ELIMINATION OF THE NODES IN PROBLEMS OF n BODIES

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Abstract. In application of the Reduction Theorem to the general problem of n ($\geqslant 3$) bodies, a Mathieu canonical transformation is proposed whereby the new variables separate naturally into (i) a coordinate system on any reduced manifold of constant angular momentum, and (ii) a quadruple made of a pair of ignorable longitudes together with their conjugate momenta. The reduction is built from a binary tree of kinetic frames Explicit transformation formulas are obtained by induction from the top of the tree down to its root at the invariable frame; they are based on the unit quaternions which represent the finite rotations mapping one vector base onto another in the chain of kinetic frames. The development scheme lends itself to automatic processing by computer in a functional language.

1. Introduction

The equations of motion in the problem of $n(\geqslant 3)$ bodies may be reduced from order $6n$ to order $6n-10$ by means of the 6 integrals of the barycenter, and the 3 integrals of the angular momentum; the reduction may be effected by canonical transformations. The practical problem of finding the canonical transformation to eliminate the total linear momentum has been solved in a definitive manner by Poincaré (1896, 1897). Among the linear mappings in Cartesian coordinates offered by Radau (1868), Poincaré showed why only two of them prove to be expedient in the General Theory of Perturbations; these are known as the *heliocentric* reduction, and the reduction in *barycentric chain* (Wintner, 1947). For the problem of 3 bodies, Whittaker (1904) supplied two canonical transformations to eliminate the angular momentum and an ascending node, one geometric and one kinematic. The geometric one is a canonical extension in which the plane of the three particles is adopted as reference element. Bennett (1904) modified the method to apply it to the problem of $n \geqslant 3$ bodies. Credited by Whittaker himself to a most penetrating study by Radau (1868), the kinematic method involves the orbital plane of each mass point; Boigey (1979, 1981) extended it to the problem of 4 bodies. All three authors, Whittaker, Bennett and Boigey, have derived the reducing canonical mapping from a generating function. That may be the reason why extension to the general case where n is > 4 appears cumbersome or unfeasible. This Note will show how Radau's kinematic reduction may be built entirely from vector constructions, and how, from that standpoint, it is easily made valid for any number $n \geqslant 3$ of particles.

The basic concept is that of a *chain of kinetic frames*, somewhat reminiscent of Jacobi's chain of barycenters. The position of a planet is determined by polar coordinates in a certain frame of the chain; the attitude of a frame is determined by its Eulerian angles in the frame preceding it in the chain. The first link in the chain is the invariable frame defined by the total angular momentum. Polar coordinates and Eulerian angles in the chain give rise to a transformation in phase space. Its

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canonical character is established by checking that it leaves invariant Cartan's 1-form in phase space. Explicit equations of such a Mathieu transformation are most readily obtained by multiplying unit quaternions. The point insisted on here is that the concept of a chain of kinetic frames, and the resulting canonical mapping is kinematic in essense; it is not tied to the Newtonian problem of n bodies, nor even for that matter to a dynamical system. Rather, it agrees with the general assumptions of the Reduction Theorem (see e.g. Abrahams and Marsden, 1978, pp. 298–306) applied to a family of n particles, assuming that the group S0(3) of rotations in $\mathbf{R}^3$ is responsible on a symplectic manifold $(\mathbf{R}^3 \times \mathbf{R}^3)^n$ for a Poisson action whose moment mapping in the sense of Souriau (1970) may be identified with the system's angular momentum. Applied to the problem of n bodies, Radau's generalized canonical mapping realizes indeed the intended reduction, without artificiality, simply by rendering ignorable the longitudes of two ascending nodes.

2. A Chain of Kinetic Frames

Consider a finite family of n mass points m_j; let $\mathbf{y}_j$ and $\mathbf{Y}_j$ stand respectively for the position and the linear momentum of m_j in an orthonormal frame $F = (\mathbf{i}, \mathbf{j}, \mathbf{k})$. For each $1 \leqslant j \leqslant n$, consider the angular momentum $\mathbf{C}_j = \mathbf{y}_j \times \mathbf{Y}_j$. Then introduce the sequence of partial sums

$$\mathbf{C}_j^* = \sum_{j+1 \leqslant k \leqslant n} \mathbf{C}_k \qquad (0 \leqslant j \leqslant n-1). \tag{1}$$

Depending on the context, the total angular momentum of the system will be designated either as the vector $\mathbf{C}$ or as the 'partial' sum $\mathbf{C}_0^*$. The following relationships

$$\mathbf{C}_j^* = \mathbf{C}_{j+1} + \mathbf{C}_{j+1}^* \qquad (0 \leqslant j \leqslant n-1), \tag{2}$$

$$\mathbf{C}_j^* \cdot (\mathbf{C}_{j+1} \times \mathbf{C}_{j+1}^*) = 0 \qquad (0 \leqslant j \leqslant n-1) \tag{3}$$

will be used throughout this note without being referenced explicitly.

On the assumption that $\mathbf{C}$ is not null, there exists a unique direction (or unit vector) $\mathbf{n}_0^*$ such that

$$\mathbf{C}_0^* = \Theta_0^* \mathbf{n}_0^* \quad \text{with} \quad \theta_0^* > 0. \tag{4}$$

Hence one can define unambiguously the angle I_0^* such that

$$\mathbf{k} \cdot \mathbf{n}_0^* = \cos I_0^* \quad \text{with} \quad 0 \leqslant I_0^* \leqslant \pi. \tag{5}$$

To simplify the exposition by ruling out special cases, assume that I_0^* mod π is $\neq 0$. Then there exists a unique direction $\mathbf{l}_0^*$ such that

$$\mathbf{k} \times \mathbf{n}_0^* = \mathbf{l}_0^* \sin I_0^*; \tag{6}$$

the direction $\mathbf{l}_0^*$ belongs to the coordinate plane $(\mathbf{i}, \mathbf{j})$ where its heading is given by an angle v_0^* such that

$$\mathbf{l}_0^* = \mathbf{i} \cos v_0^* + \mathbf{j} \sin v_0^* \quad \text{with} \quad 0 \leqslant v_0^* < 2\pi. \tag{7}$$

Finally, let $\mathbf{m}_0^* = \mathbf{n}_0^* \times \mathbf{l}_0^*$, and call F_0^* the orthonornal frame $(\mathbf{n}_0^*, \mathbf{l}_0^*, \mathbf{m}_0^*)$. In astronomy, $\mathbf{n}_0^*$ is termed the *invariable axis*, and the plane generated by the base vectors $\mathbf{l}_0^*$ and $\mathbf{m}_0^*$, the *invariable plane*. Thus I_0^* is the inclination of the invariable plane over the coordinate plane $(\mathbf{i}, \mathbf{j})$; $\mathbf{l}_0^*$ is the *ascending node* of the invariable plane in the coordinate plane $(\mathbf{i}, \mathbf{j})$, and v_0^* is the *longitude* of the ascending node. The invariable frame F_0^* is the root of a binary tree of kinetic frames.

