Calc. Var. 2, 29–48 (1994)

Multiple positive solutions of some elliptic problems via the Morse theory and the domain topology

Vieri Benci[1] and Giovanna Cerami[2]

[1] Università di Pisa, Istituto di Mathematiche Appliccata, Facoltà di Ingegneria, Via Bonanno 25/B, I-56100 Pisa, Italy
[2] Università di Palermo, Facoltà die Ingegneria, Viale delle Scienze, I-90100 Palermo, Italy

Received April 19, 1993/Accepted April 29, 1993

Abstract. We use Morse theory to estimate the number of positive solutions of an elliptic problem in an open bounded set $\Omega \subset \mathbb{R}^N$. The number of solutions depends on the topology of Ω, actually on $\mathscr{P}_t(\Omega)$, the Poincaré polynomial of Ω. More precisely, we obtain the following Morse relations:

$$\sum_{u \in \mathscr{K}} t^{\mu(u)} = t\mathscr{P}_t(\Omega) + t^2 \left[\mathscr{P}_t(\Omega) - 1\right] + t(1+t)\,\mathscr{Q}(t)\,,$$

where $\mathscr{Q}(t)$ is a polynomial with non-negative integer coefficients, $\mathscr{K}$ is the set of positive solutions of our problem and $\mu(u)$ is the Morse index of the solution u.

Mathematics Subject Classification: 35J20, 49F15, 58E05

1 Introduction

In this paper we are concerned with the following problem:

$$(P_\varepsilon) \quad \begin{cases} -\varepsilon\Delta + u = f(u) & \text{in } \Omega, \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}$$

where $\varepsilon \in \mathbb{R}^+ \backslash \{0\}$, $\Omega \subset \mathbb{R}^N$, $N \geq 3$, is a smooth bounded domain and

$$f : \mathbb{R}^+ \longrightarrow \mathbb{R}$$

is a $\mathscr{C}^{1,1}$-function with $f(0) = f'(0) = 0$.

Precisely the present research continues a study of [B.C.], [B.C.P.], and [C.P.] on the effect of the domain shape on the number of positive solutions of some semilinear elliptic problems.

During the past few years the relations between the geometry or the topology of Ω and the existence and multiplicity of solutions to problems like the following

$$\text{(1-1)}\qquad \begin{cases} -\Delta u + \lambda u = u^{p-1} & \text{in } \Omega, \quad p \in (2, 2N/(N-2)] \\ u > 0 & \text{in } \Omega \\ u = 0 & \text{in } \Omega \end{cases}$$

have been intensively investigated: we refer to [B.C.] for a detailed bibliography.
The aims of this paper are the following:

(i) to apply concentration and rescaling techniques to nonlinearities more general than u^p (see [Ca] for an other result in this direction);
(ii) to show that the Morse theory for this kind of problems gives better information than the Ljusternik-Schnirelman one.

We make the following assumptions:

(H_1) there exists $a > 0$ such that, for every $t > 0$,

$$|f(t)| \leq a + at^p$$

$$|f'(t)| \leq a + at^{p-1}$$

where a is a suitable positive constant and $p \in (1, \frac{N+2}{N-2})$

(H_2) there exists $\vartheta \in (0, 1/2)$ such that

$$F(t) \leq \vartheta\, t f(t) \quad t \geq 0$$

where

$$\text{(1-2)}\qquad F(t) = \int_0^t f(\tau) d\tau \quad \text{for} \quad t \geq 0$$

$$\text{(H}_3\text{)}\qquad \text{for every } t > 0, \ \frac{d}{dt}\left(\frac{f(t)}{t}\right) > 0\,.$$

$$\text{(H}_4\text{)}\qquad f(0) = 0; \quad f'(0) = 0\,.$$

For what follows it is useful to extend f to $\mathbb{R}^-$ in the following way:

$$\text{(H}_5\text{)}\qquad f(t) = 0 \quad \text{for} \quad t < 0\,.$$

Moreover notice that (H_2) implies that

$$\text{(H}_2'\text{)}\qquad f(t) \geq kt^{1/\vartheta - 1}, \quad k > 0, \ t > 0\,.$$

It is well known that, under the above assumptions, (P_ε) has a positive solution.
Using the Ljusternik Schnirelman theory we obtain the following result:

1.1 Theorem. *Suppose that f satisfies the assumptions (H_1), ..., (H_4) and Ω is topologically non-trivial*[1]*, then there exists $\varepsilon^* > 0$ such that, for any $\varepsilon \in (0, \varepsilon^*)$ problem (P_ε) has at least*

$$\text{cat}(\Omega) + 1$$

distinct solutions.

---

[1] We say that Ω is topologically non-trivial if cat $\Omega > 1$ where cat Ω denotes the Ljusternik Schnirelman category of $\bar{\Omega}$ in itself.

In order to state the results we obtain via the Morse theory, some notations and facts about the Morse theory (see e. g. [B2], [C]) are needed.

The solutions of (P_ε) are the critical points of the energy functional

$$E_\varepsilon(u) = \int_\Omega \left(\tfrac{1}{2}\varepsilon|\nabla u|^2 + \tfrac{1}{2}u^2 - F(u)\right) dx \ , \tag{1-3}$$

where $F(t)$, for $t \geq 0$, is defined in (1-2) and, for $t \leq 0$, is 0.

Notice that, by virtue of (H_1), E_ε is a $\mathscr{C}^2$-functional on the Sobolev space $W_0^1(\Omega)$. If u is an isolated critical point of E_ε and $E_\varepsilon(u) = c$, its (polynomial) Morse index $i_t(u)$ is defined as follows:

$$i_t(u) = \sum_k \dim\left[H_k(E_\varepsilon^c, E_\varepsilon^c \cap \setminus\{u\})\right] t^k \ ,$$

where $H_k(\cdot,\cdot)$ denotes the k-th group of homology with coefficients in some field, and

$$E_\varepsilon^c = \left\{w \in W_0^1(\Omega) | E_\varepsilon(w) \leq c\right\} \ .$$

The integer number $i_1(u)$ is called the multiplicity of u. It is well known that if u is a non-degenerate critical point, then

$$i_t(u) = t^{\mu(u)} \ ,$$

where $\mu(u)$ denotes the (numerical) Morse index of u, i.e. the dimension of the maximal subspace on which the bilinear form $E_\varepsilon''(u)[\cdot,\cdot]$ is negative-definite.

If u is a non-degenerate solution, its multiplicity is 1. If the multiplicity of u is n, a generic $\mathscr{C}^2$-small perturbation splits u into at least n non-degenerate solutions.

If $\mathscr{D}$ is any topological space, we recall that the Poincaré polynomial of $\mathscr{D}$ is defined as follows:

$$\mathscr{P}_t(\mathscr{D}) = \sum_k \dim[H_k(\mathscr{D})]\, t^k \ .$$

The main result of this paper is the following theorem:

1.2 Theorem. *Suppose that f satisfies the assumptions $(H_1), \dots, (H_4)$. Moreover suppose that*

(i) $\varepsilon \in (o, \varepsilon^*]$ where ε^* is a suitable positive constant,
(ii) the set $\mathscr{K}$ of nontrivial solutions of problem (P_ε) is discrete; then

$$\sum_{u\in\mathscr{K}} i_t(u) == t\mathscr{P}_t(\Omega) + t^2[\mathscr{P}_t(\Omega) - 1] + t(1+t)\mathscr{Q}(t) \ ,$$

where $\mathscr{Q}(t)$ is a polynomial with non-negative integer coefficients.

In the nondegenerate case, the above theorem becomes:

1.3 Corollary. *Suppose that f satisfies the assumptions $(H_1), \dots, (H_4)$. Moreover suppose that*

(i) $\varepsilon \in (o, \varepsilon^*]$ where ε^* is a suitable positive constant,
(ii) the solutions of problem (P_ε) are non-degenerate;
then

$$\sum_{u\in\mathscr{K}} t^{\mu(u)} = t\mathscr{P}_t(\Omega) + t^2[\mathscr{P}_t(\Omega) - 1] + t(1+t)\mathscr{Q}(t)$$

where $\mathcal{Q}(t)$ is a polynomial with non-negative integer coefficients and $\mathcal{K}$ is the set of solutions of (P_ε).

