If $\mathfrak{c}$ acts nontrivially on some copy of $\mathbb{C}^2$, we obtain analogously to the case above that a certain action of $\mathbb{C}^*\times\mathbb{C}^*\times\mathbb{C}^*$ on $\mathbb{C}^2\oplus\mathbb{C}^2$ must be multiplicity free, which is a contradiction by Lemma 4. This implies that $\mathfrak{c}=0$.

Case (iii). If $N(\mathfrak{sp}(2)\oplus\mathfrak{c},(\mathbb{C}^4)^k)$ is an almost-commutative space, then we take the decomposition $(\mathbb{C}^4)^k=(\mathbb{C}^4)^{k_1}\oplus\ldots\oplus(\mathbb{C}^4)^{k_r}\oplus(\mathbb{C}^4)^{k'}$ as in (i). Similarly, we obtain that the action of $\mathbb{C}^*\times\mathbb{C}^*\times\mathsf{Gl}(k_i,\mathbb{C})$ on $(\mathbb{C}^4)^{k_i}=\mathbb{C}^{k_i}\oplus\mathbb{C}^{k_i}\oplus\mathbb{C}^{k_i}\oplus\mathbb{C}^{k_i}$ given by $\rho(a,b,A)(v_1,v_2,v_3,v_4)(aAv_1,a^{-1}Av_2,bAv_3,b^{-1}Av_4)$, for all $a,b\in\mathbb{C}^*$, $A\in\mathsf{Gl}(k,\mathbb{C})$, is multiplicity free. It is easy to see that this action is equivalent to the one given in Lemma 6, (i) with $G=\mathsf{Sl}(k_i,\mathbb{C})$ and $n=4$, which is a contradiction if $k_i\geq 1$. This implies that $\mathfrak{c}=0$.

Case (iv). The only group of the form $N(\mathfrak{su}(n)\oplus\mathfrak{c},\mathbb{C}^n)$ is $N(\mathfrak{u}(n),\mathbb{C}^n)$, which is an almost-commutative space. In fact, for all $h\in\mathfrak{u}(n)$ there exists an n-dimensional torus $e^{\pi(\mathfrak{t})}\times\mathsf{S}^1\subset K_h^0|_{\tilde{V}}$ whose action on $\tilde{V}=\mathbb{C}^n$ is multiplicity free.

Case (v). Since $\mathbb{R}^n$ is of real type, there are no groups of the form $N(\mathfrak{so}(n)\oplus\mathfrak{c},\mathbb{R}^n)$ with $\mathfrak{c}\neq 0$.

Case (vi). Suppose that a group $N(\mathfrak{g}_1\oplus\ldots\oplus\mathfrak{g}_r\oplus\mathfrak{c},V_1\oplus\ldots\oplus V_r)$ is an indecomposable almost-commutative space (see Definition 3), where each $N(\mathfrak{g}_i,V_i)$ is listed in Theorem 14 and $\mathfrak{g}_i$ acts trivially on V_j for all $i\neq j$. If $\mathfrak{c}_i$ denotes the maximal subspace of $\mathfrak{c}$ acting nontrivially on V_i, then $N(\mathfrak{g}_i\oplus\mathfrak{c}_i,V_i)$ is also a commutative space. In fact, for all $h\in\mathfrak{g}_i\oplus\mathfrak{c}_i$ the group K_h^0 preserves $(\tilde{V}_i)_h$ and thus the corresponding $(K^i)_h^0$ acts multiplicity freely on $(\tilde{V}_i)_h$. Since $\mathfrak{c}_i\neq 0$ we have that $N(\mathfrak{g}_i\oplus\mathfrak{c}_i,V_i)$ must be $N(\mathfrak{u}(n),\mathbb{C}^n)$, $n\geq 3$, or $N(\mathfrak{u}(2),(\mathbb{C}^2)^k\oplus(\mathbb{C}^2)^n)$ and hence $\dim\mathfrak{c}_i=1$ (see the cases above (i)–(v)). It is easy to see that the almost-commutativity of a group satisfying the properties above follows from the almost-commutativity of the groups $N(\mathfrak{g}_i\oplus\mathfrak{c}_i,V_i)$.

