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Slopes of effective divisors on the moduli space of stable curves

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§0. Introduction

Let $\mathcal{M}_g$ be the moduli space of smooth curves of genus g and $\overline{\mathcal{M}}_g$ the moduli space of stable curves of genus g; and let $\mathbf{E}$ be the cone of effective divisor classes in $\mathbf{P} = \operatorname{Pic}(\overline{\mathcal{M}}_g) \otimes \mathbb{R}$—that is, the saturation of the semigroup of effective divisors in $\mathbf{P}$. The object of this paper is to bound the cone $\mathbf{E}$ from without and to obtain geometric information about divisor classes near its boundary.

We begin by recalling some facts about the divisor classes on $\overline{\mathcal{M}}_g$ necessary to state our results. Let $\Delta = \overline{\mathcal{M}}_g / \mathcal{M}_g$ denote the locus of singular curves in $\overline{\mathcal{M}}_g$ and δ the corresponding class in $\operatorname{Pic}(\overline{\mathcal{M}}_g)$ or in $\mathbf{P}$. The divisor Δ has $[g/2]+1$ irreducible components $\Delta_0, \ldots, \Delta_{[g/2]}$ determined by the condition that a generic point $[C]$ in Δ_i corresponds, if $i = 0$, to an irreducible curve C with a single node and, if $i > 0$, to a curve C which is the join at one point of a pair of smooth curves of genera i and $g - i$ respectively. Let δ_i be the class in $\mathbf{P}$ determined by Δ_i and let λ be the class of the Hodge line bundle (loosely put, λ is the class of the line bundle whose fibre over $[C]$ is $\Lambda^g H^0(C, \omega_C)$). A first fundamental result, due to Mumford and Harer (cf. [Ha]), asserts that $\mathbf{P}$ is generated by λ and the boundary classes δ_i, with no relations if $g > 2$. (If $g = 2$, we have the relation $10\lambda = \delta_0 + 2\delta_1$). A second is that the class λ is birationally ample: i.e. the map to projective space associated to a sufficiently large multiple of λ is birational onto its image. This follows, for example, from the construction of the Satake compactification as the projective model of $\overline{\mathcal{M}}_g$ associated to a sufficiently large multiple of λ. For technical reasons, we find it convenient to work on the moduli *functor* rather than on the moduli space (see [Mu] for a discussion of the Picard group of a moduli functor, and [HM] for an example of how this will affect our results). In practice, for $g > 3$, the

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only difference this introduces in the description above is that the divisor we refer to as δ_1 will correspond to one-half the divisor class of the locus $\Delta_1 \subset \overline{\mathcal{M}}_g$.

Let γ be an effective sum of the boundary classes. Our goal is to try to describe, for each such γ, the intersection of **E** with the plane in **P** spanned by λ and γ. Since λ is birationally ample, there will be a positive constant s_γ, with the property that for $a > s_\gamma$, the ray spanned by $a\lambda - \gamma$ contains effective divisor classes while for $a < s_\gamma$ it does not. Viewing λ as the unit vector in the y-direction and $-\gamma$ as the unit vector in the x-direction s_γ is the slope of the ray which bounds the cone $\mathbf{E} \cap \mathbf{P}$ from below. We therefore call s_γ the *slope of the effective cone* at γ. The problem we treat in this paper is that of estimating these slopes from below.

In order to motivate this study, we wish to make a few comments and a conjecture about the most important of these slopes, $s_g := s_\delta$. Let us first consider the problem of estimating s_δ from above. One approach is to find an effective divisor D, with class $[D]$, such that the difference $(a\lambda - b\delta) - [D]$ is an effective sum of boundary components. When b is chosen to be as large as possible, we shall denote the ratio a/b by s_D. By the definition of s_g, each such D yields the estimate $s_g \leqq s_D$. The difficulties arise in expressing $[D]$ in terms of the standard classes. Usually, such an expression can be obtained only by intersecting D with enough "test curves", or by a Grothendieck–Riemann–Roch computation. The success of either of these methods depends on having a sufficiently strong geometric characterization of the curves C whose moduli points lie in D. Harris and Mumford [HM] and Diaz [D] first carried this out for the divisor D which is the closure of the locus of curves of odd genus g expressible as a simply branched $(g + 1)/2$ sheeted covering of $\mathbb{P}^1$, and the divisor E which is the closure of the locus of curves with a Weierstrass point with first non-gap $g - 1$, respectively. The computation in [HM] yields the estimate $s_g \leqq 6 + 12/(g + 1)$ (the divisor E dealt with by Diaz has larger slope). Subsequently, Eisenbud and Harris [EH] refined these ideas to show that essentially the same estimate as in [HM] could be obtained for all g such that $g + 1$ is composite by using divisors D consisting of curves C possessing a g^r_d with Brill–Noether number $\rho = g - (r + 1)(g - d + r) = -1$. We worked out the simplest examples when $r = 2$ and arrived once again at the same estimate for s_g. These results lead us to conjecture:

Conjecture 0.1 $s_g \geqq 6 + 12/(g + 1)$ *with equality when* $g + 1$ *is composite.*

A principal result of this paper is to prove this conjecture for g between 2 and 5. In particular, we show that $s_4 \geqq 8.42$ and hence that s_g can be strictly greater than $6 + 12/(g + 1)$ when $(g + 1)$ is prime. We shall give a more precise statement in a moment in Theorem 0.4 but first we wish to discuss three nice consequences of (0.1) and related conjectures.

The first depends on Harris and Mumford's expression for the canonical class K on $\overline{\mathcal{M}}_g$:

$$K + \delta_1 \sim 13\lambda - 2\delta \tag{0.2}$$

In view of this and the calculations of [EH], the conjecture would show that the ray spanned by K is ineffective, and hence that the Kodaira dimension of $\overline{\mathcal{M}}_g$ is $-\infty$, exactly when $g < 23$. (This is known today except for $g = 14$ and for g between 16 and 22 [Severi, Arbarello, Sernesi, Chang-Ran 1 and 2, etc.]).

The second stems from the work of Freitag [Fr] on automorphic forms for the symplectic group $\mathrm{Sp}(2g, \mathbf{Z})$. These, as outlined by Mumford in [Mu2], may be viewed as effective divisors on partial compactifications of the moduli space of principally polarized abelian varieties and then as divisors on $\bar{\mathcal{M}}_g$ via the Torelli map. As g approaches ∞, it is conjectured that there are many such divisors with δ_0-slope arbitrarily close to 0. The related conjecture that the slope s_{δ_0} associated to the divisor class δ_0 is likewise bounded from below by $6 + 12/(g + 1)$ would imply that each such divisor must contain $\mathcal{M}_g$ and hence give rise to a new Schottky relation.

Finally, the conjecture (0.1) is related via string theory to problems in physics. In [MJNS] it is argued that the so-called cosmological constant α_g—i.e. the vacuum amplitude of the heterotic string of perturbation order g in a flat background—vanishes for any g for which the slope s_g is at least 4. The conjecture would therefore imply this statement for all orders. In fact, the vanishing of α_g follows from a weaker non-effectiveness result which Chang and Ran [Chang-Ran 3] were able to verify after this paper was completed.

The slope 4 also has a geometrical significance which was pointed out to us by Beauville. If Z is a curve of genus z lying on $\bar{\mathcal{M}}_g$ and $\pi: S \to Z$ is the universal curve over Z (i.e. its minimal semi-stable model), let τ_S denote the signature of S. We claim $\tau = 4\lambda_Z - \delta_Z$ and hence that $s_g \geqq 4$ if through a generic point $[C]$ of $\bar{\mathcal{M}}_g$ there passes a curve Z such that $\tau \leqq 0$. This follows from the equality $\tau_S = 4\chi(\mathcal{O}_S) - \chi_{\mathrm{top}}(S)$ and two straight-forward calculations: first, that $\chi(\mathcal{O}_S) = \deg_Z(\pi*(\omega_{S/Z})) + (z - 1)(g - 1) = \lambda_B + (z - 1)(g - 1)$; and second, viewing δ_Z as the number of exceptional curves in the fibres of π as in Theorem 3.1, that $\chi_{\mathrm{top}}(S) = \delta_Z + 4(z - 1)(g - 1)$.

If one believes the conjecture, then the known upper bounds for s_g are sharp. It is therefore natural to concentrate on trying to get lower bounds. Indeed, some information on the Schottky problem would follow from any positive lower bound independent of g. At this point, we must confess that we have been unable to obtain such a result. The statement of our main estimate (3.15) involves some combinatorial functions whose definitions we do not wish to pause to give here. Indeed, these functions are sufficiently intractable that we have not attempted to evaluate our estimates in closed form for general g. Their main thrust is that

$$s_g \geqq O(1/g) \text{ as } g \to \infty . \tag{0.3}$$

An heuristic argument suggests that our estimates yield, more precisely,

$$s_g = 576/5g + O(1/g)$$

but numerical evaluation shows only that $s_g > 13/2$ for $g < 10$, so we are unable to determine the Kodaira dimension of $\bar{\mathcal{M}}_g$ for any new values of g. These calculations also show that $s_g > 4$ for $g < 20$ and hence yield in this range the vanishing of the vacuum amplitude of the heterotic string mentioned above.

We are, however, able to give algorithms for these functions which enable us to evaluate our estimates when g is small. These show,

Theorem 0.4 (cf. 3.30).1) *For* $g = 2$, 3 *and* 5, s_g *has the value* $6 + 12/(g + 1)$ *predicted by Conjecture* (0.1).

2) *For* $g = 4$, $53/6 \geqq s_4 \geqq 46{,}759{,}680/5{,}550{,}633 \sim 8.42$. *In particular, when* $(g + 1)$ *is prime,* s_g *may be strictly greater than* $6 + 12/(g + 1)$. □

We also believe that irreducible divisors of slope s_g should consist of curves having some special geometric character. We will deliberately leave this last adjective vague but our prototypes here are the loci of curves possessing a linear series of Brill-Noether number -1. We are able to verify this belief in two cases.

Corollary 0.5 (cf. 3.30). 1) *For* $g = 3$, *the only effective irreducible divisor of slope less than* $28/3$ *is the divisor of hyperelliptic curves.*

2) *For* $g = 5$, *the only divisor of slope less than* $29524/3659$ (~ 8.07) *is the divisor of trigonal curves.* □

Let us next briefly explain our approach. Suppose that Z a curve in $\overline{\mathcal{M}}_g$ and set $\lambda_Z = \deg_Z \lambda$ and $\delta_Z = \deg_Z \delta$. Our method is based on the simple remark that:

$$\text{If } D \text{ is an effective divisor with class } a\lambda - b\delta \text{ and } Z^* \text{ is a deformation of } Z\text{, then either } D \cdot Z^* = a\lambda_Z - b\delta_Z \geqq 0 \text{ or } Z^* \text{ is contained in } D. \tag{0.6}$$

Hence if the union $\tilde{Z}$ of the deformations of Z is an open subset of $\mathcal{M}_g$, $a/b \geqq s_Z := \delta_Z/\lambda_Z$. But s_g is the infimum of such a/b so $s_g \geqq \delta_Z/\lambda_Z$. The curves we use in this paper are obtained by taking a general b-tuple of sections of $\mathbb{P}^1 \times \mathbb{P}^1$, blowing up their points of intersection, and letting Z be the locus of all k-sheeted branched covers, admissible in the sense of [HM], of the resulting family of b-pointed stable curves of genus 0. We find (3.22) that δ_Z/λ_Z is approximately

$$\frac{72(b-1)}{(b-1)(2k+5) - (9/2)(k)(k-1)} \tag{0.7}$$

The bound (0.3) on s_g follows by using the Riemann–Hurwitz relation

$$b = 2g + 2k - 2 \tag{0.8}$$

and choosing k to maximize (0.7) subject to the constraint that $\tilde{Z}$ be dense.

If $\tilde{Z}$ is not dense we still get some information: the argument in the preceding paragraph actually shows that if D is an effective irreducible divisor then either $s_D \geqq s_Z$ or D contains $\tilde{Z}$. Applying this to the curves Z above gives bounds which, in particular, imply

Corollary 0.8 (cf. 3.25). *If* $g \gg k$ *and* $s_D < 72/(2k + 5)$ *then* D *contains the locus of* k*-gonal curves.*

Combining this with [HM] we obtain perhaps the most picturesque result of our investigation:

Theorem 0.9 (cf. 3.29). *The common base locus of the pluricanonical linear systems on* $\overline{\mathcal{M}}_g$ *contains the hyperelliptic and trigonal loci.*

The gulf which separates our results from our conjecture for large g can be explained in several ways which we shall briefly indicate in what we feel is increasing order of likelihood. The first is that the conjecture is false and our bounds are in fact reasonably sharp. In other words, there are effective divisors of small slope but we simply do not know them. We doubt this to be the case for the same reasons which lead to make the conjecture. The second possibility is that estimates close to or equal to that of the conjecture could be obtained by using other curves Z by the method of this paper. We have tried some variants on this approach—for example, by constructing Z from a family of covers of $\mathbb{P}^1$ with other than simple branch points or from a family of plane curves with nodes—but in each case the bounds obtained were as best equal to those provided by the Z's constructed here. (Of course, this may be evidence more of our lack of ingenuity than of the non-existence of such Z's.) The final explanation, and the one which we tend to credit, is that the gulf is an inherent defect of our method. Another way to view (0.6) is as saying that the cone $\mathbf{E}$ of effective divisors on $\overline{\mathcal{M}}_g$ is contained in the dual $\mathbf{Z}^*$ to the cone $\mathbf{Z}$ spanned by those curves Z such that $\tilde{Z} = \overline{\mathcal{M}}_g$. But there is no reason to suppose that $\mathbf{Z}^*$ is not much larger than $\mathbf{E}$. Consider, for example, the analogous cones in the Neron–Severi group of a surface X. Then $\mathbf{Z}_X$ is spanned by those irreducible curves Z such that $Z \cdot Z \geqq 0$, and (0.6) says that an irreducible curve D on X must satisfy $D \cdot Z \geqq 0$ for all such Z. In fact, we know that for *any* Z, $D \cdot Z$ can be negative only if D has negative self-intersection and $Z = D$. This analogy suggests a more promising alternate strategy for proving (0.1) towards which Corollary 0.7 may be considered as a first step. First, construct other curves Z for which the degrees λ_Z and δ_Z can be computed and for which $\tilde{Z}$ is either all of $\overline{\mathcal{M}}_g$ or is a geometrically defined locus and then by estimating the s_Z's, show that any divisor $D \sim a\lambda - b\delta$ of small slope must contain lots of these geometrically defined loci. (For the Z's constructed here the $\tilde{Z}$'s are either $\overline{\mathcal{M}}_g$ or the k-gonal locus and Corollary 0.8 shows that these loci are contained in divisors of small slope). Finally, use the geometric information so obtained about divisors of small slope to estimate the coefficients a and b directly by intersecting them with test curves as in the [HM].