Let us remark that Theorem 1.2 implies that the problem (P_ε) has at least $2\mathcal{P}_1(\Omega)-1$ solutions if they are counted with their multiplicity. Of course, if Ω is topologically trivial, then $\mathcal{P}_1(\Omega)=1$, and the above theorem does not give any extra information. When Ω is topologically rich, we obtain good information on the solutions of (P_ε). An example of this is given by the following corollary.

1.4 Corollary. *Let A and C_i, $(i=1,\ldots,k)$ be contractible open non-empty sets in $\mathbb{R}^N$, smooth and bounded; suppose that*

$$\bar{C}_i \cap \bar{C}_j = \varnothing, \quad (i,j=1,\ldots,k, \quad i \neq j)$$
$$\bar{C}_i \subset A, (i=1,\ldots,k)$$

and set

$$\Omega = A \setminus \bigcup_i \bar{C}_i \, .$$

Then, there exists $\varepsilon^* > 0$ such that, for any $\varepsilon \in (0,\varepsilon^*)$, the problem (P_ε) has at least $2k+1$ solutions, if they are counted with their multiplicity.

Moreover, if the solutions are non-degenerate, k of them have Morse index N, k of them have index $N+1$, and one (the Mountain Pass Solution) has index 1.

1.5 Remark. Clearly, if we replace the equation in (P_ε) with the equation

$$-\varepsilon\Delta u + ku = f(u) \tag{1-4}$$

where k is a fixed positive constant, we obtain the same results. Now consider the equation

$$\varepsilon\Delta u = g(u) \, .$$

If we have $g'(0) < 0$, then this equation takes the form (1-4) if we set $k = -g'(0)$ and $f(u) = g(u) + ku$.

Notice that, in order to apply our techniques it is necessary to have $k > 0$ and hence $g'(0) < 0$. If $g'(0) > \lambda_1(\Omega)$ (where $\lambda_1(\Omega)$ is the first eigenvalue of $-\Delta$) we do not have any positive solution. If $g'(0) \in [0, \lambda_1(\Omega)]$ it is not yet clear what the situation is.

1.6 Remark. Using the Ljusternik Schnirelman theory, it has been proved in [BC] and [BCP] that the problem (1-1) admits at least (cat Ω) + 1 solutions if λ is big enough (or $p \uparrow 2^*; 2^* = 2N/(N-2)$) and Ω is a smooth domain topologically non-trivial. Clearly, the methods presented in this paper work for problem (1.1) which can be written in the form (1-4) with $k=1$ setting $v = (1/\varepsilon)^{1/(p-1)}u$, $\varepsilon = 1/\lambda$. So for problem (1-1), if λ is big enough, we get the conclusion of Theorems 1.2 and 1.3. Moerover, using the estimates of [BC], if p is close to 2^* and λ is fixed, theorems analogous to 1.2 and 1.3 hold for (1-1).

1.7 Remark. Suppose that Ω is topologically trivial but is close (in a suitable sense) to a set Ω_1 which has a rich topology; in this case the number of solutions of problem (P_ε) can be estimated by the topology of Ω_1. This result can be obtained using the arguments of [BCP].

2 The functional analytic setting

We set

$$J_\varepsilon(u) = E'_\varepsilon(u)[u] = \int_\Omega (\varepsilon|\nabla u|^2 + u^2)\, dx - \int_\Omega f(u)u\, dx \tag{2-1}$$

2.1 Lemma. *Suppose that* $J_\varepsilon(u) = 0$ *and that* $u \neq 0$. *Then*

$$J'_\varepsilon(u)[u] < 0$$

Proof. First notice that $J_\varepsilon(u) = 0$ and $u \neq 0$ imply

$$meas\{x \in \Omega | u(x) > 0\} > 0\ ,$$

in fact, for every $u \leq 0$, by (H_5) and $J_\varepsilon(u) = 0$, we have that

$$\int_\Omega (\varepsilon|\nabla u|^2 + u^2)\, dx = 0$$

and hence $u = 0$ a.e.

Now,

$$\begin{aligned} J'_\varepsilon(u)[u] &= 2\int_\Omega (\varepsilon|\nabla u|^2 + u^2)\, dx - \int_\Omega \left(f'(u)u^2 + f(u)u\right) dx \\ &= \int_\Omega \left(f(u)u - f'(u)u^2\right) dx\ ; \end{aligned}$$

therefore the claim follows by (H_3). □

We now set

$$\mathscr{S} = \left\{v \in W_0^1(\Omega) |\ \|v\| = 1\right\} \setminus \left\{v \in W_0^1(\Omega) | v \leq 0\ a.e.\right\} \tag{2-2}$$

(where $\|\cdot\|$ denotes the norm of $W_0^1(\Omega)$). Clearly $\mathscr{S}$ is a smooth manifold of codimension 1. Moreover define

$$M_\varepsilon = \left\{u \in W_0^1(\Omega) | u \neq 0 \quad \text{and} \quad J_\varepsilon(u) = 0\right\}\ . \tag{2-3}$$

2.2 Lemma. *M_ε is a manifold diffeomorphic to $\mathscr{S}$ and the diffeomorphism ψ_ε : $\mathscr{S} \longrightarrow M_\varepsilon$ is of class $\mathscr{C}^{1,1}$. Moreover there exist $h_\varepsilon > 0$ and $k_\varepsilon > 0$ such that for every $u \in M_\varepsilon$ the following relations hold true*

$$\|u\| \geq h_\varepsilon \tag{2-4}$$

$$E_\varepsilon(u) \geq k_\varepsilon\ . \tag{2-5}$$

Proof. For every $\bar{v} \in \mathscr{S}$, let $\xi_\varepsilon(\bar{v})$ be the positive number which realizes the maximum of the function $\lambda \longrightarrow E_\varepsilon(\lambda\bar{v})$ defined on $\mathbb{R}^+$. $\xi_\varepsilon(\bar{v})$ is well defined. In fact, $\max_{\lambda\in\mathbb{R}^+} E_\varepsilon(\lambda\bar{v})$ is achieved since, by (H_1) 0 is a local strict minimum of $E_\varepsilon(u)$ and by (H'_2),

$$\lim_{\lambda\to\infty} E_\varepsilon(\lambda\bar{v}) = -\infty\ .$$

Moreover the uniqueness of $\xi_\varepsilon(\bar{v})$ follows observing that, by (H_3), $\frac{1}{\lambda}f(\lambda\bar{v})\bar{v}$ is a strictly increasing function of λ and

$$0 = \frac{\partial}{\partial\lambda}E_\varepsilon(\lambda\bar{v}) = \lambda\int_\Omega(\varepsilon|\nabla\bar{v}|^2 + \bar{v}^2)\,dx - \int_\Omega f(\lambda\bar{v})\bar{v}\,dx$$

implies

$$\int_\Omega(\varepsilon|\nabla\bar{v}|^2 + \bar{v}^2)\,dx = \frac{1}{\lambda}\int_\Omega f(\lambda\bar{v})\bar{v}dx\,.$$

Clearly $\xi_\varepsilon(\bar{v})\bar{v} \in M_\varepsilon$. Thus M_ε is the graph of the function $\psi_\varepsilon : \mathscr{S} \longrightarrow M_\varepsilon$ defined by

$$\psi_\varepsilon(v) = \xi_\varepsilon(\bar{v})\bar{v}\,.$$

By the implicit function theorem and Lemma 2.1, ψ_ε and ξ_ε are functions of class $\mathscr{C}^{1,1}$; (2-4) is a consequence of the definition of the function ξ_ε and of the behaviour of E_ε. Lastly let us consider $u \in M_\varepsilon$, then using (H_2) and (2-4) we obtain

$$\begin{aligned}
E_\varepsilon(u) &= \int_\Omega\left(\frac{1}{2}\varepsilon|\nabla u|^2 + \frac{1}{2}u^2 - F(u)\right)dx \\
&\geq \int_\Omega\left(\varepsilon|\nabla u|^2 + u^2\right)dx - \vartheta\int_\Omega f(u)u\,dx \\
&= (1/2-\vartheta)\int_\Omega\left(\varepsilon|\nabla u|^2 + u^2\right)dx \geq (1/2-\vartheta)\,\varepsilon\,\|u\|^2 \geq k_\varepsilon\,. \qquad \square
\end{aligned}$$

2.3 Lemma. *The following statements are equivalent:*
(i) u is a critical point of E_ε;
(ii) u is a critical point of E_ε constrained on M_ε.