The next nodes in the tree are defined by induction over $1 \leqslant j \leqslant n-1$. The recursive relationship among the frames in the tree is capsuled in Figure 1. Assuming that both $\mathbf{C}_j$ and $\mathbf{C}_j^*$ are not zero, one can determine directions $\mathbf{n}_j$ and $\mathbf{n}_j^*$ such that

$$\mathbf{C}_j = \Theta_j \mathbf{n}_j \quad \text{with} \quad \theta_j > 0, \tag{8}$$

$$\mathbf{C}_j^* = \Theta_j^* \mathbf{n}_j^* \quad \text{with} \quad \theta_j^* > 0, \tag{9}$$

hence angles I_j and I_j^* satisfying the conditions

$$\mathbf{n}_{j-1}^* \cdot \mathbf{n}_j = \cos I_j \quad \text{with} \quad 0 \leqslant I_j \leqslant \pi, \tag{10}$$

$$\mathbf{n}_{j-1}^* \cdot \mathbf{n}_j^* = \cos I_j^* \quad \text{with} \quad 0 \leqslant I_j^* \leqslant \pi. \tag{11}$$

On account of (3), the directions $\mathbf{n}_{j-1}^*$, $\mathbf{n}_j$, and $\mathbf{n}_j^*$ are coplanar; the planes normal respectively to $\mathbf{n}_j$ and $\mathbf{n}_j^*$ intersect along a line which is normal to $\mathbf{n}_{j-1}^*$, hence belongs to the plane normal to $\mathbf{n}_{j-1}^*$. More precisely, the vector $\mathbf{n}_j \times \mathbf{n}_j^*$ which is not generally

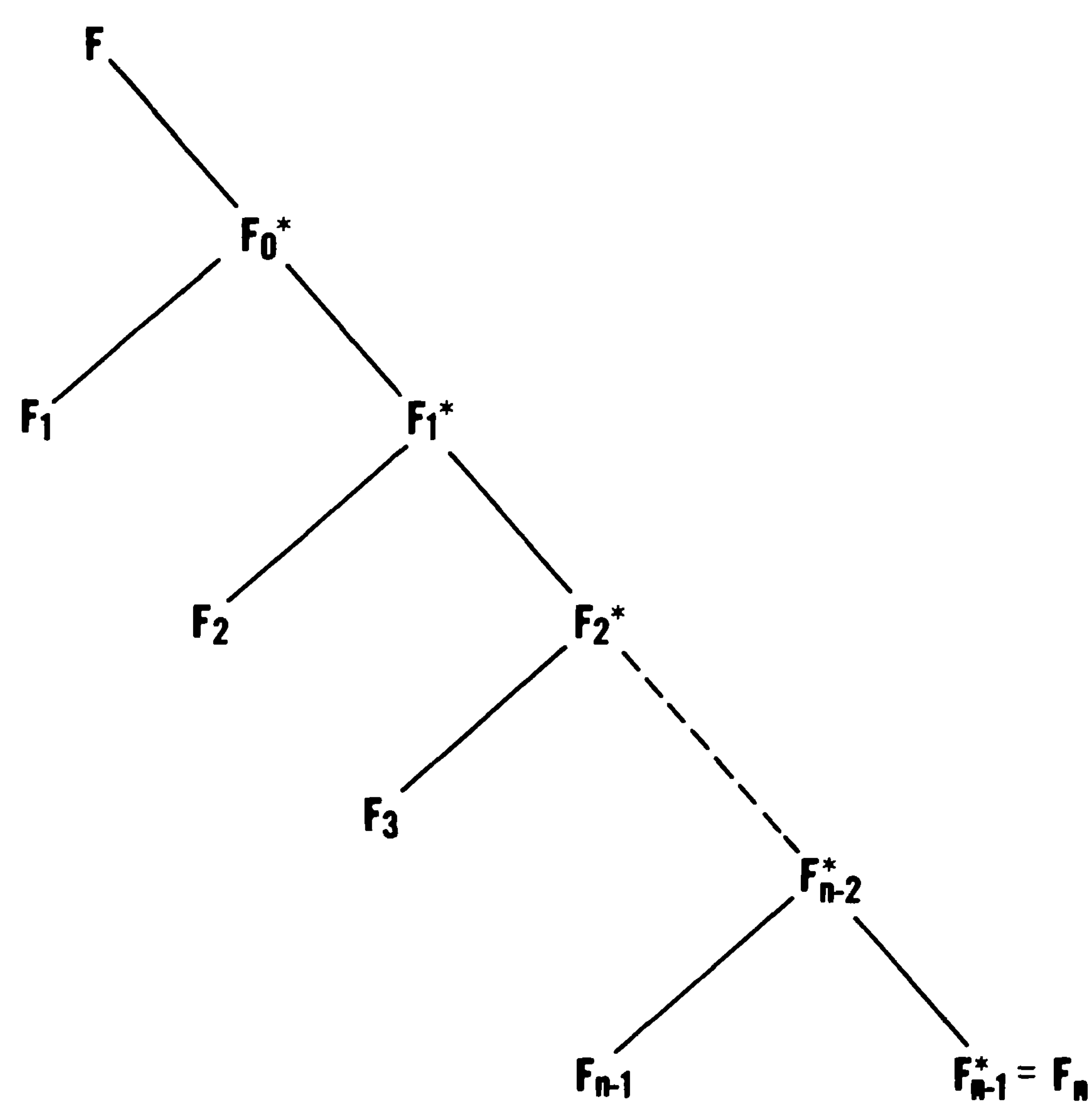

Fig. 1. The binary tree of kinetic frames.