Finally, if $\overline{\mathfrak{g}}=0$ then $\mathfrak{g}$ is abelian. Since V is a faithful representation we have that $\mathfrak{g}=\mathbb{R}$, yielding the Heisenberg group, denoted by $N(\mathbb{R},\mathbb{C}\oplus\ldots\oplus\mathbb{C})$.

Theorem 15. *The two-step nilpotent Lie groups of the form* $N(\mathfrak{g}, V)$ *with nonsemisimple* $\mathfrak{g}$ *that are almost-commutative spaces are*

(i) *the Heisenberg group* $N(\mathbb{R}, \mathbb{C}^k)$, $k \geq 1$,

(ii) $N(\mathfrak{u}(2), (\mathbb{C}^2)^k \oplus (\mathbb{C}^2)^n)$, $k \geq 1$, $n \geq 0$, *where the center of* $\mathfrak{u}(2)$ *acts nontrivially only on* $(\mathbb{C}^2)^k$,

(iii) $N(\mathfrak{u}(n), \mathbb{C}^n)$, $n \geq 3$,

(iv) $N(\mathfrak{su}(m_1)\oplus\ldots\oplus\mathfrak{su}(m_r)\oplus\mathfrak{c}, V_1\oplus\ldots\oplus V_r)$ *with the following actions:* $\mathfrak{su}(m_i)$ *acts trivially on* V_j *for all* $i\neq j$ *and* $\dim\mathfrak{c}_i=1$, *where* $\mathfrak{c}_i$ *denotes the maximal subspaces of* $\mathfrak{c}$ *acting nontrivially on* V_i. *Moreover, if* $m_i=2$, *then* $V_i=(\mathbb{C}^2)^{k_i}\oplus(\mathbb{C}^2)^{n_i}$ *as in* (ii), *and if* $m_i\geq 3$ *then* $V_i=\mathbb{C}^{m_i}$,

(v) *a direct product of some of the above groups.*

4. Classification of the groups $N(\mathfrak{g}, V)$ which are commutative spaces

We shall give in this section an explicit classification of the two-step nilpotent Lie groups $N(\mathfrak{g}, V)$ which are commutative spaces, or equivalently by Theorem 2, of the Gelfand pairs of the form $(K, N) = (G \times U, N(\mathfrak{g}, V))$.

We have obtained in Theorem 14 and Theorem 15 all the pairs of this form for which (K_x, N_x) is a Gelfand pair for any $x \in \mathfrak{z} = \mathfrak{g}$ (see Definition 2). In view of Theorem 2,(iv), we have to check along the groups listed in these theorems, which of them satisfy the condition that $(K_{x,v}, N_x)$ is a Gelfand pair for all $x \in \mathfrak{g}$, $v \in \operatorname{Ker}\pi(x)$; or equivalently, the action of $K_{x,v}$ on $\tilde{V}_x$ is multiplicity free (see Theorem 2,(v)).

We have that $K_x = C_G(x) \times U$, $K_{x,v} = \{(g,T) \in C_G(x) \times U : T\pi(g)v = v\}$, and the Lie algebra of $K_{x,v}$ is given by $\mathfrak{k}_{x,v} = \{(y, A) \in C_{\overline{\mathfrak{g}}}(x) \oplus \mathfrak{u} : (A + \pi(y))v = 0\}$, where C denotes 'centralizer', $\overline{\mathfrak{g}} = [\mathfrak{g}, \mathfrak{g}]$ and G is a Lie group with Lie algebra $\overline{\mathfrak{g}}$ (see Theorem 1).

In the next, we will consider case by case the items (i)–(v) of Theorem 14, by giving the action of $(K_{x,v})_{\mathbb{C}}$ on $\tilde{V}_x$ for a generic $x \in \mathfrak{g}$ with $\operatorname{Ker}\pi(x) \neq 0$ and a generic nonzero $v \in \operatorname{Ker}\pi(x)$.