Proof. (i) $\Rightarrow$ (ii) is immediate since $E'_\varepsilon(u) = 0 \Rightarrow u \in M_\varepsilon$.
(ii) $\Rightarrow$ (i) let u_0 be a critical point of E_ε constrained on M_ε. Then there exists $\lambda \in \mathbb{R}$ such that $E'_\varepsilon(u_0) - \lambda J'_\varepsilon(u_0) = 0$.

Thus,

$$J_\varepsilon(u_0) = E'_\varepsilon(u_0)[u_0] = \lambda J'_\varepsilon(u_0)[u_0]\,.$$

This equality and Lemma 2.1 imply that $\lambda = 0$. $\square$

By standard arguments, we have that the Palais-Smale condition holds for both the free functional E_ε and the functional E_ε constrained on M_ε, i.e. we have the following lemma.

2.4 Lemma. *(i) If $u_n \in W^1_0(\Omega)$ is a sequence such that*

$$\text{(2-6)} \qquad \|\nabla E_\varepsilon(u_n)\| \longrightarrow 0 \quad \text{and} \quad E_\varepsilon(u_n) \quad \text{is bounded}$$

then u_n has a converging subsequence.
(ii) if $u_n \in M_\varepsilon$ is a sequence such that

$$\text{(2-7)} \qquad \left\|\nabla E_\varepsilon(u_n) - \frac{(\nabla E_\varepsilon(u_n), \nabla J_\varepsilon(u_n))}{\|\nabla J_\varepsilon(u_n)\|^2}\nabla J_\varepsilon(u_n)\right\| \longrightarrow 0$$

and $E_\varepsilon(u_n)$ is bounded then u_n has a converging subsequence in M_ε.

3 Some estimates

From now on, for any function $u \in W_0^1(\mathscr{D})$ ($\mathscr{D} \subseteq \mathbb{R}^N$), we denote with the same symbol its extension to $\mathbb{R}^N$ obtained setting $u = 0$ outside of $\mathscr{D}$.

Moreover, for any $u \in W_0^1(\mathscr{D})$ ($\mathscr{D} \subseteq \mathbb{R}^N$), we denote with the symbols E_ε, J_ε, the objects corresponding to the ones we have defined by (1-3) and (2-1) for $u \in W_0^1(\Omega)$.

Also, for any $\mathscr{D} \subseteq \mathbb{R}^N$, $M_\varepsilon(\mathscr{D})$ is the submanifold of $W_0^1(\mathscr{D})$ defined as in (2-3) and we set

$$m(\varepsilon, \mathscr{D}) = \inf\{E_\varepsilon(u),\ u \in M_\varepsilon(\mathscr{D})\}\ . \tag{3-1}$$

Notice that $m(\varepsilon, \mathscr{D})$ is well defined by (2-5) of Lemma 2.2. Moreover, whenever $\mathscr{D} \subset \mathbb{R}^N$ is bounded, the infimum is achieved since E_ε satisfies (PS) on $M_\varepsilon(\mathscr{D})$ (Lemma 2.4).

If $\mathscr{D} = B_\varrho(y) = \{x \in \mathbb{R}^N : |x - y| < \varrho\}$ the number $m(\varepsilon, B_\varrho(y))$ does not depend on y; thus, for any $y \in \mathbb{R}^N$, we set

$$m(\varepsilon, \varrho) = m(\varepsilon, B_\varrho(y))\ . \tag{3-2}$$

Moreover, it is a trivial fact that

$$\varrho_1 < \varrho_2 \Rightarrow m(\varepsilon, \varrho_1) > m(\varepsilon, \varrho_2)\ . \tag{3-3}$$

If $\mathscr{D} = \mathbb{R}^N$, then (PS) for E_ε fails, however the following result holds:

3.1 Lemma. *$m(\varepsilon, \mathbb{R}^N)$ is achieved by a positive function radially symmetric $u(r)$ where r is the radial coordinate. $u(r)$ is decreasing in r and has the asymptotic behavior*

$$\begin{aligned} &\lim_{r\to\infty} r^{(N-1)/2} e^r u(r) = \eta_1 > 0 \\ &\lim_{r\to\infty} r^{(N-1)/2} e^r u'(r) = \eta_2 > 0\ . \end{aligned} \tag{3-4}$$

Proof. If we restrict our consideration to the subspace $W_r^1(\mathbb{R}^N)$ of $W^1(\mathbb{R}^N)$ consisting of functions radially symmetric about the origin, the embedding

$$j : W_r^1(\mathbb{R}^N) \longrightarrow L^p(\mathbb{R}^N)\ , \quad p \in \left(2, \frac{2N}{N-2}\right)\ ,$$

is compact [S]; so the functional E_ε satisfies the (P.S.) condition on $M_\varepsilon(\mathbb{R}^N) \cap W_r^1(\mathbb{R}^N)$ and

$$\inf\left\{E_\varepsilon(u) | u \in M(\mathbb{R}^N) \cap W_r^1(\mathbb{R}^N)\right\}$$

is achieved. Then in order to prove that $m(\varepsilon, \mathbb{R}^N)$ is achieved, it suffices to show that, for any $u \in M_\varepsilon(\mathbb{R}^N)$, there exists $w \in M_\varepsilon(\mathbb{R}^N) \cap W_r^1(\mathbb{R}^N)$ such that $E_\varepsilon(w) \leq E_\varepsilon(u)$. Let us denote by u^* the Schwartz symmetrized function about the origin of u and set $w = t^* u^*$ where $t^* = \xi_\varepsilon\left(\frac{u^*}{\|u^*\|}\right) \|u^*\|^{-1}$. Then, using the Riesz inequality and the properties of the spherical rearrangements, we obtain

$$
\begin{aligned}
E_\varepsilon(w) &= (1/2)(t^*)^2 \int_{\mathbb{R}^N} (\varepsilon|\nabla u^*|^2 + |u^*|^2)dx - \int_{\mathbb{R}^N} F(t^*u^*)dx \\
&\leq (1/2)(t^*)^2 \cdot \int_{\mathbb{R}^N} (\varepsilon|\nabla u|^2 + |u|^2)dx - \int_{\mathbb{R}^N} F(t^*u)dx \\
&= E_\varepsilon(t^*u) \leq \max_{t\in\mathbb{R}^+} E_\varepsilon(tu) = E_\varepsilon(u) \ .
\end{aligned}
$$

Then, the estimates (3-4) follow from a well known theorem of Gidas, Ni, Nirenberg [GNN]. □

In the following, $W^1_{\text{comp}}(\mathbb{R}^N)$ will denote the subspace of $W^1(\mathbb{R}^N)$ of functions whose support is compact.

For any $u \in W^1_{\text{comp}}(\mathbb{R}^N)$ we shall consider

$$
\beta(u) = \frac{\int_{\mathbb{R}^N} x \cdot |\nabla u(x)|^2 dx}{\int_{\mathbb{R}^N} |\nabla u(x)|^2 dx} \ .
$$

We define, for every $\varrho > 0$ and every $\gamma > 1$,

(3-5) $$m^*(\varepsilon, \varrho, \gamma) \equiv \inf\{E_\varepsilon(u) | u \in M_\varepsilon(B_{\gamma\varrho}(0)\backslash B_\varrho(0)) \ , \beta(u) = 0\} \ .$$

Let us point out that the number $m^*(\varepsilon, \varrho, \gamma)$ does not change if we move the center of the balls and $\beta(u)$ to any other point $x \in \mathbb{R}$.

It is clear that

$$
m^*(\varepsilon, \varrho, \gamma) > m(\varepsilon, \mathbb{R}^N)
$$

Now we set

(3-6) $$m^*(\varepsilon, \gamma) \equiv \inf_{\varrho>0} m^*(\varepsilon, \varrho, \gamma)$$

3.2 Lemma. *The relation*

(3-7) $$\lim_{\varrho\to+\infty} m(1, \varrho) = m(1, \mathbb{R}^N)$$

holds.