We conclude this introduction by describing briefly the remainder of the paper. Our geometrical results are entirely contained in §§2–3. In the former, we construct the curves Z on which we base our estimates and various related varieties which are needed to compute λ_Z and δ_Z. In the latter, we compute these degrees and obtain from them the basic estimate (3.15) and the principal results referred to above. We have written up these estimates only for the slope $s_g := s_\delta$ but our methods lead easily to estimates applicable to any effective sum γ of boundary classes. Indeed, to estimate s_γ it suffices to replace the numerator δ_Z of the basic estimate (3.15) by γ_Z which can be computed directly from the coefficients of the δ_i's in γ using 2) of Theorem 3.1. We leave the details to the interested reader.

Section 1 is concerned with certain combinatorial functions which arise in our expressions for λ_Z and δ_Z. We suggest that the reader use this only as a reference for these combinatorial definitions and results; indeed, the reader who is willing to take our arithmetic on faith needs only a few definitions which are repeated at the start of §2.

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§1. Combinatorial preliminaries

In this section, we will define certain purely combinatorial functions associated with the symmetric groups which appear in our estimates for s_g, and give an algorithm for computing them. Our treatment is somewhat more general than will be needed for most of the sequel. We have chosen to work in this generality because we feel, as outlined in the introduction, that these functions have additional applications to the geometry of moduli spaces of curves. We note also that the combinatorial analysis of certain closely related functions is the key ingredient in Harer and Zagier's beautiful computation [HZ] of the Euler characteristic of $\mathcal{M}_g$-cf. the remark following the proof of (1.18). In part A, we introduce a general class of functions $S_{C,Q}$ and show how to compute them in terms of simpler functions S_C. In part B, we show how to compute the S_C's directly and also how they may be expressed in terms of characters of symmetric groups. Lastly, in part C, we define the functions which appear in our estimates for s_g and show how to express these in terms of the $S_{C,Q}$'s.

A. The functions $S_{C,Q}$

First some notation. Let P be a *partition*: P is the set of equivalence classes of some equivalence relation $\equiv_p$ on $\mathbf{N}^+$ such that almost every equivalence class is a singleton. Let $\operatorname{supp}(P)$, the support of P, be the union of the non-singleton classes of $\equiv_p$ and let $|P|$ be the order of $\operatorname{supp}(P)$. We let $\mathcal{P}$ denote the set of all partitions and $\mathcal{P}(k)$ the set of partitions whose support lies in $\{1, \ldots, k\}$. We say $P \leqq P'$ if P is a refinement of P' and say $P \sim P'$ if there is an isomorphism of $\mathbf{N}^+$ transporting $\equiv p$ into $\equiv p'$. We call a partition *standard* and usually denote it by a Q if its cycles are intervals of non-increasing length and let $\mathcal{Q}$ and $\mathcal{Q}(k)$ respectively denote the set of all standard Q and of all standard Q in $\mathcal{P}(k)$. The set $\mathcal{Q}$ is a section of the equivalence relation $\sim$ on $\mathcal{P}$. Each $Q \in \mathcal{Q}$ is determined by the sequence $A_Q = (a_2, a_3, \ldots, a_i, \ldots,)$ in which a_i is the number of cycles of length i in Q. We shall frequently deal with the standard partition with a single non-trivial cycle of length k which we denote (k).

Let $\mathbf{S}^k$ denote the symmetric group on the letters $\{1, \ldots, k\}$, let $\mathbf{S}^\infty$ be the union of all the $\mathbf{S}^k$'s considered as a set of permutations of $\mathbf{N}^+$, and let $\mathbf{e}$ denote the common identity element of all these groups. If V is a finite subset of $\mathbf{S}^\infty$, write $i \equiv_V j$ if some element of V takes i to j. Let P_V be the partition this generates on $\mathbf{N}^+$—the P_V cycles are just the orbits in $\mathbf{N}^+$ of the subgroup generated by V—and let Q_V be the corresponding standard partition. Let C denote a conjugacy class in $\mathbf{S}^\infty$. Consider the map $q: \mathbf{S}^\infty \to \mathcal{Q}$ given by $s \to Q_{\{s\}}$. The fibre $C(s)$ of this map containing the element s is the conjugacy class of s in $\mathbf{S}^\infty$, and by restriction we get a map $q_k: \mathbf{S}^k \to \mathcal{Q}(k)$ whose fibres are the conjugacy classes of $\mathbf{S}^k$. By abuse of language, we will often speak of the conjugacy class $Q \in \mathcal{Q}$ meaning the class $q^{-1}(Q)$: especially, (k) will denote the conjugacy class of a k-cycle, and (1) that of the identity.

Let $[T(k, b)]$ be the set of ordered b-tuples of transpositions in $\mathbf{S}^k$ with typical element $t = (t_1, \ldots, t_b)$. We will be concerned with enumerating certain subsets of $[T(k, b)]$ and will adopt the convention that if such a set is denoted $[N]$, then its order is N. Thus for example

$$T(k, b) = \binom{k}{2}^b$$

If $t \in [T(k, b)]$ write

$$\Pi_t = \prod_{i=1}^{b} t_i$$

and define $P_t = P_{\{t_1, \ldots, t_b\}}$ and $Q_t = Q_{\{t_1, \ldots, t_b\}}$ as above.

If Q is a standard partition and C is a conjugacy class let

$$[\bar{S}_{C,Q}(b)] = \{t \in [T(|Q|, b)] \mid \Pi_t \in C, P_t = Q\}$$
$$[S_{C,Q}(b)] = \{t \in [T(|Q|, b)] \mid \Pi_t \in C, P_t \sim Q\}\,. \tag{1.1}$$

with the usual convention when $b = 0$ the empty product is assigned to the set $[S_{(1),(1)}(0)]$. Observe that both of these sets are empty unless $Q_C \leqq Q$ and unless b has the same parity as C. The choice, for each $P \sim Q$, of a bijection realizing this equivalence induces a surjection $^{-}$:$[S_{C,Q}(b)] \to [\bar{S}_{C,Q}(b)]$ with fibres of order $\alpha(Q)$, the number of partitions $P \in \mathscr{P}(|Q|)$ such that $P \sim Q$. Hence

$$S_{C,Q}(b) = \alpha(Q)\bar{S}_{C,Q}(b) \quad \text{for all } C \text{ and } b\,. \tag{1.2}$$

Since $\alpha(Q)$ is the order of the $\mathbf{S}^{|Q|}$ orbit of Q in $\mathscr{P}(|Q|)$ and since Q has stabilizer in $\mathbf{S}^{|Q|}$ of order

$$\prod_{i=2}^{\infty} [(i!)^{a_i}(a_i)!]$$

we obtain

$$\alpha(Q) = \frac{|Q|!}{\prod_{i=2}^{\infty} [(i!)^{a_i}(a_i)!]} \tag{1.3}$$

Now let us define

$$[S_{C,Q}(k, b)] = \{t \in [T(k, b)] \mid \Pi_t \in C, Q_t = Q\} \tag{1.4}$$

and

$$[S_C(k, b)] = \{t \in [T(k, b)] \mid \Pi_t \in C\}\,. \tag{1.5}$$

Note that $[S_{C,Q}(|Q|, b)]$ as defined by (1.4) equals $[S_{C,Q}(b)]$ as defined by (1.1). For simplicity, we use the latter notation whenever possible. Note further that each of the subsets A of $\{1, \ldots, k\}$ of order $|Q|$, the set of t in $[S_{C,Q}(k, b)]$ with $\operatorname{supp}(P_t) = A$ has order $S_{C,Q}(b)$. Therefore,

$$[S_C(k, b)] = \bigcup_{Q \leqq (k)} [S_{C,Q}(b)] \tag{1.6}$$

which yields

$$S_C(k,b) = \sum_{Q \leqq (k)} \binom{k}{|Q|} S_{C,Q}(b) \tag{1.7}$$

If P and P' are non-trivial partitions we denote by $(P|P')$ their concatenation: $i \equiv_{(Q'|Q'')} j$ if and only if either $i \equiv_{Q'} j$ or $(i - |Q'|) \equiv_{Q''} (j - |Q'|)$. If Q' and Q'' are standard we write $\{Q'|Q''\}$ for the *standard* partition equivalent to $(Q'|Q'')$. If $Q = \{Q'|Q''\}$, we claim that for any conjugacy class C there are recursion formulae (depending only on C) expressing $S_{C,Q}$ in terms of functions of the form $S_{C',Q'}$ and $S_{C'',Q''}$. We shall give such formulae in a moment for certain simple C's in a way which will make it clear how to obtain them in general.

Let us first, however, suppose that

for each C in a collection $\mathscr{I}$ of conjugacy classes, we have such formulae with C' and C'' also in $\mathscr{I}$. (1.8)

We then obtain an algorithm to compute the functions $S_{C,Q}$ for all $C \in \mathscr{I}$ as follows. First note the trivial formulae

$$S_{(1),(2)}(b) = \begin{cases} 0 \text{ if } b \text{ is even} \\ 1 \text{ if } b \text{ is even} \end{cases}$$

$$S_{C,(2)} = 0 \text{ for all other } C \tag{1.9}$$

$$S_{(2),(2)}(b) = \begin{cases} 1 \text{ if } b \text{ is odd} \\ 0 \text{ if } b \text{ is even} \end{cases}$$

Next observe that every standard partition Q except the partitions (k) may be written in the form $\{Q'|Q''\}$ with each of $|Q'|$ and $|Q''|$ smaller than $|Q|$. For example, if $Q = (1234)(56)$, then $Q = (Q'|Q'')$ with $Q' = (1234)$ and $Q'' = (12)$. Hence the set of $S_{C,Q}$'s with $C \in \mathscr{I}$ and $|Q| < k$, determine the set of $S_{C,Q}$'s with $C \in \mathscr{I}$ and $|Q| \leq k$ but $Q \neq (k)$. By (1.7), these functions together with the functions $S_C(k,b)$ for all $C \in \mathscr{I}$ determine the functions $S_{C,(k)}(b)$ for all $C \in \mathscr{I}$. Theorem 1.21 gives a formula expressing the functions $S_C(k,b)$ in terms of the characters of $\mathbf{S}^k$ and Lemma 1.16 computes these functions in elementary terms yielding the claimed algorithm.

Unfortunately the calculations required to bring the recursions for $S_{C,Q}$ given by an expression $Q = \{Q'|Q''\}$ into closed form become intractable very quickly as the complexity of the cycle structure of C increases. For our application all we need are recursions meeting the condition of (1.8) for the set $\mathscr{I} = \{(1),(3)\}$ which we now give. Let

$$\beta(Q) = \frac{|Q|!}{\prod_{i=2}^{\infty} (a_i)!} \tag{1.10}$$

Proposition 1.11. *If $Q = \{Q'|Q''\}$, then*

i) $$S_{(1),Q}(2m) = \frac{\beta(Q)}{\beta(Q')\beta(Q'')} \sum_{j=1}^{m-1} \binom{2m}{2j} S_{(1),Q'}(2j) \quad S_{(1),Q''}(2(m-j))$$

ii) $$S_{(3),Q}(2m) = \frac{\beta(Q)}{\beta(Q')\beta(Q'')} \sum_{j=1}^{m-1} \binom{2m}{2j} \Big[S_{(1),Q'}(2j) \quad S_{(3),Q''}(2(m-j)) + S_{(3),Q'}(2j) \quad S_{(1),Q''}(2(m-j)) \Big]$$

Proof. First, a general observation. If $t \in [T(|Q|, m)]$ and $Q_t = Q = \{Q'|Q''\}$, then each t_i in t transposes either two elements of $\{1, \ldots, |Q'|\}$ or two elements of $\{|Q'| + 1, \ldots, |Q'| + |Q''|\}$. Suppose there are j of the former: since Q' and Q'' are non-trivial, $1 \leqq j \leqq m - 1$. Let t' be obtained by taking the former j of the t_i's in the order they occur in t, and let t'' be the product of the latter $(m - j)$. We call the pair (t', t'') the unshuffling of t corresponding to the concatenation $Q = \{Q'|Q''\}$. The first key property of this operation is that $\Pi_t = \Pi_{t'} \Pi_{t''}$. The second is that t' is naturally an element of $[T(|Q'|, j)]$ and that, by shifting down by $|Q'|$ the indices in its transpositions, t'' may be identified with an element of $[T(|Q''|, m - j)]$. (In the sequel, we continue to tacitly make this identification.)