a unit vector, points in the direction of the intersection common to the three normal planes. Thus, considering the directions $\mathbf{l}_j$ and $\mathbf{l}_j^*$ determined by the relations

$$\mathbf{n}_{j-1}^* \times \mathbf{n}_j = \mathbf{l}_j \sin I_j, \tag{12}$$

$$\mathbf{n}_{j-1}^* \times \mathbf{n}_j^* = \mathbf{l}_j^* \sin I_j^*, \tag{13}$$

one will find that $\mathbf{l}_j$ and $\mathbf{l}_j^*$ lie both in the direction of $\mathbf{n}_j \times \mathbf{n}_j^*$, but they are of opposite sense, with $\mathbf{l}_j$ in the sense of $\mathbf{n}_j^* \times \mathbf{n}_j$, and $\mathbf{l}_j^*$ in the sense of $\mathbf{n}_j \times \mathbf{n}_j^*$. In other words, $\mathbf{l}_j + \mathbf{l}_j^* = 0$. Hence, if v_j and θ_{j-1}^* are the angles such that

$$\mathbf{l}_j = \mathbf{l}_{j-1}^* \cos v_j + \mathbf{m}_{j-1}^* \sin v_j \quad \text{with} \quad 0 \leqslant v_j < 2\pi, \tag{14}$$

$$\mathbf{l}_j^* = \mathbf{l}_{j-1}^* \cos \theta_{j-1}^* + \mathbf{m}_{j-1}^* \sin \theta_{j-1}^* \quad \text{with} \quad 0 \leqslant \theta_{j-1}^* < 2\pi, \tag{15}$$

there follows that the longitudes v_j and θ_{j-1}^* are locked in phase:

$$(v_j - \theta_{j-1}^*) \bmod 2\pi = \pi. \tag{16}$$

With the directions

$$\mathbf{m}_j = \mathbf{n}_j \times \mathbf{l}_j \quad \text{and} \quad \mathbf{m}_j^* = \mathbf{n}_j^* \times \mathbf{l}_j^*, \tag{17}$$

one builds the next link in the chain of kinetic frames, namely the frame $F_j = (\mathbf{n}_j, \mathbf{l}_j, \mathbf{m}_j)$ and the frame $F_j^* = (\mathbf{n}_j^*, \mathbf{l}_j^*, \mathbf{m}_j^*)$.

The inductive construction terminates at $j = n - 1$ with the frames F_{n-1} and F_{n-1}^*. For convenience later on, F_{n-1}^* may also be designated as the frame F_n; this convention entails the following identifications

$$\mathbf{n}_n = \mathbf{n}_{n-1}^*, \qquad \mathbf{l}_n = \mathbf{l}_{n-1}^*, \qquad \mathbf{m}_n = \mathbf{m}_{n-1}^*,$$

$$v_n = \theta_{n-2}^*, \qquad I_n = I_{n-1}^*.$$

A few remarkable geometric properties ought to be mentioned for their kinematical consequences as well as their usefulness in analytical developments. When, in the decomposition of $\mathbf{n}_j^*$ in the frame F_{j-1}^*, namely

$$\mathbf{n}_j^* = \mathbf{n}_{j-1}^* \cos I_j^* - \sin I_j^* (\mathbf{m}_{j-1}^* \cos \theta_{j-1}^* - \mathbf{l}_{j-1}^* \sin \theta_{j-1}^*),$$

the base vectors are replaced by their linear combinations

$$\mathbf{n}_{j-1}^* = \mathbf{n}_j^* \cos I_j + \mathbf{m}_j \sin I_j,$$

$$\mathbf{l}_{j-1}^* = \mathbf{l}_j \cos v_j - (\mathbf{m}_j \cos I_j - \mathbf{n}_j \sin I_j) \sin v_j,$$

$$\mathbf{m}_{j-1}^* = \mathbf{l}_j \sin v_j + (\mathbf{m}_j \cos I_j - \mathbf{n}_j \sin I_j) \cos v_j$$

in the frame F_j, one obtains that

$$\mathbf{n}_j^* = \mathbf{n}_j \cos (I_j + I_j^*) + \mathbf{m}_j \sin (I_j + I_j^*),$$

which means that, if J_j stands for the angle between the normals $\mathbf{n}_j$ and $\mathbf{n}_j^*$ then, in counterpart to the phase lock (16),

$$J_j = I_j + I_j^*. \tag{18}$$

As a major consequence of this geometric property, it will be shown how the inclinations I_j and I_j^* are unambiguously determined by the triple $(\Theta_{j-1}^*, \Theta_j, \Theta_j^*)$ of angular momenta. Indeed, there results from (2) that

$$\mathbf{C}_{j-1}^* \cdot \mathbf{C}_{j-1}^* = \mathbf{C}_j \cdot \mathbf{C}_j + \mathbf{C}_j^* \cdot \mathbf{C}_j^* + 2\mathbf{C}_j \cdot \mathbf{C}_j^*,$$

which implies that

$$2\Theta_j \Theta_j^* \cos J_j = \Theta_{j-1}^{*2} - \Theta_j^2 - \Theta_j^{*2}. \tag{19}$$

In turn, the relations

$$\mathbf{n}_j \cdot \mathbf{C}_{j-1}^* = \mathbf{n}_j \cdot (\mathbf{C}_j + \mathbf{C}_j^*) \quad \text{and} \quad \mathbf{n}_j^* \cdot \mathbf{C}_{j-1}^* = \mathbf{n}_j^* \cdot (\mathbf{C}_j + \mathbf{C}_j^*)$$

yield the formulas $(1 \leqslant j \leqslant n)$

$$\cos I_j = \frac{\Theta_j + \Theta_j^* \cos J_j}{\Theta_{j-1}^*} = \frac{\Theta_{j-1}^{*2} + \Theta_j^2 - \Theta_j^{*2}}{2\Theta_{j-1}^* \Theta_j}, \tag{20}$$

$$\cos I_j^* = \frac{\Theta_j^* + \Theta_j \cos J_j}{\Theta_{j-1}^*} = \frac{\Theta_{j-1}^{*2} + \Theta_j^{*2} - \Theta_j^2}{2\Theta_{j-1}^* \Theta_j^*} \tag{21}$$

by which I_j and I_j^* are shown to be functions solely of Θ_{j-1}^*, Θ_j, and Θ_j^*.