Proof. Let us denote by $\Psi \in W^1_r(\mathbb{R}^N)$ a positive function spherically symmetric about the origin, such that

$$
E_1(\Psi) = m(1, \mathbb{R}^N) \ , \quad \Psi \in M_1(\mathbb{R}^N)
$$

and consider the function $w_\varrho \in W^1_0(B_\varrho(0))$ defined by

$$
w_\varrho(x) = \zeta_\varrho(x)\psi(x) \ ,
$$

where $\zeta(x) : \mathbb{R} \longrightarrow [0, 1]$ is a C^∞-function defined by

$$
\zeta_\varrho(x) = \tilde{\zeta}\left(\frac{|x|}{\varrho}\right) \ .
$$

$\tilde{\zeta} : \mathbb{R}^+ \cup \{0\} \longrightarrow [0, 1]$ being a decreasing C^∞-function such that

$$
\tilde{\zeta}(t) = \begin{cases} 1 & t \leq \frac{1}{2} \\ 0 & t \geq 1 \end{cases} \ .
$$

Put

$$u_\varrho(x) = t_\varrho w_\varrho(x) ,$$

where

$$t_\varrho = \xi_1 \left(\frac{w_\varrho}{\|w_\varrho\|} \right) \|w_\varrho\|^{-1} .$$

Then $u_\varrho \in M_1(B_\varrho(0))$ and $E_1(u_\varrho) \geq m(1,\varrho) > m((1,\mathbb{R}^N)$.

Hence to prove the assertion it is sufficient to show that

(3-8)

$$\text{(a)} \quad \int_{\mathbb{R}^N \setminus B_{\varrho/2}(0)} f(\Psi)\Psi dx = o(1/\varrho)$$

$$\text{(b)} \quad \int_{\mathbb{R}^N \setminus B_{\varrho/2}(0)} F(\Psi) dx = o(1/\varrho)$$

(3-9)

$$\int_{\mathbb{R}^N} \left(|\nabla(\Psi - w_\varrho)|^2 + |\Psi - w_\varrho|^2 \right) dx = o(1/\varrho) .$$

The limit (3-8)(a) follows from the fact that, if ϱ is large enough, (3-4) and (H_1) imply

$$\begin{aligned} 0 \leq \int_{\mathbb{R}^N \setminus B_{\varrho/2}(0)} f(\Psi)\Psi dx &\leq \int_{\mathbb{R}^N \setminus B_{\varrho/2}(0)} (a\Psi + a\Psi^{p+1}) dx \\ &\leq k_1 \int_{\mathbb{R}^N \setminus B_{\varrho/2}(0)} \left(\frac{1}{e^{|x|}|x|^{(N-1)/2}} \right) dx \\ &+ k_2 \int_{\mathbb{R}^N \setminus B_{\varrho/2}(0)} \left(\frac{1}{e^{|x|}|x|^{(N-1)/2}} \right)^{p+1} dx = o\left(\frac{1}{\varrho}\right) \quad \text{as} \quad \varrho \longrightarrow +\infty . \end{aligned}$$

Analogously we prove that

$$0 \leq \int_{\mathbb{R}^N \setminus B_{\varrho/2}(0)} F(\Psi) dx \leq \int_{\mathbb{R}^N \setminus B_{\varrho/2}(0)} (b\Psi + b\Psi^{p+1}) dx = o(1/\varrho)$$

Also, using (3-4), if ϱ is large enough, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^N} \left(|\nabla(\Psi - w_\varrho)|^2 + |\Psi - w_\varrho|^2 \right) dx \\ &\leq k_3 \int_{\mathbb{R}^N \setminus B_{\varrho/2}(0)} |\nabla\Psi|^2 dx + k_3 \int_{\mathbb{R}^N \setminus B_{\varrho/2}(0)} |\Psi|^2 dx \\ &\leq k_4 \int_{\mathbb{R}^N \setminus B_{\varrho/2}(0)} \left(\frac{1}{e^{|x|}|x|^{(N-1)/2}} \right)^2 dx = o\left(\frac{1}{\varrho}\right) . \end{aligned}$$

□

3.3 Lemma. *The inequality*

(3-10)

$$m^*(1,\gamma) > m(1,\mathbb{R}^N)$$

holds for any fixed $\gamma > 1$.

Proof. It is obvious that $m^*(1,\gamma) \geq m((1,\mathbb{R}^N)$. To prove the strict inequality, we argue by contradiction and we suppose that the equality holds. In this case there exists a sequence ϱ_n such that

$$m^*(1,\varrho_n,\gamma) \longrightarrow m(1,\mathbb{R}^N) \quad \text{for} \quad n \longrightarrow +\infty \,. \tag{3-11}$$

We can exclude at once that $\{\varrho_n\}$ is bounded. In fact if it were bounded by L, we should have

$$m^*(1,\varrho_n\gamma) \geq m(1,\gamma L) > m(1,\mathbb{R}^N)\,.$$

So we can assume that (3-11) holds with $\varrho_n \uparrow +\infty$. Then there exists a sequence of functions $\{u_n\}$ such that

$$u_n \in W_0^1(B_{\gamma\varrho_n}(0)\backslash B_{\varrho_n}(0))\,, \quad u_n \neq 0\,, \quad \beta(u_n) = 0\,,$$

$$\frac{1}{2}\int_{B_{\gamma\varrho_n}(0)\backslash B_{\varrho_n}(0)} \left(|\nabla u_n|^2 + u_n^2\right) dx - \int_{B_{\gamma\varrho_n}(0)\backslash B_{\varrho_n}(0)} F(u_n)dx \longrightarrow m(1,\mathbb{R}^N)$$

$$\int_{B_{\gamma\varrho_n}(0)\backslash B_{\varrho_n}(0)} \left(|\nabla u_n|^2 + u_n^2\right) dx = \int_{B_{\gamma\varrho_n}(0)\backslash B_{\varrho_n}(0)} f(u_n)u_n dx\,.$$

On the other hand it is known (see [L]) that any minimizing sequence in $W_0^1(\mathbb{R}^N\backslash B_{\varrho_1}(0))$ has the form

$$w_n(x) + \Psi(x - y_n)\,, \tag{3-12}$$

where $\{w_n(x)\} \subset W^1(\mathbb{R}^N)$ is a sequence going strongly to 0 in $W^1(\mathbb{R}^N)$, $\{y_n\} \subset \mathbb{R}^N$ is such that $|y_n| \longrightarrow +\infty$ and $\Psi \in W^1(\mathbb{R}^N)$ is a positive function, spherically symmetric about the origin, such that $E_1(\Psi) = m(1,\mathbb{R}^N)$, $\Psi \in M_1(\mathbb{R}^N)$. Thus, in particular, u_n has the form (3-12).

Since any regular solution φ of $-\Delta u + u = f(u)$ in $\mathbb{R}^N$ satisfies the following Pohozaev type inequality

$$\int_{\mathbb{R}^N} |\nabla\varphi|^2 dx = \frac{2N}{N-2}\int_{\mathbb{R}^N}\left[-\frac{\varphi^2}{2} + F(\varphi)\right] dx$$

we have

$$\int_{\mathbb{R}^N} |\nabla\Psi|^2 dx = Nm(1,\mathbb{R}^N) = b > 0\,.$$

Since $\|w_n\|_{W^1(\mathbb{R}^N)} \longrightarrow 0$, it follows that

$$\int_{B_{\varrho_n/4}(y_n)} |\nabla(w_n(x) + \Psi(x - y_n))|^2 dx \underset{n\to+\infty}{\longrightarrow} b\,.$$

Set $C_n = B_{\gamma\varrho_n}(0)\backslash B_{\varrho_n}(0)$; clearly

$$\int_{C_n} |\nabla u_n|^2 dx = \int_{C_n\cap[B_{\varrho_n/4}(y_n)]} |\nabla u_n|^2 dx + \int_{C_n\backslash[B_{\varrho_n/4}(y_n)]} |\nabla u_n|^2 dx$$

and

$$(3\text{-}13)\qquad \begin{aligned} 0 &\le \int_{C_n\setminus[B_{\varrho_n/4}(y_n)]} |\nabla u_n|^2 dx \\ &\le \int_{\mathbb{R}^N\setminus[B_{\varrho_n/4}(y_n)]} |\nabla w_n(x)+\Psi(x-y_n))|^2 dx \underset{n\to+\infty}{\longrightarrow} 0 \end{aligned}$$

$$(3\text{-}14)\qquad \begin{aligned} &\int_{C_n\cap[B_{\varrho_n/4}(y_n)]} |\nabla u_n|^2 dx \\ &= \int_{C_n\cap[B_{\varrho_n/4}(y_n)]} |\nabla w_n(x)+\Psi(x-y_n))|^2 dx \underset{n\to+\infty}{\longrightarrow} b\,; \end{aligned}$$

thus $C_n \cap [B_{\varrho_n/4}(y)] \neq \varnothing$ and $|y_n| > (3/4)\varrho_n$ for n big enough.