Now suppose that $t \in [\bar{S}_{(1),Q}(2m)]$; that is $\Pi_t = \mathbf{e}$. Then we must have $\Pi_{t'} = \Pi_{t''} = \mathbf{e}$, $Q_{t'} = Q'$, and $Q_{t''} = Q''$. Since $\mathbf{e}$ is even, each of t' and t'' must involve an even number of transpositions. Therefore, unshuffling gives a map

$$[\bar{S}_{(1),Q}(2m)] \to \bigcup_{j=1}^{m-1} ([\bar{S}_{(1),Q'}(2j)] \times [\bar{S}_{(1),Q''}(2(m-j))]) \tag{1.12}$$

This map is surjective and the order of its fiber over an element of the j^{th} term of the union is

$$\binom{2m}{2j}$$

the number of shufflings of t' and t''. Now observe that $(\alpha(Q)/\beta(Q)) = (\alpha(Q')/\beta(Q'))\cdot(\alpha(Q'')/\beta(Q''))$. Using this and (1.2), we compute

$$\begin{aligned} S_{(1),Q}(2m) &= \alpha(Q)\bar{S}_{(1),Q}(2m) \\ &= \beta(Q)\frac{\alpha(Q')\alpha(Q'')}{\beta(Q')\beta(Q'')} \sum_{j=1}^{m-1} \binom{2m}{2j} \bar{S}_{(1),Q'}(2j) \quad \bar{S}_{(1),Q''}(2(m-j)) \\ &= \frac{\beta(Q)}{\beta(Q')\beta(Q'')} \sum_{j=1}^{m-1} \binom{2m}{2j} S_{(1),Q'}(2j) \quad S_{(1),Q''}(2(m-j)) \end{aligned}$$

which is i).

If $t \in [S_{(3),Q}(2m)]$ and t unshuffles to (t', t''), then as above $Q_{t'} = Q'$ and $Q_{t''} = Q''$. But now either $\Pi_{t'} = \mathbf{e}$ and $\Pi_{t''} \in (3)$, or $\Pi_{t'} \in (3)$ and $\Pi_{t''} = \mathbf{e}$. Unshuffling therefore surjects $[\bar{S}_{(3),Q}(2m)]$ onto

$$\bigcup_{j=1}^{m-1} \{([\bar{S}_{(1),Q'}(2j)] \times [\bar{S}_{(3),Q''}(2(m-j))]) \cup ([\bar{S}_{(3),Q'}(2j)] \times [\bar{S}_{(1),Q''}(2(m-j))])\}$$

with fibers of order $\binom{2m}{2j}$ over the jth term in the union. Now an argument exactly like the one above proves ii). □

B. The functions S_C

We now wish to study the functions $S_C(k, b)$. It will be convenient to introduce the space V_k of $\mathbb{Z}$-valued class functions on $\mathbf{S}^k$ which we identify as usual with the center of the integral group ring of $\mathbf{S}^k$. Let γ_C be the characteristic function of the conjugacy class C and if $f \in V_k$, write

$$f = \sum_{C \in \mathcal{Q}(k)} f_C \gamma_C .$$

Now, viewing C and C' as conjugacy classes in $\mathbf{S}^k$ define

$$U_{C,C'} = |\{(t, c') \in (2) \times C' | tc' \in C\}|$$
$$W_{C,C'} = U_{C,C'}/|C'| . \tag{1.13}$$

Then for any $c' \in C'$, $W_{C,C'} = |\{t \in (2) | tc' \in C\}|$. We let $W(k)$ be the $|\mathcal{Q}(k)|$ by $|\mathcal{Q}(k)|$ matrix whose (C, C') entry is $W_{C,C'}$, and define a vector $S(k, b)$ by

$$S(k, b) = W(k)^b \gamma_{(1)} . \tag{1.14}$$

Lemma 1.15. $S(k, b) = \sum_{C \in \mathcal{Q}(k)} S_C(k, b) \gamma_C$

Proof. The case $b = 0$ is clear as $S_C(k, b)$ is 1 if $C = (1)$ and is 0 otherwise. The lemma will therefore follow by induction if we show that

$$\sum_{C \in \mathcal{Q}(k)} S_C(k, b)\gamma_C = W(k) \sum_{C \in \mathcal{Q}(k)} S_C(k, b-1)\gamma_C$$

or equating coordinates that

$$S_C(k, b) = \sum_{C' \in \mathcal{Q}(k)} w_{C,C'} S_{C'}(k, b-1) .$$

If $t \in [S_C(k, b)]$ write $t = (t_1, t')$. The map $t \to t'$ sends $[S_C(k, b)]$ to

$$\bigcup_{C' \in \mathcal{Q}(k)} [S_{C'}(k, b-1)] .$$

The fiber of this map over an element t' of the C' term of the union has order $|\{t \in [T(k, 1)] | t\Pi_{t'} \in C\}|$. As $\Pi_{t'} \in C'$, this order is $w_{C,C'}$ and summing completes the induction. □

The lemma reduces the computation of the S_C's to that of the $w_{C,C'}$'s. To find these, let $Q = Q_C$ and $Q' = Q_{C'}$ with Q and Q' in $\mathcal{Q}(k)$. Write $A_Q = (a_2, \ldots, a_i, \ldots,)$ and $A_{Q'} = (a'_2, \ldots, a'_i, \ldots,)$ and set $a_1 = k - |Q|$. We omit the straightforward proof of the

Lemma 1.16. *Unless we are in one of the four cases below,* $w_{C,C'} = 0$.

Case	*Relation of* A_C *and* $A_{C'}$	$w_{C,C'}$
i)	$a_i = a'_i + 1$, $a_j = a'_j + 1$ ($i \neq j$), $a_{i+j} = a'_{i+j} - 1$, $a_l = a'_l$ $l \neq i, j, i+j$	$(i+j)a'_{i+j}$
ii)	$a_i = a'_i - 1$, $a_j = a'_j - 1$ ($i \neq j$), $a_{i+j} = a'_{i+j} + 1$, $a_l = a'_l$ $l \neq i, j, i+j$	$ija'_i a'_j$
iii)	$a_i = a'_i + 2$, $a_{2i} = a'_{2i} - 1$, $a_l = a'_l$ $l \neq i, 2i$	ia'_{2i}
iv)	$a_i = a'_i - 2$, $a_{2i} = a'_{2i} + 1$, $a_l = a'_l$ $l \neq i, 2i$	$i^2 a'_i(a'_i - 1)/2$

□

This result completes our algorithm for computing the functions S_C and hence the functions $S_{C,Q}$. We would like to conclude with one complement which allows us to express $S_C(k, b)$ in a closed form involving only the characters of $\mathbf{S}^k$. We index the irreducible characters χ of $\mathbf{S}^k$ as usual by the elements of $\mathcal{Q}(k)$—cf. [Jam–Ker]—and considering these as elemets of V_k, write

$$\chi_Q = \sum_{C \in \mathcal{Q}(k)} \chi_Q(C)\gamma_C$$

Then set

$$\xi_Q = \sum_{C \in \mathcal{Q}(k)} |C|\chi_Q(C)\gamma_C \quad \text{and} \quad \mu_Q = \frac{\binom{k}{2}\chi_Q((2))}{\chi_Q((1))} \tag{1.17}$$

Proposition 1.18. $W(k)\xi_Q = \mu_Q \xi_Q$.

Proof. This is a mild twist on standard results in character theory; we sketch the proof referring the reader to [Burnside, §223] for more details. For a moment let S be any finite group and let $\mathcal{Q}$ index its conjugacy classes with (1) denoting the class of the identity and $\gamma_C \in \mathbb{Z}[S]$ the characteristic function of the class C. In $\mathbb{Z}[S]$, the equations

$$\gamma_{C''}\gamma_C = \sum_{C' \in \mathcal{Q}} \omega_{C,C',C''}\gamma_{C'} \tag{1.19}$$

are uniquely solvable because the γ_C's form a basis of the center of $\mathbb{Z}[S]$. In fact, $\omega_{C,C',C''} = (1/|C'|)|\{(g, g', g'') \in C \times C' \times C'' \,|\, g''g = g'\}|$ is easily seen to be an integral solution. Since the γ_C's are central they act as scalars on any irreducible S-module. Applying an irreducible representation with character χ to (1.19) and taking traces gives

$$\left(\frac{|C|\chi(C)}{\chi((1))}\right)\left(\frac{|C''|\chi(C'')}{\chi((1))}\right)=\sum_{C'\in\mathcal{Q}}\omega_{C,C',C''}\left(\frac{|C'|\chi(C')}{\chi((1))}\right)$$

which after rearrangement becomes

$$\sum_{C'\in\mathcal{Q}}\omega_{C,C',C''}|C'|\chi(C')=\left(\frac{|C''|\chi(C'')}{\chi((1))}\right)(|C|\chi(C)) \tag{1.20}$$

Now suppose that C'' is a class of elements of order 2 and $g''\in C''$. Then the equations $g''g=g'$ and $g=g''g'$ are equivalent so

$$\begin{aligned}\omega_{C,C',C''}&=(1/|C'|)|\{(g,g',g'')\in C\times C'\times C''|g=g''g'\}|\\&=(1/|C'|)|\{(g',g'')\in C'\times C''|g''g'\in C\}|\,.\end{aligned}$$

In particular, if $S=\mathbf{S}^k$ and $C''=(2)$, this shows that $\omega_{C,C',C''}=w_{C,C'}$ and if we substitute in (1.20) the quantities defined in (1.17) it becomes the assertion of the proposition. □

Remark. The ω's have appeared in other geometric contexts. In $\mathbf{S}^{2k}$, suppose that C denotes the class of a $(2k)$ cycle and C'' that of k disjoint transpositions. Then if we sum $\omega_{C,C',C''}$ over those C' which have exactly $k+1-2g$ cycles, we obtain, up to trivial factors, the function $\varepsilon_g(k)$ analysed by Harer and Zagier in [HZ].

Let us now obtain the promised closed form for $S_C(k,b)$. Let, for a moment, [,] denote the inner product on V_k for which the γ_C's are of length $1/|C|$ and pairwise orthogonal. Then

$$\begin{aligned}[\xi_Q,\xi_{Q'}]&=\left[\sum_{C\in\mathcal{Q}(k)}|C|\chi_Q(C)\gamma_C,\ \sum_{C'\in\mathcal{Q}(k)}|C'|\chi_{Q'}(C')\gamma_{C'}\right]\\&=\sum_{C\in\mathcal{Q}(k)}|C|\chi_Q(C)\chi_{Q'}(C)=(k!)\delta_{Q,Q'}\end{aligned}$$

by the usual orthogonality relations for irreducible characters. The ξ_Q therefore form a basis of V_k. Let a_Q for $Q\in\mathcal{Q}(k)$ be the coordinates of $\gamma_{(1)}$ in this basis. Then

$$\begin{aligned}\gamma_{(1)}=\sum_{Q\in\mathcal{Q}(k)}a_Q\xi_Q&=\sum_{Q\in\mathcal{Q}(k)}a_Q\left(\sum_{C\in\mathcal{Q}(k)}|C|\chi_Q(C)\gamma_C\right)\\&=\sum_{C\in\mathcal{Q}(k)}\left(\sum_{Q\in\mathcal{Q}(k)}a_Q\chi_Q(C)\right)|C|\gamma_C\,.\end{aligned}$$

In other words, we may characterize the a_Q as the unique solutions of the equations

$$\sum_{Q\in\mathcal{Q}(k)}a_Q\chi_Q((1))=1\quad\text{and}\quad\sum_{Q\in\mathcal{Q}(k)}a_Q\chi_Q(C)=0\text{ for }C\neq(1)\,.$$

Now recall that

$$\gamma_{(1)} = \frac{1}{k!}\chi_{\text{reg}} = \frac{1}{k!}\sum_{Q\in\mathcal{Q}(k)}\chi_Q((1))\chi_Q$$
$$= \frac{1}{k!}\sum_{C\in\mathcal{Q}(k)}\left(\sum_{Q\in\mathcal{Q}(k)}\chi_Q((1))\chi_Q(C)\right)\gamma_C .$$

We must therefore have $a_Q = \chi_Q((1))/k!$, and hence

$$S(k,b) = W(k)^b\gamma_{(1)}$$
$$= W(k)^b\sum_{Q\in\mathcal{Q}(k)}\frac{\chi_Q((1))}{k!}\xi_Q$$
$$= \sum_{Q\in\mathcal{Q}(k)}\mu_Q^b\frac{\chi_Q((1))}{k!}\xi_Q \text{ by (1.18)}$$
$$= \frac{\binom{k}{2}^b}{k!}\sum_{C\in\mathcal{Q}(k)}|C|\left(\sum_{Q\in\mathcal{Q}(k)}\left[\frac{\chi_Q((2))}{\chi_Q((1))}\right]^b\chi_Q((1))\chi_Q(C)\right)\gamma_C$$

Equating the coefficients of γ_C in this expression and in Lemma 1.15 yields

Theorem 1.21. $S_C(k,b) = \dfrac{\binom{k}{2}^b}{k!}|C|\displaystyle\sum_{Q\in\mathcal{Q}(k)}\left[\frac{\chi_Q((2))}{\chi_Q((1))}\right]^b\chi_Q((1))\chi_Q(C)$ □

Remark. After we had completed this work, Basil Gordon pointed out to us that this result for $S_{(1)}$ was already known to Hurwitz [Hu]. He was actually interested—as we will be in part C and in the applications—in the function $S_{(1),(k)}(k,b)$ which counts the number of k-sheeted *connected* coverings of $\mathbb{P}^1$ simply branched at b fixed points; the function $S_{(1)}(k,b)$ counts all such covers connected or not. A now standard combinatorial argument (cf. [Gou–Ja]) shows that in such a situation the exponential generating function for $S_{(1),(k)}$ is the natural logarithm of the exponential generating function for $S_{(1)}$. (Warning: the corresponding statement is *not* true for any other conjugacy class C.) Hurwitz discovered this relation by elementary considerations and to our knowledge his paper is the earliest example of this principle in the literature. Our proof of (1.21) is shorter than Hurwitz's and is couched in terms more familiar to the modern reader so we have chosen to include it here.