For the sake of completeness, one should mention the analogous relations

$$\Theta_{j-1}^* \sin I_j = \Theta_j^* \sin J_j, \tag{22}$$

$$\Theta_{j-1}^* \sin I_j^* = \Theta_j \sin J_j, \tag{23}$$

which can be derived from the equalities of cross products

$$\mathbf{n}_j^* \times \mathbf{C}_{j-1}^* = \mathbf{n}_j^* \times (\mathbf{C}_j + \mathbf{C}_j^*) \quad \text{and} \quad \mathbf{n}_j \times \mathbf{C}_{j-1}^* = \mathbf{n}_j \times (\mathbf{C}_j + \mathbf{C}_j^*)$$

equivalent to the relationships

$$-\Theta_{j-1}^* \sin I_j^* \mathbf{I}_j^* = \Theta_j \sin I_j \mathbf{I}_j \quad \text{and} \quad -\Theta_{j-1}^* \sin I_j \mathbf{I}_j = \Theta_j^* \sin I_j^* \mathbf{I}_j^*.$$

It must be emphasized that identities of this kind, known for a long time in the Newtonian problem of three bodies, are valid for any $n \geqslant 2$. They are the key used by Boigey to interpret the canonical variables which she introduced for problems of four bodies in an analytical manner by *conditioned* canonical transformations (Boigey, 1977).

3. A Mathieu Transformation by Chained Rotations

The purpose of this section is to build a canonical mapping from the Cartesian components of the positions $\mathbf{y}_j$ and linear momenta $\mathbf{Y}_j (1 \leqslant j \leqslant n)$ in the original frame

F to coordinates and momenta determining these quantities in the frames F_j^* $(0 \leqslant j \leqslant \leqslant n-1)$ and $F_j (1 \leqslant j \leqslant n-1)$. The construction is merely geometric, requiring, as one goes down the chain of kinetic frames, that the Cartan 1-form

$$\omega = \sum_{1 \leqslant j \leqslant n} \mathbf{Y}_j \cdot \mathrm{d}\mathbf{y}_j \tag{24}$$

be kept invariant. This requirement will make the intended mapping a Mathieu transformation.

Let r_j and θ_j be the polar coordinates of the j-th particle in the frame F_j:

$$\mathbf{y}_j = r_j(\mathbf{l}_j \cos\theta_j + \mathbf{m}_j \sin\theta_j). \tag{25}$$

Consider then the 1-form

$$\delta\mathbf{y}_j = (\mathbf{l}_j \cos\theta_j + \mathbf{m}_j \sin\theta_j)\,\mathrm{d}r_j + r_j(-\mathbf{l}_j \sin\theta_j + \mathbf{m}_j \cos\theta_j)\,\mathrm{d}\theta_j \tag{26}$$

corresponding to a virtual displacement of the j-th particle in the frame F_j while ignoring the rotation of F_j with respect to the initial frame F. With the displacement (26) written in the form

$$\delta\mathbf{y}_j = \frac{1}{r_j}\mathbf{y}_j \delta r_j + (\mathbf{n}_j \times \mathbf{y}_j)\,\mathrm{d}\theta_j,$$

it is easy to obtain the inner product

$$\mathbf{Y}_j \cdot \delta\mathbf{y}_j = \frac{1}{r_j}(\mathbf{y}_j \cdot \mathbf{Y}_j)\,\mathrm{d}r_j + (\mathbf{n}_j \cdot \mathbf{C}_j)\,\mathrm{d}\theta_j.$$

Recalling (8), and introducing the momentum R_j such that

$$r_j R_j = \mathbf{y}_j \cdot \mathbf{Y}_j, \tag{27}$$

one finally arrives at the form

$$\mathbf{Y}_j \cdot \delta\mathbf{y}_j = R_j\,\mathrm{d}r_j + \Theta_j\,\mathrm{d}\theta_j \qquad (1 \leqslant j \leqslant n). \tag{28}$$

There remains to evaluate the contributions made to the Cartan form ω by the virtual rotations of the frames F_j.

According to Cartan's theorem of the moving frame, the infinitesimal generators of the rotations of F_j and F_j^* relative to the frame F_{j-1}^* correspond to the vector 1-forms

$$\mathrm{d}\omega_j = \mathbf{n}_{j-1}^*\,\mathrm{d}v_j + \mathbf{l}_j\,\mathrm{d}I_j \qquad (1 \leqslant j \leqslant n-1), \tag{29}$$

$$\mathrm{d}\omega_j^* = \mathbf{n}_{j-1}^*\,\mathrm{d}\theta_{j-1}^* + \mathbf{l}_j^*\,\mathrm{d}I_j^* \qquad (1 \leqslant j \leqslant n-1), \tag{30}$$

whereas the rotation of the invariable frame F_0^* relative to the coordinate frame F is associated with the vector 1-form

$$\mathrm{d}\omega_0 = \mathbf{k}\,\mathrm{d}v_0^* + \mathbf{l}_0^*\,\mathrm{d}I_0^*.$$

By construction of the chain of kinetic frames,

$$\mathrm{d}\mathbf{y}_j = \delta\mathbf{y}_j + (\mathrm{d}\boldsymbol{\omega}_j + \sum_{0 \leqslant k \leqslant j-1} \mathrm{d}\boldsymbol{\omega}_k^*) \times \mathbf{y}_j \qquad (1 \leqslant j \leqslant n-1),$$

$$\mathrm{d}\mathbf{y}_n = \delta\mathbf{y}_n + (\sum_{0 \leqslant k \leqslant j-1} \mathrm{d}\boldsymbol{\omega}_k^*) \times \mathbf{y}_n,$$

hence the Cartan 1-form may be decomposed into a sum

$$\omega = \omega' + \sum_{1 \leqslant j \leqslant n} (R_j\,\mathrm{d}r_j + \Theta_j\,\mathrm{d}\theta_j) \tag{31}$$

whose first term is the 1-form

$$\omega' = \sum_{1 \leqslant j \leqslant n-1} (\mathrm{d}\boldsymbol{\omega}_j + \sum_{0 \leqslant k \leqslant j-1} \mathrm{d}\boldsymbol{\omega}_k^*)\cdot\mathbf{C}_j + (\sum_{0 \leqslant k \leqslant n-1} \mathrm{d}\boldsymbol{\omega}_k^*)\cdot\mathbf{C}_n.$$