Now

$$\frac{\int_{C_n\cap[B_{\varrho_n/4}(y_n)]} x|\nabla u_n|^2 dx}{\int_{C_n\cap[B_{\varrho_n/4}(y_n)]} |\nabla u_n|^2 dx} = y_n + \frac{\int_{C_n\cap[B_{\varrho_n/4}(y_n)]}(x-y_n)|\nabla u_n|^2 dx}{\int_{C_n\cap[B_{\varrho_n/4}(y_n)]} |\nabla u_n|^2 dx}$$

so

$$\left|\frac{\int_{C_n\cap[B_{\varrho_n/4}(y_n)]} x|\nabla u_n|^2 dx}{\int_{C_n\cap[B_{\varrho_n/4}(y_n)]} |\nabla u_n|^2 dx}\right| \ge \varrho_n/2\,.$$

On the other hand

$$\left|\frac{\int_{C_n\setminus[B_{\varrho_n/4}(y_n)]} x|\nabla u_n|^2 dx}{\int_{C_n\setminus[B_{\varrho_n/4}(y_n)]} |\nabla u_n|^2 dx}\right| < \gamma\varrho_n\,.$$

Since $\beta(u_n) = 0$, we have

$$\int_{C_n\cap[B_{\varrho_n/4}(y_n)]} x|\nabla u_n|^2 dx + \int_{C_n\setminus[B_{\varrho_n/4}(y_n)]} x|\nabla u_n|^2 dx = 0\,.$$

then

$$\begin{aligned} (\varrho_n/2)\cdot\left|\int_{C_n\cap[B_{\varrho_n/4}(y_n)]} |\nabla u_n|^2 dx\right| &\le \left|\int_{C_n\cap[B_{\varrho_n/4}(y_n)]} x|\nabla u_n|^2 dx\right| \\ = \left|\int_{C_n\setminus[B_{\varrho_n/4}(y_n)]} x|\nabla u_n|^2 dx\right| &< \gamma\varrho_n\left|\int_{C_n\setminus[B_{\varrho_n/4}(y_n)]} |\nabla u_n|^2 dx\right| \end{aligned}$$

and using (3-13) and (3-14), we obtain:

$$(\varrho_n/2)\cdot(b+o(1)) \le \gamma\varrho_n\cdot o(1)$$

and this inequality gives

$$b/(2\gamma) < o(1)$$

which is a contradiction. □

3.4 Lemma. *For any* $\gamma > 1$, *there exists* $\bar R \equiv \bar R(\gamma) > 0$ *such that*

$$(3\text{-}15)\qquad m(1,R) < m^*(1,R,\gamma)$$

for every $R \geq \bar{R}(\gamma)$.

Proof. From Lemmas 3.2 and 3.3, we deduce that there exists $\bar{R} > 0$ such that for every $R \geq \bar{R}$

$$m(1, R) < m^*(1, \gamma) .$$

Then, (3-15) holds since, by definition,

$$m^*(1, R, \gamma) \geq m^*(1, \gamma)$$

for every R. □

3.5 Corollary. *For every* $r > 0$ *and* $\gamma > 1$, *there exists* $\bar{\varepsilon} \equiv \bar{\varepsilon}(\gamma, \varrho)$, *such that the relation*

$$\text{(3-16)} \qquad m(\varepsilon, \varrho) < m^*(\varepsilon, \varrho, \gamma)$$

holds for every $\varepsilon \in (o, \bar{\varepsilon}]$.

Proof. We assert that $\bar{\varepsilon} = (\varrho/\bar{R})^2$ where $\bar{R}$ is the number found in Lemma 3.4. Indeed it is possible to define a one to one map

$$T : W_0^1(B_\varrho(0)) \longrightarrow W_0^1(B_{\varrho/\sqrt{\varepsilon}}(0))$$

by $T(u) = u_\varepsilon(x) \equiv u(\sqrt{\varepsilon}x)$ for every $u \in W_0^1(B_\varrho(0))$. Then a simple calculation shows that

$$\int_{B_{\varrho/\sqrt{\varepsilon}}(0)} \left(|\nabla u_\varepsilon|^2 + u_\varepsilon^2\right) dx - \int_{B_{\varrho/\sqrt{\varepsilon}}(0)} f(u_\varepsilon)u_\varepsilon dx$$
$$= \varepsilon^{-N/2} \left[\int_{B_\varrho(0)} (\varepsilon|\nabla u|^2 + u^2)dx - \int_{B_\varrho(0)} f(u)u\, dx\right] ;$$

and

$$(1/2)\int_{B_\varrho/\sqrt{\varepsilon}(0)} (|\nabla u|^2 + u_\varepsilon^2)dx - \int_{B_\varrho/\sqrt{\varepsilon}(0)} F(u_\varepsilon)dx$$
$$= \varepsilon^{-N/2} \left[(1/2)\int_{B_\varrho(0)} (\varepsilon|\nabla u|^2 + u^2)dx - \int_{B_\varrho(0)} F(u)dx\right] .$$

So

$$m(\varepsilon, \varrho) = \varepsilon^{N/2} m(1, \varrho/\sqrt{\varepsilon}) .$$

Analogously it is easy to verify that

$$m^*(\varepsilon, \varrho, \gamma) = \varepsilon^{N/2} m^*(1, \varrho/\sqrt{\varepsilon}, \gamma) ;$$
$$m(\varepsilon, \mathbb{R}^N) = \varepsilon^{N/2} m(1, \mathbb{R}^N) .$$

Hence if $\varrho/\sqrt{\varepsilon} \geq \bar{R}$, that is if $\varepsilon \leq (\varrho/\bar{R})^2$, the relation (3-16) follows from (3-15).

4 Proof of Theorem 1.1

In what follows, without any loss of generality, we shall assume $O \in \Omega$. Moreover, we denote by $r > 0$, a number such that the sets

$$\Omega^+ = \{x \in \mathbb{R}^N | d(x, \Omega) \leq r\} \quad \text{and} \quad \Omega^- = \{x \in \Omega | d(x, \partial\Omega) \geq 2r\}$$

are homotopically equivalent to Ω and $B_r(0) \subset \Omega$. Finally we fix

$$\gamma = \frac{\text{diam}\, \Omega}{r} .$$

4.1 Lemma. *There exists* ε^* *such that*

$$u \in M_\varepsilon , \quad E_\varepsilon(u) \leq m(\varepsilon, r) \Rightarrow \beta(u) \in \Omega^+ \tag{4-1}$$

for every $\varepsilon \in (0, \varepsilon^*]$.

Proof. First of all we observe that by the choice of r,

$$m(\varepsilon, \Omega) < m(\varepsilon, r)$$

for any $\varepsilon > 0$.

Thus the set of the functions verifying the condition on the left side of (4-1) is non-empty. Also let us notice that our choice of r and γ implies that

$$\gamma r = \text{diam}\, \Omega .$$

Now let ε^* be the number, depending of course on γ and r, which satisfies the assertion of Corollary 3.5.

In order to prove (4-1), let us suppose $\varepsilon \in (0, \varepsilon^*]$ and let $u^* \in M_\varepsilon$ be a function such that $E_\varepsilon(u^*) \leq m(\varepsilon, r)$.