One corollary of the Theorem is that the asymptotic behaviour in b of $S_C(k,b)$ is what one would expect.

Corollary 1.22. *As b increases keeping the same parity as C,*

$$\frac{S_C(k,b)}{\binom{k}{2}^b} \text{ approaches } \frac{2|C|}{k!}.$$

Proof. When b and C have the same parity, the terms in the sum in 1.21 corresponding to the trivial and sign characters of $\mathbf{S}^k$ each contribute 1 to that sum. If χ is any irreducible character of $\mathbf{S}^k$ then $|\chi_C| \leqq \deg \chi = \chi((1))$ with equality if and only if elements of C act by homotheties in the corresponding representation ρ. The characters of $\mathbf{S}^k$ are integer valued [JK] so such a homothety can only be $\pm$ Id. The transpositions generate $\mathbf{S}^k$ so if ρ acts by $\pm$ Id on (2), then it does so on all of $\mathbf{S}^k$: i.e. ρ is the trivial or sign character. Hence for all other χ, the χ^{th} term of the sum in 1.21 tends to zero as b increases from which the corollary follows. □

C. Related functions that count admissible covers

Here we shall define the functions which arise in our estimates for s_g from counting certain types of admissible covers and express them in terms of the functions $S_{C,Q}$. We shall be interested in subsets of $[S_{(1),(k)}(b)]$ which to simplify notation we shall rename $[N(k,b)]$. While our approach will be purely combinatorial, the geometric interpretation of these functions being postponed until §2, we mention that $N(k,b)/k!$ counts the k-sheeted connected covers of $\mathbb{P}^1$ simply branched at b fixed points. Since this set is empty unless b is even we shall suppose this henceforth. We make three other notational conventions. First, in equations between sets or functions depending on k and b, we suppress these variables if they are equal for all terms. Second, when a conjugacy class or partition (k) is used to subscript a subset of N we drop the brackets, writing, for example, N_3 for $N_{(3)}$. Lastly, in a similar vein we will use (k,j) to denote the concatenation of a k-cycle and a j-cycle and abuse this notation by allowing k or j to equal 0 or 1: when, say, j is 0 or 1, (k,j) is to be interpreted as a k-cycle. Before turning to these new functions, we note one fact which will be needed in §2.

Lemma 1.24. *The action of* $\mathbf{S}^k$ *on* $[N(k,b)]$ *by simultaneous conjugation of the b-tuples of transpositions is trivial for* $k=2$ *and has trivial stabilizers for* $k \geqq 3$.

Proof. The case $k=2$ is clear. If $k \geqq 3$ and $t \in [N(k,b)]$, then $Q_t = (k)$ so the subgroup of $\mathbf{S}^k$ generated by $\{t_1, \ldots, t_b\}$ is transitive. Now consider the graph Γ with vertices $1, \ldots, k$ in which an edge joins i and j if and only if some $t_1 = (ij)$. Transitivity means that Γ is connected and since $k \geqq 3$ this forces Γ to have a vertex i of valence more than 1. If a permutation stabilizes t then it must either fix or exchange the ends of each edge of Γ. Therefore any vertex i of valence more than 1 must be fixed and so must any vertex connected to i. Since Γ is connected, the permutation is the identity. □

We begin to break up $[N(k,b)]$ by classifying the elements t according to the conjugacy class of the product of the first pair of transpositions in t. We fix k and b once and for all. Write $t = (t_1, t_2, t^*)$ and let

$$\begin{aligned} [N_1] &= \{t \in [N] \mid t_1 t_2 \in (1)\} = \{t \in [N] \mid t_1 = t_2\} \\ [N_{2,2}] &= \{t \in [N] \mid t_1 t_2 \in (2,2)\} = \{t \in [N] \mid t_1 \cap t_2 = \varnothing\} \qquad (1.25) \\ [N_3] &= \{t \in [N] \mid t_1 t_2 \in (3)\} = \{t \in [N] \mid |t_1 \cap t_2| = 1\} \end{aligned}$$

and say that the t in each set are of type 1, (2, 2) or 3 respectively. (In the right hand column of (1.25), we have abused notation by writing t to mean $\operatorname{supp}(P_t)$. By definition

$$N = N_1 + N_{2,2} + N_3 . \tag{1.26}$$

If $t = (t_1, t_2, t^*)$ is of type (2, 2) and $t_i = (p_i q_i)$ then $\Pi_{t*}(p_i) = q_i$. Therefore $\equiv_t$ and $\equiv_{t*}$ are the same and $Q_{t*} = (k)$. Since $\Pi_{t*} = (t_1 t_2)^{-1} \in (2, 2)$, t^* lies in $[S_{(2,2),(k)}(b-2)]$. Conversely, such a t^* determines t_1 and t_2 up to order so

$$N_{2,2}(k, b) = 2 \cdot S_{(2,2),(k)}(k, b-2) \tag{1.27}$$

Remark. To use (1.27) to compute $N_{2,2}$, we would need to give an analogue of Proposition 1.11 for the class (2, 2). Although this is not difficult, we have chosen to proceed as follows: we shall see below how to express N_1 and N_3 in terms of the functions treated in (1.11) and N is by definition such a function; we can therefore use (1.26) to evaluate $N_{2,2}$.

If t is of type 3, arguments like those in the preceding paragraph show that $Q_{t*} = (k)$ and that $\Pi_{t*} \in (3)$, hence that $t^* \in [S_{(3),(k)}(b-2)]$. For each such t^*, there are three pairs (t_1, t_2) such that $t_1 t_2 = (\Pi_{t*})^{-1}$ so

$$N_3(k, b) = 3S_{(3),(k)}(b-2) . \tag{1.28}$$

The t of type 1 are more complicated and to analyse them we introduce, for $0 \leqq j \leqq [k/2]$, the sets

$$[M_j] = \{t \in [N_1] \mid Q_{t*} = (k-j, j)\} . \tag{1.29}$$

Note that if t is of type 1, then $\Pi_{t*} = \mathbf{e}$ and that since $t_1 = t_2$ at most one pair of cycles in Q_{t*} can coalesce in Q_t. Either $Q_{t*} = (k)$ and $t \in [M_0]$ or two cycles of lengths j and $(k-j)$ with $1 \leqq j \leqq [k/2]$ do coalesce and then $t \in [M_j]$.

i.e.
$$[N_1] = \bigcup_{j=0}^{[k/2]} [M_j] \tag{1.30}$$

Now if $t \in [M_j]$ then $t^* \in [S_{(1),(k-j,j)}(k, b-2)]$. If $j = 0$, t_1 is arbitrary so

$$M_0(k, b) = \binom{k}{2} S_{(1),(k)}(b-2) \tag{1.31}$$

If $j \geqq 1$, t_1 must connect the j-cycle and the $(k-j)$-cycle of P_{t*} so there are $j(k-j)$ choices for t_1. If $j = 1$, then $S_{(1),(k-1,1)}(k, b-2) = S_{(1),(k-1)}(k, b-2) = kS_{(1),(k-1)}(b-2)$ so

$$M_1(k, b) = k(k-1)S_{(1),(k-1)}(b-2) . \tag{1.32}$$

If $j > 1$, $S_{(1),(k-j,j)}(k, b-2) = S_{(1),(k-j,j)}(b-2)$, so

$$M_j(k, b) = j(k-j)S_{(1),(k-j,j)}(b-2), \quad \text{for } 2 \leqq j \leqq [k/2] . \tag{1.33}$$

Combining these we find that

$$N_1(k, b) = \binom{k}{2} S_{(1),(k)}(b-2) + k(k-1)S_{(1),(k-1)}(b-2) + \sum_{j=2}^{[k/2]} j(k-j)S_{(1),(k-j,j)}(b-2) \tag{1.34}$$

In §2, we will need to partition N_1 in one other way. Define a function $g = g(k, b) = b/2 - k + 1$—recall b is even. In the applications, g will be the genus of a connected k-sheeted cover of $\mathbb{P}^1$ simply branched at b points so this definition just translates the Riemann–Hurwitz formula. Conversely, each element of $[N(k, b)]$ defines, by Hurwitz's construction, an irreducible cover of $\mathbf{P}^1$ of genus $g = g(k, b)$. Since such a cover has non-negative genus, we see that if $[N(k, b)]$ is non-empty—and, a fortiori, if $[N_1(k, b)]$ is non-empty—then $b \geqq 2(k-1)$. If $t \in [M_j]$ and $j \geqq 2$, let (t', t'') be the unshuffling of t^* corresponding to the concatenation $Q_{t*} = (k-j, j)$ as in the proof of Proposition 1.11. Suppose that t' has length b' and t'' has length b'' and let $g' = g(k-j, b') = b'/2 + (k-j) + 1$ and $g'' = g(j, b'') = b''/2 - j + 1$. Note that $b' + b'' = b - 2$ and that $g' + g'' = g$. Now if $0 \leqq i \leqq g$, let

$$\begin{aligned} [M_{j,i}] &= \{t \in [M_j] \mid g' = i, g'' = g - i\} \\ &= \{t \in [M_j] \mid b' = 2(k-j) + 2i - 2, b'' = 2j + 2(g-i) - 2\}\,. \end{aligned} \tag{1.35}$$

Then $[M_0] = [M_{0,g}]$, $[M_1] = [M_{1,g}]$ and for $j \geqq 2$

$$[M_j] = \bigcup_{i=0}^{g} [M_{j,i}] \tag{1.36}$$

If $1 \leqq i < [g/2]$, let

$$[O_i] = \bigcup_{j=2}^{[k/2]} ([M_{j,i}] \cup [M_{j,g-i}]) \tag{1.37}$$

and let

$$[O_{g/2}] = \bigcup_{j=2}^{[k/2]} [M_{j,i}] \quad \text{if } g \text{ is even and}$$

$$[O_0] = [M_1] \cup \left[\bigcup_{j=2}^{[k/2]} ([M_{j,0}] \cup [M_{j,g}]) \right]. \tag{1.38}$$

Finally, let

$$[N_{\text{sing}}] = [M_0] \cup \left(\bigcup_{i=1}^{[g/2]} [O_i] \right). \tag{1.39}$$

Recall that the unshuffling corresponding to the concatenation $Q = (k-j, j)$ takes $[\bar{S}_{(1),(k-j,j)}(k, b-2)]$ onto

$$\bigcup_{i=0}^{g} [\bar{S}_{(1),(k-j)}(2(k-j+i-1))] \times [\bar{S}_{(1),(j)}(2(j+g-i-1))]$$

with fiber over the elements of the ith term of the union of order

$$\binom{b-2}{2(k-j+i-2)}$$

(In the proof of (1.11), we took the union over a larger range for i, but the discussion above shows that all the terms we have dropped are empty.) If $t \in [M_{j,i}(k, b)]$, then

t^* will lie in $[S_{(1),(k-j,j)}(k, b-2)]$. For any choice of the surjection $\bar{\ }$: $[S_{(1),(k,-j,j)}(k, b-2)] \to [\bar{S}_{(1),(k-j,j)}(k, b-2)]$ discussed above (1.2) the unshuffling of $\bar{t}^*$ will lie in the ith term of the union above. Define $\gamma(k,j) := j(k-j)\alpha((j, k-j))$. Then we have shown for $j \geqq 2$

$$M_{j,i}(k,b) = \gamma(k,j)\binom{b-2}{2(k-j+i-1)} N(k-j, 2(k-j+i-1))N(j, 2(j+g-i-1))$$

Reassembling these values we obtain

$$\begin{aligned} O_i(k,b) = \sum_{j=2}^{[k/2]} \gamma(k,j)\Bigg[&\binom{b-2}{2(k-j+i-1)} N(k-j, 2(k-j+i-1)) \\ &\times N(j, 2(j+g-i-1)) + \binom{b-2}{2(k-j+g-i-1)} \\ &\times N(k-j, 2(k-j+g-i-1))N(j, 2(j+i-1))\Bigg] \end{aligned} \tag{1.40}$$

if $i < g/2$ and to one half this sum if g is even and $i = g/2$. Finally,

$$\begin{aligned} N_{\text{sing}}(k,b) = \binom{k}{2} N(k, b-2) \sum_{i=1}^{g-1} \Bigg(\sum_{j=2}^{[k/2]} \gamma(k,j)\Bigg[&\binom{b-2}{2(k-j+i-1)} \\ &\times N(k-j, 2(k-j+i-1))N(j, 2(j+g-i-1))\Bigg]\Bigg) \end{aligned} \tag{1.41}$$

since the i^{th} and $(g-i)^{\text{th}}$ terms in the last sum give the right hand side of (1.40).