The intermediate 1-form ω' is simplified first by interverting the summations over j and k so as to obtain that

$$\omega' = \mathrm{d}\boldsymbol{\omega}_0^*\cdot\mathbf{C}_0^* + \sum_{1 \leqslant k \leqslant n-1} (\mathrm{d}\boldsymbol{\omega}_k^*\cdot\mathbf{C}_k^* + \mathrm{d}\boldsymbol{\omega}_k\cdot\mathbf{C}_k).$$

However, on account of (16), $\mathrm{d}v_k = \mathrm{d}\theta_k^*\,(1 \leqslant k \leqslant n-1)$; hence, in view of (29) and (30),

$$\omega' = (\mathbf{C}_0^*\cdot\mathbf{k})\,\mathrm{d}v_0^* + \sum_{1 \leqslant k \leqslant n-1} (\mathbf{C}_{k-1}^*\cdot\mathbf{n}_{k-1}^*)\,\mathrm{d}\theta_{k-1}^*.$$

At this stage is introduced the momentum

$$N_0^* = \mathbf{C}_0^*\cdot\mathbf{k} = \Theta_0^*\cos I_0^*, \tag{32}$$

which is the projection of the total angular momentum on the coordinate axis $\mathbf{k}$. In view of (9), ω' is then set in its definitive form as the expression

$$\omega' = N_0^*\,\mathrm{d}v_0^* + \sum_{0 \leqslant j \leqslant n-2} \Theta_j^*\,\mathrm{d}\theta_j^*,$$

which leads to the canonical type for Cartan's 1-form

$$\omega = \sum_{1 \leqslant j \leqslant n} (R_j\,\mathrm{d}r_j + \Theta_j\,\mathrm{d}\theta_j) + N_0^*\,\mathrm{d}v_0^* + \sum_{0 \leqslant j \leqslant n-2} \Theta_j^*\,\mathrm{d}\theta_j^*. \tag{33}$$

In sum, geometric constructions in the chain of kinetic frames suffice to define a Mathieu transformation from the Cartesian components in F of the family of pairs $(\mathbf{y}_j, \mathbf{Y}_j)$ into the $3n$ coordinates

$$r_1, r_2, \ldots, r_n, \theta_1, \theta_2, \ldots, \theta_n, v_0^*, \theta_0^*, \theta_1^*, \ldots, \theta_{n-2}^* \tag{34}$$

and their conjugate momenta

$$R_1, R_2, \ldots, R_n, \Theta_1, \Theta_2, \ldots, \Theta_n, N_0^*, \Theta_0^*, \Theta_1^*, \ldots, \Theta_{n-2}^*. \tag{35}$$

These are the variables introduced for $n = 2$ by Radau (1868) and Whittaker (1904) in problems of three bodies. For $n = 3$, they correspond to the elements proposed by Boigey (1981) in the problem of four bodies, except for the angle θ_0^*, and its conjugate momentum Θ_0^*, and with due consideration to the fact that, in this Note unlike in Boigey's dissertation, the angles θ_2 and θ_3 are reckoned from diametrically opposite directions.

4. Quaternions along the Chain of Kinetic Frames

Substituting the coordinates (34) and their conjugate momenta (35) in the Hamiltonian characterizing a problem of n bodies is usually an akward task. But the job is alleviated to a considerable extent if the finite rotations linking the frames in the kinetic chain are represented by unit quaternions. Doing away with spherical trigonometry, one composes rotations by multiplying their representative quaternions.

In order to establish the notations, let it be recalled that a quaternion is a real linear combination

$$q = \chi_0 q_0 + \chi_1 q_1 + \chi_2 q_2 + \chi_3 q_3$$

in a base whose elements satisfy the following relations

$$q_0^2 = -q_1^2 = -q_2^2 = -q_3^2 = 1,$$

$$q_0 q_i = q_i q_0 = q_i \qquad (1 \leqslant i \leqslant 3),$$

$$q_i q_j = \sum_{1 \leqslant k \leqslant 3} \varepsilon_{i,j,k} q_k,$$

$\varepsilon_{i,j,k}$ being the Levi–Civita symbol relative to the triple $(1, 2, 3)$, thus equal to $+1$ if (i, j, k) is an even permutation of $(1, 2, 3)$, -1 if (i, j, k) is an odd permutation, and zero otherwise. The combination

$$\bar{q} = \chi_0 q_0 - \chi_1 q_1 - \chi_2 q_2 - \chi_3 q_3$$

is called the *conjugate* of the quaternion q. The *norm* of a quaternion is defined as the real number $\|q\|$ such that

$$\|q\|^2 = q\bar{q} = \bar{q}q = \chi_0^2 + \chi_1^2 + \chi_2^2 + \chi_3^2.$$

A *unit* quaternion is a quaternion whose norm is equal to 1.

Consider two orthonormal frames $F = (\mathbf{i}, \mathbf{j}, \mathbf{k})$ and $F' = (\mathbf{i}', \mathbf{j}', \mathbf{k}')$, and let ρ be the finite rotation such that.

$$\rho(\mathbf{i}) = \mathbf{i}', \qquad \rho(\mathbf{j}) = \mathbf{j}', \qquad \rho(\mathbf{k}) = \mathbf{k}'.$$

The rotation ρ may be parametrized by the Eulerian angles of the frame F' with respect to the frame F. Thus, let θ be the angle satisfying the conditions

$$\mathbf{k} \cdot \mathbf{k}' = \cos\theta \quad \text{with} \quad 0 \leqslant \theta \leqslant \pi;$$

then consider a unit vector $\mathbf{l}$ such that $\mathbf{k} \times \mathbf{k}' = \mathbf{l} \sin \theta$. If $\theta \bmod \pi$ is zero, then $\mathbf{l}$ is an arbitrary direction in the plane $(\mathbf{i}, \mathbf{j})$; otherwise it is uniquely determined. In any case, let ϕ and ψ denote the angles satisfying the conditions

$$\mathbf{l} = \mathbf{i} \cos \phi + \mathbf{j} \sin \phi \qquad \text{with} \quad 0 \leqslant \phi < 2\pi,$$

$$\mathbf{l} = \mathbf{i}' \cos \psi - \mathbf{j}' \sin \psi \qquad \text{with} \quad 0 \leqslant \psi < 2\pi.$$