We argue by contradiction and we assume that $x^* = \beta(u^*) \notin \Omega^+$. Then $\Omega \subset B_{\text{diam}\, \Omega}(x^*) \backslash B_r(x^*) = B_{\gamma r}(x^*) \backslash B_r(x^*)$.

Therefore

$$m^*(\varepsilon, r, \gamma) = \inf \left\{ \int_{B_{\gamma r}(x^*) \backslash B_r(x^*)} \left[\frac{1}{2} \varepsilon |\nabla u|^2 + \frac{1}{2} u^2 - F(u) \right] dx : \right.$$
$$\int_{B_{\gamma r}(x^*) \backslash B_r(x^*)} \left[\varepsilon |\nabla u|^2 + u^2 - f(u) u \right] dx = 0 ;$$
$$\left. u \in W_0^1(B_{\gamma r}(x^*) \backslash B_r(x^*)) ; \qquad \beta(u) = x^* \right\} \leq E_\varepsilon(u^*) \leq m(\varepsilon, r) .$$

that contradicts, by our choice of ε, (3-16). □

For any $\varepsilon > 0$, let us define the operator

$$\phi_\varepsilon : \Omega^- \longrightarrow W_0^1(\Omega)$$

by

$$[\phi_\varepsilon(y)](x) = \begin{cases} u_\varepsilon(|x - y|) & \forall x \in B_{2r}(y) \\ 0 & \forall x \in \Omega \backslash B_{2r}(y) \end{cases}$$

where $u_\varepsilon(|x|)$ is a positive function, radially symmetric about the origin, such that $u_\varepsilon \in M_\varepsilon(B_{2r}(0))$, $E_\varepsilon(u_\varepsilon) = m(\varepsilon, 2r) < m(\varepsilon, r)$.

Note that ϕ_ε is continuous and that

$$\phi_\varepsilon(y) \in M_\varepsilon\ , \quad \beta\left(\phi_\varepsilon(y)\right) = y\ .$$

Also we set

$$M_\varepsilon^c = \{u \in M_\varepsilon | E_\varepsilon(u) \leq c\}\ .$$

4.2 Lemma. *Let $\varepsilon^* > 0$ be as in Lemma 4.1, then for every $\varepsilon \in (0, \varepsilon^*]$*

$$\beta\left(M_\varepsilon^{m(\varepsilon,r)}\right) \subseteq \Omega^+\ , \quad \phi_\varepsilon(\Omega^-) \subset M_\varepsilon^{m(\varepsilon,r)}\ ;$$

and

$$\beta \circ \phi_\varepsilon = j$$

where $j : \Omega^- \longrightarrow \Omega^+$ denotes the embedding map, i.e.,

$$j(x) = x\ , \quad \forall x \in \Omega^-\ .$$

Proof. The proof is an immediate consequence of the relation (4-1) and the definition of ϕ_ε. □

4.3 Lemma. *Let ε be as in Lemma 4.1, then for every $\varepsilon \in (0, \varepsilon^*]$*

$$\operatorname{cat}\left(M_\varepsilon^{m(\varepsilon,r)}\right) \geq \operatorname{cat} \Omega\ .$$

Proof. Suppose that $\operatorname{cat}(M_\varepsilon^{m(\varepsilon,r)}) = n$; this means that n is the smallest positive integer such that

$$M_\varepsilon^{m(\varepsilon,r)} \subseteq \mathrm{A}_1 \cup \mathrm{A}_2 \cup \ldots \cup \mathrm{A}_n\ ,$$

where A_i, $i = 1, \ldots, n$, are closed and contractible in $M_\varepsilon^{m(\varepsilon,r)}$, i.e. there exist

$$\mathscr{H}_1 \in \mathscr{C}\left([0,1] \times \mathrm{A}_i\ ,\ M_\varepsilon^{m(\varepsilon,r)}\right)\ , \quad i = 1, 2, \ldots, n$$

such that

$$\begin{aligned} &\mathscr{H}_1(0, u) = u && \text{for every} \quad u \in \mathrm{A}_i \\ &\mathscr{H}(1, u) = w_i \in M_\varepsilon^{m(\varepsilon,r)} && \text{for every} \quad u \in \mathrm{A}_i\ . \end{aligned}$$

Now, we set

$$K_i = \phi_\varepsilon^{-1}(A_i)\ .$$

The sets K_i are closed subsets of Ω^- and

$$\Omega^- \subseteq K_1 \cup \ldots \cup K_n$$

Moreover K_i, $i = 1, \ldots, n$, is contractible in Ω^+, in fact if we consider the maps h_i, $i = 1, \ldots, n$, defined by

$$h_i(t, x) = \beta \circ \mathscr{H}_i(t, \phi_\varepsilon(x))$$

we have

$$\begin{aligned} &h_i \in \mathscr{C}\left([0,1] \times K_i, \Omega^+\right) \\ &h_i(0, x) = \beta \circ \mathscr{H}_i(0, \phi_\varepsilon(x)) = \beta \circ \phi_\varepsilon(x) = x\ ,\ \forall x \in K_i \\ &h_i(1, x) = \beta \circ \mathscr{H}_i(1, \phi_\varepsilon(x)) = \beta(w_i) = x_i \in \Omega^*\ ,\ \forall x \in K_i \end{aligned}$$

Then

$$\mathrm{cat}_{\Omega^+}(\Omega^-) \leq n\,.$$

Since $\mathrm{cat}_\Omega(\Omega) = \mathrm{cat}_{\Omega^+}(\Omega^-)$, the lemma is proved. □

We are now ready to give the

Proof of Theorem 1.1 Choose ε^* as in Lemma 4.1 and consider $\varepsilon \in (0, \varepsilon^*]$. If, for every $\varrho \in [r, 2r)$, $m(\varepsilon, \varrho)$ is a critical level, we are done. Otherwise, we can suppose that $m(\varepsilon, r)$ is not a critical level. Since the functional E_ε satisfies the Palais-Smale condition on the set $M_\varepsilon^{m(\varepsilon,r)}$, applying a classical result of the Ljusternik-Schnirelman theory, we deduce that:

$$\#\left\{u \in M_\varepsilon : E_\varepsilon(u) \leq m(\varepsilon, r),\ \nabla E_\varepsilon\big|_{M_\varepsilon}(u) = 0\right\} \geq \mathrm{cat}\ M_\varepsilon^{m(\varepsilon,r)}$$

and, using (4-2), we conclude that the functional E_ε has at least (cat Ω) critical points in M_ε having energy less than $m(\varepsilon, r)$.

Since cat $\Omega > 1$, then the set $\mathfrak{G} = \overline{\phi_\varepsilon(\Omega^-)}$ is non-contractible in $M_\varepsilon^{m(\varepsilon,r)}$. Then in order to prove the existence of an other critical point, it is sufficient to construct an energy level $c > m(\varepsilon, r)$ such that $\mathfrak{G}$ is contractible in M_ε^c.

Take $u^* \in M_\varepsilon (u^* \geq 0)$ such that $u^* \notin \mathfrak{G}$, and define the set

$$\Theta = \{\vartheta u^* + (1 - \vartheta)u | \vartheta \in [0, 1],\ u \in \mathfrak{G}\}$$

Θ is compact and contractible, moreover $0 \notin \Theta$ (since every u in $\mathfrak{G}$ is positive on a set of positive measure); hence the set

$$\Lambda = \left\{t(w)w | w \in \Theta,\ t(w) = \xi_1\left(\frac{w}{\|w\|}\right)\|w\|^{-1}\right\}$$

is well defined and $\mathfrak{G} \subseteq \Lambda \subseteq M_\varepsilon$. Then, setting

$$c = \max\left\{E_\varepsilon(w), w \in \Lambda\right\}$$

we have that $\mathfrak{G}$ is contractible in M_ε^c. □

5 Morse theory for the functional E_ε

First of all we recall some notation: if (X, Y) is a couple of topological spaces, we set

$$\mathcal{P}_t(X, Y) = \sum_k \dim[H_k(X, Y)]t^k$$

where $H_k(X, Y)$ is the k-th homology group with coefficients in some field; moreover we set

$$\mathcal{P}_t(X) = \mathcal{P}_t(X, \phi) = \sum_k \dim H_k(X)t^k$$

5.1 Lemma. *Let ε^* be as in Lemma 4.1 ; then for any $\varepsilon \in (0, \varepsilon^*]$*

(5-1) $$\mathcal{P}_t\left(M_\varepsilon^{m(\varepsilon,r)}\right) = \mathcal{P}_t(\Omega) + \mathcal{Z}(t)\,,$$

where $\mathcal{Z}(t)$ is a polynomial with non-negative coefficients.