§2. The basic construction

In this section, our goal is to outline the construction of certain curves in Z in $\overline{\mathcal{M}}_g$. We maintain the notation developed in §1 for the combinatorial functions discussed there. (For the benefit of the reader who has turned directly to this section, we repeat a few definitions. We denote by $[N]$ and N a finite set and its order respectively. If $t = (t_1, \dots, t_b)$ is an ordered b-tuple of simple transpositions in $\mathbf{S}^k$, the symmetric group on k-letters, then let $\Pi_t = t_1 \cdot \ldots \cdot t_b$, let P_t be the partition of $\{1, \dots, k\}$ into orbits under the action of the subgroup generated by the t_i's, and let Q_t be the corresponding standard partition obtained by conjugating P_t so that its cycles are intervals of decreasing length. Let (k) denote a k-cycle and define

$$[N(k,b)] = \{t = (t_1, \dots, t_b) \mid \Pi_t = \mathbf{e}, Q_t = (k)\}.$$

We fix k and b once and for all and will suppress them when possible, writing, for example, N for $N(k,b)$.)

If $[M]$ is a subset of $[N]$ and is invariant under the action of $\mathbf{S}^k$ by simultaneous conjugation of the t_i's, we will denote the quotient $[M]/\mathbf{S}^k$ by $[\tilde{M}]$. By Lemma 1.24, this action is free if $k \geqq 3$ and trivial for $k = 2$ so $M = k!\tilde{M}$ if $k \geqq 3$ and $M = \tilde{M}$ if $k = 2$.

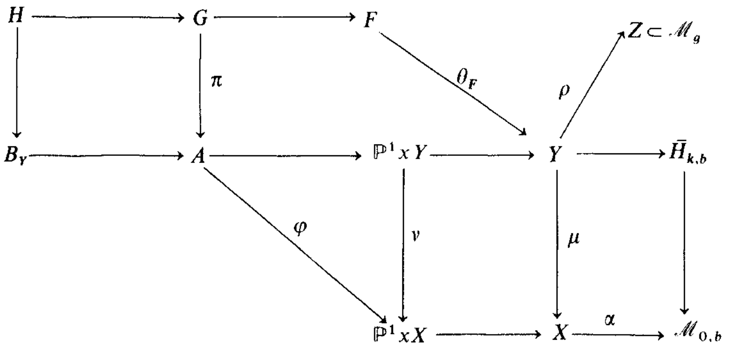
To begin with we take X to be any smooth complete curve. On $\mathbb{P}^1 \times X$, let f_x denote the fibre over a point $x \in X$, let f be the numerical equivalence class of a general fiber and let s be the class of a horizontal section (so $s^2 = 0$). Fix an even integer $b = 2m$ and let $\sigma_i \sim s + c_i f$, $i = 1, \ldots, b$ be b sections of $\mathbb{P}^1 \times X$ which meet transversely everywhere and such that each fibre f_x contains at most one of these intersections. Let $c = \sum_{i=1}^{b} c_i$ and $\sigma = \sum_{i=1}^{b} \sigma_i \sim bs + cf$. Let $[I_X]$ be the set of nodes of σ, or equivalently the set of intersections of the σ_i's. Then

$$I_X = \sum_{i<j} (c_i + c_j) = (b-1)c \tag{2.1}$$

We let B_X be the blow-up of $\mathbb{P}^1 \times X$ at $[I_X]$ and denote by B_x its fibre over $x \in X$ and by $\tilde{\sigma}_i$ the proper transform of σ_i on B_X.

Let $\mathcal{M}_{0,b}$ be the moduli space of b-pointed stable curves of genus 0—as usual, the b marked points are required to be smooth – and let $\alpha\colon X \to \bar{\mathcal{M}}_{0,b}$ be the map which sends a point $x \in X$ to the class of the fiber B_x marked by its b points of intersection with the $\tilde{\sigma}_i$'s. Next let $\bar{\beta}\colon \bar{H}_{k,b} \to \bar{\mathcal{M}}_{0,b}$ be the Hurwitz scheme of admissible covers of stable b-pointed curves of genus 0 constructed in [HM] to which we refer for details on what follows. The points of $\bar{H}_{k,b}$ over the moduli point $[B]$ in $\bar{\mathcal{M}}_{0,b}$ of a curve B are the *connected* k-sheeted covers of $\pi\colon C \to B$ which are unbranched except over the nodes and marked points of B and which have a single simple branch point over each of the b marked points of B.

To see what the covering β looks like, suppose that B is a smooth curve with marked points $P_1, \ldots, P_b$. We call the data consisting of a choice of an unmarked base point $P \in B$, pairwise disjoint loops $\gamma_1, \ldots, \gamma_b$ so that γ_i has winding number δ_{ij} about P_j, and a numbering $Q_1, \ldots, Q_k$ of the fiber of C over P a *description* of π. We then view the monodromy of π around γ_i as an element t_i in the symmetric group $\mathbf{S}^k$ on k letters: the simple branching hypothesis on π means that each t_i is a simple transposition. A covering with monodromy data $t = (t_1, \ldots, t_b)$ exists if and only if, in the notation of §1, $\Pi_t := t_1 \cdot \ldots \cdot t_b = \mathbf{e}$ and is connected if and only $Q_t = (k)$. Moreover, two such sets of data yield isomorphic covers if and only if the corresponding transpositions are simultaneously conjugate in $\mathbf{S}^k$. In fact, the restriction $\beta\colon H_{k,b} \to \mathcal{M}_{0,b}$ of β to the locus of smooth curves is an $\tilde{N} = \tilde{N}(k,b)$ sheeted unramified cover whose general fibre is naturally identified with $[\tilde{N}]$. Note



(2.2)

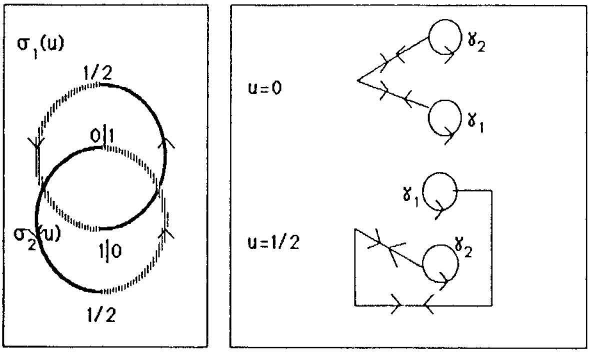
however that β is branched over the boundary of $\overline{\mathcal{M}}_{0,b}$: in the next paragraphs we will in effect determine the branching over codimension one strata. At this point we pause to insert a diagram of our basic construction after which we will precis it.

First we let Y be the fiber product of α and β so that the map $\mu\colon Y \to X$ is again an $\tilde{N}$-sheeted cover. We let $[I_Y] := \nu^*([I_X])$ be the set of singularities of $\tau := \nu^*(\sigma)$ on $\mathbb{P}^1 \times Y$ and let $[J_Y]$ be the projection to Y of $[I_Y]$. Now, over the complement of $[J_Y]$ in Y, we have a nice family of branched covers of $\mathbb{P}^1$; we would like to complete this family of covers to a family of curves $\theta_g\colon G \to Y$ over Y inducing the map $\rho\colon Y \to Z \subset \overline{\mathcal{M}}_g$, and we would like to do so in a way that makes it possible for us to calculate readily the numerical invariants of the family G: in practice, this means we want the family $G \to Y$ to factor as a finite branched cover $G \to A \to Y$ of a family $A \to Y$ of rational curves over Y.

There are a priori two "extreme" ways to do this. The largest would be the family $\theta_H\colon H \to B_Y \to Y$ of admissible covers, where B_Y was the blow-up of $\mathbb{P}^1 \times Y$ at the points of $[I_Y]$ and where the fiber of $H \to B_Y$ over each y was the admissible cover $C_y \to B_y$ represented by the image of y in $\overline{H}_{k,b}$. The smallest (with smooth G, at any rate) would be the family $\theta_F\colon F \to Y$ of semistable curves obtained by taking the surface whose fiber over y was the stable model of C_y (that is, the curve obtained by repeatedly blowing down all rational curves of C_y passing only once or twice through nodes of C_y) and minimally resolving the singularities (all rational double points) of this surface. There are, unfortunately, difficulties involved with each of these choices: H has the defect that (as we shall see) it does not exist; and F, though it does (as we shall see) exist, is not readily described as a branched cover of a family of rational curves over Y. Instead, we will construct and describe a family $\theta_G\colon G \to A \to Y$ that will be intermediate between these two (that is, G will be obtained either by blowing up F or blowing down "H", and A by blowing up $\mathbb{P}^1 \times Y$ at a subset of the points of $[I_Y]$).

The main task in working out this description will be to understand the geometry of diagram (2.2) near a point of $[J_Y]$. Because this geometry is *independent* of which point we choose, all the formulae will also be of a *local* character on Y. This local character can be varied by allowing more special singularities on σ or equivalently by allowing X to approach more special boundary points of $\overline{\mathcal{M}}_{0,b}$. Our bound comes from an analysis of the unique codimension 1 stratum of $\overline{\mathcal{M}}_{0,b}$ but generalizes straightforwardly to yield bounds for other strata. We worked out the case where σ has an ordinary point of total ramification but found the bound obtained inferior to the one we shall now outline and we suspect that this is more generally true.

The first step in our analysis is to determine the branching of Y over X. We know that μ is unramified except over points of X in $[J_X]$. Suppose therefore that σ_1 and σ_2 meet on f_x and that $\mu(y) = x$. We will abuse language slightly and call such points *boundary points* of Y. Pick a small loop Γ, with parameter $u \in [0, 1]$ based at a point x^* near x in X and going once positively around x, a point y^* near y on Y and lying over x^*, and a lift θ of Γ to a path based at y^*. Choose a continuous family of descriptions of the cover C_u of $B_{\Gamma(u)}$ parameterized by $\theta(u)$. As we move around Γ in X, the points $P_1(u)$ and $P_2(u)$ in $B_{\Gamma(u)}$ will circle each other once as shown on the left in Diagram 2.4.



(2.4)

The resulting pictures of the loops γ_1 and γ_2 at times $u = 0$ and $u = 1/2$ is then shown on the right. Homotopically,

$$\gamma_1(1/2) \sim \gamma_2(0)$$

and (2.5)

$$\gamma_2(1/2) \sim \gamma_2(0)^{-1} * \gamma_1(0) * \gamma_2(0) .$$

Going completely around Γ has the effect of iterating this transformation twice so that

$$\gamma_1(1) \sim \gamma_2(0)^{-1} * \gamma_1(0) * \gamma_2(0) \tag{2.6}$$

and

$$\gamma_2(1) \sim (\gamma_2^{-1} * \gamma_1 * \gamma_2)^{-1}(0) * \gamma_2(0) * (\gamma_2^{-1} * \gamma_1 * \gamma_2)(0)$$

Let t_1 and t_2 be the transpositions which describe the branching of the cover C_0 parameterized by $y^* = \theta(0)$ over the points where B_{x*} meets σ_1 and σ_2, and let t'_1 and t'_2 be the new transpositions obtained by following the descriptions of the covers C_u to time $u = 1$. The equations above then say that

$$t'_1 = t_2^{-1} t_1 t_2 \tag{2.7}$$

and

$$t'_2 = (t_2^{-1} t_1 t_2)^{-1} t_2 (t_2^{-1} t_1 t_2) .$$

We say that an element $t \in [N]$ is of type Π if the product $t_1 t_2$, or equivalently $t_3 \cdot \ldots \cdot t_b$, lies in the conjugacy class Π and let $[N_\Pi]$ denote the set of such t. Corresponding to the cases where t_1 and t_2 have 0, 1, or 2 letters in common we have a decomposition by $\mathbf{S}^k$-invariant subsets

$$[N] = [N_1] \cup [N_3] \cup [N_{2,2}] \tag{2.8}$$

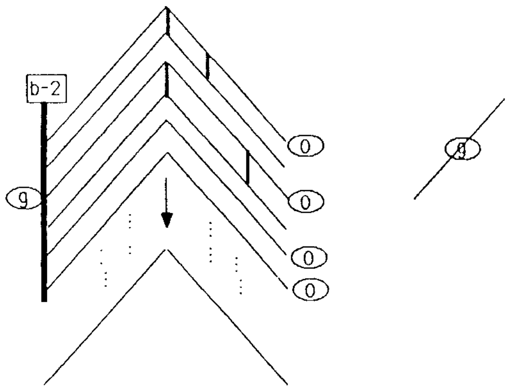
For any point $y \in [J_Y] \subset Y$, if the monodromy data near y is ordered with t_1 and t_2 as above, we will say that y is of *type* Π if its monodromy data is of type Π. This property is independent of the choice of a description of the cover associated to y. We will denote by $[J_{Y,\Pi}]$ the set of such points, and corresponding by $[I_{Y,\Pi}]$ the set of points of $[I_Y] \subset \mathbb{P}^1 \times Y$ lying over points of $[J_{Y,\Pi}]$. The branching of Y over X is then described by

Proposition 2.9. 1. *The cover* μ: $Y \to X$ *is unramified at boundary points of* Y *of types* (1) *and* (2, 2) *and is triply branched at points of type* (3).
2. $I_Y = (\tilde{N}_1 + \tilde{N}_{2,2} + \tilde{N}_3/3)(b-1)c$.