By means of the Eulerian angles (ϕ, θ, ψ) is defined the unit quaternion q whose components are

$$\chi_0 = \cos \tfrac{1}{2}\theta \cos \tfrac{1}{2}(\phi + \psi),$$

$$\chi_1 = \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}(\phi - \psi),$$

$$\chi_2 = \sin \tfrac{1}{2}\theta \sin \tfrac{1}{2}(\phi - \psi),$$

$$\chi_3 = \cos \tfrac{1}{2}\theta \sin \tfrac{1}{2}(\phi + \psi).$$

The components are chosen so that the axis of the finite rotation ρ has for direction the unit vector

$$(1 - \chi_0^2)^{1/2}(\chi_1 \mathbf{i} + \chi_2 \mathbf{j} + \chi_3 \mathbf{k}),$$

the amplitude $0 \leqslant \chi \leqslant \pi$ of the rotation being given by the relation

$$\chi_0 = \cos \tfrac{1}{2}\chi.$$

A quick calculation will show that

$$\begin{aligned}
\mathbf{i}'\cdot\mathbf{i} &= \chi_1^2 - \chi_2^2 - \chi_3^2 + \chi_0^2, & \mathbf{j}'\cdot\mathbf{i} &= 2(\chi_2\chi_1 - \chi_3\chi_0), & \mathbf{k}'\cdot\mathbf{i} &= 2(\chi_3\chi_1 + \chi_2\chi_0),\\
\mathbf{i}'\cdot\mathbf{j} &= 2(\chi_1\chi_2 + \chi_3\chi_0), & \mathbf{j}'\cdot\mathbf{j} &= \chi_2^2 - \chi_3^2 - \chi_1^2 + \chi_0^2, & \mathbf{k}'\cdot\mathbf{j} &= 2(\chi_3\chi_2 - \chi_1\chi_0),\\
\mathbf{i}'\cdot\mathbf{k} &= 2(\chi_1\chi_3 - \chi_2\chi_0), & \mathbf{j}'\cdot\mathbf{k} &= 2(\chi_2\chi_3 + \chi_1\chi_0), & \mathbf{k}'\cdot\mathbf{k} &= \chi_3^2 - \chi_1^2 - \chi_2^2 + \chi_0^2.
\end{aligned} \tag{36}$$

Now, for the vector

$$\mathbf{x} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} = x'\mathbf{i}' + y'\mathbf{j}' + z'\mathbf{k}',$$

create the quaternions

$$\xi = xq_1 + yq_2 + zq_3 \quad \text{and} \quad \xi' = x'q_1 + y'q_2 + z'q_3.$$

It is easy to verify from (36) that

$$\xi' = \bar{q}\xi q. \tag{37}$$

The basic relationship translates in quaternion multiplications the action of the rotation ρ on the three-dimensional vector space $\mathbf{R}^3$. In the sense of (37), the unit quaternion q is said to *represent* the rotation ρ. There follows immediately from (37) that $\bar{q}$ is the unit quaternion representing the inverse rotation ρ^{-1}; also if q_1 and q_2

at the top level (for $j = 1$) of the tree of kinetic frames, which implies that the one-parameter group of rotations about the invariable axis $\mathbf{n}_0^*$ constitutes the *isotropy* group responsible for the elimination of the longitude θ_0^* of the ascending node $\mathbf{l}_1^*$ on any manifold of constant angular momentum (see e.g. Abraham and Marsden, 1978, p. 392).

With the quaternions α_j^*, α_j, and β_j ($1 \leqslant j \leqslant n-1$), one can design an algorithm for computing the mutual distances $r_{j,k}$ ($1 \leqslant j < k \leqslant n$) in an inductive manner. If $\sigma_{j,k}$ designates the angle from position $\mathbf{y}_j$ to position $\mathbf{y}_k$, then

$$r_{j,k}^2 = (\mathbf{y}_j - \mathbf{y}_k)\cdot(\mathbf{y}_j - \mathbf{y}_k) = r_j^2 + r_k^2 - 2r_j r_k \cos\sigma_{j,k};$$

hence the problem of calculating the distances $r_{j,k}$ amounts to developing the cosine of their angular distances. But

$$\cos\sigma_{j,k} = (\mathbf{l}_j\cdot\mathbf{l}_k)\cos\theta_j\cos\theta_k + (\mathbf{l}_j\cdot\mathbf{m}_k)\cos\theta_j\sin\theta_k + $$
$$+ (\mathbf{m}_j\cdot\mathbf{l}_k)\sin\theta_j\cos\theta_k + (\mathbf{m}_j\cdot\mathbf{m}_k)\sin\theta_j\sin\theta_k \tag{38}$$

which means that the problem is solved once the quaternion has been obtained which represents the rotation from the frame F_j to the frame F_k. The construction begins at $j = n-1$ with the quaternion β_{n-1} for the rotation from F_{n-1} to $F_{n-1}^* = F_n$. In that case, by application of the formulas (36), one finds immediately that

$$\mathbf{l}_{n-1}^*\cdot\mathbf{l}_{n-1} = -1,$$

$$\mathbf{l}_{n-1}^*\cdot\mathbf{m}_{n-1} = \mathbf{l}_{n-1}\cdot\mathbf{m}_{n-1}^* = 0,$$

$$\mathbf{m}_{n-1}^*\cdot\mathbf{m}_{n-1} = -\cos J_{n-1},$$

hence the result, well known in the problem of three bodies,

$$\cos\sigma_{n-1,n} = -\cos\theta_{n-1}\cos\theta_n - \sin\theta_{n-1}\sin\theta_n\cos J_{n-1}. \tag{39}$$

Note however that both arguments of latitude θ_{n-1} and θ_n are reckoned from the ascending nodes of the orbital planes for particles m_{n-1} and m_n respectively, that is, from directions which are diametrically at the opposite of one another; this explains why, in contrast with the classical formula found in Whittaker (1904) and Boigey (1981), both terms in the right hand member of (39) appear with a minus sign.

In the next step of the induction are computed the quaternions

(i) $\beta_{n-2}\alpha_{n-1}$ for the rotation from the frame F_{n-2} to the frame F_{n-1},

(ii) $\beta_{n-2}\alpha_{n-1}^*$ for the rotation from F_{n-2} to F_{n-1}^*.