Case (i). $\mathbb{C}^* \times \mathsf{SO}(k-1,\mathbb{C}) \times \mathsf{Sp}(n,\mathbb{C})$ on $\mathbb{C}^k \oplus \mathbb{C}^{2n}$. This action is multiplicity free if and only if $k = 0, 1$.

Case (ii). $\mathbb{C}^* \times \mathsf{Sp}(k_i,\mathbb{C})$ on $\mathbb{C}^{2k_i} \oplus \mathbb{C}$, which is always multiplicity free.

Case (iii). $\mathbb{C}^* \times \mathsf{Sp}(k-1,\mathbb{C})$ on $\mathbb{C} \oplus \mathbb{C}^{2(k-1)}$. It is multiplicity free for every $k \geq 1$.

Case (iv). $(\mathbb{C}^*)^k$ on $\mathbb{C}^k$ for some $k < n$, and it always satisfies the multiplicity free condition of Lemma 4.

Case (v). $(\mathbb{C}^*)^k$ on $\mathbb{C}^k$ for some $k < [\frac{n}{2}]$, always satisfying the multiplicity free condition of Lemma 4.

Now, we shall do the same for items (i)–(iv) of Theorem 15.

Case (i). $\operatorname{Ker}\pi(x) = 0$ for any nonzero $x \in \mathfrak{g} = \mathbb{R}$.

Case (ii). $\mathbb{C}^* \times \mathsf{U}(k-1) \times \mathsf{Sp}(n,\mathbb{C})$ on $\mathbb{C} \oplus \mathbb{C}^{k-1} \oplus \mathbb{C}^{2n}$. This action is always multiplicity free.

Case (iii). $(\mathbb{C}^*)^k$ on $\mathbb{C}^k$ for some $k < n$, and it always satisfies the multiplicity free condition of Lemma 4.

Case (iv). $H_1 \times \ldots \times H_r$ on $\tilde{V}_1 \oplus \ldots \oplus \tilde{V}_r$, where $H_i = \mathbb{C}^* \times \mathsf{U}(k_i - 1) \times \mathsf{Sp}(n_i,\mathbb{C})$ and $\tilde{V}_i = \mathbb{C}\oplus\mathbb{C}^{k_i-1}\oplus\mathbb{C}^{2n_i}$ if $m_i = 2$, and $H_i = (\mathbb{C}^*)^{j_i}$, $\tilde{V}_i = \mathbb{C}^{j_i}$ for some $j_i < m_i$ when $m_i \geq 3$. Thus this action is always multiplicity free.

We then have finished the proof of the following result.

Theorem 16. *The two-step nilpotent Lie groups $N(\mathfrak{g},V)$ that are commutative spaces are*

(i) *all the groups listed in Theorem* 14, *except for the case* (i), $k \geq 2$,
(ii) *all the groups listed in Theorem* 15,
(iii) *direct products.*

Remark 2. The analysis we have made in this section on which of the almost-commutative spaces classified in Theorems 14,15 are really commutative spaces, can be considerably simplified in the following way: the elements $\nu = x + v \in \mathfrak{n} = \mathfrak{z} \oplus V$ which appears in Theorem 2, (iv), are identified via the fixed inner product on $\mathfrak{n}$ with elements in $\mathfrak{n}^*$, and so by Kirillov theory with unitary representations of N. It is well known (see [BJR2]) that for the commutativity of $(N, \langle\ ,\ \rangle)$ it suffices that $(K_\nu, N_\nu) = (K_{x,v}, N_x)$ is a Gelfand pair for any family of functionals $\nu \in \mathfrak{n}^*$ that yield a set of representations π_ν with full Plancherel measure in $\hat{N}$. Since for all the spaces listed in Theorems 14,15, except for case (i) of Theorem 14, there is an open dense subset of $\mathfrak{n}^*$ with $\operatorname{Ker} \pi(x) = 0$, this implies that we can assume $v = 0$ in all the cases except one. In other words, we only should check case (i) of Theorem 14, all the other almost-commutative spaces listed are automatically commutative by the observation given above.