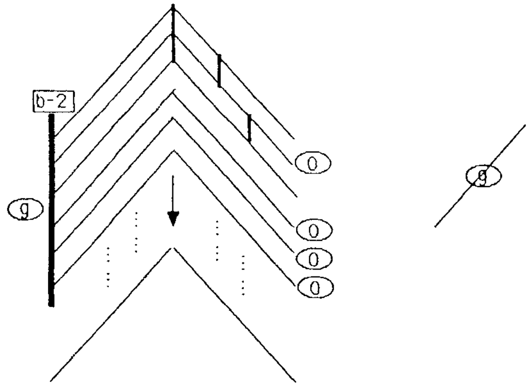
Proof. The equality in 2) follows directly from 1), the definition of I_Y, and the decomposition (2.8). As for 1), the order of branching of μ at y is the number of times that we must iterate the map $(t_1, t_2) \to (t'_1, t'_2)$ before we again obtain (t_1, t_2). Equations (2.7) show immediately that if t_1 and t_2 commute—i.e. in types (1) and (2, 2)—then this order is 1. For type (3) points, (2.7) shows that if $t_1 = (12)$ and $t_2 = (23)$, then the monodromy is $\{(12), (23)\} \to \{(13), (12)\} \to \{(23), (13)\} \to \{(12), (23)\}$. □

Next we would like to sketch the admissible covers $C_y \to B_y$ for $y \in [J_Y]$. The fiber B_y is then always the join of two smooth rational curves, one with $(b-2)$ marked points which we have placed on the left and one with 2 marked points which we have placed on the right. The diagrams below show the branching over the right component and in a neighborhood of the node on the left one in the style developed by Diaz. An oblique line segment represents each sheet and the abutment of two such segments represents a node at which the left and right curves meet. Thin vertical strokes denote ramification between the corresponding sheets of the indicated fiber. A thick vertical bar joins sheets which are connected by ramifications occurring outside the drawing on the left component and a boxed number on top of such a bar indicate the number of simple ramifications it represents. (The branch points in B_y are not indicated). At the same time we show the semi-stable reductions of the curves C_y obtained by contracting all *extraneous components*, that is, chains of smooth rational components meeting the remainder of the curve in only one point. For these, we give a graph describing the topological type. In both pictures we indicate the genus of each component of the curve sketched by a numbered oval. For each of the types (2, 2) and (3) all the covers are described by a single picture—Diagrams 2.10 and 2.11 below—and the semi-stable reduction is a smooth curve.

(2,2)



(2.10)

(3)  (2.11)

The situation for type (1) covers is more complicated. Write the monodromy data of such a cover as $t = ((\mathrm{nm}), (\mathrm{nm}), t^*)$ with $t^* = (t_3, \ldots, t_b)$. We recall first the combinatorial decompositions (1.30)

$$[N_1] = [M_0] \cup [M_1] \cup \ldots \cup [M_{[g/2]}]$$

according to whether the partition Q_{t*} is $(k), (k-1, 1), \ldots,$ or $(k - [g/2], [g/2])$ respectively. Next for $j \geqq 2$ recall the decomposition (1.35)

$$[M_j] = [M_{j,0}] \cup [M_{j,1}] \cup \ldots \cup [M_{j,g}]$$

where $[M_{j,i}]$ is the subset of $[M_j]$ for which number of transpositions in t^* with support in the j-cycle of Q_{t*} is $(2i + 2j - 2)$, or equivalently for which the number with support in the $(k-j)$ cycle is $(2(g-i) + 2(k-j) - 2)$. (When $j = 1$, i is always 0). These sets are all $\mathbf{S}^k$-invariant and we will say that a cover is of type (1_j) or type $(1_{j,i})$ if its monodromy data lies in $[M_j]$ or $[M_{j,i}]$. From the Riemann–Hurwitz formula it follows that at a point of type (1_0), the left hand side of the cover is a curve of genus $(g-1)$ expressed as a k-sheeted cover of $\mathbb{P}^1$. Similarly, at a point of type $(1_{j,i})$ with $j > 0$, the left hand side becomes the disjoint union of two curves one of genus i expressed as a j-sheeted cover and the other of genus $(g-i)$ expressed as a $(k-j)$ sheeted cover. This leads to the pictures in Diagram 2.12. Recalling the definitions of the sets $[O_i]$ (1.37) and $[N_{\mathrm{sing}}]$ (1.39), we summarize the foregoing discussion in the

Corollary 2.13. 1. *The number of covers C_y of a fixed B_x with $x \in [J_X]$ whose stable model lies in Δ_0 is $\tilde{M}_0$.*

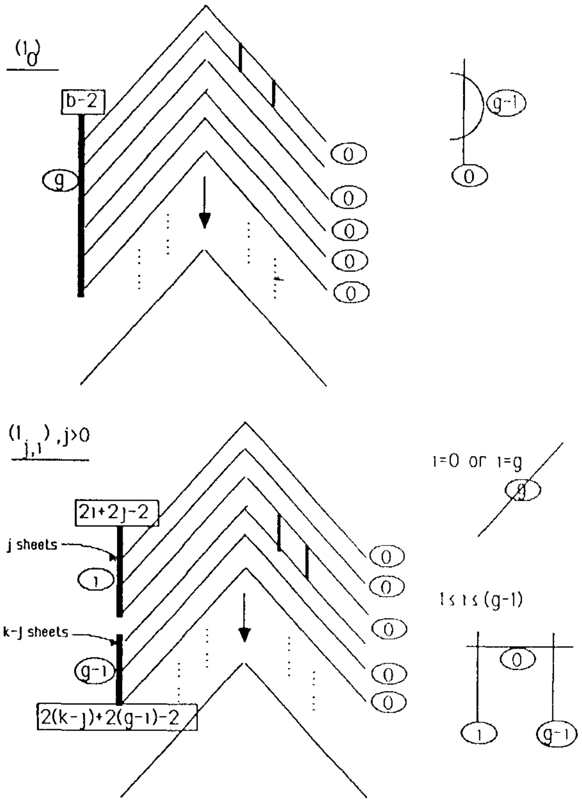
2. *If $1 \leqq i \leqq [g/2]$, the number of C_y whose stable model lies in Δ_i is $\tilde{O}_i$.*

3. *The number of C_y whose semi-stable reduction is singular is $\tilde{N}_{\mathrm{sing}}$ and each such reduction contains precisely two nodes.* □

We now turn to the description of the family of curves $\theta_G\colon G \to A \to Y$. Over the complement Y' of $[J_Y]$ in Y the surface H may be constructed by pulling back the smooth universal admissible cover

$$\theta\colon \mathscr{G}_{k,b} \to \mathbb{P}^1 \times H_{k,b} \to H_{k,b}$$

parameterized by $H_{k,b}$ whose existence is shown in [HM]; the fibers of H being



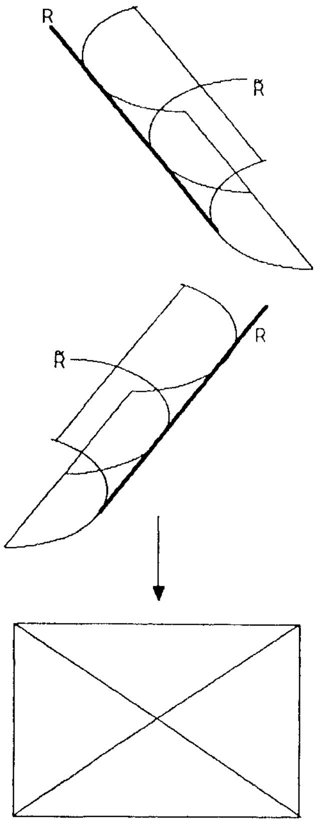
(2.12)

stable, F has to be equal to H over Y'; we will take $G_{Y'}$ equal to both and $A_{Y'}$ equal to $\mathbb{P}^1 \times Y'$. But the map θ does not extend to a family of admissible covers parameterized by $\bar{H}_{k,b}$. We shall therefore construct G and A by providing local analytic descriptions G_U and A_U over neighborhoods U of points of $[J_Y]$.

Say that τ_1 and τ_2 meet at a point $(t, y) \in [I_Y]$ lying over $y \in [J_Y]$. Let U be a neighborhood of y in Y and V a neighborhood of (t, y) in $\mathbb{P}^1 \times U$ containing no other nodes of τ. Let u denote a coordinate in U and also its pullback to $\mathbb{P}^1 \times Y$; and let v be a coordinate in the fiber directions on $\mathbb{P}^1 \times U$.

We begin with the case where y is of type (2, 2). Here we see clearly why we don't want to work with admissible covers all the time. The family F has to be smooth here. We can express F as a branched cover of the product $\mathbb{P}^1 \times U$; the branch divisor τ of the cover has a node, of course, but the ramification divisor in F is smooth over V, consisting of curves R_i lying over τ_i, $i = 1, 2$, with R_1 and R_2 lying on two disjoint pairs of sheets. Nothing is going on here except that two ramification points of the cover $F_y \to \mathbb{P}^1$ happen to lie over the same point of $\mathbb{P}^1$; and we will take $G_U \simeq F_U$ and $A_U \simeq \mathbb{P}^1 \times U$ over U.

Note for future reference that if the equation of $\tau_1 \tau_2$ near (t, y) is $u^2 - v^2 = 0$, then the equation of the cover G at one of the points of R lying over (t, y) will be $w^2 = u - v$, and at the other point it will be $w^2 = u + v$. The equation of R will be $w = 0$, of course, and the equation of the complement $\tilde{R}$ of R in the pullback of τ will be $u \pm w^2 = 0$; in particular we see that $\tilde{R}$ will meet R transversely twice over (t, y) as indicated in the diagram below.



Observe that by contrast *it is not possible* to extend H to a family of admissible covers of the blow-up at the point (t, y) of $\mathbb{P}^1 \times U$ (that is, the inverse image of U in B_Y) with central fiber as pictured in Diagram 2.10. This corresponds to the fact that the admissible cover of Diagram 2.10 has automorphisms (the involution exchanging sheets of either of the components that are 2-sheeted covers of the exceptional divisor and fixing the remaining components of the cover); we can see it directly by noting that such a cover would have to have an isolated branch point at the node of the fiber of B_Y over y. If we insisted on having a family of admissible covers, we would have to make a base change of order 2 around y, so that τ_1 and τ_2 would be simply tangent at (t, y), blow up twice to separate them, and then blow down the first exceptional divisor to an A_1 singularity.

Type (3) points are more interesting. Since Y is triply branched over X at such points the sections τ_1 and τ_2 will no longer meet transversely at such points; indeed they will have local equations $v = \pm u^3$. As in the previous case the family F of stable curves should be smooth here, and we see again that we can express it as a branched cover of $\mathbb{P}^1 \times U$: if we rescale the coordinates so that the equation defining $\tau_1 \tau_2$ in U becomes $27v^2 - 4u^6 = 0$, then a local equation for such a cover has the form $w^3 - u^2 w + v = 0$, where w completes u to a system of coordinates on F. This equation already defines a smooth surface whose fibre over $u = 0$ has an ordinary triple branch point at the origin $v = 0$. Therefore again in this case we will take $G_U \simeq F_U$ and $A_U \simeq \mathbb{P}^1 \times U$ over U.

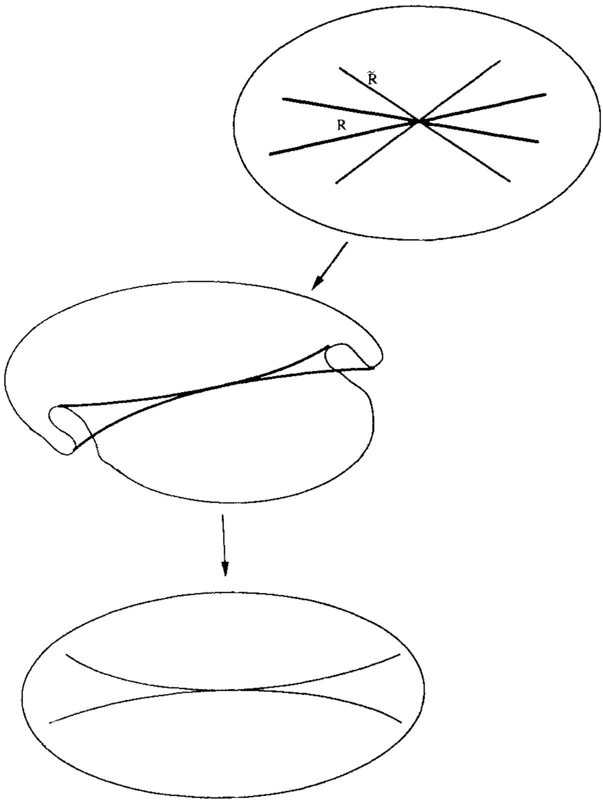
We note for future reference that the ramification divisor R of $G \to A$ is given in coordinates w and u by the equation

$$0 = \frac{\partial v}{\partial w} = \frac{\partial}{\partial w}(-w^3 + u^2 w) = -3w^2 + u^2 \tag{2.13.i}$$

and so consists of two smooth arcs meeting transversely; we also observe that the complement of R in the inverse image of the branch divisor τ of G over A is given by

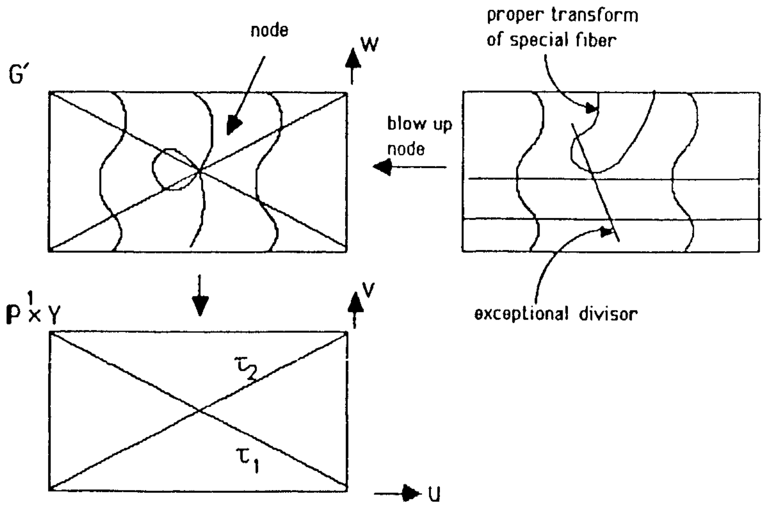
$$\frac{27v^2 - 4u^6}{(-3w^2 + u^2)^2} = \frac{27(-w^3 + u^2 w)^2 - 4u^6}{(-3w^2 + u^2)^2} = 3w^2 - 4u^2 \tag{2.13.ii}$$

and so in a neighborhood of the point $u = w = 0$ consists of two smooth arcs meeting the two arcs of R transversely, as in the picture:



Observe that here we could, if we wanted, construct a family of admissible covers without base change (the admissible cover pictured in Diagram 2.11 has no automorphisms); we would, however, have to blow up $\mathbb{P}^1 \times U$ three times to separate τ_1 and τ_2, then blow down the first two exceptional divisors to create an A_2 singularity.