A straightforward calculation yields that

$$(\beta_{n-2}\alpha_{n-1})_0 = -\cos\tfrac{1}{2}(J_{n-2} - I_{n-1})\sin\tfrac{1}{2}v_{n-1},$$

$$(\beta_{n-2}\alpha_{n-1})_1 = \sin\tfrac{1}{2}(J_{n-2} - I_{n-1})\sin\tfrac{1}{2}v_{n-1},$$

$$(\beta_{n-2}\alpha_{n-1})_2 = \sin\tfrac{1}{2}(J_{n-2} + I_{n-1})\cos\tfrac{1}{2}v_{n-1},$$

$$(\beta_{n-2}\alpha_{n-1})_3 = \cos\tfrac{1}{2}(J_{n-2} + I_{n-1})\cos\tfrac{1}{2}v_{n-1};$$

it is worth observing that, in the style of XIXth century celestial mechanics, such relationships were obtained by chasing Delambre analogies around spherical triangles. With this quaternion, it is then easy to find the angles between base vectors in the frames F_{n-2} and F_{n-1} through the formulas

$$\begin{aligned}
\mathbf{l}_{n-1}\cdot\mathbf{l}_{n-2} &= -\cos v_{n-1},\\
\mathbf{l}_{n-1}\cdot\mathbf{m}_{n-2} &= -\sin v_{n-1}\cos J_{n-2},\\
\mathbf{m}_{n-1}\cdot\mathbf{l}_{n-2} &= \sin v_{n-1}\cos I_{n-1},\\
\mathbf{m}_{n-1}\cdot\mathbf{m}_{n-2} &= \sin J_{n-2}\sin I_{n-1} - \cos J_{n-2}\cos I_{n-1}\cos v_{n-1}.
\end{aligned}$$

Hence, in application of the general expression (38),

$$\begin{aligned}
\cos\sigma_{n-2,n-1} = {} & \cos\theta_{n-2}\cos\theta_{n-1}\cos\theta^*_{n-2} + \\
& + \sin\theta_{n-2}\cos\theta_{n-1}\sin\theta^*_{n-2}\cos J_{n-2} - \\
& - \cos\theta_{n-2}\sin\theta_{n-1}\sin\theta^*_{n-2}\cos I_{n-1} + \\
& + \sin\theta_{n-2}\sin\theta_{n-1}\sin I_{n-1}\sin J_{n-2} + \\
& + \sin\theta_{n-2}\sin\theta_{n-1}\cos\theta^*_{n-2}\cos I_{n-1}\cos J_{n-2}.
\end{aligned}\tag{40}$$

The same calculations are repeated for the rotation from F_{n-2} to F^*_{n-1}; so one obtains in turn the quaternion product

$$\begin{aligned}
(\beta_{n-2}\alpha^*_{n-1})_0 &= -\cos\tfrac{1}{2}(J_{n-2} - I^*_{n-1})\sin\tfrac{1}{2}\theta^*_{n-2},\\
(\beta_{n-2}\alpha^*_{n-1})_1 &= \sin\tfrac{1}{2}(J_{n-2} - I^*_{n-1})\sin\tfrac{1}{2}\theta^*_{n-2},\\
(\beta_{n-2}\alpha^*_{n-1})_2 &= \sin\tfrac{1}{2}(J_{n-2} + I^*_{n-1})\cos\tfrac{1}{2}\theta^*_{n-2},\\
(\beta_{n-2}\alpha^*_{n-1})_3 &= \cos\tfrac{1}{2}(J_{n-2} + I^*_{n-1})\cos\tfrac{1}{2}\theta^*_{n-2},
\end{aligned}$$

then, from directives in (36), the direction cosines

$$\begin{aligned}
\mathbf{l}^*_{n-1}\cdot\mathbf{l}_{n-2} &= -\cos\theta^*_{n-2},\\
\mathbf{l}^*_{n-1}\cdot\mathbf{m}_{n-2} &= -\sin\theta^*_{n-2}\cos J_{n-2},\\
\mathbf{m}^*_{n-1}\cdot\mathbf{l}_{n-2} &= \sin\theta^*_{n-2}\cos I^*_{n-1},\\
\mathbf{m}^*_{n-1}\cdot\mathbf{m}_{n-2} &= \sin J_{n-2}\sin I^*_{n-1} - \cos J_{n-2}\cos I^*_{n-1}\cos\theta^*_{n-2}.
\end{aligned}$$

and, finally, in application of (38),

$$\begin{aligned}
\cos\sigma_{n-2,n} = {} & -\cos\theta_{n-2}\cos\theta_n\cos\theta^*_{n-2} - \\
& - \sin\theta_{n-2}\cos\theta_n\sin\theta^*_{n-2}\cos J_{n-2} + \\
& + \cos\theta_{n-2}\sin\theta_n\sin\theta^*_{n-2}\cos I^*_{n-1} + \\
& + \sin\theta_{n-2}\sin\theta_n\sin I^*_{n-1}\sin J_{n-2} - \\
& - \sin\theta_{n-2}\sin\theta_n\cos\theta^*_{n-2}\cos I^*_{n-1}\cos J_{n-2}.
\end{aligned}\tag{41}$$

It is now clear how one should proceed in passing from a problem of four bodies to a problem of five bodies. The tree of kinetic frames is one level deeper; the additional work is done by traversing the tree from the leftmost node of the additional particle to the subtree of the problem of four bodies. Thus to the previous results, one would add the evaluations of $\cos \sigma_{n-3,n}$, $\cos \sigma_{n-3,n-1}$, and $\cos \sigma_{n-3,n-2}$ based on the quaternions:

(i) $\beta_{n-3}\alpha_{n-2}$ for the rotation from F_{n-3} to F_{n-2},
(ii) $\beta_{n-3}\alpha^*_{n-2}\alpha_{n-1}$ for the rotation from F_{n-3} to F_{n-1}, and
(iii) $\beta_{n-3}\alpha^*_{n-2}\alpha^*_{n-1}$ for the rotation from F_{n-3} to F_n.

The final expressions increase in complexity; it serves no purpose to enter the results in this Note.