Remark 3. (*A note on Gelfand pairs*) We now enumerate the indecomposable Gelfand pairs $(G \times U^0, N(\mathfrak{g}, V))$ obtained in the classification:

(I) $\big(\mathsf{SU}(2) \times \mathsf{Sp}(n), N(\mathfrak{su}(2), (\mathbb{C}^2)^n)\big)$, $n \geq 1$ (Heisenberg-type),
(II) $\big(\mathsf{SU}(2) \times \mathsf{Sp}(n), N(\mathfrak{su}(2), \mathbb{R}^3 \oplus (\mathbb{C}^2)^n)\big)$, $n \geq 0$,
(III) $\big(\mathsf{Spin}(4) \times \mathsf{Sp}(k_1) \times \mathsf{Sp}(k_2), N(\mathfrak{su}(2) \oplus \mathfrak{su}(2), (\mathbb{C}^2)^{k_1} \oplus \mathbb{R}^4 \oplus (\mathbb{C}^2)^{k_2})\big)$, $k_1 + k_2 \geq 1$,
(IV) $\big(\mathsf{Sp}(2) \times \mathsf{Sp}(k), N(\mathfrak{sp}(2), (\mathbb{C}^4)^k)\big)$, $k \geq 1$.
(V) $\big(\mathsf{SU}(n) \times \mathsf{S}^1, N(\mathfrak{su}(n), \mathbb{C}^n)\big)$, $n \geq 3$,
(VI) $(\mathsf{SO}(n), N(\mathfrak{so}(n), \mathbb{R}^n))$, $n \geq 4$ (free two-step nilpotent Lie groups),
(VII) $\big(\mathsf{U}(k), N(\mathbb{R}, \mathbb{C}^k)\big)$, $k \geq 1$ (Heisenberg groups),
(VIII) $\big(\mathsf{SU}(2) \times \mathsf{U}(k) \times \mathsf{Sp}(n), N(\mathfrak{u}(2), (\mathbb{C}^2)^k \oplus (\mathbb{C}^2)^n)\big)$, $k \geq 1$, $n \geq 0$,
(IX) $\big(\mathsf{SU}(n) \times \mathsf{S}^1, N(\mathfrak{u}(n), \mathbb{C}^n)\big)$, $n \geq 3$,
(X) $(\mathsf{SU}(m_1) \times \ldots \times \mathsf{SU}(m_r) \times U_1 \times \ldots \times U_r$, $N(\mathfrak{su}(m_1) \oplus \ldots \oplus \mathfrak{su}(m_r) \oplus \mathfrak{c}, V_1 \oplus \ldots \oplus V_r))$ with $U_i = \mathsf{S}^1$ if $m_i \geq 3$ and $U_i = \mathsf{U}(k_i) \times \mathsf{Sp}(n_i)$ if $m_i = 2$.

It is proved in [BJR1] that a free two-step nilpotent Lie group $N(\mathfrak{so}(n), \mathbb{R}^n)$ does not admit any proper subgroup $K' \subset \mathsf{SO}(n)$ such that the corresponding pair $(K', N(\mathfrak{so}(n), \mathbb{R}^n))$ is a Gelfand pair. This is also true in cases (V) and (IX). In fact, in these cases we have that K_h^0 is a maximal torus of $\mathsf{U}(n)$ for all regular $h \in \mathfrak{su}(n)$ and $\tilde{V}_h = \mathbb{C}^n$. Thus, if (K', N) is a Gelfand pair, then K' should contain all maximal tori of $\mathsf{U}(n)$, and this implies that $K' = \mathsf{U}(n)$.

However, this does not hold in general. For example, we can take $K' = \mathsf{SU}(2) \times \mathsf{Sp}(n_1) \times \ldots \times \mathsf{Sp}(n_r)$ in case (I) if $n \geq 2$, with $n_1 + \ldots + n_r = n$, and analogously for cases (III), (VIII), (X). Since (VII) corresponds to the Heisenberg groups, any $K' \subset \mathsf{U}(k)$ acting multiplicity freely on $\mathbb{C}^n$ can be taken.

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