Finally, consider the case of points of type (1). Here if we just take the family of branched covers over $Y' = Y - [J_Y]$ and complete it as a family $G' \to \mathbb{P}^1 \times Y \to Y$ of branched covers of $\mathbb{P}^1$ the total space G' of the family will be singular: over $V \subset \mathbb{P}^1 \times U$ we find $d-2$ smooth sheets and one component that has local equation $w^2 = v^2 - u^2$, i.e., an A_1 singularity. If we now blow up the point (t, y) in V and its inverse image in G' we arrive at a family of admissible covers $H_U \to \mathbb{P}^1 \times U \to U$ with fiber over y as pictured in Diagram 2.12; observe in particular that blowing up the singularity of G' has resolved it, so that H_U is smooth. Diagram (2.14) below shows the local picture around the node of G' before and after the blowup. Here it will be handy to take G to be the family of admissible covers; we accordingly take A_U to be the blow-up of $\mathbb{P}^1 \times U$ at the point (t, y), and $G_U = H_U$ the family of admissible covers over U. Observe that the ramification divisor R of G over A in U is disjoint from its complement in the inverse image of the branch divisor τ.



(2.14)

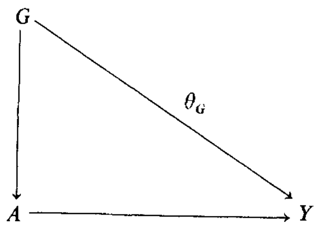
To describe the family F near such a point, note that in diagram (2.12), the exceptional curve of (2.16) is the unique rational component of the right hand cover meeting the left hand side in two points, so all the others will have to be blown down. The fiber of the resulting surface over y is the admissible cover in (2.12) with all extraneous $\mathbb{P}^1$'s on the *right* contracted. But this surface is not relatively minimal over y if the subtype is $(1_{j,0})$ or $(1_{j,g})$. For the genus zero component of the left side meets the remainder of the curve in only one point, and hence together with the exceptional curve forms an extraneous chain of rational components.

Contracting these we see that the fibre of the relatively minimal model $\theta_F\colon F \to Y$ is the semi-stable reduction shown in (2.12).

This completes our construction of the cover $G \to A \to Y$, and our proof that the smooth family $F \to Y$ of semistable curves exists. We summarize our description in the

Theorem 2.15. 1). *There is a smooth relatively minimal family of semi-stable curves of genus g, $\theta_F\colon F \to Y$ whose fiber over a point $y \in Y$ is a semi-stable model of the corresponding admissible cover shown in Diagrams* 2.10–2.12.

2) *Let A be the surface obtained by blowing up $\mathbb{P}^1 \times Y$ at the nodes of τ of type* (1). *Then there is a surface G obtained from F by a sequence of blowups of points which lie over $[I_{Y,(1)}]$ and are smooth in their θ_F-fibers, and a finite covering $G \to A$ such that over points $y \in [J_Y]$ of type* (1) *the fiber of the triangle*



(2.16)

is the admissible cover $H_y \to B_y$ pictured in Diagram 2.12.

§3. Computation of degrees and estimates for s_g

In this section, we will compute the degrees λ_Z and δ_Z of the Hodge bundle Λ and the boundary Δ restricted to the curve Z constructed in the previous section, and the ratio $s_Z := \delta_Z/\lambda_Z$. In fact, it will be no more work to determine the degree of each component of Δ on Z. We begin by determining these degrees.

Theorem 3.1. 1). $\delta_Z = 2(b-1)c\tilde{N}_{\text{sing}}$.

2). i) *The degree of the restriction of the class δ_0 to the curve Z is*

$$2(b-1)c\tilde{M}_0 \tag{3.2}$$

ii) *For $1 \leqq i \leqq [g/2]$, the degree of the restriction of the class δ_i to the curve Z is*

$$2(b-1)c\tilde{O}_i \tag{3.3}$$

Proof. All these assertions follow directly from the descriptions of the fiber of the family of semistable curves $\theta_F\colon F \to Y$ in §2. We showed there that over each of the $(b-1)c$ points of $[I_X]$ there are $\tilde{N}_{\text{sing}}$ points $y \in Y$ for which the fiber F_y is singular. The point in $\bar{\mathcal{M}}_g$ corresponding to each of these singular fibers lies in a unique component of the boundary: in Δ_0 if the combinatorial data corresponding to y lies in one of the $\tilde{M}_0$ orbits of type 1_0, and in Δ_i if it lies in one of the $\tilde{O}_i$ orbits of type $1_{j,i}$ or $1_{j,(g-i)}$ for some $j \geqq 2$. Moreover, since the family $\theta_F\colon F \to Y$ is smooth and relatively minimal, the local intersection number of the image $Z = \rho(Y)$ of Y in moduli with the corresponding boundary component at the point $\rho(y)$ is simply the number of nodes in the corresponding fiber F_y. As Diagram 2.12 shows, this number is always 2 which yields the set of degrees above. □

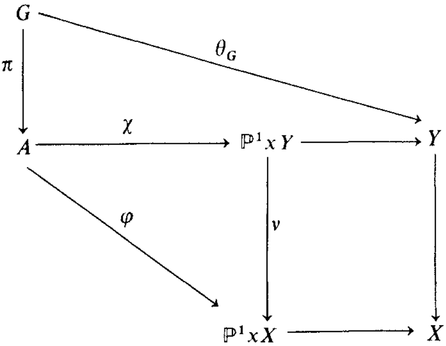
The computation of the degree λ_Z is more involved; it is for this that we have introduced in the previous section the family of curves $G \to A \to Y$. Our basic tool is a result due to Arakelov.

Theorem 3.4. ([Arakelov]). *Suppose that $\theta_G\colon G \to Y$ is a smooth family of nodal curves of genus g parameterized by a complete curve Y, for which the induced map $\rho\colon Y \to Z \subset \bar{\mathcal{M}}_g$ is finite. Then the degree λ_Z of the pullback to Z via ρ of the Hodge bundle Λ is given by*

$$\deg \lambda_Z = \frac{\omega^2 + \delta}{12} \tag{3.5}$$

where δ is the sum over the points $y \in Y$ of the number of nodes of the fiber G_y, and $\omega = c_1(\omega_{G/Y})$ is the first Chern class of the relative dualizing sheaf of G over Y. □

We now proceed to carry out this computation. The approach will be to express the surface G as a branched cover of the family $A \to Y$ of rational curves and use the Riemann–Hurwitz formula; in order to fix some notation, let us recall the basic set-up from §2.



(3.6)

On $\mathbb{P}^1 \times Y$, we let s denote the class of a horizontal section (i.e. $s^2 = 0$); we let σ be a fixed collection of b sections of $\mathbb{P}^1 \times X$ over X and $\tau = \nu^{-1}(\sigma)$ their pullback to $\mathbb{P}^1 \times Y$ as in section 2. The surface A was obtained by blowing up the set $[I_{Y,(1)}]$ of nodes of τ of type (1); let us denote by F the exceptional divisor of this blow-up, and by $\tilde{\tau}$ the proper transform of τ on A. The map $\pi\colon G \to A$ is a k-sheeted branched cover. We denote by R the ramification divisor in G of this covering so that $\pi*(R) = \tilde{\tau}$.

We now compute. Let $\omega = c_1(\omega_{G/Y})$ be the Chern class of the relative dualizing sheaf of our family. We have then

$$\begin{aligned}\omega &= \pi^*(\omega_{A/Y}) + R \\ &= \pi^*(\chi^*(\omega_{\mathbb{P}^1 \times Y/Y}) + F) + R \\ &= \pi^*(-2\chi^*(s) + F) + R\end{aligned}$$

Taking self-intersections, we find that

$$\omega^2 = (\pi^*(-2\chi^*(s) + F))^2 + 2(\pi^*(-2\chi^*(s) + F))\cdot R + R^2 \tag{3.7}$$

We will compute the three terms on the right side of (3.7) in succession. The first term may be rewritten

$$\deg(\pi)\cdot(-2\chi^*(s) + F)^2 = kF^2 \tag{3.8}$$

since $s^2 = 0$ and F is orthogonal to the image of χ^*. The second we may rewrite using the push-pull formula as

$$\begin{aligned} 2(-2\chi^*(s) + F)\cdot\pi_*(R) &= 2(-2\chi^*(s) + F)\cdot\tilde{\tau} \\ &= 2(-2\chi^*(s) + F)\cdot(\chi^*(\tau) - 2F) \\ &= -4(s\cdot\tau) - 4F^2 \\ &= -4\deg(\nu)\cdot(s\cdot\sigma) - 4F^2 \\ &= -4\tilde{N}c - 4F^2 \end{aligned} \tag{3.9}$$

In order to compute the third term we will use the description given in §2 of the ramification divisor R of π, and of the complement $\tilde{R}$ of R in the inverse image of the branch divisor $\tilde{\tau}$ of π. In §2 we saw that R and $\tilde{R}$ are disjoint, except over points of $[I_Y]$ of type (2, 2) and (3); over a point of type (2, 2) we saw that R and $\tilde{R}$ had two points of transverse intersection and over a point of type (3) they had a common node, with all branches meeting pairwise transversely for an intersection multiplicity of 4. Over each of the $(b-1)c$ points $x \in [J_X]$, there lie $\tilde{N}_1$ points of Y of type (1), $\tilde{N}_{2,2}$ of type (2, 2) and $\tilde{N}_3/3$ of type (3), so that we have

$$(R\cdot R) = (b-1)c(2\tilde{N}_{2,2} + 4\tilde{N}_3/3)$$

and using this and the fact that as divisors,

$$\pi^*(\tilde{\tau}) = 2R + \tilde{R}$$

and

$$\pi_*(R) = \tilde{\tau}$$

we compute that

$$\begin{aligned} R^2 &= R\cdot(\pi^*(\tilde{\tau}) - \tilde{R})/2 \\ &= \tilde{\tau}^2/2 - (b-1)c(\tilde{N}_{2,2} + 2\tilde{N}_3/3) \end{aligned} \tag{3.10}$$

Now, we have

$$\begin{aligned} \tilde{\tau}^2 &= (\tau - 2F)^2 \\ &= (\varphi^*(\sigma))^2 + 4F^2 \\ &= 2\tilde{N}bc + 4F^2 \end{aligned}$$

since φ is an $\tilde{N}$-sheeted cover and $\sigma^2 = 2bc$. This yields

$$R^2 = \tilde{N}bc + 2F^2 - (b-1)c(\tilde{N}_{2,2} + 2\tilde{N}_3/3) \tag{3.11}$$

Note that since F is the exceptional divisor associated to the blow-up of $\mathbb{P}^1 \times Y$ at the points of $[I_{Y,(1)}]$, $F^2 = -I_{Y,(1)} = -(b-1)c\tilde{N}_1$. Thus, plugging in the values from (3.8), (3.9) and (3.11) into (3.7), we find that

$$\begin{aligned}\omega^2 &= (b-4)\tilde{N}c + (k-2)F^2 - (b-1)c(\tilde{N}_{2,2} + 2\tilde{N}_3/3)\\ &= (b-4)Nc - (b-1)c(k-2)\tilde{N}_1 - (b-1)c(\tilde{N}_{2,2} + 2\tilde{N}_3/3) \qquad (3.12)\end{aligned}$$

Next we compute $\delta = \delta_{G/Y}$. By the construction of the family G, the fiber G_y of G over each point of y of type (1) contains k nodes, while the fibers over each point of type (2, 2) or (3) are smooth. Hence

$$\delta = (b-1)ck\tilde{N}_1. \qquad (3.13)$$

We have thus

$$\begin{aligned}\lambda_Z &= (\delta + \omega^2)/12\\ &= ((b-4)\tilde{N}c + (b-1)c(2\tilde{N}_1 - \tilde{N}_{2,2} - 2\tilde{N}_3/3))/12\end{aligned}$$

since $\tilde{N} = \tilde{N}_1 + \tilde{N}_{2,2} + \tilde{N}_3$. We express this as the

Theorem 3.14. $\lambda_Z = ((b-1)c(3\tilde{N}_1 + \tilde{N}_3/3) - 3c\tilde{N})/12$ □

Corollary 3.15. $s_Z = \dfrac{72(b-1)N_{\text{sing}}}{(b-1)(9N_1 + N_3) - 9N}$ □

(We note that adding tildas to the various N's in this expression leaves its value unchanged since by Lemma 1.24 each term would be multiplied by the same factor: 1 if $k = 2$ and $k!$ if $k \geqq 3$. We shall leave them out for the remainder of this section.)