A similar induction is applicable for developing scalar products of the type $\mathbf{Y}_j \cdot \mathbf{Y}_k$ since

$$Y_j \cdot Y_k = \left(R_j \cos \theta_j \mathbf{l}_j - \frac{\Theta_j}{r_j} \sin \theta_j \mathbf{m}_j \right) \cdot \left(R_k \cos \theta_k \mathbf{l}_k - \frac{\Theta_k}{r_k} \sin \theta_k \mathbf{m}_k \right).$$

A crucial conclusion of this analysis is that the longitudes v^*_0 and θ^*_0 of the ascending nodes $\mathbf{l}^*_0$ and $\mathbf{l}^*_1$ will be ignorable in an Hamiltonian that depends exclusively on the mutual distances $r_{j,k}$ or on the scalar products $\mathbf{Y}_j \cdot \mathbf{Y}_k$.

5. Elimination of the Nodes

In the coordinates (34) and their conjugate momenta (35), the reduction of a problem of n bodies which admits a vector integral of angular momentum is conditioned to reflect nicely the mathematical structures at play while adhering closely to the kinematical mechanisms at work. On the one hand, the system is invariant for the three-parameter group S0(3) of rotations about the origin of coordinates, and this geometric property implies in particular that the longitude v^*_0 of the ascending node $\mathbf{l}^*_0$ of the invariable plane is an ignorable coordinate in the Hamiltonian. On the other hand, on any manifold of constant angular momentum, the system is also invariant with respect to the one-parameter group S^1 of rotations around the invariable axis $\mathbf{n}^*_0$, which symmetry means that the longitude θ^*_0 of the node $\mathbf{l}^*_1 = -\mathbf{l}_1$ is ignorable in the Hamiltonian. From this standpoint, there is no need for arguing, as Whittaker does after Jacobi, that invariant relations based on the integral of angular momentum authorize giving a special value to a variable before taking partial derivatives.

Take for instance the Newtonian problem of $n+1$ ($n \geqslant 2$) bodies. To emphasize that the method of this Note applies to any form of the problem, rather than considering the Hamiltonian in chained barycentric coordinates as was done by Boigey (1979, 1981), one will start here from the heliocentric reduction with the Hamiltonian

$$H = \sum_{1 \leqslant j \leqslant n} \left[\frac{1}{2}\left(\frac{1}{m_0} + \frac{1}{m_1}\right) \mathbf{Y}_j \cdot \mathbf{Y}_j - G\frac{m_0 m_1}{r_j} \right] - $$
$$ - G \sum_{1 \leqslant j < k \leqslant n} \left(\frac{m_j m_k}{r_{j,k}} - \frac{1}{m_0} \mathbf{Y}_j \cdot \mathbf{Y}_k \right) \tag{42}$$

From the developments carried in Section 4, there follows that this Hamiltonian depends neither on the angles ν_0^* and θ_0^* nor on the momentum N_0^*, hence that N_0^*, Θ_0^* and ν_0^* are integrals of the problem, which implies also, by reason of (32), that the inclination I_0^* is a constant. But in the system of phase variables (34) and (35), $(\theta_0^*, \nu_0^*, I_0^*)$ are the spherical coordinates of the angular momentum. In the new variables, the canonical system based on (42) separates into a system of order $6n-4$ made of the equations $(1 \leqslant j \leqslant n)$

$$\dot{r}_j = \frac{\partial H}{\partial R_j}, \qquad \dot{\theta}_j = \frac{\partial H}{\partial \Theta_j}, \qquad \dot{R}_j = -\frac{\partial H}{\partial R_j}, \qquad \dot{\Theta}_j = -\frac{\partial H}{\partial \theta_j},$$

and of the equations $(1 \leqslant j \leqslant n-2)$

$$\dot{\theta}_j^* = \frac{\partial H}{\partial \Theta_j^*}, \qquad \dot{\Theta}_j^* = -\frac{\partial H}{\partial \theta_j},$$

to be followed by the quadratures indicated by the derivatives

$$\dot{\nu}_0^* = \frac{\partial H}{\partial N_0^*}, \quad \text{and} \quad \dot{\theta}_0^* = \frac{\partial H}{\partial \Theta_0^*}.$$

Whether the new phase variables (34) and (35) are practical in the General Theory of Perturbations is an open question. At least, for planetary theories, the answer is likely to be in the negative: the tree of kinetic frames imposes a recursive hierarchy without physical correspondance in the solar system. But finding a natural system of coordinates for eliminating the nodes in a planetary cluster was not the purpose of this Note. The intention was to show how the global symmetry with respect to the group S0(3) triggers a chain of partial rotations from one particle in the system to the next one, and why this chaining of rotations affords a suitable coordinate system leading without artificiality to the elimination of the nodes.

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References

Abraham, R. and Marsden J. E.: 1978, *Foundations of Mechanics*, The Benjamin/Cummings Publishing Company, Inc., Reading MA.

Bennett, T. L.: 1904, *Mess. Math.* (2) **34**, 113.

Boigey, F.: 1977, *Compt. Rend. Acad. Sci. Paris* **284A**, 911.

Boigey, F.: 1979, *Compt. Rend. Acad. Sci. Paris* **289A**, 295.

Boigey, F.: 1981, 'Transformations canoniques à variables imposées. Applications à la mécanique céleste', Thèse de Doctorat d'Etat, Université Pierre et Marie Curie (Paris VI), 168p.

Boigey, F.: 1982, *Celest. Mech.* **27**, 399.

Poincaré, H.: 1896, *Compt. Rend. Acad. Sci. Paris* **123**, 1030; reprinted in J. Lévi (ed.), *Oeuvres complètes de Poincaré*, **7**, (1952), Gauthier–Villars, Paris, pp. 496–491.

Poincaré, H.: 1897, *Bull. Astron.* **14**, 53; reprinted in *Acta Math.* **21**, 83; also in J. Lévi (ed.), *Oeuvres complètes de Poincaré* **7**, (1952), Gauthier–Villars, Paris, pp. 500–511.

Radau, R.: 1868, *Ann Scient. Ec. Norm. Sup.* (1) **5**, 321.

Souriau, J. M.: 1970, *Structures des systèmes dynamiques*, Dunod, Paris.

Whittaker, E. T.: 1904, *Analytical Dynamics of Particles and Rigid Bodies*, Cambridge University Press; reprinted by Dover, New York 1961.

Wintner, A.: 1947, *The Analytical Foundations of Celestial Mechanics*, Princeton University Press, Princeton – London – Oxford.