Proof. Let us denote with $\phi_{\varepsilon,k}$ and β_k the homomorphisms induced by ϕ_ε and β respectively between the k-th homology groups, i.e.

$$H_k(\Omega^-) \xrightarrow{\phi_{\varepsilon,k}} H_k\left(M_\varepsilon^{m(\varepsilon,r)}\right) \xrightarrow{\beta_k} H_k(\Omega^+) .$$

Since $\beta_k \circ \phi_{\varepsilon,k} = Id_k$, $H_k(\Omega^-)$ is homotopic to a subspace of $H_k(M_\varepsilon^{m(\varepsilon,r)})$ and hence

$$\dim\left(H(\Omega^-)\right) \le \dim\left(H_k\left(M_\varepsilon^{m(\varepsilon,r)}\right)\right) ;$$

so, from the fact that Ω and Ω^- are homotopically equivalent, it follows that

$$\dim\left(H_k(\Omega)\right) = \dim\left(H_k(\Omega^-)\right) \le \dim\left(H_k\left(M_\varepsilon^{m(\varepsilon,r)}\right)\right) .$$

Then, we get (5-1). □

5.2 Lemma. *Let ε^* and ε be as in Lemma 5.1, $\delta \in (o, k_\varepsilon/2)$ (k_ε is defined in Lemma 2.2) and let $c \in (\delta, +\infty]$ be a noncritical level of E_ε; then*

$$\mathscr{P}_t(E_\varepsilon^c, E_\varepsilon^\delta) = t\mathscr{P}_t(M_\varepsilon^c) . \tag{5-2}$$

Before proving Lemma 5.2, we need to recall some results.

5.3 Lemma. *Let $\mathfrak{M}$ be a manifold and let $\mathfrak{N} \subset \mathfrak{M}$ be a closed oriented submanifold of codimension d. If W is a subset of $\mathfrak{N}$ closed in $\mathfrak{N}$, then*

$$\mathscr{P}(\mathfrak{M}, \mathfrak{M}\backslash W) = t^d \mathscr{P}(\mathfrak{N}, \mathfrak{N}\backslash W) .$$

Proof. This is an immediate consequence of the Thom isomorphism theorem (see e.g. [D] pag. 321, Corollary 11.14), since $\mathfrak{N}$ is a strong deformation retract in $\mathscr{M}$. Moreover notice that the Thom theorem, in this form, holds even if the dimension of $\mathfrak{M}$ is infinite. □

Now set

$$\Sigma_\varepsilon(a,b) = \left\{u \in W_0^1(\Omega) | a < E_\varepsilon(u) < b\right\} .$$

5.4 Lemma. *If a and b are not critical levels, then*

$$\mathscr{P}(E_\varepsilon^b, E_\varepsilon^a) = \mathscr{P}(\Sigma_\varepsilon(a,b), \Sigma_\varepsilon(a,b)\backslash M_\varepsilon)$$

Proof. Take two open neighboroods U and V of M_ε, $\bar{U} \subset V$, and let χ be a C^1-function which is 1 in U and 0 out of V. Then define

$$F(u) = \nabla E_\varepsilon(u) - \chi(u) \frac{(\nabla E_\varepsilon(u), \nabla J_\varepsilon(u))}{\|\nabla J_\varepsilon(u)\|^2} \nabla J_\varepsilon(u)$$

and let be $\eta(t,u)$ be flow relative to the Cauchy problem

$$\begin{cases} \dfrac{d}{dt}\eta(t,u) = -\dfrac{F(u)}{1 + \|F(u)\|} \\ \eta(0,u) = u \end{cases} .$$

It is easy to check that the Cauchy problem is well posed and $\eta(t,u)$ is defined for every $t \in \mathbb{R}$ and every $u \in W_0^1(\Omega)$. Moreover, if V is sufficiently small and $\nabla E(u) \neq 0$, then

$$\frac{d}{dt}E_\varepsilon(\eta(t,u)) < 0\ .$$

Now set

$$W = \left\{u \in \overline{\Sigma_\varepsilon(a,b)} \mid \forall t \geq 0,\ \eta(t,u) \in \overline{\Sigma_\varepsilon(a,b)}\right\}\ .$$

By the Theorem I.3.8.(i) of [B$_2$] (cfr. also [B-G]), we get

$$\mathscr{P}_t(\overline{\Sigma_\varepsilon(a,b)}, \overline{\Sigma_\varepsilon(a,b)}\backslash W) = \mathscr{P}_t(\overline{\Sigma_\varepsilon(a,b)},\ E_\varepsilon^{-1}(a))\ .$$

Since

$$\mathscr{P}_t(E_\varepsilon^b, E_\varepsilon^a) = \mathscr{P}_t(\overline{\Sigma_\varepsilon(a,b)}, E_\varepsilon^{-1}(a))\ ,$$

and $W = \overline{\Sigma_\varepsilon(a,b)} \cap M_\varepsilon = M_\varepsilon^b$, we have that

$$\mathscr{P}_t(E_\varepsilon^b, E_\varepsilon^a) = \mathscr{P}_t\left(\overline{\Sigma_\varepsilon(a,b)}, \overline{\Sigma_\varepsilon(a,b)}\backslash M_\varepsilon\right)\ .$$

Now, since a and b are not critical values,

$$\mathscr{P}_t\left(\overline{\Sigma_\varepsilon(a,b)}, \overline{\Sigma_\varepsilon(a,b)}\backslash M_\varepsilon\right) = \mathscr{P}_t(\Sigma_\varepsilon(a,b), \Sigma_\varepsilon(a,b)\backslash M_\varepsilon)$$

from which the conclusion follows. □

Proof of Lemma 5.2. We apply Lemma 5.3 with $\mathfrak{M} = \Sigma_\varepsilon(a,b)$ and $\mathfrak{N} = \Sigma_\varepsilon(a,b) \cap M_\varepsilon$. The conclusion follows from Lemma 5.4. □

5.5 Corollary. *Let ε^*, ε and δ be as in Lemma 5.2; then*

$$\mathscr{P}_t(E_\varepsilon^{m(\varepsilon,r)}, E_\varepsilon^\delta) = t\mathscr{P}_t(\Omega) + t\mathscr{Z}(t) \tag{5-3}$$

$$\mathscr{P}_t(W_0^1(\Omega), E_\varepsilon^\delta) = t\mathscr{P}_t(M_\varepsilon) = t\ , \tag{5-4}$$

where $\mathscr{Z}(t)$ is a polynomial with non-negative integer coefficients.

Proof. As we have already observed in the proof of Th. 1.1, we can assume without loss of generality that $m(\varepsilon,r)$, is not a critical level of E_ε. Then (5-3) follows from (5-1) and (5-2). (5-4) follows from (5-2) and the fact that M_ε is contractible, so dim $H_k(M_\varepsilon) = 1$ if $k = 0$ and dim $H_k(M_\varepsilon) = 0$ if $k \neq 0$. □

5.6 Lemma. *Let ε^* and ε be as in Lemma 5.2; then*

$$\mathscr{P}_t(W_0^1\left(\Omega), E_\varepsilon^{m(\varepsilon,r)}\right) = t^2\left[\mathscr{P}_t(\Omega) + \mathscr{Z}(t) - 1\right] \tag{5-5}$$

where $\mathscr{Z}(t)$ is a polynomial with non-negative integer coefficients.