We now wish to discuss what Corollary 3.15 implies about effective divisors in $\bar{\mathcal{M}}_g$. The basic tool is the observation, discussed in the introduction, that

If $D \sim a\lambda - b\delta$ is an effective divisor with $s_D := a/b < s_Z$, then D contains $\tilde{Z}$. (3.16)

For the curves Z we are using s_Z depends only on k and b or equivalently on k and g, in view of the Riemann–Hurwitz formula which in our case reads

$$b = 2(k + g - 1). \qquad (3.17)$$

To make this dependence clear we shall denote the slope s_Z by $s_g(k)$ defining b implicitly where it is used by (3.17).

Two types of conclusions result from (3.16). The first is that if $\tilde{Z} = \bar{\mathcal{M}}_g$ then $s_g \geqq s_Z$. To say that $\tilde{Z} = \bar{\mathcal{M}}_g$ for the curves Z constructed in §2 simply means that the generic curve of genus g is expressible as a k-sheeted branched cover of $\mathbb{P}^1$. A standard result from Brill–Noether theory is that this is possible if and only if $k \geqq [(g+3)/2]$. (Here $[a]$ denotes the greatest integer in a). In other words,

$$s_g \geqq \sup_{k \geqq [(g+3)/2]} (s_g(k)) \qquad (3.18)$$

Remark. When g is odd and $k = (g+1)/2$, $\tilde{Z}$ is the divisor of k-gonal curves considered in [HM]. In this case, (3.16) implies that an effective irreducible divisor has slope greater than $s_g(k)$ or is the k-gonal divisor. In the latter case [HM] shows that the divisor has slope greater than 6 which, as we shall shortly see is much larger than $s_g(k)$. Therefore, in (3.18) we could take the sup over all $k \geqq [(g+2)/2]$.

One immediate consequence of (3.18) is that:

Corollary 3.19. *For all $g \geqq 2$, $s_g \geqq 0$.* □

In view of the connection to the Schottky problem mentioned in the introduction, it is natural to ask whether (3.18) leads to a positive bound for s_g uniform in g. The answer is almost certainly no. We believe that

$$s_g = O\left(\frac{1}{g}\right). \tag{3.20}$$

Based on calculations of $s_g(k)$ for small g and k done on an IBM-370 at the University of Toronto and on a Ridge-32 in the Mathematics Department at Columbia University we would guess the implied constant to be about 60. Since we have no application for (3.20) we shall not try to give a proof here. Instead we give an heuristic argument which brings out some ideas we will need for later estimates. Fix two transpositions t_1 and t_2 in $\mathbf{S}^k$ and let $L = L(k, b)$ be the number of elements of $[N(k, b)]$ of the form $t = (t_1, t_2, t^*)$. The key assumption we will use is that L/N is independent of the choice of t_1 and t_2 to first order in k. More precisely,

$$\frac{N}{L} = \binom{k}{2}^2 + O(k^3). \tag{3.21}$$

Assume that $b \gg k$ for a moment. Then the corresponding statement about $[S_{(1)}(k, b)]$ is clear: indeed, the ratio

$$\binom{k}{2}^2$$

is just that predicted asymptotically by Corollary 1.22. On the other hand, it follows easily from Proposition 1.11 that for $b \gg k$,

$$\frac{S_{(1),(k)}(k, b)}{S_{(1)}(k, b)} = 1 + O\left(\frac{1}{k}\right).$$

Recalling that N is by definition equal to $S_{(1),(k)}$, this leads quickly to a proof of (3.21) when $b \gg k$. The difficulty is that in the geometric application at hand we want $k \geqq [(g+3)/2]$ and $b = 2(k + g - 1)$ which together force b to lie roughly between $2k$ and $6k$. In this range, the asymptotically trivial terms are quite significant (cf. Remark 3.23), but while (3.21) fails the estimates it yields are in rough agreement with our numerical evidence.

We will also need to assume that $N_{\text{sing}}/N_1 = 1 + O(1/k)$. For $b \gg k$, this again follows by combining the combinatorial descriptions of N_{sing} and N from §1 with Theorems 1.11 and 1.21.

These estimates quickly yield (3.20) as follows. The number of pairs (t_1, t_2) whose product is trivial is $(k)(k-1)/2$ and the number whose product is a 3-cycle is $(2k-4)(k)(k-1)$. Hence (3.21) immediately implies that to first order $N(k, b) = (k)(k-1)N_1(k, b)/2$ and that $N_3(k, b) = (k-2)N_1(k, b)$. Likewise the second assumption allows us to replace N_{sing} by N_1 to first order. Inserting these estimates into (3.15) and cancelling the common factor of N_1 gives,

$$s_g(k) = \frac{72(b-1)}{(b-1)(2k+5) - (9/2)(k)(k-1)} + O\left(\frac{1}{k^2}\right). \qquad (3.22)$$

Setting k equal to the minimum permissible value $[(g+3)/2]$ allowed in (3.18) immediately gives $s_g = O(1/g)$.

Remark 3.23. With a little more work one can see that for fixed g, the leading term on the right hand side of (3.22) is a decreasing function of k for $k \geqq [(g+3)/2]$ and for this k equals $576/5g$. Comparing this with the corresponding numerical values for the right hand side of (3.15) which are about $60/g$ gives a rough idea of how far off (3.21) is in the geometric range. The difference is only apparently nugatory for it means that our numerical results give a proof that $\overline{\mathscr{M}}_g$ has Kodaira dimension $-\infty$ only for $g \leqq 10$, not for $g \leqq 15$ as would follow from the heuristics.

The second type of conclusion to which (3.16) leads is that divisors of sufficiently small slope contain the locus $\tilde{Z}$ of deformations of certain of our curves Z. In particular, we find immediately that

Corollary 3.24. *If $s_D < s_g(k)$, then D contains the k-gonal locus.* □

Here we can use asymptotic information to better effect. The results from §1 which we used above to determine the asymptotic behaviour in k of certain of the N's under the assumption that $b \gg k$ yield very nice estimates for k fixed and b tending to ∞. For example, Corollary 1.22 immediately implies that the proportion of the elements of $[S_{(1)}(k, b)]$ which begin with a fixed pair of transpositions t_1 and t_2 can be made as nearly independent of this pair as we wish by taking b sufficiently large. With a little more work based on Proposition 1.11 we obtain the corresponding result for $[N]$. Similarly, as b increases the ratio of N_{sing} to N_1 approaches 1. But assuming $b \gg k$ is equivalent, in view of Riemann–Hurwitz, to assuming that $g \gg k$. Therefore the argument preceding (3.22) shows that if we increase g keeping k fixed, $s_g(k)$ and

$$\frac{72(2g+2k-3)}{(2g+2k-3)(2k+5) - (9/2)(k)(k-1)}$$

approach one another. Raising g again if necessary we find that

Corollary 3.25. *If $g \gg k \geqq 2$, than any divisor on $\overline{\mathscr{M}}_g$ of slope less than $72/(2k+5)$ contains the $k-$gonal locus.* □

For small k the results of §1 permit us to compute $s_g(k)$ in closed form and get sharper results valid for all g. We have carried this out below for k equal to 2, 3 and 4 below and have summarized the results in Table 3.26:

Table 3.26

$k =$	2	3	4
$S_{(1)}$	1	3^{b-1}	$\frac{1}{12}6^b + \frac{3}{4}2^b$
N	1	$3^{b-1} - 3$	$\frac{1}{12}6^b - \frac{4}{3}3^b - \frac{3}{4}2^b$
N_1	1	$3^{b-2} - 3$	$\frac{1}{72}6^b - \frac{4}{9}3^b + \frac{3}{8}2^b - 60$
N_{sing}	1	$3^{b-1} - 9$	$\frac{1}{72}6^b - \frac{8}{9}3^b + \frac{3}{8}2^b - 12(b^2 - b + 2)$
S_3	0	$2 \cdot 3^{b-1}$	$\frac{2}{3}6^b$
N_3	0	$2 \cdot 3^{b-2}$	$\frac{1}{18}6^b - \frac{8}{9}3^b$

Substituting these values in the formula for $s_g(k)$ we find that

Corollary 3.27. 1) $s_g(2) = 8 + 4/g$

$$2)\ s_g(3) = \frac{72(2g+3)(3^{2g+2} - g)}{(2g+3)(3^{2g+4} + 2 \cdot 3^{2g+2} - 27) - (3^{2g+5} - 27)}$$

$$3)\ s_g(4) = \frac{(2g+5)(6^b - 64(3^b) + 27(2^b) - 864(b^2 - b + 2))}{(2g+5)(\frac{13}{72}6^b - \frac{44}{9}3^b + \frac{27}{8}2^b - 540) - \frac{3}{4}6^b + 12(3^b) + \frac{27}{4}2^b} \qquad \square$$

Observe that the case $k = 2$ agrees with what we know a priori, since the relation

$$\delta = (8 + 4/g) \cdot \lambda$$

holds modulo the classes $\delta_1, \delta_2, \ldots$ in the Picard group of the moduli functor for hyperelliptic curves (cf. [$C - H$]). Note that the leading term $8 = 72/(2 \cdot 2 + 5)$ in g of $s_g(2)$ is as predicted by Corollary 3.25; as is the leading term of $s_g(3)$:

$$\frac{72(2g+3)\, 3^{2g+2}}{(2g+3)(3^{2g+4} + 2(3^{2g+2}))} \sim \frac{72}{11}$$

and the leading term of $s_g(4)$. A moment's inspection of this formula will convince the reader that $s_g(3)$ is always at least as big as this estimate. We therefore have shown that

Corollary 3.28. *Let $g \geqq 3$ and let D be an effective divisor on $\overline{\mathcal{M}}_g$.*
1) *If $s_D < 8$, then D contains the hyperelliptic locus.*
2) *If $s_D < 72/11$, then D contains the trigonal locus.* $\square$

When $g \geqq 23$, we may apply this result to divisors in the pluricanonical linear series on $\overline{\mathcal{M}}_g$ which by [HM] are of slope $13/2$ to get

Corollary 3.29. *The common base locus of the pluricanonical linear series on $\overline{\mathcal{M}}_g$ contains the hyperelliptic and trigonal loci.* $\square$

Finally for small genera these results combined with the upper bounds for s_g discussed in the introduction allow us either to determine s_g or to give very sharp estimates for it.

Corollary 3.30. 0) *For* $g = 2$, $s_g = 10$.

1) *For* $g = 3$ *we have* $s_g = 9$. *Moreover the only effective irreducible divisor of slope less than 28/3 on* $\bar{\mathcal{M}}_3$ *is the divisor of hyperelliptic curves.*

2) *For* $g = 4$, *we have*

$$8.8333\ldots = \frac{53}{7} \geqq s_4 \geqq \frac{46{,}759{,}680}{5{,}550{,}633} \sim 8.4242$$

3) *For* $g = 5$, *we have* $s_g = 8$. *Moreover, the only effective irreducible divisor of slope less than* 29524/3659 ($\sim$ 8.07) *on* $\bar{\mathcal{M}}_5$ *is the trigonal divisor.*

4) *For* $g = 6$, *we have*

$$8.08333\ldots = \frac{97}{12} \geqq s_6 \geqq \frac{15982387645104}{2180880964555} \sim 7.328409$$

Proof. For genus 2, the lower bound follows from (3.18) and (3.27) and the upper bound from [C–H]. For genus 3, the remark following (3.18) shows that if D is an effective irreducible divisor on $\bar{\mathcal{M}}_3$ then either $s_D \geqq s_3(2)$ or D is the hyperelliptic divisor. By (3.27.1), $s_3(2) = 28/3$. Since [D] and [HM] show that the hyperelliptic divisor has slope 9, this proves both assertions. In genus 5, the same argument shows that an effective irreducible divisor D on $\bar{\mathcal{M}}_5$ of slope less than $s_5(3)$ is the trigonal divisor. By (3.27.2), $s_5(3) = 29524/3659$ while by [HM] the trigonal divisor has slope 8 yielding both assertions.

The lower bounds in genera 4 and 6 follow in each case by straightforward application of (3.18) and (3.27). As for the upper bound, this comes by exhibiting an explicit effective divisor in each moduli space. In the case $g = 4$, we use the divisor in $\mathcal{M}_4$ of curves whose canonical model lies on a singular quadric surface in $\mathbb{P}^3$ (or, equivalently, that have a vanishing theta-null; or, equivalently, have one rather than two pencils of degree 3) and in case $g = 6$, we use the divisor in $\mathcal{M}_6$ of curves C possessing a pencil $|D|$ of degree 4 satisfying $H^0(C, \omega_C(-2D)) \neq 0$ (equivalently, the closure of the locus of curves possessing fewer than 5 pencils of degree 4). The classes of both divisors are computed in the paper [E–H]. □

We conclude with two remarks. First, that the values found for s_2, s_3 and s_5 are exactly those of Conjecture (0.1). Secondly, that our lower bound for s_4 is greater than $8.4 = 6 + 12/5$: i.e. when $g + 1$ is prime we cannot in general hope for equality in (0.1).

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