Proof. Let δ be as in Corollary 5.5 and consider the exact sequence,

$$\begin{aligned}&\to H_k(W_0^1(\Omega), E_\varepsilon^\delta) \xrightarrow{j_k} H_k(W_0^1(\Omega), E_\varepsilon^{m(\varepsilon,r)}) \to \\ &\xrightarrow{\partial_k} H_{k-1}(E_\varepsilon^{m(\varepsilon,r)}, E_\varepsilon^\delta) \xrightarrow{i_{k-1}} H_{k-1}(W_0^1(\Omega), E_\varepsilon^\delta) \longrightarrow\ .\end{aligned}$$

Thus, for $k > 2$,

$$\dim\left[H_k(W_0^1(\Omega), E_\varepsilon^{m(\varepsilon,r)})\right] = \dim\left[H_{k-1}(E_\varepsilon^{m(\varepsilon,r}, E_\varepsilon^\delta)\right]$$

For $k = 2$, we have

$$\to H_2(W_0^1(\Omega), E_\varepsilon^\delta) \xrightarrow{j_2} H_2(W_0^1(\Omega), E_\varepsilon^{m(\varepsilon,r)}) \xrightarrow{\partial_2}$$
$$\xrightarrow{\partial_2} H_1(E_\varepsilon^{m(\varepsilon,r)}, E_\varepsilon^\delta) \xrightarrow{i_1} H_1(W_0^1(\Omega), E_\varepsilon^\delta) \to$$

since, i_1 is an isomorphism,

$$H_2(W_0^1(\Omega), E_\varepsilon^{m(\varepsilon,r)}) = j_2\left(H_2\left(W_0^1(\Omega), E_\varepsilon^\delta\right)\right) = 0 .$$

For $k = 1$, we have

$$\xrightarrow{\partial_2} H_1(E_\varepsilon^{m(\varepsilon,r)}, E_\varepsilon^\delta) \xrightarrow{i_1} H_1(W_0^1(\Omega), E_\varepsilon^\delta) \xrightarrow{j_1} H_1(W_0^1(\Omega), E_\varepsilon^{m(\varepsilon,r)}) \xrightarrow{\partial_1}$$
$$H_1(W_0^1(\Omega), E_\varepsilon^{m(\varepsilon,r)}) = 0 .$$

Moreover

$$H_0(W_0^1(\Omega), E_\varepsilon^{m(\varepsilon,r)}) = 0 .$$

By the above formulas and (5-2), we conclude that

$$\mathscr{P}_t(W_0^1(\Omega), E_\varepsilon^{m(\varepsilon,r)}) = t\left[\mathscr{P}_t(E_\varepsilon^{m(\varepsilon,r)}), E_\varepsilon^\delta) - t\right]$$
$$= t^2\left[\mathscr{P}_t(\Omega) + \mathscr{Z}(t) - 1\right] .$$ □

5.7 Lemma. *Let ε be as in Lemma 5.1; suppose that*
(i) $\varepsilon \in (o, \varepsilon^*]$
(ii) the set $\mathscr{K}$ of nontrivial solutions of problem (P_ε) is discrete;
then

(5-6) $$\sum_{u\in\mathfrak{C}_1} i_t(u) = t\mathscr{P}_t(\Omega) + t\mathscr{Z}(t) + (1+t)\mathscr{Q}_1(t)$$

(5-7) $$\sum_{u\in\mathfrak{C}_2} i_t(u) = t^2\left[\mathscr{P}_t(\Omega) + \mathscr{Z}(t) - 1\right] + (1+t)\mathscr{Q}_2(t) ,$$

where

$$\mathfrak{C}_1 = \{u \in \mathscr{K} | \delta < E_\varepsilon(u) \leq m(\varepsilon, r)\}$$

and

$$\mathfrak{C}_2 = \{u \in \mathscr{K} | E_\varepsilon(u) > m(\varepsilon, r)\} .$$

Proof. Since E_ε satisfies the Palais-Smale condition, by the Morse theory we have that

$$\sum_{u\in\mathfrak{C}_1} i_t(u) = \mathscr{P}_t(E_\varepsilon^{m(\varepsilon,r)}, E_\varepsilon^\delta) + (1+t)\mathscr{Q}_1(t) .$$

Then the (5-6) follows from (5-3). Analogously, (5-7) follows from (5-5). □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Choose ε^* and ε as in Lemma 5.7. Since E_ε does not have any non-zero solution below the level δ, $\mathscr{K} = \mathfrak{C}_1 \cup \mathfrak{C}_2$, then

$$\sum_{u\in\mathscr{K}} i_t(u) = \sum_{u\in\mathfrak{C}_1} i_t(u) + \sum_{u\in\mathfrak{C}_2} i_t(u) .$$

The conclusion follows by Lemma 5.7. □

Proof of Theorem 1.3. Theorem 1.3 is an immediate consequence of Theorem 1.2,; in fact, if u is a nondegenerate critical point,

$$i_t(u) = t^{\mu(u)} . \qquad \square$$

Proof of Corollary 1.4. The assertion is a consequence of the Theorems 1.2, 1.3 and the fact that

$$\mathcal{P}_t(\Omega) = 1 + kt^{N-1} . \tag{5-6}$$

The computation of $\mathcal{P}_t(\Omega)$ is an easy application of algebraic topology techniques which we will show for completeness.

Using the excision property and the fact that C_1's are ENR, we have:

$$H_q(A,\Omega) \cong H_q\left(\bigcup_i \bar{C}_i, \bigcup_i \partial C_i\right) \cong \bigoplus_{i=1}^{k} H_q(\bar{C}_i, \partial C_i) \cong \bigoplus_{i=1}^{k} H_q(B_N, \partial B_N) ,$$

hence $\mathcal{P}_t(A,\Omega) = kt^N$.

From the exactness of the following sequence,

$$\longrightarrow H_q(A) \xrightarrow{j_q} H_q(A,\Omega) \xrightarrow{\partial_q} H_{q-1}(\Omega) \xrightarrow{i_{q-1}} H_{q-1}(A) \longrightarrow$$

it follows that, for $q = N$,

$$\dim\left[H_{N-1}(\Omega)\right] = \dim\left[H_N(A,\Omega)\right] = k$$

for $q \geq 2$ and $q \neq N$

$$\dim\left[H_{q-1}(\Omega)\right] = \dim\left[H_q(A,\Omega)\right] = 0$$

moreover, since Ω is connected,

$$\dim\left[H_0(\Omega)\right] = 1 .$$

Concluding, we get (5-6). $\square$

References

[B₁] Benci, V.: A new approach to the Morse-Conley theory and applications. Ann. Mat. Pura Appl. **158**, 231–305 (1991)

[B₂] Benci, V.: Introduction to Morse theory. A new approach. (to appear)

[B-C] Benci, V., Cerami, G.: The effect of the domain topology on the number of positive solutions of nonlinear elliptic problems, Arch. Ration. Mech. Anal. **114**, 79–93 (1991)

[B-C-P] Benci, V., Cerami, G., Passaseo, D.: On the number of the positive solutions of some nonlinear elliptic problems. In: Nonlinear analisys. A tribute in honour of G. Prodi. Quaderno Scuola Normale Superiore, Pisa, 1991, pp. 93–109

[B-G] Benci, V., Giannoni, F.: In preparation.

[Ca] Candela, A.M.: Remarks on the number of solutions for a class of nonlinear elliptic problems. Preprint.

[C-P] Cerami, G., Passaseo, D.: Existence and multiplicity of positive solutions for nonlinear elliptic problems in exterior domains with rich topology. J. Nonlinear Anal. T.M.A. **18**, 103–119 (1992).

[C] Chang, K.C.: Infinite dimensional Morse theory and its applications. Semin. Mat. Supér. **97**, Press of the University of Montreal (1986)

[D] Dold, A.; Lectures on algebraic topology. Berllin, Heidelberg, New York: Springer 1972
[G.N.N.] Gidas, B., Ni, W.M., Nirenberg, L.: Symmetry of positive solutions of nonlinear elliptic equations in $\mathbb{R}^N$. Mathematical analysis and applications. Part A: Advances in mathematics supplementary studies, vol. 7A, pp. 369–402 (1980)
[L] Lions, P.L.: The concentration – compactness principle in the calculus of variations. The locally compact case. I–II. Ann. Inst. Henri Poincaré, Anal. Non Lineaire **1**, 223–283, 109–145 (1984)
[MP] Marino, A., Prodi, G.: Metodi perturbativi nella teoria di Morse. Boll. U.M.I. **11** (4), 1–32 (1975)
[S] Strauss, W.: Existence of solitary waves in higher dimensions. Commun. Math. Phys. **55**, 149–162 (1977)